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Note on Intuitionistic Fuzzy (Normal) Subgroups or Vague (Normal) Subgroups

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K. Lakshmi [α] & Dr. G. Vasanti [σ]

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I. Introduction

Zadeh, in his pioneering paper, introduced the notion of Fuzzy Subset of a set X as a function μ from X to the closed interval [0,1] of real numbers. The function μ, he called, the membership function which assigns to each memebr x of X its membership value, μx in $[0,1]$.

In 1983, Atanassov[1] generalized the notion of Zadeh fuzzy subset of a set further by introducing an additional function ν which he called a nonmembership function with some natural conditions on μ and ν, calling these new generalized fuzzy subsets of a set, intutionistic fuzzy subsets. Thus according to him an intutionistic fuzzy subset of a set X, is a pair $A = (\mu_A, \nu_A)$, where μ_A, ν_A are functions from the set X to the closed interval $[0,1]$ of real numbers such that for each $x \in X$, $\mu x + \nu x \leq 1$, where μ_A is called the membership function of A and ν_A is called the nonmembership function of A.

Later on in 1984, Atanassov and Stoeva[3], further generalized the notion intuitionistic fuzzy subset to L-intuitionistic fuzzy subset, where L is any complete lattice with a complete order reversing involution N. Thus an L-intutionistic fuzzy subset A of a set X, is a pair (μ_A, ν_A) where μ_A, ν_A: $X \to L$ are such that $\mu_A \leq N\nu_A$. Let us recall that a complete order reversing involution is a map N: $L \to L$ such that (1) $N0_L = 1_L$ and $N1_L = 0_L$ (2) $\alpha \leq \beta$ implies $N\beta \leq N\alpha$ (3) $NN\alpha = \alpha$ (4) $N(\vee_{i \in I}\alpha_i) = \wedge_{i \in I}N\alpha_i$ and $N(\wedge_{i \in I}\alpha_i) = \vee_{i \in I}N\alpha_i$.

Interestingly the same notion of intutionistic fuzzy subset of set was also introduced by Gau and Buehrer[6] in 1993 under a different name called Vague subset. Thus whether we called intutionistic fuzzy subset of a set or if-subset of a set for short, or vague subset of a set, they are one and the same.

In order to make the document more readable, hereonwards we use the phrase if-subset for intuitionistic fuzzy or vague subset of a set. Obviously, if/v-subset only means intuitionistic fuzzy/vague subset, if/v-(normal)subgroup only means intuitionistic fuzzy/vague (normal) subgroup.

Coming to generalizations of algebraic structures on to the intuitionitic fuzzy/vague sets:

as early as 1989, Biswas[7] introduced the notion of if/v-subgroup of a group and studied some properties of the same.

Ref

1. Atanassov, K., Intuitionistic fuzzy sets, in: V. Sgurev, Ed., VII ITKR's Session, Sofia, June 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

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In 2004, Hur-Jang-Kang[15] introduced and studied if/v-normal subgroup of a group and Hur etal.[10,11,16] continued their studies of the same. In Hur etal.[16], they established a one-one correspondence between, if/v-normal subgroups and if/v-congrunces.

In 2003, Banergee-Basnet[6] introduced and studied the notions of if/v-subrings and if/v-ideals of a ring. The same year Hur-Jang-Kang[10] introduced and studied the notion if/v-subring of a ring. In Hur etal.[17,18] continued their studies of if/v-ideals. In Hur etal.[18], they introduced and studied the notions of if/v-prime ideals, if/v-completely prime ideals and if/v-weakly completely prime ideals.

Coming back to the studies of intuitionistic fuzzy/vague subgroups of a group, Feng[8] and Palaniappan etal.[22] initiated the study intuitionistic L-fuzzy/L-vague subgroups of a group.

In this paper we studied some propertities of intuitionistic fuzzy subgrups and intuitionistic fuzzy normal subgrups of an intuitionistic fuzzy subset.

For any set X, the set of all if/v-subsets of X be denoted by $A(X)$. By defining, for any pair of if/v-subsets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ of X, $A \leq B$ iff $\mu_A \leq \mu_B$ and $\nu_B \leq \nu_A$, $A(X)$ becomes a complete infinitely distributive lattice. In this case for any family $(A_i)_{i\in I}$ of if/v-subsets of X, $(\vee_{i\in I} A_i)x = \vee_{i\in I} A_i x$ and $(\wedge_{i\in I} A_i)x = \wedge_{i\in I} A_i x$.

For any set X, one can naturally associate, with X, the if/v-subset $(\mu_X, \nu_X) = (1_X, 0_X)$, where 1_X is the constant map assuming the value 1 for each $x \in X$ and 0_X is the constant map assuming the value 0 for each $x \in X$, which turns out to be the largest element in $A(X)$. Observe that then, the if/v-empty subset ϕ of X gets naturally associated with the if/v-subset $(\mu_\phi, \nu_\phi) = (0_X, 1_X)$, which turns out to be the least element in $A(X)$.

Let $A = (\mu_A, \nu_A)$ be an if/v-subset of X. Then the if/v-complement of A, denoted by A^c is defined by (ν_A, μ_A). Observe that $A^c = X - A = X \wedge A^c$.

Throughout this paper the capital letters X, Y Z stand for arbitrary but fixed (crisp) sets, the small letters f, g stand for arbitrary but fixed (crisp) maps $f : X \to Y$ and $g : Y \to Z$, the capital letters A, B, C, D, E, F together with their suffixes stand for if/v-subsets and the capital letters I and J stand for the index sets. Ingeneral whenever P is an if-subset of a set X, *always* μ_P and ν_P denote the membership and nonmembership function of the if-subset P respectively. Also we frequently use the standard convention that $\vee\phi = 0$ and $\wedge\phi = 1$.

II. Intuitionistic Fuzzy/Vague-subgroups

In this section, first we give some definitions and statements. In the Lemma that follows this, we give equivalent statements which are quite frequently used in several prepositions later on without an explicit mention. Then analogues of some crisp theoretic results are established. In the end, Lagranges theorem is generalized to fuzzy setup.

Definitions and Statements 2.1 *(a) Let A, B be a pair of if/v-subsets of G. Let C be defined by, $\mu_C x = \vee_{x=yz}\{\mu_A y \wedge \mu_B z\}$ and $\nu_C x = \wedge_{x=yz}\{\nu_A y \vee \nu_B z\}$, for each $x \in G$. Then the if/v-subset C of G is called the if/v-product of A by B and is denoted by $A \circ B$.*

(b) For any if/v-subset A of G, the if/v-inverse of A, denoted by A^{-1}, defined by $(\mu_{A^{-1}}, \nu_{A^{-1}})$ is in fact an if/v-subset of G, where for each $x \in G$ $\mu_{A^{-1}}(x) = \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1})$.

(c) For any $y \in G$ and for any pair α, β of [0, 1], the if/v-point of G, denoted by $y_{\alpha,\beta}$, is defined by the if/v-subset $y_{\alpha,\beta} = (\chi_y^\alpha, \chi_y^\beta)$ where $\chi_y^\alpha(x) = \alpha$, $\chi_y^\beta(x) = \beta$ when $x = y$ and $\chi_y^\alpha(x) = \chi_y^\beta(x) = 0$ when $x \neq y$.

(d) An if/v-subset A of G is called an if/v-subgroup of G iff:

(1) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$, for each $x, y \in G$.

(2) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$, for each $x \in G$.

Ref

6. Banerjee B. and Basnet D.Kr., Intuitionistic fuzzy subrings and ideals, Journal of Fuzzy Mathematics Vol.11(1)(2003), 139-155.

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(e) For any if/v-subgroup A of a group G, $A_* = \{x \in G/\mu_A(x) = \mu_A(e)$ *and* $\nu_A(x) = \nu_A(e)\}$ *and* $A^* = \{x \in G/\mu_A(x) > 0$ *and* $\nu_A(x) < 1\}$.
(f) For any if/v-subset A of G and for any $\alpha, \beta \in [0,1]$*, the* (α,β)*-level subset of A, denoted by* $A_{\alpha,\beta}$*, is defined by* $A_{\alpha,\beta} = \{g \in G/\mu_A g \geq \alpha, \nu_A g \leq \beta\}$.

The following Lemma, which provides alternative equivalent statements for some of the above definitions and statements, is quite useful and is frequently used without an explicit mention of it in several proofs in later chapters.

Notes

Lemma 2.2 *Let* $A, B, (A_i)_{i\in I}$ *be if/v-subsets of a group G. Let* $\alpha = \vee\mu_A G$, $\beta = \wedge\nu_A G$, $y_{\alpha,\beta} = (\chi_y^\alpha, \chi_y^\beta)$. *Then the following are true:*

1. $(\mu_{A\circ B})(x) = \vee_{y\in G}(\mu_A(y) \wedge \mu_B(y^{-1}x)) = \vee_{y\in G}(\mu_A(y^{-1}) \wedge \mu_B(yx))$ $= \vee_{y\in G}(\mu_A(xy^{-1}) \wedge\mu_B(y)) = \vee_{y\in G}(\mu_A(xy) \wedge \mu_B(y^{-1}))$ *and* $(\nu_{A\circ B})(x) = \wedge_{y\in G}(\nu_A(y) \vee \nu_B(y^{-1}x)) = \wedge_{y\in G}(\nu_A(y^{-1}) \vee \nu_B(yx))$ $= \wedge_{y\in G}(\nu_A(x y^{-1}) \vee \nu_B(y)) = \wedge_{y\in G}(\nu_A(xy) \vee \nu_B(y^{-1}))$, *for each* $x \in G$. *In particular,* $(\mu_{A\circ B})(xy) = \vee_{z\in G}(\mu_A(xz) \wedge \mu_B(z^{-1}y)) = \vee_{z\in G}(\mu_A(xz^{-1}) \wedge\mu_B(zy))$ *and* $(\nu_{A\circ B})(xy) = \wedge_{z\in G}(\nu_A(xz) \vee \nu_B(z^{-1}y)) = \wedge_{z\in G}(\nu_A(xz^{-1}) \vee\nu_B(zy))$.
2. $A \circ (B \circ C) = (A \circ B) \circ C$.
3. $y_{\alpha,\beta} \circ A = (\chi_y^\alpha \circ \mu_A, \chi_y^\beta \circ \nu_A)$, $(\chi_y^\alpha \circ \mu_A)x = \mu_A(y^{-1}x)$ *and* $(\chi_y^\beta \circ \nu_A)x = \nu_A(y^{-1}x)$, *for each* $x, y \in G$.
   In particular $e_{\alpha,\beta} \circ A = A$.
4. $A \circ y_{\alpha,\beta} = (\mu_A \circ \chi_y^\alpha, \nu_A \circ \chi_y^\beta)$, $(\mu_A \circ \chi_y^\alpha)x = \mu_A(xy^{-1})$ *and* $(\nu_A \circ \chi_y^\beta)x = \nu_A(xy^{-1})$, *for each* $x, y \in G$. *In particular* $A \circ e_{\alpha,\beta} = A$.
5. $(A^{-1})^{-1} = A$;
6. $A \leq A^{-1}$ *iff* $A^{-1} \leq A$ *iff* $A = A^{-1}$;
7. $A \leq B$ *iff* $A^{-1} \leq B^{-1}$;
8. $(\vee_{i\in I}A_i)^{-1} = \vee_{i\in I}A_i^{-1}$;
9. $(\wedge_{i\in I}A_i)^{-1} = \wedge_{i\in I}A_i^{-1}$;
10. $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$;
11. $g_{\alpha,\beta} \circ h_{\gamma,\delta} = (gh)_{\alpha\wedge\gamma,\beta\vee\delta}$.

