Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 3 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

# General Definition of the Mean for Non-Additive Variables

By Vaclav Studeny
*Masaryk University, Czech Republic*

*Abstract-* This short note aims to offer the conceptual approach to the definition of the mean value, with reection to the prewious very interesting attempts with the same goal. The definition is at once simple and powerfull and its advantage are demonstrated with aplications in financial mathematics. The definition of the man value involves the term objective function, which characterize the nature of object, which is taken to average.

*Keywords: mean value, average, aggregate interest rate.*

GJSFR-F Classification : *FOR Code : MSC 2010: 26E60*

GENERALDEFINITIONOFTHEMEANFORNONADDITIVEVARIABLES

*Strictly as per the compliance and regulations of :*

Ref

3. Billingsley, Patrick (1995), Probability and Measure (Third ed.), John Wiley sons, ISBN 0-471-00710-2.

# General Definition of the Mean for Non-Additive Variables

Vaclav Studeny

*Abstract-* This short note aims to offer the conceptual approach to the definition of the mean value, with reection to the prewious very interesting attempts with the same goal. The definition is at once simple and powerfull and its advantage are demonstrated with aplications in financial mathematics. The definition of the man value involves the term objective function, which characterize the nature of object, which is taken to average.

*Keywords: mean value, average, aggregate interest rate.*

## I. Introduction

### a) *General observation*

If we sum some deterministic variable, we obtain the same result as if we substitute all of the variable by its arithmetical mean. In stochastic case it is true approximately and with a big probability if we sum big enough number of independent and identically distributed variables with finite second moment as central limit theorem show us (see [3], [5]). But in life we do not deal with summing only. If we will multiply them, for instance, and if we sustitue the values by arithmetical mean, the result will differ more if the values of the variables will vary from the mean more. But if we used geometrical mean except of arithmetical ones, the situation will be exactly as in previous case of summation. This is an observation which bring us to our simple, but very useful concept.

There are a lot of other reasons why we need to represent a big set of data by only one number. Expected value or arithmetical mean is the most popular way how to do it, but not the best in all of the cases.

To make the problem of mean value more clear let us consider investment funds. They can measure the profit in different years by the rate of return, but rate of return is not additive variable too, this means, that the sum of the interest rate is not quantity with the sens. What information gives to us the arithmetical mean of the rates of return per different year. The answer is not very heartwarming.

Let us suppose that the number of time periods (say of years to be in the common practical situation) is $N$, and rate of profit in $i$-th period is $\xi_i$. let us suppose, that mean value of rates is equal to K, hence:

$$\frac{1}{N}\sum_{i=1}^{N}\xi_i = K \tag{1}$$

overall rate of profit by all of the $N$ time periodes is

$$\zeta = \prod_{i=1}^{N}(1+\xi_i) - 1 \tag{2}$$

If $N \geq 2$ equations (1) and (2) does not determine $\zeta$ — which is the value we are actually interested in — in unique way. On the set of all $(\xi_i)_{i=1}^{N}$ fulfilling (1) take the function $\zeta$ different values: its maximum is $(1+K)^N - 1$, and it is not bounded below indeed, hence: *The knowledge of arithmetical mean admit*

*Author: Department of Applied Mathematics, Faculty of Economics and Administration, Masaryk University Lipova, Brno.*
*e-mail: Vclv.St@gmail.com*

*to bounded the rate of the whole return above but not below. With the same arithmetical mean the rate of the whole return can be arbitrarily small.*

*1.1 Proof:* If $N = 2$ then if $\xi_1$ a $\xi_2$ are rates of return in a two successive time periods and $\zeta$ rate of return per two period and if $K$ is the arithmetical mean of $\xi_1$ a $\xi_2$ we have:

$$(1+\xi_1)\cdot(1+\xi_2) = 1+\zeta$$

$$\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 = K$$

hence

$$\xi_2 = -\xi_1 + 2\,K$$

$$\zeta = 2\,K - {\xi_1}^2 + 2\,\xi_1\,K \tag{3}$$

The last equation gives attitude toward actual rate of return per two periods and rate of return per one of the period in the case of constant average rate of return $K$. The dependence is analytical so that we can find the maximum as the zero point of the derivative:

$$\frac{\partial}{\partial \xi_1}\zeta = -2\,\xi_1 + 2\,K = 0 \Longleftrightarrow \xi_1\xi_2 = K \tag{4}$$

and

$$\frac{\partial^2}{\partial {\xi_1}^2}\zeta = -2 \tag{5}$$

so that if average rate of return is constant equal to $K$ then:

- rate of return $\zeta$ have maximum $\zeta = (K+1)^2 - 1$ if $\xi_1 = \xi_2 = K$, i. e. if all of the rates are equal.
- If one of the rate is equal to zero, $\xi_1 = 0$, and the second is equal to $\xi_2 = 2K$ then the whole rate of return is $\zeta = 2K$.
- If one of the rate $\xi_1 = -1$ i. e. we lost all of the capital, it is enough, to second rate of return be $\xi_2 = 1 + 2\,K$ and the mean value will stay $K$, while whole rate of return will be $\zeta = -1$. It means that the investor lost all of the invested resources, but arithmetical mean of rates of the return is still $K$. This fact show the marketing potential of arithmetical mean in the case. You can report the average rate of return 10% while you bereave your clients for all of the money.
- further if $\xi_1$ goes to $-\infty$, then $\xi_2$ goes to $-\infty$ and total rate of return goes to $-\infty$ but we can still keep the positive average.

A similar situation occurs also in the case when the number of time periods is greater. If we know arithmetical mean $K$ of the rates of return the rate of return per $N$ time periods is

$$\left(1 + N\,K - \left(\sum_{i=1}^{N-1}\xi_i\right)\right)\left(\prod_{i=1}^{N-1}(1+\xi_i)\right) = 1+\zeta \tag{6}$$

and the function $\zeta$, with arguments made of $N-1$ rates of returns $\xi_i$ has again maximum

$$\zeta = (1 + K/N)^N - 1 \tag{7}$$

in the point

$$(\xi_i)_{i=1}^N = (K)_{i=1}^n \tag{8}$$

and $\zeta$ is not bounded above.

- Present value of invested capital should be equal to 0 independently on the arithmetical mean of rates of return. It is necessary and sufficient if one of the rates is equal to $-1$.
- If one of a fund has (arithmetical) average of rate returns higher than other, it does not necessary mean, that it has higher total rate of return per all of the periods (that it brings higher profit to its savers). For the marketing point of view to show mean o rates of returns is better advertisement for the funds which have higher diversity between rates of return.

We can see, that arithmetical mean does not gives to us any substantive information in that case.

Similar situation is, of course, nowadays popular average temperature of the earth (see [4]). What rush people make about it notwithstanding this quantity is not satisfactorily defined. The number, which

Ref

4. Climate Change 2013: The Physical Science Basis. Contribution of Working Group I to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change.

should be a result of some integration does not realize themselves anywhere. This fact will goes more clear, if we imagine that we calculated the average temperatures of stars that we see. The result should not be the temperature of any place in the universe. /it There exists average of temperatures, but not average temperature.

In the case of investment funds, one can say, that we only used bad mean. That usage of the geometrical mean, brings to us much more better result. If we should compute the mean using the rule:

$$\zeta = \left( \prod_{i=1}^{N} (1 + \xi_i) \right)^{\left(\frac{1}{N}\right)} - 1 \tag{9}$$

expect of

$$\zeta = \frac{1}{N} \sum_{i=1}^{N} \xi_i \tag{10}$$

we should obtain something, connected with the rate of return by the unique and the simple way: $\left( \prod_{i=1}^{N} (1 + \xi_i) \right) = (1 + \zeta)^N$. Particularly funds with higher mean rate of return should have higher whole return.

In the history there are more cases of attempts to generalize the notion of mean value. Two of them I find the most interesting and the most deep.

## II. Two General Conceptions of Mean

With one generalization come János Dezs Aczél, the great Hungarian mathematician living in Canada. As en expert on the functional equations, he watch the question: how big family of possible means of two values one can obtain if he compute aritmetical mean of images of this values by some mapping and then he returned back with inverse mapping. To be more precise on one side we can imagine the mean as a binary operation $\circ$ an on the other hand as a value

$$(x, y) \mapsto k^{-1} \left( \frac{k(x) + k(y)}{2} \right) \tag{11}$$

General properties of this mean are described in [Azzel p. 229]. Main result, which shows quite general character of this definition is:

*2.2. Theorem:* There exists a continuous and strictly monotonic function $k$ which gives a value of a mean; (11) holds if, and only if, $\circ : I^2 \mapsto I$ is continuous and strictly increasing in booth variables, idempotent, commutative (symmetric) and medial (bisymetric) which means:

$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w) \tag{12}$$

It is rely beautiful result but not practical enough.

