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Hypergeometric Solutions of Certain Definite Integrals

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Ref

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Hypergeometric Solutions of Certain Definite Integrals

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Abstract- We have developed certain definite integrals involving Hypergeometric functions. This integrals are new.

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I. Introduction

a) *Generalized Hypergeometric Function*

The generalized hypergeometric function of one variable [Prudnikov, p.437; see also E.D., p.73(2)] is defined as follows:

$$ {}_AF_B\left[\begin{matrix} a_1, a_2, \ldots.., a_A; \\ b_1, b_2, \ldots.., b_B; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots\ldots (a_A)_n}{(b_1)_n (b_2)_n \ldots\ldots (b_B)_n} \frac{z^n}{n!} $$

or
$$ {}_AF_B\left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{[(a_A)]_n}{[(b_B)]_n} \frac{z^n}{n!} \qquad (1.1) $$

where for the sake of convenience (in the contracted notation), (a_A) denotes the array of A number of parameters given by $a_1, a_2, \ldots., a_A$. The denominator parameters are neither zero nor negative integers. The numerator parameters may be zero and negative integers. A and B are positive integers or zero.

b) *Kampé De Fériet Double Hypergeometric Function*

In 1921 Appell's four double hypergeometric function[Appell, p.296(1); Bailey, p.73(1,2,3,4)] F_1 F_2 F_3 F_4 and their confluents forms[4] $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet [5].

We recall the definition of general double hypergeometric function of Kampé de Fériet [Appell, p.150(29)], in slightly modified notation of Srivastava and Panda [Srivastava and Panda, pp.423-424(26,27)]

$$ F^{A:B;D}_{E:G;H}\left[\begin{matrix} (a_A) : (b_B); (d_D); \\ (e_E) : (g_G); (h_H); \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m+n} [(b_B)]_m [(d_D)]_n}{[(e_E)]_{m+n} [(g_G)]_m [(h_H)]_n} \frac{x^m y^n}{m!\ n!} \qquad (1.2) $$

c) *Wright's Generalized Hypergeometric Function*

Wright's generalized hypergeometric function [Srivastava & Manocha, p.50(1.5.21), p.179(34 iii), p.395(23)], is defined by

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$$
{}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix}\,z\right]=\frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\ldots\Gamma(\alpha_p)}{\Gamma(\beta_1)\,\Gamma(\beta_2)\,\ldots\Gamma(\beta_q)}\times\;{}_p\Psi_q^*\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix}\,z\right] \tag{1.3}
$$

$$
=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1n)\,\Gamma(\alpha_2+A_2n)\,\ldots\Gamma(\alpha_p+A_pn)}{\Gamma(\beta_1+B_1n)\,\Gamma(\beta_2+B_2n)\,\ldots\Gamma(\beta_q+B_qn)}\frac{z^n}{n!} \tag{1.4}
$$

$$
{}_p\Psi_q^*\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p);\\(\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix}\,z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{A_1n}(\alpha_2)_{A_2n}\ldots(\alpha_p)_{A_pn}}{(\beta_1)_{B_1n}(\beta_2)_{B_2n}\ldots(\beta_q)_{B_qn}}\frac{z^n}{n!} \tag{1.5}
$$

Notes

II. Main Integrals

$$
\int_0^{\frac{\pi}{2}}\frac{\sin^{-1}(x\cos^4\phi)\mathrm{d}\phi}{\sqrt{(1-x^2\cos^6\phi)}}=\frac{3}{16}\pi x\;{}_6F_5\left[\begin{matrix}1,\,1,\,\frac{5}{8},\,\frac{7}{8},\,\frac{9}{8},\,\frac{11}{8}\;;\\ \frac{3}{2},\,\frac{3}{4},\,1,\,\frac{5}{4},\,\frac{3}{2}\;;\end{matrix}\;x^2\right] \tag{2.1}
$$

$$
\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\phi}{\sqrt{(1-x\cos^6\phi)}}=\frac{\pi}{2}\;{}_4F_3\left[\begin{matrix}\frac{1}{2},\,\frac{1}{6},\,\frac{1}{2},\,\frac{5}{6}\;;\\ 1,\,\frac{1}{3},\,\frac{2}{3}\;;\end{matrix}\;x\right] \tag{2.2}
$$

$$
\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta\mathrm{d}\phi}{\sqrt{(1-x\cos^4\theta)(1-x\cos^4\theta\cos^4\phi)}}=\frac{\pi^2}{4}F^{2:1;3}_{2:0;2}\left[\begin{matrix}\frac{1}{4},\,\frac{3}{4}:\frac{1}{2}\;;\frac{1}{2},\,\frac{1}{4},\,\frac{3}{4}\;;\\ \frac{1}{2},\,\frac{3}{2}:\,—;\,\frac{1}{2},\,\frac{3}{2}\;;\end{matrix}\;x,\,x\right] \tag{2.3}
$$