Proof: (1): Since G is a group and hence for each $x \in G$, $\{(a, b) \in G\times G/x = ab\} = \{(a, a^{-1}x) \in G \times G/a \in G\} = \{(a^{-1}, ax) \in G \times G/a \in G\} = \{(xb^{-1}, b) \in G \times G/b \in G\} = \{(xb, b^{-1}) \in G \times G/b \in G\}$, this assertion follows.
(2): $\mu_{A\circ(B\circ C)}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_{(B\circ C)}(y)) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge (\vee_{z\in G} (\mu_B(yz^{-1}) \wedge \mu_C z))) = \vee_{y\in G} \vee_{z\in G} (\mu_A(xy^{-1}) \wedge \mu_B(yz^{-1}) \wedge \mu_C(z))$ and
$\mu_{(A\circ B)\circ C}(x) = \vee_{z\in G}(\mu_{(A\circ B)}(xz^{-1}) \wedge \mu_C z) = \vee_{z\in G}(\vee_{y\in G}(\mu_A(xy^{-1})\wedge\mu_B(yz^{-1})) \wedge\mu_C(z)) = \vee_{z\in G} \vee_{y\in G} (\mu_A(xy^{-1}) \wedge \mu_B(yz^{-1}) \wedge \mu_C(z))$, since $\alpha_i \wedge (\vee_{j\in J}\beta_j) = \vee_{j\in J}(\alpha_i \wedge \beta_j)$, when [0,1] is a complete infinite meet distributive lattice. Hence $\mu_{A\circ(B\circ C)}(x) = \mu_{(A\circ B)\circ C}(x)$.
Similarly, $\nu_{A\circ(B\circ C)}(x) = \wedge_{y\in G}(\nu_A(xy^{-1})\vee\nu_{(B\circ C)}(y)) = \wedge_{y\in G}(\nu_A(xy^{-1})\vee(\wedge_{z\in G} (\nu_B\ (yz^{-1}) \vee\nu_C z))) = \wedge_{y\in G}\wedge_{z\in G}(\nu_A(xy^{-1})\vee\nu_B(yz^{-1})\vee\nu_C(z)) = \nu_{(A\circ B)\circ C}(x)$, since $\alpha_i \vee (\wedge_{j\in J}\beta_j) = \wedge_{j\in J}(\alpha_i \vee \beta_j)$, when [0, 1] is a complete infinite join distributive lattice. Therefore $A \circ (B \circ C) = (A \circ B) \circ C$.
(3): $(\chi_y^\alpha \circ \mu_A)x = \vee_{x=ba}(\chi_y^\alpha(b) \wedge \mu_A(a)) = \vee_{b\in G}(\chi_y^\alpha(b) \wedge \mu_A(b^{-1}x)) = \alpha \wedge \mu_A(y^{-1}x) = (\vee\mu_A G) \wedge (\mu_A(y^{-1}x)) = \mu_A(y^{-1}x)$.
Similarly $(\chi_y^\beta \circ \nu_A)x = \wedge_{x=ba}(\chi_y^\beta(b) \vee \nu_A(a)) = \wedge_{b\in G}(\chi_y^\beta(b) \vee \nu_A(b^{-1}x)) = \beta \vee \nu_A(y^{-1}x) = (\wedge\nu_A G) \vee (\nu_A(y^{-1}x)) = \nu_A(y^{-1}x)$.
Letting $y = e$, $e_{\alpha,\beta} \circ A = A$.

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(4): $(\mu_A \circ \chi_y^\alpha)x = \vee_{x=ab}(\mu_A(a) \wedge \chi_y^\alpha(b)) = \vee_{b\in G}(\mu_A(xb^{-1}) \wedge \chi_y^\alpha(b)) = \mu_A(xy^{-1}) \wedge \alpha = \mu_A(xy^{-1}) \wedge (\vee \mu_A G) = \mu_A(xy^{-1})$.
Similarly $(\nu_A \circ \chi_y^\beta)x = \wedge_{x=ab}(\nu_A(a) \vee \chi_y^\beta(b)) = \wedge_{b\in G}(\nu_A(xb^{-1}) \vee \chi_y^\beta(b)) = (\nu_A(xy^{-1})) \vee \beta = (\nu_A(xy^{-1})) \vee (\wedge \nu_A G) = \nu_A(xy^{-1})$.
Letting $y = e$, $A \circ e_{\alpha,\beta} = A$.
(5): For each $x \in G$, $\mu_{A^{-1}}(x) = \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1})$. $\mu_{(A^{-1})^{-1}}(x) = \mu_{A^{-1}}(x^{-1}) = \mu_A(x^{-1})^{-1} = \mu_A x$ and $\nu_{(A^{-1})^{-1}}(x) = \nu_{A^{-1}}(x^{-1}) = \nu_A(x^{-1})^{-1} = \nu_A x$. Hence $(A^{-1})^{-1} = A$.
(6): Let $A \leq A^{-1}$. Then for each $x \in G$, $\mu_A(x) \leq \mu_{A^{-1}}(x) = \mu_A(x^{-1})$ and $\nu_A(x) \geq \nu_{A^{-1}}(x) = \nu_A(x^{-1})$. Hence $\mu_{A^{-1}}(x^{-1}) = \mu_A x \leq \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x^{-1}) = \nu_A x \geq \nu_A(x^{-1})$ implies $\mu_{A^{-1}} \leq \mu_A$ and $\nu_{A^{-1}} \geq \nu_A$ or $A^{-1} \leq A$. Thus $A \leq A^{-1}$ implies $A^{-1} \leq A$.
Similarly $A^{-1} \leq A$ implies for each $x \in G$, $\mu_{A^{-1}}(x^{-1}) \leq \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x^{-1}) \geq \nu_A(x^{-1})$ which implies $\mu_A(x) = \mu_{A^{-1}}(x^{-1}) \leq \mu_A(x^{-1}) = \mu_{A^{-1}}(x)$ and $\nu_A(x) = \nu_{A^{-1}}(x^{-1}) \geq \nu_A(x^{-1}) = \nu_{A^{-1}}(x)$. or $A \leq A^{-1}$. Thus $A^{-1} \leq A$ implies $A \leq A^{-1}$. Now $A \leq A^{-1}$ iff $A^{-1} \leq A$ iff $A = A^{-1}$ is clear.
(7): ($\Rightarrow$): Let $A \leq B$. Then for each $x \in G$, $\mu_A(x^{-1}) \leq \mu_B(x^{-1})$ and $\nu_A(x^{-1}) \geq \nu_B(x^{-1})$. Hence $\mu_{A^{-1}}(x) = \mu_A(x^{-1}) \leq \mu_B(x^{-1}) = \mu_{B^{-1}}(x)$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1}) \geq \nu_B(x^{-1}) = \nu_{B^{-1}}(x)$ or $A^{-1} \leq B^{-1}$.
($\Leftarrow$): Let $A^{-1} \leq B^{-1}$. Then for each $x \in G$, $\mu_{A^{-1}}(x) \leq \mu_{B^{-1}}(x)$ and $\nu_{A^{-1}}(x) \geq \nu_{B^{-1}}(x)$. Hence $\mu_A(x^{-1}) = \mu_{A^{-1}}(x) \leq \mu_{B^{-1}}(x) = \mu_B(x^{-1})$ and $\nu_A(x^{-1}) = \nu_{A^{-1}}(x) \geq \nu_{B^{-1}}(x) = \nu_B(x^{-1})$ or $A \leq B$.
(8): Let $A_i = (\mu_{A_i}, \nu_{A_i})$, $A_i^{-1} = (\mu_{A_i^{-1}}, \nu_{A_i^{-1}})$. Then for each $x \in G$, $(\vee_{i\in I}\mu_{A_i})^{-1}(x) = (\vee_{i\in I}\mu_{A_i})(x^{-1}) = \vee_{i\in I}\mu_{A_i}(x^{-1}) = \vee_{i\in I}\mu_{A_i^{-1}}(x) = (\vee_{i\in I}\mu_{A_i^{-1}})(x)$ and $(\wedge_{i\in I}\nu_{A_i})^{-1}(x) = (\wedge_{i\in I}\nu_{A_i})(x^{-1}) = \wedge_{i\in I}\nu_{A_i}\ (x^{-1}) = \wedge_{i\in I}\nu_{A_i^{-1}}(x) = (\wedge_{i\in I}\ \nu_{A_i^{-1}})(x)$.
Hence $(\vee_{i\in I}A_i)^{-1} = \vee_{i\in I}A_i^{-1}$.
(9): Let $A_i = (\mu_{A_i}, \nu_{A_i})$, $A_i^{-1} = (\mu_{A_i^{-1}}, \nu_{A_i^{-1}})$. Then for each $x \in G$, $(\wedge_{i\in I}\mu_{A_i})^{-1}(x) = (\wedge_{i\in I}\mu_{A_i})(x^{-1}) = \wedge_{i\in I}\mu_{A_i}(x^{-1}) = \wedge_{i\in I}\mu_{A_i^{-1}}(x) = (\wedge_{i\in I}\ \mu_{A_i^{-1}})(x)$ and $(\vee_{i\in I}\nu_{A_i})^{-1}(x) = (\vee_{i\in I}\nu_{A_i})(x^{-1}) = \vee_{i\in I}\nu_{A_i}\ (x^{-1}) = \vee_{i\in I}\nu_{A_i^{-1}}(x) = (\vee_{i\in I}\ \nu_{A_i^{-1}})(x)$.
Hence $(\wedge_{i\in I}A_i)^{-1} = \wedge_{i\in I}A_i^{-1}$.
(10): Let $(A \circ B)^{-1} = (\mu_{(A\circ B)^{-1}}, \nu_{(A\circ B)^{-1}})$, $B^{-1} \circ A^{-1} = (\mu_{B^{-1}\circ A^{-1}}, \nu_{B^{-1}\circ A^{-1}})$.
Then for each $x \in G$, $\mu_{(A\circ B)^{-1}}(x) = \mu_{A\circ B}(x^{-1}) = \vee_{y\in G}(\mu_A(x^{-1}y) \wedge \mu_B(y^{-1}))$ and $\nu_{(A\circ B)^{-1}}(x) = \nu_{A\circ B}(x^{-1}) = \wedge_{y\in G}\ (\nu_A(x^{-1}y) \vee \nu_B(y^{-1}))$.
On the other hand, $\mu_{(B^{-1}\circ A^{-1})}(x) = \vee_{y\in G}(\mu_{B^{-1}}(y) \wedge \mu_{A^{-1}}(y^{-1}x))$
$=\vee_{y\in G}(\mu_B\ (y^{-1}) \wedge \mu_A(y^{-1}x)^{-1}) = \vee_{y\in G}(\mu_B(y^{-1}) \wedge \mu_A(x^{-1}y))$
$= \vee_{y\in G}(\mu_A(x^{-1}y) \wedge \mu_B(y^{-1})) = \mu_{(A\circ B)}(x^{-1}) = \mu_{(A\circ B)^{-1}}(x)$ and
$(\nu_{B^{-1}\circ A^{-1}})(x) = \wedge_{y\in G}(\nu_{B^{-1}}(y) \vee \nu_{A^{-1}}(y^{-1}x)) = \wedge_{y\in G}(\nu_B(y^{-1}) \vee \nu_A(y^{-1}x)^{-1})$
$= \wedge_{y\in G}(\nu_B(y^{-1}) \vee \nu_A(x^{-1}y)) = \wedge_{y\in G}(\nu_A(x^{-1}y) \vee \nu_B(y^{-1})) = \nu_{(A\circ B)}(x^{-1}) = \nu_{(A\circ B)^{-1}}(x)$. Therefore $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$.
(11): $(\chi_g^\alpha \circ \chi_h^\gamma)(x) = \vee_{z\in G}(\chi_g^\alpha(xz^{-1}) \wedge \chi_h^\gamma(z)) = \chi_g^\alpha(xh^{-1}) \wedge \gamma = \chi_{gh}^\alpha(x) \wedge \gamma = \alpha \wedge \gamma = \chi_{gh}^{\alpha\wedge\gamma}(x)$ where the third equality follows because $g = xh^{-1}$ or $gh = x$ and $\chi_g^\beta \circ \chi_h^\delta(x) = \wedge_{z\in G}(\chi_g^\beta(xz^{-1}) \vee \chi_h^\delta(z)) = \chi_g^\beta(xh^{-1}) \vee \delta = \chi_{gh}^\beta(x) \vee \delta = \beta \vee \delta = \chi_{gh}^{\beta\vee\delta}(x)$.
Hence $g_{\alpha,\beta} \circ h_{\gamma,\delta} = (gh)_{\alpha\wedge\gamma,\beta\vee\delta}$.