On the other way, in the Book Means and Their Inequalities, P. S. Bullen, D. S. Mitrinović P. M. Vasić (see [2] p. 372) attempt to axiomatic definition of the mean:

- it is symmetric,
- homogeneous of degree 1,
- reflexive
- associative, $\mathrm{f}(a_1, \ldots, a_n) = \mathrm{f}(\mathrm{f}(a_1, \ldots, a_p), \ldots, \mathrm{f}(a_1, \ldots, a_p), a_{p+1}, \ldots, a_n)$
- increasing in each variable.

This definition is very good of generaliyation od the term, but not good enough for us, from a lot of reasons. One of the reason should be that a average interest rate of saving is not symmetrical (and the symmetry is not disrupt only by adding of some weights). By latest interest are interested all of the deposits, but by the first one only the first deposit.

The drawback of these conceptions is, that the mean is related only to the values, it is computed from but not to their meaning, which means to this what we are going to do with the values. This fact has a deep impact. There are lot of monez ion the

(For instance: arithmetical mean has sense only for additive variables it means variables we are going to sum but common definition does not respect this fact.) Our conception is at once more simple,

Ref

2. P.S. Bullen, D.S. Mitrinovic, P.M. Vasic, Means and Their Inequalities, D. Reidel Publishing Com-any, Dordrecht, 1988.

more general. It is, in some way, the generalization of the Azzel's conception. The central role plays the *objective function* in it. This function characterize what are the values, we deal with, relevant for.

## III. General Definition of the Mean

3.3. *Difinition:* $Z$ is the mean value of $(z_i)_{i=1}^n$ *with respect to function* $\mathrm{F}\colon \coprod_{i\in I} X^i \to Y$ if

$$\mathrm{F}\left((z_i)_{i=1}^n\right) = \mathrm{F}\left((Z)_{i=1}^n\right) \tag{13}$$

(i. e. $F(z_1, z_2, \ldots, z_n) = F(Z, Z, \ldots, Z)$).
If $F$ is injection, then the mean value is determined by unique way.
If $\mathrm{Im}(f) = \mathrm{Im}(f|_\Delta)$, where $\Delta$ is diagonal: $\Delta = \{(x)_{i=1}^n | x \in X\}$, mean exists for every input values.

3.4. *Example:* If $F$ is sumation, we obtain arithmetical mean. If $F$ is multiplying, we obtain geometrical mean.

In the case of an election, the votes are summed. Arithmetical mean of number of votes corespondents to proportional representation in parliment.
Rates of inflation does multiply to each other. If

$$\iota := [.01,\ .03,\ .02,\ .01,\ .03] \tag{14}$$

is the rate of inflation per 5 time periodes, the rate of inflation per all of the time is

$$\left(\prod_{j=1}^{5} (1+\iota_j)\right) - 1 = .10386857 \tag{15}$$

and if the rate of inflation should be the same per all of the time subintervals an should be equal to

$$\kappa := \left(\prod_{j=1}^{5} (1+\iota_j)\right)^{1/5} - 1 = .019960783 \tag{16}$$

then the rate of inflation per all of the time should be the same:

$$\left(\prod_{j=1}^{5} (1+\kappa)\right) - 1 = .10386857 \tag{17}$$

Hence if the objective function will be

$$\iota \mapsto \left(\prod_{j} (1+\iota_j)\right) - 1 \tag{18}$$

then the mean with respect to this function is (except for some addition of unit) geometrical mean.

## IV. Mean Rate of Return of Pension Funds

Let us suppose, that we know rate of return of pension funds. We are looking for mean rate of return, which admits to us compare this funds.

Let us suppose, that deposits of the savers are constant for a long of the time. We can suppose, that, if the deposit per year (together with state grants) are equal to $x$ and if the rate of return, in consequence of dividing of profit in the year $i$ is equal to $\xi_i$ saver should have after the time $N$:

$$x\left(\sum_{j=1}^{N} \prod_{i=1}^{j} (1+\xi_{N-i+1})\right) \tag{19}$$

For saver is important mean rate of return with respect to the function

Notes

Global Journal of Science Frontier Research ( F ) Volume XV Issue III Version I Year 2015

$$(\xi)_{i=1}^{T} \to \left( \sum_{j=1}^{N} \prod_{i=1}^{j} (1+\xi_{N-i+1}) \right) \tag{20}$$

using our definition, this rate of the return is $\zeta$, fulfilling the equation

$$K = \sum_{j=1}^{N} \left( \prod_{i=1}^{j} (1+\xi_{N-i+1}) \right) = \sum_{j=1}^{N} \left( \prod_{i=1}^{j} (1+\zeta) \right) = -\frac{-(1+\zeta)^{(N+1)}+1+\zeta}{\zeta} \tag{21}$$

Notes

It is an algebraical equation of the degree $N$.
If the time period is only one, the mean rate of return is equal to the rate of return per this period:

$$\zeta = -1 + K \tag{22}$$

If the number of time periods is equal to 2, the equation has two solutions, in general.