III. Use of Series Iteration Technique in Evaluation of Integrals

Derivation of (2.1)

$$
\int_0^{\frac{\pi}{2}}\frac{\sin^{-1}(x\cos^4\phi)\mathrm{d}\phi}{\sqrt{(1-x^2\cos^6\phi)}}=\int_0^{\frac{\pi}{2}}(x\cos^4\phi)\;{}_2F_1\left[\begin{matrix}1,\,1\;;\\ \frac{3}{2}\;;\end{matrix}\;x^2\cos^8\phi\right]\mathrm{d}\phi
$$

$$
=x\int_0^{\frac{\pi}{2}}\cos^4\phi\sum_{m=0}^{\infty}\frac{(1)_m(1)_m}{(\frac{3}{2})_m\;m!}(x^2\cos^8\phi)^m\mathrm{d}\phi
$$

$$
=x\sum_{m=0}^{\infty}\frac{(1)_m(1)_m}{(\frac{3}{2})_m\;m!}(x^2)^m\int_0^{\frac{\pi}{2}}\cos^{8m+4}\mathrm{d}\phi
$$

$$
=x\sum_{m=0}^{\infty}\frac{(1)_m(1)_m}{(\frac{3}{2})_m\;m!}(x^2)^m\frac{\Gamma(\frac{8m+5}{2})\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{8m+6}{2})}
$$

$$
=\frac{3\pi x}{16}\sum_{m=0}^{\infty}\frac{(1)_m(1)_m(\frac{5}{8})_m(\frac{7}{8})_m(\frac{9}{8})_m(\frac{11}{8})_m}{(\frac{3}{2})_m(\frac{3}{4})_m(1)_m(\frac{5}{4})_m(\frac{3}{2})_m}
$$

$$
=\frac{3}{16}\pi x\;{}_6F_5\left[\begin{matrix}1,\,1,\,\frac{5}{8},\,\frac{7}{8},\,\frac{9}{8},\,\frac{11}{8}\;;\\ \frac{3}{2},\,\frac{3}{4},\,1,\,\frac{5}{4},\,\frac{3}{2}\;;\end{matrix}\;x^2\right]
$$

Notes

Derivation of (2.2)

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1-x\cos^6\phi)}} = \int_0^{\frac{\pi}{2}} \left(1-x\cos^6\phi\right)^{-\frac{1}{2}} d\phi = \int_0^{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m x^m \cos^{6m}\phi}{m!} d\phi$$

$$= \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m x^m}{m!} \left(\frac{\Gamma(\frac{6m+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{6m+2}{2})} \right) = \frac{\pi}{2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{1}{6})_m (\frac{3}{6})_m (\frac{5}{6})_m x^m}{(\frac{1}{3})_m (\frac{2}{3})_m (1)_m m!}$$

$$= \frac{\pi}{2}\, {}_4F_3 \left[\begin{array}{ccc} \frac{1}{2}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6} & ; & \\ & & x \\ 1, \frac{1}{3}, \frac{2}{3} & ; & \end{array} \right]$$

Derivation of (2.3)

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta d\phi}{\sqrt{(1-x\cos^4\phi)(1-x\cos^4\theta\cos^4\phi)}}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (1-x\cos^4\theta)^{-\frac{1}{2}} (1-x\cos^4\phi\cos^4\theta)^{-\frac{1}{2}} d\theta d\phi$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{1}{2})_n \, x^{m+n}}{m!\; n!} \quad \int_0^{\frac{\pi}{2}} \cos^{4m+4n}\theta d\theta \Bigg) \Bigg(\int_0^{\frac{\pi}{2}} \cos^{4n}\phi d\phi \Bigg)$$

$$= \frac{\pi}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m (\frac{1}{2})_n}{m!\; n!} x^{m+n} \frac{\sqrt{\pi}(\frac{1}{2})_{2m+2n}}{(1)_{2m+2n}} \frac{\sqrt{\pi}(\frac{1}{2})_{2n}}{(1)_{2n}}$$

$$= \frac{\pi^2}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_m \, (\frac{1}{2})_n \, (\frac{1}{4})_{m+n} \, (\frac{3}{4})_{m+n} \, (\frac{1}{4})_m \, (\frac{3}{4})_n}{(\frac{1}{2})_{m+n} \, (\frac{3}{2})_{m+n} \, (\frac{1}{2})_n \, (\frac{3}{2})_n \; m! \; n1} x^m \; x^n$$

$$= \frac{\pi^2}{4} F^{2:1;3}_{2:0;2} \left[\begin{array}{ccc} \frac{1}{4}, \frac{3}{4} : \frac{1}{2} \; ; \frac{1}{2}, \frac{1}{4}, \frac{3}{4} & ; & \\ & & x, x \\ \frac{1}{2}, \frac{3}{2} : \text{—} ; \frac{1}{2}, \frac{3}{2} & ; & \end{array} \right]$$

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Notes