Lemma 2.3 For any if/v-subset A of a group G such that $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$:
(1) $\mu_A(x^n) \geq \mu_A(x)$ (2) $\nu_A(x^n) \leq \nu_A(x)$
for each $x \in G$ and $n \in N$.

Proof: (1): $\mu_A(x^n) = \mu_A(x^{n-1}x) \geq \mu_A(x^{n-1}) \wedge \mu_A(x) \geq \mu_A(x) \wedge \mu_A(x) ... \wedge \mu_A(x) = \mu_A(x)$ for each $x \in G$.
(2): $\nu_A(x^n) = \nu_A(x^{n-1}x) \leq \nu_A(x^{n-1}) \vee \nu_A(x) \leq \nu_A(x) \vee \nu_A(x) ... \vee \nu_A(x) = \nu_A(x)$ for each $x \in G$.

Notes

Notes

Lemma 2.4 *Whenever A is an if/v-subgroup of a group G, for each $x \in G$, $\mu_A(x^{-1}) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(x)$.*

Proof: Let A be an if/v-subgroup of G. Then for each $x \in G$, $\mu_A(x^{-1}) \geq \mu_A(x)$, $\nu_A(x^{-1}) \leq \nu_A(x)$. $\mu_A(x) = \mu_A((x^{-1})^{-1}) \geq \mu_A(x^{-1})$ and $\nu_A(x) = \nu_A((x^{-1})^{-1}) \leq \nu_A(x^{-1})$.
Hence $\mu_A(x^{-1}) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(x)$.

Corollary 2.5 *For any if/v-subgroup A of a group G, the following are true for each $x \in G$:*

1. *$\mu_A(e) \geq \mu_A(x)$ and $\nu_A(e) \leq \nu_A(x)$;*
2. *$\mu_{A\circ A} \geq \mu_A$ and $\nu_{A\circ A} \leq \nu_A$.*

Proof: (1): $\mu_A(e) = \mu_A(xx^{-1}) \geq \mu_A(x) \wedge \mu_A(x^{-1}) = \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(e) = \nu_A(xx^{-1}) \leq \nu_A(x) \vee \nu_A(x^{-1}) = \nu_A(x) \vee \nu_A(x) = \nu_A(x)$.
(2): $\mu_{A\circ A}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_A(y)) \geq \mu_A(xe) \wedge \mu_A(e) \geq \mu_A(x)$ and $\nu_{A\circ A}(x) = \wedge_{y\in G}(\nu_A(xy^{-1}) \vee \nu_A(y)) \leq \nu_A(xe) \vee \nu_A(e) \leq \nu_A x$ for each $x \in G$.

Lemma 2.6 *For any if/v-subset A of a group G, A is an if/v-subgroup iff $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$.*

Proof: ($\Rightarrow$): Suppose A is an if/v-subgroup. Then by 2.4, $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y^{-1}) = \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y^{-1}) = \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$.
($\Leftarrow$): First, by hypothesis and 2.5(1), $\mu_A(x^{-1}) = \mu_A(ex^{-1}) \geq \mu_A(e) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(ex^{-1}) \leq \nu_A(e) \vee \nu_A(x) = \nu_A x$ for each $x \in G$.

Letting x^{-1} inplace of x, $\mu_A(x) \geq \mu_A(x^{-1})$ and $\nu_A(x) \leq \nu_A(x^{-1})$ for each $x \in G$ or $\mu_A(x) = \mu_A(x^{-1})$ and $\nu_A(x) = \nu_A(x^{-1})$ for each $x \in G$.
Next, $\mu_A(xy) = \mu_A(x(y^{-1})^{-1}) \geq \mu_A(x) \wedge \mu_A(y^{-1}) = \mu_A(x) \wedge \mu_A(y)$.
Similarly $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. Therefore A is an if/v-subgroup of G.

Lemma 2.7 *For any if/v-subgroup A of a group G,*

1. *$A_* = \{x \in G/\mu_A(x) = \mu_A(e),\ \nu_A(x) = \nu_A(e)\}$ is a subgroup of G;*
2. *$A^* = \{x \in G/\mu_A(x) > 0,\ \nu_A(x) < 1\}$ is a subgroup of G whenever L is strongly regular.*

Proof: (1): Let $x, y \in A_*$. Then $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(e)$, $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y) = \nu_A(e)$. By 2.5(1), $\mu_A(xy^{-1}) \leq \mu_A e$, $\nu_A(xy^{-1}) \geq \nu_A e$ for each $x, y \in G$. So, $\mu_A(xy^{-1}) = \mu_A e$ and $\nu_A(xy^{-1}) = \nu_A e$ or $xy^{-1} \in A_*$ implying A_* is a subgroup of G.
(2): Since L is strongly regular, by 2.1(f), for each $x, y \in A^*$, $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y) > 0$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y) < 1$ or $xy^{-1} \in A^*$ implying, A^* is a subgroup of G.

Lemma 2.8 *For any if/v-subset A of a group G, A is an if/v-subgroup of G iff A satisfies the following conditions:*
(1) $\mu_{A\circ A} = \mu_A$ and $\nu_{A\circ A} = \nu_A$ or equivalently $A \circ A = A$.
(2) $\mu_{A^{-1}} = \mu_A$ and $\nu_{A^{-1}} = \nu_A$ or equivalently $A^{-1} = A$.

Proof: ($\Rightarrow$): Let A be an if/v-subgroup of G. Then for each $x, y \in G$, $\mu_A x = \mu_A(xy^{-1}y) \geq \mu_A(xy^{-1}) \wedge \mu_A(y)$, $\nu_A x = \nu_A(xy^{-1}y) \leq \nu_A(xy^{-1}) \vee \nu_A(y)$, $\mu_A(x^{-1}) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(x)$.
(1): $\mu_{A\circ A}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_A(y)) \leq \vee_{y\in G}\mu_A(x) = \mu_A(x)$ or $\mu_{A\circ A} \leq \mu_A$ and $(\nu_{A\circ A})(x) = \wedge_{y\in G}(\nu_A(xy^{-1}) \vee \nu_A(y)) \geq \wedge_{y\in G}\nu_A(x) = \nu_A(x)$ or $\nu_{A\circ A} \geq \nu_A$.
Now by 2.5(2), we get that $\mu_{A\circ A} = \mu_A$ and $\nu_{A\circ A} = \nu_A$.
(2): 2.4 implies for each $x \in G$, $\mu_{A^{-1}}(x) = \mu_A(x^{-1}) = \mu_A(x)$ or $\mu_A = \mu_{A^{-1}}$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1}) = \nu_A(x)$ or $\nu_{A^{-1}} = \nu_A$.

(⇐): 2.2(1) and the facts that $\mu_{A^{-1}} = \mu_A$, $\nu_{A^{-1}} = \nu_A$, $\mu_{A\circ A} \leq \mu_A$, $\nu_{A\circ A} \geq \nu_A$ imply, for each $x \in G$ $\mu_A(xy^{-1}) \geq \mu_{A\circ A}(xy^{-1}) \geq \mu_A(xy^{-1}y) \wedge \mu_A(y^{-1}) = \mu_A x \wedge \mu_A y$ and $\nu_A(xy^{-1}) \leq \nu_{A\circ A}(xy^{-1}) \leq \nu_A(xy^{-1}y) \vee \nu_A(y^{-1}) = \nu_A x \vee \nu_A y$.

Lemma 2.9 For any pair of if/v-subgroups A and B of a group G, $A \circ B$ is an if/v-subgroup of G iff $A \circ B = B \circ A$.

Proof: (⇒): Since A, B and $A\circ B$ are if/v-subgroups of G, $A^{-1} = A$, $B^{-1} = B$, $A\circ B = (A\circ B)^{-1} = B^{-1}\circ A^{-1} = B\circ A$.
(⇐): Let $A\circ B = B\circ A$. Then (a) $(A\circ B)\circ(A\circ B) = A\circ(B\circ A)\circ B = A\circ(A\circ B)\circ B = (A\circ A)\circ(B\circ B) = A\circ B$ and (b) $(A\circ B)^{-1} = (B\circ A)^{-1} = A^{-1}\circ B^{-1} = A\circ B$. By 2.8, $A\circ B$ is an if/v-subgroup of G.

Lemma 2.10 For any pair of groups G and H and for any crisp homomorphism $f{:}G\rightarrow H$ the following are true:

1. *A is an if/v-subgroup of G implies $f(A)$ is an if/v-subgroup of H, whenever [0, 1] is a complete infinite distributive lattice;*
2. *B is an if/v-subgroup of H implies $f^{-1}(B)$ is an if/v-subgroup of G.*

Proof: (1): Let $fA = B$. Then $\mu_B y = \vee \mu_A f^{-1}y$, $\nu_B y = \wedge \nu_A f^{-1}y$. Now we show that $\mu_B(xy^{-1}) \geq \mu_B(x) \wedge \mu_B(y)$ and $\nu_B(xy^{-1}) \leq \nu_B(x) \vee \nu_B(y)$. Let us recall that $\mu_B(x) = \vee\mu_A f^{-1}x = \vee_{a\in f^{-1}x}\mu_A a$, $\mu_B y = \vee\mu_A f^{-1}y = \vee_{b\in f^{-1}y}\mu_A b$ and $\nu_B(x) = \wedge\nu_A f^{-1}x = \wedge_{a\in f^{-1}x}\nu_A a$, $\nu_B y = \wedge\nu_A f^{-1}y = \wedge_{b\in f^{-1}y}\nu_A b$. If one of $f^{-1}x$ or $f^{-1}y$ is empty, we are done because $\vee\phi = 0_L$ and $\wedge\phi = 1_L$. So, let both of them be non-empty. $a \in f^{-1}x$, $b \in f^{-1}y$ imply $fa = x$, $fb = y$ which implies $fab^{-1} = fafb^{-1} = xy^{-1}$ which in turn implies $c = ab^{-1} \in f^{-1}(xy^{-1})$. Since A is an if/v-subgroup of G, $\mu_B(xy^{-1}) = \vee_{c\in f^{-1}(xy^{-1})}\mu_A c \geq \mu_A(ab^{-1}) \geq \mu_A(a) \wedge \mu_A(b)$ and similarly $\nu_B(xy^{-1}) \leq \nu_A(a) \vee \nu_A(b)$ for each $a \in f^{-1}x$, $b \in f^{-1}y$.
Observe that in any complete infinite distributive lattice,
(1) $\gamma \geq \alpha\wedge\beta$ for each $\alpha \in M \subseteq [0,\ 1]$, for each $\beta \in N \subseteq [0,\ 1]$ implies $\gamma \geq (\vee_{\alpha\in M}\alpha) \wedge (\vee_{\beta\in N}\beta) = (\vee M) \wedge (\vee N)$,
(2) $\gamma \leq \alpha\vee\beta$ for each $\alpha \in M \subseteq [0,\ 1]$, for each $\beta \in N \subseteq [0,\ 1]$ implies $\gamma \leq (\wedge_{\alpha\in M}\alpha) \vee (\wedge_{\beta\in N}\beta) = (\wedge M) \vee (\wedge N)$.
So, we will get that $\mu_B(xy^{-1}) \geq \mu_B x \wedge \mu_B y$ and $\nu_B(xy^{-1}) \leq \nu_B x \vee \nu_B y$.
Hence $fA = B$ is an if/v-subgroup of G.