If the total remittance will be higher than $\frac{1}{4}$ of all saved remittances all of the solution will be real. In the case — it seem to be reasonable — when total remittance will be higher than double of saved remittances one solution will be positive an will be equal to

$$\zeta = -\frac{3}{2} + \frac{1}{2}\sqrt{1+4\,K} \tag{23}$$

In case of 3 periods, there is the only one real mean rate of return:

$$\zeta = 1/6\,\frac{\left(28+108\,K+12\,\sqrt{9+42\,K+81\,K^2}\right)^{2/3} - 8 - 8\sqrt[3]{28+108\,K+12\,\sqrt{9+42\,K+81\,K^2}}}{\sqrt[3]{28+108\,K+12\,\sqrt{9+42\,K+81\,K^2}}} \tag{24}$$

If the number of time periods is higher than four, we are not able to solve the equation (21) algebraically, and we have to solve it numerically.

4.5. *Example:* There was some pension funds in Czech republic during the five year between the years 1999 and 2003. This was an period with very wildly changed rates of return of the pension funds. We will concentrate onto these:

(1) ČSOB Progres, (2) Zemský PF, (3) PF KB, (4) ING, (5) Credit Suisse, (6) PF ČP, (7) Allianz, (8) Generali, (9) Nový ČP, (10) ČSOB Stabilita, (11) PF ČS, (12) Hornický PF; The following table show us their rate of return:

| **Funds** | **rate of return (in %)** | | | | |
|---|---|---|---|---|---|
| | 1999 | 2000 | 2001 | 2002 | 2003 |
| ČSOB Progres | 7.7 | 5.6 | 3.9 | 4.3 | 4.3 |
| Zemský PF | 7.0 | 5.0 | 4.6 | 4.1 | 4.01 |
| PF KB | 7.2 | 4.9 | 4.4 | 4.6 | 3.4 |
| ING | 6 | 4.4 | 4.8 | 4 | 4 |
| Credit Suisse | 6.5 | 4.1 | 4.3 | 3.4 | 3.4 |
| PF ČP | 6.6 | 4.5 | 3.8 | 3.2 | 3.1 |
| Allianz | 6 | 3.8 | 4.4 | 3.7 | 3 |
| Generali | 5.3 | 3.6 | 4.6 | 4.1 | 3 |
| Nový ČP | 5.6 | 3.8 | 4.1 | 3.5 | 3.34 |
| ČSOB Stabilita | 6.1 | 4.2 | 3.2 | 3.0 | 2.34 |
| PF ČS | 4.4 | 4.2 | 3.8 | 3.5 | 2.64 |
| Hornický PF | 4.4 | 2 | 2.8 | 3.2 | 2.44 |

(25)

Českomoravská stavební spořitelna published for agents middleman papers for information. They ordered the pension funds by the arithmetical mean of their rate of return, which are, as we have explained, irrelevant (second row of following table, values are in percents). If we computed the mean with respect to objective function ($\xi_i^k$ is rate of return of $k$-th fund at $i$-th time period):

$$\Phi\colon \left(\xi^k\right)_{i=1}^{5} \longmapsto \sum_{j=1}^{5} \prod_{i=1}^{j} \left(1+\xi^k_{5-i+1}\right)\Bigg) \tag{26}$$

Notes

(third column of following table, values are in percents) we can see, that it happens in some cases, that funds with higher rate of return are behind funds with smaller rate of return:

| **Funds** | **mean values of the rates of return** | |
|---|---|---|
| | **arithemtical** (*inappropriate*) | **vith respect to the $\Phi$** (*appropriate*) |
| ČSOB Progres | 5.16 | 4.638 |
| Zemský PF | 4.94 | 4.501 |
| PF KB | 4.90 | 4.394 |
| ING | 4.64 | 4.358 |
| Credit Suisse | 4.34 | 3.895 |
| PF ČP | 4.24 | † 3.704 |
| Allianz | 4.18 | † 3.787 |
| Generali | 4.12 | † 3.857 |
| Nový ČP | 4.06 | 3.757 |
| ČSOB Stabilita | 3.76 | ‡ 3.203 |
| PF ČS | 3.70 | ‡ 3.438 |
| Hornický PF | 2.96 | 2.789 |

(27)