(2): Let $f^{-1}B = A$. Then $\mu_A x = \mu_B fx$, $\nu_A x = \nu_B fx$. Now we show that $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$.
Since f is a homomorphism and B is an if/v-subgroup of H, $\mu_A(xy^{-1}) = \mu_B f(xy^{-1}) = \mu_B(fx)(fy)^{-1} \geq \mu_B fx \wedge \mu_B fy = \mu_A x \wedge \mu_A y$ and
$\nu_A(xy^{-1}) = \nu_B f(xy^{-1}) = \nu_B(fx)(fy)^{-1} \leq \nu_B fx \vee \nu_B fy = \nu_A x \vee \nu_A y$.
Hence $f^{-1}B = A$ is an if/v-subgroup of G.

Lemma 2.11 For any family of if/v-subgroups $(A_i)_{i\in I}$ of a group G, $\wedge_{i\in I}A_i$ is an if/v-subgroup of G.

Proof: Let $C = \wedge_{i\in I}A_i$. Then $\mu_C = \wedge_{i\in I}\mu_{A_i}$, $\nu_C = \vee_{i\in I}\nu_{A_i}$. Now we show that, $\mu_C(xy^{-1}) \geq \mu_C(x) \wedge \mu_C(y)$ and $\nu_C(xy^{-1}) \leq \nu_C(x) \vee \nu_C(y)$.
Let us recall that in any complete lattice,
(1) $\wedge_{i\in I}(\alpha_i \wedge \beta_i) = (\wedge_{i\in I}\alpha_i) \wedge (\wedge_{i\in I}\beta_i)$
(2) $\vee_{i\in I}(\alpha_i \vee \beta_i) = (\vee_{i\in I}\alpha_i) \vee (\vee_{i\in I}\beta_i)$
(3) $\alpha_i \leq \beta_i$ for each $i \in I$ implies $\wedge_{i\in I}\ \alpha_i \leq \wedge_{i\in I}\ \beta_i$
(4) $\alpha_i \leq \beta_i$ for each $i \in I$ implies $\vee_{i\in I}\ \alpha_i \leq \vee_{i\in I}\ \beta_i$
Now the above and A_i is an if/v-subgroup of G imply, $\mu_C(xy^{-1}) = (\wedge_{i\in I}\mu_{A_i})(xy^{-1}) = \wedge_{i\in I}\mu_{A_i}(xy^{-1}) \geq \wedge_{i\in I}(\mu_{A_i}(x)\wedge\mu_{A_i}(y)) = (\wedge_{i\in I}\mu_{A_i}x) \wedge (\wedge_{i\in I}\mu_{A_i}y) = \mu_C x \wedge \mu_C y$ and $\nu_C(xy^{-1}) = (\vee_{i\in I}\nu_{A_i})(xy^{-1}) = \vee_{i\in I}\nu_{A_i}(xy^{-1}) \leq \vee_{i\in I}(\nu_{A_i}(x) \vee \nu_{A_i}(y)) = (\vee_{i\in I}\nu_{A_i}x) \vee (\vee_{i\in I}\nu_{A_i}y) = \nu_C x \vee \nu_C y$. Hence $C = \wedge_{i\in I}A_i$ is an if/v-subgroup of G.

Notes

Notes

It may so happen that the $\wedge_{i\in I}A_i$ may be the empty if/v-subset which is trivially an if/v-subgroup of G as shown in the following example:

Example 2.12 $A_n = (\frac{1}{n}, 1-\frac{1}{n})$, $\wedge_{n=1}^{\infty}A_n = (0,1) = \phi$, *the empty subgroup of* G.

The $A \vee B$ of if/v-subgroups A, B of a group G need *not* be an if/v-subgroup as shown in the following example:

Example 2.13 *Let* $A = (\chi_{2z}, 1-\chi_{2z})$, $B = (\chi_{3z}, 1-\chi_{3z})$ *be the* I*-if/v-subgroups of* Z*, the additive group of integers, where* I *= [0,1], the closed interval of real numbers. Then* $A \vee B = (\chi_{2z} \vee \chi_{3z}, (1-\chi_{2z}) \wedge (1-\chi_{3z}))$ *and* $\mu_{A\vee B}(5) = (\chi_{2z} \vee \chi_{3z})5 = 0 \vee 0 = 0$.
If $A \vee B$ *is an if/v-subgroup of* G*, then* $0 = \mu_{A\vee B}(3+2) \geq \mu_{A\vee B}(3) \wedge \mu_{A\vee B}(2) = (\chi_{2z} \vee \chi_{3z})3 \wedge (\chi_{2z} \vee \chi_{3z})2 = (0 \vee 1) \wedge (1 \vee 0) = 1 \wedge 1 = 1$*, a contradiction. So* $A \vee B$ *is* not *an if/v-subgroup of* G.

Lemma 2.14 *For any family of if/v-subgroups* $(A_i)_{i\in I}$ *of* G, $\vee_{i\in I}A_i$ *is an if/v-subgroup of* G *whenever* $(A_i)_{i\in I}$ *is a sup/inf assuming chain of if/v-subgroups.*

Proof: Let $A = \vee_{i\in I}A_i$. Then $\mu_A = \vee_{i\in I}\mu_{A_i}$, $\nu_A = \wedge_{i\in I}\nu_{A_i}$. Now we show that $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$. If one of $\mu_A x$ or $\mu_A(y) = 0$ and one of $\nu_A x$ or $\nu_A(y) = 1$ then anyway the inequalities hold good.

Let $\mu_A x$, $\mu_A y > 0$ and $\nu_A x$, $\nu_A y < 1$. Then $\vee_{i\in I}(\mu_{A_i}x)$, $\vee_{i\in I}(\mu_{A_i}y) > 0$ and $\wedge_{i\in I}(\nu_{A_i}x)$, $\wedge_{i\in I}(\nu_{A_i}y) < 1$. Then there exists $i_0 \in I$ such that $\mu_{A_{i_0}}x = \vee_{i\in I}\mu_{A_i}x$, $\nu_{A_{i_0}}x = \wedge_{i\in I}\nu_{A_i}x$ and there exists $j_0 \in I$ such that $\mu_{A_{j_0}}y = \vee_{i\in I}\mu_{A_i}y$, $\nu_{A_{j_0}}y = \wedge_{i\in I}\nu_{A_i}y$ because $(A_i)_{i\in I}$ is a sup/inf assuming chain.
Now (1) $A_{i_0} \leq A_{j_0}$ or (2) $A_{j_0} \leq A_{i_0}$ because $(A_i)_{i\in I}$ is a chain.
(1) Suppose $A_{i_0} \leq A_{j_0}$ or $\mu_{A_{i_0}} \leq \mu_{A_{j_0}}$ and $\nu_{A_{j_0}} \leq \nu_{A_{i_0}}$. Then $\mu_A(xy^{-1}) \geq \mu_{A_{j_0}}(xy^{-1}) \geq \mu_{A_{j_0}}x \wedge \mu_{A_{j_0}}y \geq \mu_{A_{i_0}}x \wedge \mu_{A_{j_0}}y = (\vee_{i\in I}\mu_{A_i}x) \wedge (\vee_{i\in I}\mu_{A_i}y) = \mu_A x \wedge \mu_A y$ and $\nu_A(xy^{-1}) \leq \nu_{A_{j_0}}(xy^{-1}) \leq \nu_{A_{j_0}}x \vee \nu_{A_{j_0}}y \leq \nu_{A_{i_0}}x \vee \nu_{A_{j_0}}y = (\wedge_{i\in I}\nu_{A_i}x) \vee (\wedge_{i\in I}\nu_{A_i}y) = \nu_A x \vee \nu_A y$.
2) Suppose $A_{j_0} \leq A_{i_0}$ or $\mu_{A_{j_0}} \leq \mu_{A_{i_0}}$ and $\nu_{A_{i_0}} \leq \nu_{A_{j_0}}$. Then $\mu_A(xy^{-1}) \geq \mu_{A_{i_0}}(xy^{-1}) \geq \mu_{A_{i_0}}x \wedge \mu_{A_{i_0}}y \geq \mu_{A_{i_0}}x \wedge \mu_{A_{j_0}}y = (\vee_{i\in I}\mu_{A_i x}) \wedge (\vee_{i\in I}\mu_{A_i}y) = \mu_A x \wedge \mu_A y$ and $\nu_A(xy^{-1}) \leq \nu_{A_{i_0}}(xy^{-1}) \leq \nu_{A_{i_0}}x \vee \nu_{A_{i_0}}y \leq \nu_{A_{i_0}}x \vee \nu_{A_{j_0}}y = (\wedge_{i\in I}\nu_{A_i}x) \vee (\wedge_{i\in I}\nu_{A_i}y) = \nu_A x \vee \nu_A y$.

If/V-Cosets And If/V-Index Of An If/V-Subgroup

Definitions 2.15 *(1) For any if/v-subgroup* A *of a group* G *and for any* $g \in G$*, the if/v-subset* $gA = (\mu_{gA}, \nu_{gA})$ *of* G*, where* μ_{gA}, $\nu_{gA} : G \to$ *[0, 1], are defined by* $\mu_{gA}x = \mu_A(g^{-1}x)$ *and* $\nu_{gA}x = \nu_A(g^{-1}x)$*, is called the if/v-left coset of* A *by* g *in* G*. The if/v-subset* $Ag = (\mu_{Ag}, \nu_{Ag})$ *of* G*, where* $\mu_{Ag}x = \mu_A(xg^{-1})$ *and* $\nu_{Ag}x = \nu_A(xg^{-1})$ *is called the if/v-right coset of* A *by* g *in* G.
(2) The set of all if/v-left cosets of A *in* G *is denoted by* $(G/A)_L$*. The set of all if/v-right cosets of* A *in* G *is denoted by* $(G/A)_R$.
(3) (Later on we show, as in the crisp set up, that) The number of if/v-left cosets of A *in* G *is the same as the number of if/v-right cosets of* A *in* G *and this common number, denoted by* $(G : A)$*, is called the if/v-index of* A *in* G.

Theorem 2.16 *For any if/v-subgroup* A *of a group* G *and for any pair of elements* g, h *of* G*, the following are true:*

1. $gA = g_{\mu_A e, \nu_A e} \circ A$ *and* $Ag = A \circ g_{\mu_A e, \nu_A e}$.
2. $gA = hA$ *iff* $gA_* = hA_*$.
3. $Ag = Ah$ *iff* $A_* g = A_* h$.

Proof: (1): From 2.2(3) and 2.15(1), $(\chi_g^{\mu_A e} \circ \mu_A)(x) = \mu_A(g^{-1}x) = \mu_{gA}x$ and $(\chi_g^{\nu_A e} \circ \nu_A)(x) = \nu_A(g^{-1}x) = \nu_{gA}x$ or $\mu_{gA} = \chi_g^{\mu_A e} \circ \mu_A$ and $\nu_{gA} = \chi_g^{\nu_A e} \circ \nu_A$.
Hence $gA = g_{\mu_A e, \nu_A e} \circ A$.
From 2.2(4) and 2.15(1), $(\mu_A \circ \chi_g^{\mu_A e})(x) = \mu_A(xg^{-1}) = \mu_{Ag}x$ and $(\nu_A \circ \chi_g^{\nu_A e})(x) = \nu_A(xg^{-1}) = \nu_{Ag}x$ or $\mu_{Ag} = \mu_A \circ \chi_g^{\mu_A e}$ and $\nu_{Ag} = \nu_A \circ \chi_g^{\nu_A e}$.
Hence $Ag = A \circ g_{\mu_A e, \nu_A e}$.
(2): ($\Rightarrow$): Suppose $gA = hA$. Then $\mu_{gA} = \mu_{hA}$ and $\nu_{gA} = \nu_{hA}$ or for each $x \in G$, $\mu_{gA}(x) = \mu_{hA}(x)$ and $\nu_{gA}x = \nu_{hA}x$ which implies $\mu_A(g^{-1}x) = \mu_A(h^{-1}x)$ and $\nu_A(g^{-1}x) = \nu_A(h^{-1}x)$.
Choosing $x = h$, $\mu_A(g^{-1}h) = \mu_A(h^{-1}h) = \mu_A(e)$ and $\nu_A(g^{-1}h) = \nu_A(h^{-1}h) = \nu_A(e)$ implying $g^{-1}h \in A_*$, where $A_* = \{x \in G/\mu_A(x) = \mu_A(e), \nu_A(x) = \nu_A(e)\}$.
Hence $gA_* = hA_*$.
($\Leftarrow$): From 2.7, A is an if/v-subgroup of G implies A_* is a subgroup of G. Suppose $gA_* = hA_*$. Then $g^{-1}h \in A_*$ or $\mu_A(g^{-1}h) = \mu_A(e)$, $\nu_A(g^{-1}h) = \nu_A(e)$. Hence for each $z \in G$, $\mu_A(g^{-1}z) = \mu_A(g^{-1}hh^{-1}z) \geq \mu_A(g^{-1}h) \wedge \mu_A(h^{-1}z) = \mu_A(e) \wedge \mu_A(h^{-1}z) = \mu_A(h^{-1}z)$ and $\nu_A(g^{-1}z) = \nu_A(g^{-1}hh^{-1}z) \leq \nu_A(g^{-1}h) \vee \nu_A(h^{-1}z) = \nu_A(e) \wedge \nu_A(h^{-1}z) = \nu_A(h^{-1}z)$ because, $\mu_A e$ is the largest of $\mu_A G$ and $\nu_A e$ is the least of $\nu_A G$.
Similarly, for each $z \in G$, $\mu_A(h^{-1}z) \geq \mu_A(g^{-1}z)$ and $\nu_A(h^{-1}z) \leq \nu_A(g^{-1}z)$.
Hence for each $z \in G$, $\mu_A(g^{-1}z) = \mu_A(h^{-1}z)$, $\nu_A(g^{-1}z) = \nu_A(h^{-1}z)$ or $\mu_{gA}(z) = \mu_{hA}(z)$, $\nu_{gA}(z) = \nu_{hA}(z)$ for each z or $\mu_{gA} = \mu_{hA}$, $\nu_{gA} = \nu_{hA}$ or $gA = hA$.
(3) ($\Rightarrow$): Suppose $Ag = Ah$. Then $\mu_{Ag} = \mu_{Ah}$, $\nu_{Ag} = \nu_{Ah}$ or for each $x \in G$, $\mu_{Ag}(x) = \mu_{Ah}(x)$ and $\nu_{Ag}x = \nu_{Ah}x$ which implies $\mu_A(xg^{-1}) = \mu_A(xh^{-1})$ and $\nu_A(xg^{-1}) = \nu_A(xh^{-1})$.
Choosing $x = h$, $\mu_A(hg^{-1}) = \mu_A(hh^{-1}) = \mu_A(e)$ and $\nu_A(hg^{-1}) = \nu_A(hh^{-1}) = \nu_A(e)$ implying $hg^{-1} \in A_*$ or $A_*g = A_*h$.
($\Leftarrow$): Suppose $A_*g = A_*h$. Then $hg^{-1} \in A_*$ or $\mu_A(hg^{-1}) = \mu_A(e)$ and $\nu_A(hg^{-1}) = \nu_A(e)$.
Hence for each $z \in G$, $\mu_A(zg^{-1}) = \mu_A(zh^{-1}hg^{-1}) \geq \mu_A(zh^{-1}) \wedge \mu_A(hg^{-1}) = \mu_A(zh^{-1}) \wedge \mu_A(e) = \mu_A(zh^{-1})$ and $\nu_A(zg^{-1}) = \nu_A(zh^{-1}hg^{-1}) \leq \nu_A(zh^{-1}) \vee \nu_A(hg^{-1}) = \nu_A(zh^{-1})$.
Similarly, for each $z \in G$, $\mu_A(zh^{-1}) \geq \mu_A(zg^{-1})$ and $\nu_A(zh^{-1}) \leq \nu_A(zg^{-1})$.
Hence for each $z \in G$, $\mu_A(zg^{-1}) = \mu_A(zh^{-1})$, $\nu_A(zh^{-1}) = \nu_A(zg^{-1})$ or $\mu_{Ag}(z) = \mu_{Ah}(z)$, $\nu_{Ag}(z) = \nu_{Ah}(z)$ for each z or $Ag = Ah$.

Corollary 2.17 *For any if/v-subgroup A of a group G, the following are true: (1) The number of if/v-left(right) cosets of A in G is the same as the number of left(right) cosets of A_* in G.*

(2) $(G : A) = (G : A_*)$.

Proof: (1): Let $\Im$ be the set of all if/v-left cosets of A in G and $\aleph$ be the set of all if/v-left cosets of B in G. Define $\phi : \Im \to \aleph$ by $\phi(gA) = gA_*$. Then by 2.16(2), ϕ is both well defined and one-one. But clearly, ϕ is onto. Thus ϕ is a bijection implying our assertion.
(2): For any subgroup H of a group G, the number of left coset of H in G is the same as the number of right coset of H in G. Now the assertion follows from (1).

In the crisp set up, when G is a finite group, for any subgroup H of G, $|H| = \frac{|G|}{(G:H)}$. If one were to define the order for an if/v-subgroup of a finite group, the preceeding equation suggests that $|A| = \frac{|G|}{(G:A)}$. But $(G : A) = (G : A_*)$ and consequently $|A| = |A_*|$. Thus the definition of if/v-order of an if/v-subgroup is as follows:

Definition 2.18 *For any if/v-subgroup A of a group G, the* order of A*, denoted by $|A|$, is defined to be the order of A_* or $|A_*|$. In other words $|A| = |A_*|$.*

An if/v-subgroup A of a group G is *finite* or *infinite* according as its order $|A|$ is finite or infinite.

Notes

Lagranges Theorem

Theorem 2.19 *For any finite group G and for any if/v-subgroup A, order of A, $|A|$ divides the order of G, $|G|$.*

III. Intuitionistic Fuzzy/Vague-Normal Subgroups

Notes

In this section, we begin with equivalent conditions for if/v-normality for a subgroup and several of these conditions will be used in some subsequent results, sometimes, without an explicit mention. Later on we proceed to generalize various crisp theoretic results mentioned in the beginning of this chapter.

The following is a theorem which gives equivalent statements for an if/v-normal subgroup, some what similarly as in crisp set up.

Theorem 3.1 *Let A be an if/v-subgroup of G. Then the following are equivalent:*

1. $\mu_A(xy) = \mu_A(yx)$ *and* $\nu_A(xy) = \nu_A(yx)$ *for each* $x, y \in G$,
2. $\mu_A(xyx^{-1}) = \mu_A(y)$ *and* $\nu_A(xyx^{-1}) = \nu_A(y)$ *for each* $x, y \in G$,
3. $\mu_A[xy] \geq \mu_A x$ *and* $\nu_A[xy] \leq \nu_A x$ *for each* $x, y \in G$, *where* $[x, y] = x^{-1}y^{-1}xy$ *is the commutator of* x, y,
4. $\mu_A(xyx^{-1}) \geq \mu_A(y)$ *and* $\nu_A(xyx^{-1}) \leq \nu_A(y)$ *for each* $x, y \in G$,
5. $\mu_A(xyx^{-1}) \leq \mu_A(y)$ *and* $\nu_A(xyx^{-1}) \geq \nu_A(y)$ *for each* $x, y \in G$,
6. $A \circ B = B \circ A$ *for each if/v-subset* B *of* G,
7. $Ag \circ Ah = Agh$, $gA \circ hA = ghA$, $Agh = ghA$ *and* $Ag \circ Ah = Ah \circ Ag$ *for each* $g, h \in G$,
8. $gA = Ag$ *for each* $g \in G$,
9. $A = g_{\mu_A e, \nu_A e} \circ A \circ g^{-1}_{\mu_A e, \nu_A e}$ *for each* $g \in G$.

Proof: Let $x, y \in G$.

(1)⇒(2): $\mu_A(xyx^{-1}) = \mu_A(x^{-1} \cdot xy) = \mu_A(y)$ and $\nu_A(xyx^{-1}) = \nu_A(x^{-1} \cdot xy) = \nu_A(y)$.

(2)⇒(3): $\mu_A(x^{-1}y^{-1}xy) = \mu_A(x^{-1}(y^{-1}xy)) \geq \mu_A(x^{-1}) \wedge \mu_A(y^{-1}xy) = \mu_A(x^{-1}) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(x^{-1}y^{-1}xy) = \nu_A(x^{-1}(y^{-1}xy)) \leq \nu_A(x^{-1}) \vee \nu_A(y^{-1}xy) = \nu_A(x^{-1}) \vee \nu_A(x) = \nu_A(x)$, by 2.4 and 2.6.

(3)⇒(4): $\mu_A(y^{-1}xy) = \mu_A(xx^{-1}y^{-1}xy) \geq \mu_A(x) \wedge \mu_A(x^{-1}y^{-1}xy) \geq \mu_A(x)$ and $\nu_A(y^{-1}xy) = \nu_A(xx^{-1}y^{-1}xy) \leq \nu_A(x) \vee \nu_A(x^{-1}y^{-1}xy) \leq \nu_A(x)$.

(4)⇒(5): $\mu_A(xyx^{-1}) \leq \mu_A(x^{-1} \cdot xyx^{-1} \cdot (x^{-1})^{-1}) = \mu_A(y)$ and $\nu_A(xyx^{-1}) \geq \nu_A(x^{-1} \cdot xyx^{-1} \cdot (x^{-1})^{-1}) = \nu_A(y)$.

(5)⇒(1): $\mu_A(xy) = \mu_A(xyxx^{-1}) = \mu_A(x \cdot yx \cdot x^{-1}) \leq \mu_A(yx)$ and $\mu_A(yx) = \mu_A(y \cdot xy \cdot y^{-1}) \leq \mu_A(xy)$, implying $\mu_A(xy) = \mu_A(yx)$.
$\nu_A(xy) = \nu_A(xyxx^{-1}) = \nu_A(x \cdot yx \cdot x^{-1}) \geq \nu_A(yx)$ and $\nu_A(yx) = \nu_A(y \cdot xy \cdot y^{-1}) \geq \nu_A(xy)$, implying $\nu_A(xy) = \nu_A(yx)$.

(1)⇒(6): $\mu_{A\circ B}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_B(y)) = \vee_{y\in G}(\mu_A(y^{-1}x) \wedge \mu_B(y)) = \vee_{y\in G}(\mu_B(y) \wedge \mu_A(y^{-1}x)) = \mu_{B\circ A}(x)$ and $\nu_{A\circ B}(x) = \wedge_{y\in G}(\nu_A(xy^{-1}) \vee \nu_B(y)) = \wedge_{y\in G}(\nu_A(y^{-1}x) \vee \nu_B(y)) = \wedge_{y\in G}(\nu_B(y) \vee \nu_A(y^{-1}x)) = \nu_{B\circ A}(x)$, implying $A \circ B = B \circ A$.

(6)⇒(7): 2.16, 2.8, 2.2(11) imply $Ag \circ Ah = A \circ g_{\mu_A e,\nu_A e} \circ A \circ h_{\mu_A e,\nu_A e} = A \circ A \circ g_{\mu_A e,\nu_A e} \circ h_{\mu_A e,\nu_A e} = A \circ g_{\mu_A e,\nu_A e} \circ h_{\mu_A e,\nu_A e} = A \circ (gh)_{\mu_A e,\nu_A e} = Agh$. Similarly $gA \circ hA = ghA$.

Now letting $B = (gh)_{\mu_A e,\nu_A e}$, by the hypothesis, the above implies $Agh = ghA$. Again by hypothesis, $Ag \circ Ah = A \circ g_{\mu_A e,\nu_A e} \circ h_{\mu_A e,\nu_A e} = A \circ h_{\mu_A e,\nu_A e} \circ g_{\mu_A e,\nu_A e} = Ahg = Ah \circ Ag$.

(7)⇒(8): $h = e$ implies $Ag = gA$.