(†) *We can see, that the fund Allianz, which stay on the 7th place had the rate of return better, than PF ČP which is on the 6th place and better rate of return than this two had thwe fund Generali, which stay behind booth others. Booth are in the papers presented as the funds with the same awarage rate of return equal to 4.2.*

(‡) *The fund ČSOB Stabilita, the papers in question are written for its propagation too, is presented as the fund with mean rate of return (arithmetic mean 3.8) higher than rate of return of fund PF ČS (arithmetic mean 3.7) but it should be presented (with mean 3.2) behind the fund PF ČS (mean 3.4)!*

If we are going to interpreted the result, we have to determine the sensitivity on the change of interest rate. We can substitute the rates of return by mean rates of return we have computed. The relative change of saved money after time $T$ if the interest rate is changed from $\xi$ onto $\zeta$, (i. e.. rate of the profit or loss on the whole saved quontity) is

$$\frac{(1+\zeta)^T\,\xi-\xi-(1+\xi)^T\,\zeta+\zeta}{\left((1+\xi)^T-1\right)\,\zeta}. \tag{28}$$

In our case is $T=5$ and the values for differen funds are in the same order as in the previous tables, they are in percents:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | −0.27 | −0.49 | −0.56 | −1.5 | −1.8 | −1.7 | −1.5 | −1.7 | −2.8 | −2.4 | −3.6 |
| 0.27 | 0 | −0.21 | −0.29 | −1.2 | −1.6 | −1.4 | −1.3 | −1.5 | −2.6 | −2.1 | −3.4 |
| 0.49 | 0.21 | 0 | −0.072 | −0.99 | −1.4 | −1.2 | −1.1 | −1.3 | −2.4 | −1.9 | −3.2 |
| 0.56 | 0.29 | 0.072 | 0 | −0.92 | −1.3 | −1.1 | −1.0 | −1.2 | −2.3 | −1.8 | −3.1 |

Notes

$$
\begin{array}{cccccccccccc}
1.5 & 1.2 & 1.0 & 0.93 & 0 & -0.38 & -0.21 & -0.075 & -0.27 & -1.4 & -0.91 & -2.2 \\
1.9 & 1.6 & 1.4 & 1.3 & 0.38 & 0 & 0.17 & 0.31 & 0.11 & -1.0 & -0.53 & -1.8 \\
1.7 & 1.4 & 1.2 & 1.1 & 0.21 & -0.17 & 0 & 0.14 & -0.061 & -1.2 & -0.70 & -2.0 \\
1.6 & 1.3 & 1.1 & 1.0 & 0.075 & -0.31 & -0.14 & 0 & -0.20 & -1.3 & -0.83 & -2.1 \\
1.8 & 1.5 & 1.3 & 1.2 & 0.28 & -0.11 & 0.061 & 0.20 & 0 & -1.1 & -0.64 & -1.9 \\
2.9 & 2.6 & 2.4 & 2.3 & 1.4 & 1.0 & 1.2 & 1.3 & 1.1 & 0 & 0.47 & -0.83 \\
2.4 & 2.1 & 1.9 & 1.9 & 0.92 & 0.53 & 0.70 & 0.84 & 0.64 & -0.47 & 0 & -1.3 \\
3.8 & 3.5 & 3.3 & 3.2 & 2.2 & 1.8 & 2.0 & 2.2 & 2.0 & 0.83 & 1.3 & 0
\end{array} \tag{29}
$$

*In the $i$-th row $j$-th column the percentage of how much should be the whole amount higher if we invested into $j$-th fund except of into $i$-th fund is written. So in the proper choose of the fund we should have approximately 4 percent more than in the improper choose.*

## V. Mean Rate of Return of Simultaneous Savings

Suppose, that we have two accounts, each interested with different interest rate $\xi_1$ and $\xi_2$. We divide the capital in amount $x_1 + x_2$ between them this way, than we will have in he first capital in amount $x_1$ and in the second $x_2$. Objective function — future value in time $t$ is the function $\Psi_{x_1,x_2,t}\colon (\xi_1, \xi_2) \mapsto x_1 (1+\xi_1)^t + x_2 (1+\xi_2)^t$ and it depende on three parameters. Mean interest rate with respect to this function is the solution $\zeta$ of the equation:

$$x_1 (1+\xi_1)^t + x_2 (1+\xi_2)^t = (x_1 + x_2)\,(1+\zeta)^t \tag{30}$$

hence

$$\zeta = \left. \frac{x_1\,(1+\xi_1)^t + x_2\,(1+\xi_2)^t}{x_1 + x_2} \right)^{\left(\frac{1}{t}\right)} - 1 \tag{31}$$

And it is generalized exponential mean. Worthy to note is its dependence on the time $t$, especially the limits $t \to \infty$.

$$\lim_{t\to\infty} \left. \frac{x_1 (1+\xi_1)^t + x_2 (1+\xi_2)^t}{x_1 + x_2} \right)^{\left(\frac{1}{t}\right)} - 1 = \lim_{x\to\infty} e^{\ln \left( \frac{x_1\,(1+\xi_1)^t + x_2\,(1+\xi_2)^t}{x_1 + x_2} \right)^{\left(\frac{1}{t}\right)}} - 1 =$$

$$= \lim_{x\to\infty} e^{\dfrac{\ln\left(x_1\,(1+\xi_1)^t + x_2\,(1+\xi_2)^t\right) - \ln(x_1+x_2)}{t}} - 1 \tag{32}$$

using the L'Hospital rule we obtain

$$\lim_{t\to\infty} \frac{\ln\left(x_1\,(1+\xi_1)^t + x_2\,(1+\xi_2)^t\right) - \ln(x_1+x_2)}{t} =$$

$$= \lim_{t\to\infty} \frac{x_1\,(1+\xi_1)^t \ln(1+\xi_1) + x_2\,(1+\xi_2)^t \ln(1+\xi_2)}{x_1\,(1+\xi_1)^t + x_2\,(1+\xi_2)^t} \tag{33}$$

and after canceling by the term $(1+\xi_2)^t$we obtain:

$$\lim_{t\to\infty} \frac{x_1 \ln(1+\xi_1) \left(\dfrac{1+\xi_1}{1+\xi_2}\right)^t + x_2 \ln(1+\xi_2)}{x_1 \left(\dfrac{1+\xi_1}{1+\xi_2}\right)^t + x_2}. \tag{34}$$

Supposing: $\xi_1 < \xi_2$

$$\lim_{t\to\infty} \left. \frac{x_1\ (1+\xi_1)^t + x_2\ (1+\xi_2)^t}{x_1+x_2} \right)^{\left(\frac{1}{t}\right)} - 1 = \xi_2, \tag{35}$$

we obtain the greatest of both number. It looks mysteriously: If we save divide capital into two parts and save booth with some interest rates per unit of time $\xi_1$ and $\xi_2$ respectively , we obtain the same result as to save booth parts with interest rate per unit of time $\zeta$, and if the time goes to infinity the difference bvetween $\zeta$ and the maximum of $\xi_1$ and $\xi_2$ goes to the zero On the other hand:

$$\lim_{t\to 0} \zeta = t\,(1+\xi_1)^{\frac{x_1}{x_1+x_2}}\ (1+\xi_2)^{\frac{x_2}{x_1+x_2}} - 1 \tag{36}$$

which is generalized geometrical mean.

The same result we obtain also in the case of more than two accounts. If the rates of interest will be $(\xi_i)$ an the initial state of accounts will be $(x_i)$ then the state function will be

$$(\xi_i) \longmapsto \sum_{i=1}^{n} x_i\,(1+\xi_i)^t \tag{37}$$

and mean value $(\xi_i)$ with respect to the function will be solution $\zeta$ of the equation

$$\sum_{i=1}^{n} x_i\,(1+\xi_i)^t = \sum_{i=1}^{n} x_i\,(1+\zeta)^t = (1+\zeta)^t \sum_{i=1}^{n} x_i \tag{38}$$

hence

$$\zeta = \left. \frac{\sum_{i=1}^{n} x_i\,(1+\xi_i)^t}{\sum_{i=1}^{n} x_i} \right)^{\frac{1}{t}} - 1 \tag{39}$$

and if $\xi_n = \max_i(\xi_i)$ we can show in the same way, that

$$\lim_{t\to\infty} \zeta = e^{\left(\lim_{t\to\infty} \frac{\left(\sum_{i=1}^{n-1} x_i \ln(1+\xi_i) \left(\frac{1+\xi_i}{1+\xi_n}\right)^t\right) + x_n \ln(1+\xi_n)}{\left(\sum_{i=1}^{n-1} x_i \left(\frac{1+\xi_i}{1+\xi_n}\right)^t\right) + x_n}\right)} - 1 = \xi_n = \max_i(\xi_i) \tag{40}$$

a

$$\lim_{t\to 0} \zeta = \prod_{i=1}^{n} (1+\xi_i)^{\frac{x_i}{\sum_{j=1}^{n} x_j}} \tag{41}$$