(8)⇒(9): By 2.16, 2.2(4) and 2.2(11), $g_{\mu_A e,\nu_A e} \circ A \circ g^{-1}_{\mu_A e,\nu_A e} = gA \circ g^{-1}_{\mu_A e,\nu_A e} = Ag \circ g^{-1}_{\mu_A e,\nu_A e} = A \circ g_{\mu_A e,\nu_A e} \circ g^{-1}_{\mu_A e,\nu_A e} = A \circ (gg^{-1})_{\mu_A e,\nu_A e} = A \circ (e)_{\mu_A e,\nu_A e} = Ae = A$.

(9)⇒(1): By 2.15, $\mu_A(xy) = \mu_A(y^{-1}yxy) = \mu_{yAy^{-1}}(yx) = \mu_A(yx)$ and $\nu_A(xy) = \nu_A(y^{-1}yxy) = \nu_{yAy^{-1}}(yx) = \nu_A(yx)$.

Definition and Statements 3.2 *(1) For any if/v-subgroup A of a group G, A is an L-if/v-normal subgroup of G iff it satisfies any one of the previous nine equivalent conditions. In particular, A is an if/v-normal subgroup of G iff for each $g \in G$, $Ag = gA$.*

(2) The set of all if/v-cosets of G, denoted by G/A or $\frac{G}{A}$, whenever A is an if/v-normal subgroup of G, is called the if/v-quotient set of G by A.

(3) Whenever G is a finite group and A is an if/v-normal subgroup of G, from the generalized Lagranges Theorem 2.19, $|(G/A)| = \frac{|G|}{|A|}$.

Proposition 3.3 *The following are true for any group G:*

(a) If G is abelian then every if/v-subgroup of G is if/v-normal subgroup of G, but not conversely.

(b) For an if/v-subgroup A of G and for any $z \in G$, the if/v-subset $zAz^{-1} = (\mu_{zAz^{-1}}, \nu_{zAz^{-1}})$ where $\mu_{zAz^{-1}}x = \mu_A(z^{-1}xz)$ and $\nu_{zAz^{-1}}x = \nu_A(z^{-1}xz)$ for each $x \in G$, is an if/v-subgroup of G.

(c) For any if/v-subgroup A of G, for each $z \in G$, $zAz^{-1} = z_{\mu_A e,\nu_A e} \circ A \circ z^{-1}_{\mu_A e,\nu_A e}$.

Proof: (a): It follows from 3.1(1) and 3.2(1).

(b): Since $\mu_{zAz^{-1}}x = \mu_A(z^{-1}xz) \leq N\nu_A(z^{-1}xz) = N\nu_{zAz^{-1}}x$, it follows that zAz^{-1} is an if/v-subset of G.

$\mu_{zAz^{-1}}(xy) = \mu_A(z^{-1}xyz) = \mu_A(z^{-1}xzz^{-1}yz) \geq \mu_A(z^{-1}xz) \wedge \mu_A(z^{-1}yz) = \mu_{zAz^{-1}}(x) \wedge \mu_{zAz^{-1}}(y)$ and $\nu_{zAz^{-1}}(xy) = \nu_A(z^{-1}xyz) = \nu_A(z^{-1}xzz^{-1}yz) \leq \nu_A(z^{-1}xz) \vee \nu_A(z^{-1}yz) = \nu_{zAz^{-1}}(x) \vee \nu_{zAz^{-1}}(y)$.

$\mu_{zAz^{-1}}(x) = \mu_A(z^{-1}xz) = \mu_A(z^{-1}x^{-1}z) = \mu_{zAz^{-1}}(x^{-1})$ and $\nu_{zAz^{-1}}(x) = \nu_A(z^{-1}xz) = \nu_A(z^{-1}x^{-1}z) = \nu_{zAz^{-1}}(x^{-1})$ for each $z \in G$.Hence zAz^{-1} is an if/v-subgroup of G.

(c): It follows from 2.16(1).

Definition 3.4 *For any pair of if/v subgroups A and B of a group G, A is said to be an -if/v-conjugate of B iff there exists $y \in G$ such that $A = yBy^{-1}$ or simply $A = B_y$.*

It is easy to see that being conjugate to an arbitrary but fixed if/v-subgroup A, is an equivqlence relation on the set of all if/v-subgroups of G.

Theorem 3.5 *For any if/v-normal subgroup A of G, the following are true:*

(1) $A_* = \{x/\mu_A(x) = \mu_A(e),\ \nu_A(x) = \nu_A(e)\}$ is a normal subgroup of G.

(2) $A^* = \{x \in G/\mu_A(x) > 0,\ \nu_A(x) < 1\}$ is a normal subgroup of G, whenever L is a strongly regular complete lattice.

Proof: By 2.7, A_* is subgroup of G and A^* is a subgroup of G when L is a strongly regular complete lattice. Since A is an if/v-normal subgroup of G, by 3.1(2),

Notes

Notes

(1): For each $y \in A_*$, $\mu_A(xyx^{-1}) = \mu_A y = \mu_A(e)$ and $\nu_A(xyx^{-1}) = \nu_A y = \nu_A(e)$ or $xyx^{-1} \in A_*$ or A_* is a normal subgroup of G.
(2): For each $y \in A^*$, $\mu_A(xyx^{-1}) = \mu_A y > 0$ and $\nu_A(xyx^{-1}) = \nu_A(y) < 1$ or $xyx^{-1} \in A^*$ or A^* is a normal subgroup of G.

If/V-Normalizer

Theorem 3.6 For any if/v-subgroup A of a group G, $N_G(A) = \{x \in G/\mu_A(xy) = \mu_A(yx),\ \nu_A(xy) = \nu_A(yx),$ for each $y \in G\}$ is a subgroup of G and the restriction of A to $N_G(A)$, denoted by $A|N_G(A)$, defined by $(\mu_A|N_G(A), \nu_A|N_G(A))$, is an if/v-normal subgroup of $N_G(A)$.

Proof: Since $\mu_A(ey) = \mu_A(y) = \mu_A(ye)$ and $\nu_A(ey) = \nu_A(ye)$ for each $y \in G$, $e \in N_G(A)$.
Let $x, y \in N_G(A)$ and $z \in G$. Then $x \in N_G(A)$ implies $\mu_A(x \cdot y^{-1}z) = \mu_A(y^{-1}z \cdot x)$, $\nu_A(x \cdot y^{-1}z) = \nu_A(y^{-1}z \cdot x)$ and $y \in N_G(A)$ implies $\mu_A(x^{-1}z^{-1} \cdot y) = \mu_A(y \cdot x^{-1}z^{-1})$, $\nu_A(x^{-1}z^{-1} \cdot y) = \nu_A(y \cdot x^{-1}z^{-1})$.
From the above, $\mu_A(xy^{-1} \cdot z) = \mu_A(x \cdot y^{-1}z) = \mu_A(y^{-1}z \cdot x) = \mu_A((y^{-1}zx)^{-1}) = \mu_A(x^{-1}z^{-1} \cdot y) = \mu_A(y \cdot x^{-1}z^{-1}) = \mu_A((z \cdot xy^{-1})^{-1}) = \mu_A(z \cdot xy^{-1})$ and $\nu_A(xy^{-1} \cdot z) = \nu_A(x \cdot y^{-1}z) = \nu_A(y^{-1}z \cdot x) = \nu_A((y^{-1}zx)^{-1}) = \nu_A(x^{-1}z^{-1} \cdot y) = \nu_A(y \cdot x^{-1}z^{-1}) = \nu_A((z \cdot xy^{-1})^{-1}) = \nu_A(z \cdot xy^{-1})$.
Thus $xy^{-1} \in N_G(A)$ and $N_G(A)$ is a subgroup of G.
Now we show that $A|N_G(A)$ is an if/v-normal subgroup of $N_G(A)$.
But first $A|N_G(A)$ is an if/v-subgroup of $N_G(A)$ because for each $x, y \in N_G(A)$, $(\mu_A|N_G(A))(xy^{-1}) = \mu_A(xy^{-1}) \geq \mu_A x \wedge \mu_A y = (\mu_A|N_G(A))x \wedge (\mu_A|N_G(A))y$ and
$(\nu_A|N_G(A))(xy^{-1}) = \nu_A(xy^{-1}) \leq \nu_A x \vee \nu_A y = (\nu_A|N_G(A))x \vee (\nu_A|N_G(A))y$.
Next for each $x, y \in N_G(A)$,
$(\mu_A|N_G(A))(xy) = \mu_A(xy) = \mu_A(yx) = (\mu_A|N_G(A))(yx)$ and $(\nu_A|N(A))(xy) = \nu_A(xy) = \nu_A(yx) = (\nu_A|N_G(A))(yx)$ implying $A|N_G(A)$ is an if/v-normal subgroup of $N_G(A)$.

Definition 3.7 For any if/v-subgroup A of a group G, the subgroup $N_G(A)$ of G defined as above is called the normalizer of A in G and $A|N_G(A)$ is called the if/v-normalizer of A.

lemma 3.8 For any if/v-subgroup A of a group G, A is an if/v-normal subgroup of G iff $N_G(A) = G$.

Proof: ($\Rightarrow$): Always $N_G(A) \subseteq G$. On the other hand, $x \in G$ implies for each $y \in G$, by 3.1(1), $\mu_A(xy) = \mu_A(yx)$ and $\nu_A(xy) = \nu_A(yx)$. So, $x \in N_G(A)$.
($\Leftarrow$): Again by 3.1(1), we get that A is an if/v-normal subgroup of G.

Theorem 3.9 For any if/v-subgroup B of a group G, the number of if/v-conjugates of B in G is equal to the index $(G : N_G(B))$ of the normalizer $N_G(B)$ in G.

Proof: Let $u, v \in G$. Then $v^{-1}Gu = G$. Now $uBu^{-1} = vBv^{-1}$ iff for each $x \in G$, $\mu_B(u^{-1}xu) = \mu_B(v^{-1}xv)$ and $\nu_B(u^{-1}xu) = \nu_B(v^{-1}xv)$ iff (put $x = vxu^{-1}$) $\mu_B(u^{-1}v \cdot x) = \mu_B(x \cdot u^{-1}v)$ and $\nu_B(u^{-1}v \cdot x) = \nu_B(x \cdot u^{-1}v)$ iff $u^{-1}v \in N_G(B)$ iff $u^{-1}N_G(B) = v^{-1}N_G(B)$. Hence $B_u \rightarrow u^{-1}N_G(B)$ is a bijection from $\{uBu^{-1}/u \in G\}$ onto $\{uN_G(B)/u \in G\}$.

Theorem 3.10 For any if/v-subgroup B of a group G, $\wedge_{u \in G} uBu^{-1}$ is an if/v-normal subgroup of G and is the largest if/v-normal subgroup of G that is contained in B.

Proof: First observe that uBu^{-1} is an if/v-subgroup of G for each $u \in G$ by 6.1.3(b). So $\wedge_{u \in G} uBu^{-1}$ is an if/v-subgroup of G, by 2.11.
Since $\{uBu^{-1}/u \in G\} = \{(xu)B(xu)^{-1}/u \in G\}$ for each $x \in G$,
$\wedge_{u \in G}\mu_{uBu^{-1}}(x^{-1}yx) = \wedge_{u \in G}\mu_B(u^{-1}\ x^{-1}yx\ u) = \wedge_{u \in G}\mu_B((xu)^{-1}y(xu)) = \wedge_{u \in G}\ \mu_{(xu)B(xu)^{-1}}(y) = \wedge_{u \in G}\mu_{uBu^{-1}}(y)$ and

$\vee_{u\in G}\nu_{uBu^{-1}}(x^{-1}yx) = \vee_{u\in G}\nu_B(u^{-1}\ x^{-1}yx\ u) = \vee_{u\in G}\nu_B((xu)^{-1}y(xu))$
$= \vee_{u\in G}\nu_{(xu)B(xu)^{-1}}(y) = \vee_{u\in G}\nu_{uBu^{-1}}(y)$ for each $x, y \in G$.
Hence $\wedge_{u\in G}uBu^{-1}$ is an if/v-normal subgroup of G.
Next, let A be an if/v-normal subgroup of G, with $A \leq B$. Since A is an if/v-normal subgroup of G, $A = uAu^{-1}$ for each $u \in G$. Since $A \leq B$, $A = uAu^{-1} \leq uBu^{-1}$ for each $u \in G$ or $A \leq \wedge_{u\in G}uBu^{-1}$ or $\wedge_{u\in G}uBu^{-1}$ is the largest if/v-normal subgroup of G that is contained in B.

lemma 3.11 For any if/v-normal subgroup A of a group G and for any $x, y \in G$ such that $xA = yA$, $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.

Proof: By 2.16(2), $xA = yA$ implies $xA_* = yA_*$ which implies $x^{-1}y \in A_*$ and $y^{-1}x \in A_*$ or $\mu_A(x^{-1}y) = \mu_Ae = \mu_A(y^{-1}x)$ and $\nu_A(x^{-1}y) = \nu_Ae = \nu_A(y^{-1}x)$. Since A is an if/v-normal subgroup of G, $\mu_A(x) = \mu_A(y^{-1}xy) \geq \mu_A(y^{-1}x) \wedge \mu_A(y) = \mu_A(e) \wedge \mu_A(y) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y^{-1}xy) \leq \nu_A(y^{-1}x) \vee \nu_A(y) = \nu_A(e) \vee \nu_A(y) = \nu_A(y)$. Similarly, $\mu_A(y) = \mu_A(x^{-1}yx) \geq \mu_A(x^{-1}y) \wedge \mu_A(x) = \mu_A(e) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(y) = \nu_A(x^{-1}yx) \leq \nu_A(x^{-1}y) \vee \nu_A(x) = \nu_A(e) \vee \nu_A(x) = \nu_A(x)$. Hence $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.

Theorem 3.12 For any if/v-normal subgroup A of a group G. The following are true in G/A:

1. *$(xA) \circ (yA) = (xy)A$ for each $x, y \in G$;*
2. *$(G/A, \circ)$ is a group;*
3. *$G/A \cong G/A_*$;*
4. *Let $A^{(*)}$ be an if/v-subset of G/A be defined by $\mu_{A^{(*)}}(xA) = \mu_A(x)$ and $\nu_{A^{(*)}}(xA) = \nu_A(x)$ for each $x \in G$. Then $A^{(*)}$ is an if/v-normal subgroup of G/A.*

Proof: (1): Since A is an if/v-normal subgroup, by 3.1(7), this follows.

(2): By (1), G/A is closed under the operation $\circ$.
For each $x, y, z \in G$, $xA \circ (yA \circ zA) = xA \circ (yz)A = (xyz)A = (xy)A \circ zA = (xA \circ yA) \circ zA$. So G/A is associative under the operation $\circ$.
By 2.2(3), $eA = A$. Further by (1), for each $x \in G$, $A \circ xA = eA \circ xA = exA = xA$ and $xA \circ A = xA \circ eA = xeA = xA$ or A is the identity element for G/A. $(x^{-1}A) \circ (xA) = (x^{-1}x)A = eA = A = (xA) \circ (x^{-1}A)$ or $x^{-1}A$ is the inverse of xA in G/A. Hence $(G/A, \circ)$ is a group.
(3): Let $\eta : G/A \to G/A_*$, defined by $\eta(xA) = xA_*$. Then η is well defined and 1-1 because $xA = yA$ iff $xA_* = yA_*$.
Now we show that η is a homomorphism or $xyA_* = xA_*\ yA_*$. But by 3.5(1), A_* is a normal subgroup of G and so it follows that η is a homomorphism.

Now we show that η is onto. $\beta \in G/A_*$ implies $\beta = gA_*$, $g \in G$. Then $gA \in \frac{G}{A}$ such that $\eta(gA) = gA_* = \beta$ or η is onto.
(4): First we show that $A^{(*)}$ is an if/v-subgroup of G/A.
Since A be an if/v-subgroup of G,
(a): $\mu_{A^{(*)}}(gA \circ hA) = \mu_{A^{(*)}}(ghA) = \mu_A(gh) \geq \mu_A(g) \wedge \mu_A(h) = \mu_{A^{(*)}}(gA) \wedge \mu_{A^{(*)}}(hA)$ and $\nu_{A^{(*)}}(gA \circ hA) = \nu_{A^{(*)}}(ghA) = \nu_A(gh) \leq \nu_A(g) \vee \nu_A(h) = \nu_{A^{(*)}}(gA) \vee \nu_{A^{(*)}}(hA)$.
(b): $\mu_{A^{(*)}}((gA)^{-1}) = \mu_{A^{(*)}}(g^{-1}A) = \mu_A(g^{-1}) = \mu_A(g) = \mu_{A^{(*)}}(gA)$ and $\nu_{A^{(*)}}((gA)^{-1}) = \nu_{A^{(*)}}(g^{-1}A) = \nu_A(g^{-1}) = \nu_A(g) = \nu_{A^{(*)}}(gA)$.
Therefore $A^{(*)}$ is an if/v-subgroup of G/A.
Now we show that $A^{(*)}$ is an if/v-normal subgroup of G/A.
Since A is an if/v-normal subgroup of G, for each $g, h \in G$, $\mu_{A^{(*)}}((gA)^{-1} \circ (hA) \circ (gA)) = \mu_{A^{(*)}}(g^{-1}A \circ hA \circ gA) = \mu_{A^{(*)}}(g^{-1}hgA) = \mu_A(g^{-1}hg) \geq \mu_A(h) = \mu_{A^{(*)}}(hA)$ and $\nu_{A^{(*)}}((gA)^{-1} \circ (hA) \circ (gA)) = \nu_{A^{(*)}}(g^{-1}A \circ hA \circ gA) = \nu_{A^{(*)}}(g^{-1}hgA) = \nu_A(g^{-1}hg) \leq \nu_A(h) = \nu_{A^{(*)}}(hA)$. Hence $A^{(*)}$ is an if/v-normal subgroup of G/A.

Notes

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Theorem 3.13 *For any if/v-subgroup B of a group G and for any normal subgroup N of G, the if/v-subset C: $G/N \to L$ where for each $x \in G$, $\mu_C(xN) = \vee\mu_B(xN)$ and $\nu_C(xN) = \wedge\nu_B(xN)$, is an if/v-subgroup of G/N when L is a complete infinite distributive lattice.*

Proof: Since B is an if/v-subgroup of G and N is a normal subgroup of G and hence for each $x \in G$, $(xN)^{-1} = x^{-1}N$,

$$\mu_C((xN)^{-1}) = \mu_C(x^{-1}N) = \vee\ \mu_B(x^{-1}N) = \vee_{z\in x^{-1}N}\mu_B z$$
$$= \vee_{w^{-1}\in x^{-1}N(=(xN)^{-1})}\mu_B w^{-1} = \vee_{w\in xN}\mu_B w = \vee\ \mu_B(xN) = \mu_C(xN)$$
and $\nu_C((xN)^{-1}) = \nu_C(x^{-1}N) = \wedge\ \nu_B(x^{-1}N) = \wedge_{z\in x^{-1}N}\nu_B z$
$$= \wedge_{w^{-1}\in x^{-1}N(=(xN)^{-1})}\ \nu_B w^{-1} = \wedge_{w\in xN}\nu_B w = \wedge\ \nu_B(xN) = \nu_C(xN)$$

where the 5^{th} equality in both cases is due to the fact that $w \in xN$ iff $w^{-1} \in (xN)^{-1}$. Hence $C(xN)^{-1} = C(xN)$.

Since [0,1] is a complete infinite distributive lattice and N is a normal subgroup of G, for each $x, y \in G$

$$\mu_C((xN)(yN)) = \vee\mu_B(xyN) = \vee_{z\in xyN}\mu_B z = \vee_{u\in xN,v\in yN}\mu_B(uv)$$
$$\geq \vee_{u\in xN,v\in yN}(\mu_B(u)\wedge\mu_B(v)) = (\vee_{u\in xN}\mu_B(u))\wedge(\vee_{v\in yN}\ \mu_B(v))$$
$$= (\vee\mu_B(xN))\wedge(\vee\mu_B(yN)) = (\mu_C(xN))\wedge(\mu_C(yN))$$ and
$$\nu_C((xN)(yN)) = \wedge\nu_B(xyN) = \wedge_{z\in xyN}\nu_B z = \wedge_{u\in xN,v\in yN}\nu_B(uv)$$
$$\leq \wedge_{u\in xN,v\in yN}(\nu_B(u)\vee\nu_B(v)) = (\wedge_{u\in xN}\nu_B(u))\vee(\wedge_{v\in yN}\nu_B(v))$$
$$= (\wedge\nu_B(xN))\vee(\wedge\nu_B(yN)) = (\nu_C(xN))\vee(\nu_C(yN)).$$

Hence C is an if/v-subgroup of G/N.

Definition 3.14 *For any if/v-subgroup B of a group G and for any normal subgroup N of G, the if/v-subgroup C:$G/N \to L$, where L is a complete infinite distributive lattice, defined by $\mu_C(xN) = \vee\mu_B(xN)$ and $\nu_C(xN) = \wedge\nu_B(xN)$ for each $x \in G$, is called the if/v-quotient subgroup of G/N relative to B and is denoted by B/N or $\frac{B}{N}$.*
In other words when N is a normal subgroup of G and B is any if/v-subgroup of G, and [0,1] is a complete infinite distributive lattice, $\frac{B}{N}$: $\frac{G}{N} \to$ [0, 1] is defined by $\mu_{\frac{B}{N}}(gN) = \vee\mu_B(gN)$ and $\nu_{\frac{B}{N}}(gN) = \wedge\nu_B(gN)$ for each $g \in G$.

Lemma 3.15 *For any pair of groups G and H and for any crisp homomorphism $f : G \to H$, the following are true:*

1. *A is an if/v-normal subgroup of G implies $f(A)$ is an if/v-normal subgroup of H when f is onto.*
2. *B is an if/v-normal subgroup of H implies $f^{-1}(B)$ is an if/v-normal subgroup of G.*

Proof: (1): A is an if/v-normal subgroup of G implies $\mu_A(g^{-1}hg) \geq \mu_A(h)$ and $\nu_A(g^{-1}hg) \leq \nu_A(h)$ for each $h, g \in G$.

Let $fA = B$. Then $\mu_B y = \vee\mu_A f^{-1}y$ and $\nu_B y = \wedge\nu_A f^{-1}y$. Since the if/v-image of an if/v-subgroup is an if/v-subgroup, we only show that $\mu_B(g^{-1}hg) \geq \mu_B(h)$ and $\nu_B(g^{-1}hg) \leq \nu_B(h)$ for each $g, h \in G$.
Since f is onto, for each $y \in H$, $f^{-1}y \neq \phi$. Let $a \in f^{-1}g$, $b \in f^{-1}h$. Then $fa = g$, $fb = h$ and $fa^{-1} = g^{-1}$. Since f is a homomorphism, $g^{-1}hg = f(a^{-1}ba)$ and $a^{-1}ba \in f^{-1}(g^{-1}hg)$. So, for each $b \in f^{-1}h$, $\mu_B(g^{-1}hg) = \vee\mu_A f^{-1}(g^{-1}hg)$ $= \vee_{c\in f^{-1}(g^{-1}hg)}\mu_A c \geq \mu_A(a^{-1}ba) \geq \mu_A(b)$ and $\nu_B(g^{-1}hg) = \wedge\nu_A f^{-1}(g^{-1}hg) =$ $\wedge_{c\in f^{-1}(g^{-1}hg)}\nu_A c \leq \nu_A(a^{-1}ba) \leq \nu_A(b)$ implying $\mu_B(g^{-1}hg) \geq \vee_{b\in f^{-1}h}\ \mu_A(b)$ $= \mu_B(h)$ and $\nu_B(g^{-1}hg) \leq \wedge_{b\in f^{-1}h}\ \nu_A(b) = \nu_B(h)$
or $B = f(A)$ is an if/v-normal subgroup of H when f is onto.