*Aggregate interest rate per unit of time converges to maximal interest rate per unit of time with time going to infinity.*

The similar situation is the saving annuities:

## VI. Mean Interest Rate of Saving

Suppose, savings with annuities $x_i$ in the moments $t \in \mathbb{N}$ with constants interest rates $\xi_i$. Objective function is sum of future values of savings with different interest rates:

$$\Phi_{(x_i)_{i=1}^k, N} := \left((\xi_i)_{i=1}^k\right) \longmapsto \sum_{i=1}^{k} \left(\sum_{j=0}^{N-1} x_i\ (1+\xi_i)^j\right) = \sum_{i=1}^{k} \frac{x_i\ \left((1+\xi_i)^N - 1\right)}{\xi_i} \tag{42}$$

saved amount, $(x_i)_{i=1}^k$ and number of savings with different interest rates $k$ are parametrs $\Phi$.

Mean interest rate of savings $\Xi = \Xi\left((x_i), N\right)$ with respect to function $\Phi_{(x_i)_{i=1}^k, N}$ is solution of equation:

$$xxx := \frac{\left.\sum_{i=1}^{k} x_i\right)\ \left((1+\Xi)^N - 1\right)}{\Xi} = \sum_{i=1}^{k} \frac{x_i\ \left((1+\xi_i)^N - 1\right)}{\xi_i} \tag{43}$$

Notes

Notes

We will investigate dependence on parametr $N$, i. e. on the number of saved ammounts. And again:

$$\lim_{t\to\infty} \Xi = \max_i (\xi_i) \tag{44}$$

*6.6. Proof:* Let

$$\eta = \eta(\xi_i, x_i, n, T) \tag{45}$$

be mean interest rate with respect to the function

$$\Phi := \xi \to \sum_{\tau=1}^{n} \left( \sum_{j=1}^{k} x_j \ (1+\xi_j)^{(T-\tau)} \right) \tag{46}$$

i. e. solutiobn of the equation

$$\sum_{\tau=1}^{n} \left( \sum_{j=1}^{k} x_j \ (1+\xi_j)^{(T-\tau)} \right) = \left( \sum_{j=1}^{k} x_j \right) \quad \sum_{\tau=1}^{n} (1+\eta)^{(T-\tau)} \Big) \tag{47}$$

and let

$$\zeta(\xi_i, x_i, \tau, T) \tag{48}$$

be a mean interest rate with respect to the function

$$\Psi := \xi \to \sum_{j=1}^{k} x_j \ (1+\xi_j)^{(T-\tau)} \tag{49}$$

i. e. solutioon of the equation

$$\sum_{j=1}^{k} x_j \ (1+\xi_j)^{(T-\tau)} = (1+\zeta)^{(T-\tau)} \left( \sum_{j=1}^{k} x_j \right) \tag{50}$$

It is clear, that

$$\min\left(\zeta(\xi_i, x_i, \tau, T)\right)_{\tau=1}^{n} \le \eta(\xi_i, x_i, n, T) \le \max\left(\zeta(\xi_i, x_i, \tau, T)_{\tau=1}^{n}\right) \tag{51}$$

And using (40) we have

$$\lim_{T\to\infty} \min\left(\zeta(\ldots, \tau, T)_{\tau=1}^{n}\right) = \lim_{T\to\infty} \max\left(\zeta(\ldots, \tau, T)_{\tau=1}^{n}\right) = \max_i (\xi) \tag{52}$$

hence

$$\lim_{T\to\infty} \eta(\xi_i, x_i, n, T) \tag{53}$$

that is why

$$\lim_{T\to\infty} \eta(\xi_i, x_i, T, T) = \lim_{n\to\infty} \lim_{T\to\infty} \eta(\xi_i, x_i, n, T) = \lim_{n\to\infty} \max_i (\xi_i) = \max_i (\xi_i) \tag{54}$$

Q. e. d.

Mean interest rate of saving is less than mean iterest rate of interesting, but it has the same limit $t \to 0$ a $t \to \infty$.

The oprevious example vas the mean, which has already a name. Let us finished vith quite eotic mean, the mean which *is not increasing function in all of its arguments*.

*6.7. Example:* One of the specific bank in Czech republic, which offer mortgages is so called building societies. The common product they ofer has three parts, one used at once (it's happening in order to accommodate certain Czech laws): saving with one interest rate — saved money is then used for the redemption of debt — so called bridging credit with the second interest — this credit is used at the time of saving — and credit with third interest rat, which is used for amortization of the rest of the debt. Unnecessarily complicated? Mabe. And now imagine, that someone, who wont to buy a house will comes to the mortgage market try to compare all of the mortgages by their interest rate and here he has three interest rates except of one. Of course, he need to substitute them by only one comparable number and I think that this number should be called, in accordance with tradition, mean of this three. But it is clear, that while customer wishes the interest rate of the loan and bridge loan to be the as low as posiible he wants to have the interest rate of savings by contrast the greatest possible. So the mean will be increasing in the first two interest rates, but in the third will be decrasing.