(2): Let $f^{-1}B = A$. Then for each $g \in G$, $\mu_A g = \mu_B fg$ and $\nu_A g = \nu_B fg$. Since the if/v-inverse image of an if/v-subgroup is an if/v-subgroup we only show that $\mu_A(g^{-1}hg) \geq \mu_A(h)$ and $\nu_A(g^{-1}hg) \leq \nu_A(h)$.

Since f is a homomorphism and B is an if/v-normal subgroup of H, for each $g, h \in G$, $\mu_A(g^{-1}hg) = \mu_B f(g^{-1}hg) = \mu_B((fg)^{-1}(fh)(fg)) \geq \mu_B fh = \mu_A h$ and $\nu_A(g^{-1}hg) = \nu_B f(g^{-1}hg) = \nu_B((fg)^{-1}(fh)(fg)) \leq \nu_B fh = \nu_A h$ or $A = f^{-1}B$ is an if/v-normal subgroup of G.

Definition 3.16 *For any pair of if/v-subgroups A and B of a group G such that $A \leq B$, A is called an if/v-normal subgroup of B iff for each $x, y \in G$, $\mu_A(xyx^{-1}) \geq \mu_A(y) \wedge \mu_B(x)$ and $\nu_A(xyx^{-1}) \leq \nu_A(y) \vee \nu_B(x)$.*

Theorem 3.17 *For any pair of if/v-subgroups A and B of a group G such that $A \leq B$, the following are equivalent:*

1. *A is an if/v-normal subgroup of B.*
2. *$\mu_A(yx) \geq \mu_A(xy) \wedge \mu_B(x)$ and $\nu_A(yx) \leq \nu_A(xy) \vee \nu_B(x)$ for each $x, y \in G$.*
3. *$(\chi_x^{\mu_A e} \circ \mu_A) \geq (\mu_A \circ \chi_x^{\mu_A e}) \wedge \mu_B$ and $(\chi_x^{\nu_A e} \circ \nu_A) \leq (\nu_A \circ \chi_x^{\nu_A e}) \vee \nu_B$ for each $x \in G$.*

Proof: (1)⇒(2): Since A is an if/v-normal subgroup of B, for each $x, y \in G$, $\mu_A(yx) = \mu_A(x^{-1}xyx) = \mu_A(x^{-1}(xy)x) \geq \mu_A(xy) \wedge \mu_B(x)$ and $\nu_A(yx) = \nu_A(x^{-1}xyx) = \nu_A(x^{-1}(xy)x) \leq \nu_A(xy) \vee \nu_B(x)$.
(2)⇒(3): By 2.2(3) and 2.2(4), we have $(\chi_x^{\mu_A e} \circ \mu_A)y = \mu_A(x^{-1}y) \geq \mu_A(yx^{-1}) \wedge \mu_B(y) = (\mu_A \circ \chi_x^{\mu_A e})y \wedge \mu_B y = ((\mu_A \circ \chi_x^{\mu_A e}) \wedge \mu_B)y$ and $(\chi_x^{\nu_A e} \circ \nu_A)y = \nu_A(x^{-1}y)$ $\leq \nu_A(yx^{-1}) \vee \nu_B(y) = (\nu_A \circ \chi_x^{\nu_A e})y \vee \nu_B y = ((\nu_A \circ \chi_x^{\nu_A e}) \vee \nu_B)y$ or for each $x \in G$, $(\chi_x^{\mu_A e} \circ \mu_A) \geq (\mu_A \circ \chi_x^{\mu_A e}) \wedge \mu_B$ and $(\chi_x^{\nu_A e} \circ \nu_A) \leq (\nu_A \circ \chi_x^{\nu_A e}) \vee \nu_B$.
(3)⇒(1): Letting $z^{-1} = x^{-1}y$ and by 2.2(3) and 2.2(4), we have
$\mu_A(x^{-1}yx) = \mu_A(z^{-1}x\ ) = (\chi_z^{\mu_A e} \circ \mu_A)x \geq (\mu_A \circ \chi_z^{\mu_A e})x \wedge \mu_B x = \mu_A(xz^{-1}) \wedge \mu_B(x) = \mu_A(xx^{-1}y) \wedge \mu_B(x) = \mu_A(y) \wedge \mu_B(x)$ and
$\nu_A(x^{-1}yx) = \nu_A(z^{-1}x) = (\chi_z^{\nu_A e} \circ \nu_A)x \leq (\nu_A \circ \chi_z^{\nu_A e})x \vee \nu_B x = \nu_A(xz^{-1}) \vee \nu_B(x)$ $= \nu_A(xx^{-1}y) \vee \nu_B(x) = \nu_A(y) \vee \nu_B(x)$ or for each $x, y \in G$, $\mu_A(x^{-1}yx) \geq \mu_A(y)$ $\wedge\ \mu_B(x)$ and $\nu_A(x^{-1}yx) \leq \nu_A(y) \vee \nu_B(x)$ or A is an if/v-normal subgroup of B.

Theorem 3.18 *For any pair of if/v-subgroups A and B of a group G such that A is an if/v-normal subgroup of B:*

1. *A_* is a normal subgroup of B_*.*
2. *A^* is a normal subgroup of B^* whenever [0, 1] is strongly regular.*

Proof: (1): Since $\mu_A e$ is the largest of $\mu_A G$, $\nu_A e$ is the smallest of $\nu_A G$ and A is an if/v-subgroup of G, we get for each $x, y \in A_*$, $\mu_A(xy^{-1}) \geq \mu_A x \wedge \mu_A y = \mu_A e$ and $\nu_A(xy^{-1}) \leq \nu_A x \vee \nu_A y = \nu_A e$, so we have $\mu_A xy^{-1} = \mu_A e$ and $\nu_A xy^{-1}$ $= \nu_A e$ or $xy^{-1} \in A_*$. Hence A_* is a subgroup of B_*.
Again since $\mu_A e$ is the largest of $\mu_A G$, $\nu_A e$ is the smallest of $\nu_A G$, A is an if/v-normal subgroup of B; we get for each $b \in B_*$ and $a \in A_*$, $\mu_A(bab^{-1}) \geq \mu_A a \wedge \mu_B b = \mu_A e \wedge \mu_B e = \mu_A e$ and $\nu_A(bab^{-1}) \leq \nu_A a \vee \nu_B b = \nu_A e \vee \nu_B e = \nu_A e$, so we have $\mu_A(bab^{-1}) = \mu_A e$ and $\nu_A(bab^{-1}) = \nu_A e$ or $bab^{-1} \in A_*$. Therefore A_* is a normal subgroup of B_*.
(2): Since [0, 1] is strongly regular, for each $x, y \in A^*$, $\mu_A(xy^{-1}) \geq \mu_A x \wedge \mu_A y$ > 0 and $\nu_A(xy^{-1}) \leq \nu_A x \vee \nu_A y < 1$ or $xy^{-1} \in A^*$. Hence A^* is a subgroup of B^*.
Again, since [0,1] is strongly regular, for each $b \in B^*$ and $a \in A^*$, we get $\mu_A(bab^{-1}) \geq \mu_A a \wedge \mu_B b > 0$ and $\nu_A(bab^{-1}) \leq \nu_A a \vee \nu_B b < 1$ or $bab^{-1} \in A^*$. Hence A^* is a normal subgroup of B^* when [0,1] is strongly regular.

Lemma 3.19 *For any pair of if/v-subgroups A and B of a group G such that A is an if/v normal subgroup of B, the if/v-subset C: $\frac{B^*}{A^*} \rightarrow$[0, 1] defined by, for each $b \in B^*$ $\mu_C bA^* = \vee \mu_B bA^*$ and $\nu_C bA^* = \wedge \nu_B bA^*$, is an if/v-subgroup of $\frac{B^*}{A^*}$, whenever [0,1] is a strongly regular complete infinite distributive lattice.*

Notes

Proof: Since [0,1] is strongly regular, by 3.18, A^* is a normal subgroup of B^*. Now in 3.13 set $G = B^*$, $N = A^*$, $B = B$. Then since [0,1] is a complete infinite distributive lattice, C is an if/v-subgroup of $\frac{B^*}{A^*}$.

Definition 3.20 For any pair of if/v-subgroups A and B of a group G such that A is an if/v normal subgroup of B and L is a strongly regular complete infinite distributive lattice, the if/v-quotient subgroup of $B|B^*$ relative to A^*, denoted by B/A or $\frac{B}{A}$, is defined by B/A: $B^*/A^* \to L$ with $\mu_{B/A}(bA^*) = \vee\mu_B(bA^*)$ and $\nu_{B/A}(bA^*) = \wedge\nu_B(bA^*)$ for each $b \in B^*$ and is called L-if/v-quotient subgroup of B relative to A.

In what follows we prove a natural relation between $(\frac{B}{A})^*$ and $\frac{B^*}{A^*}$ which is used in the Third Isomorphism Theorem.

Lemma 3.21 For any pair of if/v-subgroups A and B of a group G such that A is an if/v-normal subgroup of B, $(B/A)^* = B^*/A^*$.

Proof: Let us recall that $(B/A)^* = \{bA^* \in (B^*/A^*)\ /b \in B^*, \mu_{B/A}(bA^*) > 0$ and $\nu_{B/A}(bA^*) < 1\}$. So always, $(B/A)^* \subseteq B^*/A^*$.
$\alpha \in B^*/A^*$ implies $\alpha = bA^*$ for some $b \in B^*$. Now as $e \in A^*$ and $b \in B^*$, $\mu_{B/A}(bA^*) = \vee\mu_B(bA^*) \geq \mu_B b > 0$ and $\nu_{B/A}(bA^*) = \wedge\nu_B(bA^*) \leq \nu_B b < 1$ implying $\alpha \in (B/A)^*$. Hence $(B/A)^* = B^*/A^*$.

Theorem 3.22 For any if/v-normal subgroup A of G and an if/v-subgroup B of G, $A \wedge B$ is an if/v-normal subgroup of B.

Proof: By 3.11, if A, B are if/v-subgroups of G then $A \wedge B$ is an if/v-subgroup of G and $A \wedge B \leq B$. Now we show that $C = A \wedge B$ is an if/v-normal subgroup of B or for each $x, y \in G$, $\mu_C(xyx^{-1}) \geq \mu_C(y) \wedge \mu_B(x)$ and $\nu_C(xyx^{-1}) \leq \nu_C(y) \vee \nu_B(x)$. Since A is an if/v-normal subgroup of G, for each $x, y \in G$,

$$\mu_C(xyx^{-1}) = (\mu_A \wedge \mu_B)(xyx^{-1}) = \mu_A(xyx^{-1}) \wedge \mu_B(xyx^{-1})$$
$$\geq \mu_A(y) \wedge \mu_B(xyx^{-1}) \geq \mu_A(y) \wedge \mu_B(x) \wedge \mu_B(y) \wedge \mu_B(x^{-1})$$
$$= (\mu_A(y) \wedge \mu_B(y)) \wedge \mu_B(x) = \mu_{A\wedge B}(y) \wedge\mu_B(x) = \mu_C(y) \wedge \mu_B(x) \text{ and}$$
$$\nu_C(xyx^{-1}) = (\nu_A \vee \nu_B)(xyx^{-1}) = \nu_A(xyx^{-1}) \vee \nu_B(xyx^{-1})$$
$$\leq \nu_A(y) \vee \nu_B(xyx^{-1}) \leq \nu_A(y) \vee \nu_B(x) \vee \nu_B(y) \vee \nu_B(x^{-1})$$
$$= (\nu_A(y) \vee \nu_B(y)) \vee \nu_B(x) = \nu_{A\wedge B}(y) \vee\nu_B(x) = \nu_C(y) \vee \nu_B(x).$$

Therefore $\mu_C(xyx^{-1}) \geq \mu_C(y) \wedge \mu_B(x)$ and $\nu_C(xyx^{-1}) \leq \nu_C(y) \vee \nu_B(x)$ or $C = A \wedge B$ is an if/v-normal subgroup of B.

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Notes