Let us choose some denotation: suppose, that we pay the dept by payments $x_1$ with interest rate $\xi_1$ and at the same time we save money (as saving or insurance,...) by remittance at quantity $x_2$ with the interest rate $\xi_2$ both the time of duration $N$. After that we use the saved money to amortize part of the depth, than we pay the rest of the depth with interest rate $\xi_3$ by payments $x_3$ time of the length $K$.

Notes

For the evaluation of an expediency of a product like this is good to know the suitable mean value of the values $\xi_1$, $\xi_2$ and $\xi_3$. Present value of all of the payments, we are going to pay is

$$PV = x_1 \sum_{t=0}^{N-1} (1+\xi_1)^t (1+\xi_2)^{-N} + x_2 \sum_{t=1}^{N} (1+\xi_2)^{-t} + x_3 \sum_{t=1}^{K} (1+\xi_3)^{-t} (1+\xi_2)^{-N} =$$

$$= \frac{(1+\xi_2)^{-N} x_1 \left((1+\xi_1)^N - 1\right)}{\xi_1} - \frac{x_2 \left(\left((1+\xi_2)^{-1}\right)^N - 1\right)}{\xi_2} - \frac{\left(\left((1+\xi_3)^{-1}\right)^K - 1\right)(1+\xi_2)^{-N} x_3}{\xi_3}$$

hence we are interested in mean value with respect to objective function

$$(\xi_1, \xi_2, \xi_3) \mapsto PV$$

and this is the solution of the equation

$$x_1 \sum_{t=0}^{N-1} (1+\xi_1)^t (1+\xi_2)^{-N} + x_2 \sum_{t=1}^{N} (1+\xi_2)^{-t} + x_3 \sum_{t=1}^{K} (1+\xi_3)^{-t} (1+\xi_2)^{-N} =$$

$$= x_1 \sum_{t=0}^{N-1} (1+\zeta)^t (1+\zeta)^{-N} + x_2 \sum_{t=1}^{N} (1+\zeta)^{-t} + x_3 \sum_{t=1}^{K} (1+\zeta)^{-t} (1+\zeta)^{-N}$$

If no one of the interest rate is equal to 0, the equation is equivalent to

$$\frac{(1+\xi_2)^{-N} x_1 \left((1+\xi_1)^N - 1\right)}{\xi_1} - \frac{x_2 \left(\left((1+\xi_2)^{-1}\right)^N - 1\right)}{\xi_2} - \frac{\left(\left((1+\xi_3)^{-1}\right)^K - 1\right)(1+\xi_2)^{-N} x_3}{\xi_3} =$$

$$= \frac{(1+\zeta)^{-N} x_1 \left((1+\zeta)^N - 1\right)}{\zeta} - \frac{x_2 \left(\left((1+\zeta)^{-1}\right)^N - 1\right)}{\zeta} - \frac{\left(\left((1+\zeta)^{-1}\right)^K - 1\right)(1+\zeta)^{-N} x_3}{\zeta}$$

We are not able solve this equation algebraically in general so we cannot write the explicit formula for mean, we are looking for, but for all possible choice of parameters we can solve it numerically.

There is an interesting fact, that this mean is reflective (mean value of $(\zeta, \zeta, \zeta)$ is $(\zeta)$), but *it is not increasing in the first variable* $(\xi_1)$: no other conception mentioned above reckon on that fact!

## Literature


1. Aczel, J. and Dhombres, J., Functional Equations in Several Variables, Cambridge University Press, 1989. ISBN-10: 0521352762 — ISBN-13: 9780521352765
2. P.S. Bullen, D.S. Mitrinovic, P.M. Vasic, Means and Their Inequalities, D. Reidel Publishing Com-any, Dordrecht, 1988.

3. Billingsley, Patrick (1995), Probability and Measure (Third ed.), John Wiley sons, ISBN 0-471-00710-2.
4. Climate Change 2013: The Physical Science Basis. Contribution of Working Group I to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change.
5. Levy, Paul: Theorie de I' Addition des Variables ALeatoires, Paul Uvy. GauthierVillars, Paris, 1937.

Notes

This page is intentionally left blank