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Some Relationships between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities

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Some Relationships between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities

M. P. Chaudhary[α], Diriba Kejela Geleta[σ] & Gedefa Negassa Feyissa[ρ]

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I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{(n-1)} \tag{1.2}$$

$$(a_1, a_2, a_3, ..., a_k; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

In [1], Jacobi's triple product identity is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\Phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{1.6}$$

$$\Psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

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Ref

1. M. P. Chaudhary : *Generalization for character formulas in terms of continued fraction identities*, Malay J. Mat. 1(1)(2014) 24-34.

$$f(-q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}=(q;q)_\infty \tag{1.8}$$

Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)-1_\infty=(-q;q)_\infty \tag{1.9}$$

In [1], Roger-Ramanujan identities are given as

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.10}$$

$$H(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.11}$$

Roger-Ramanujan function is given by

$$R(q)=q^{\frac{1}{5}}\frac{H(q)}{G(q)}=q^{\frac{1}{5}}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \tag{1.12}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty(q^b;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(q^a,q^b,q^c\cdots q^t;q^n)_\infty \tag{1.13}$$

$$(q^a;q^n)_\infty(q^a;q^n)_\infty(q^c;q^n)_\infty\cdots(q^t;q^n)_\infty=(q^a,q^a,q^c\cdots q^t;q^n)_\infty \tag{1.14}$$

In [1, p-28(2.2)], following identity is given

$$(q^2;q^2)_\infty(-q;q)_\infty=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

$$=\frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-\cdots}\,;(|q|<1) \tag{1.15}$$

In [1, p-27(2.1)], following identity is given

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+\cdots}\,;(|q|<1) \tag{1.16}$$

In [1, p-28(2.4)], following identity is given

$$C(q)=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}=1+\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+\cdots}\,;(|q|<1) \tag{1.17}$$

Lastly, we turn to the recent investigation by Andrews et. al. [2], involving combinatorial partition identities associated with the following general family

$$R(s,t,l,u,v,w):=\sum_{n=0}^{\infty}q^{s(n\ 2)+tn}r(l,u,v,w;n) \tag{1.18}$$

where

$$r(l,u,v,w:n):=\sum_{j=0}^{[\frac{n}{u}]}(-1)^j\frac{q^{uv(j\ 2)+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j} \tag{1.19}$$

Ref

2. George Andrews, Kathrin Bringman, and Karl Mahlburg : *Double series Representations for Schur's partition function and related identities*, pre-print.

In particular, we recall the following combinatorial partition identities [2, p.106, Th.3]

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty \tag{1.20}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty \tag{1.21}$$

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty} \tag{1.22}$$

Ref

1. M. P. Chaudhary : *Generalization for character formulas in terms of continued fraction identities*, Malay J. Mat. 1(1)(2014) 24-34.

Computation of q-product identities

Chaudhary[1], has computed several q-product identities. Here we are giving some identities from [1], and some new identities have been computed, are useful for next section of this paper, as given below

$$(q^2;q^2)_\infty = \prod_{n=0}^{\infty}(1-q^{2n+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2})$$

$$= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4}) \times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8})$$

$$= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty = (q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.23}$$

$$(q^4;q^4)_\infty = \prod_{n=0}^{\infty}(1-q^{4n+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4})$$

$$= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12})$$

$$= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty = (q^4,q^8,q^{12};q^{12})_\infty \tag{1.24}$$

Similarly we can compute following, as

$$(q^1;q^1)_\infty = (q^1;q^2)_\infty(q^2;q^2)_\infty = (q^1,q^2;q^2)_\infty \tag{1.25}$$

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty(q^4;q^4)_\infty = (q^2,q^4;q^4)_\infty \tag{1.26}$$

$$(q^2;q^2)_\infty = (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty$$
$$= (q^2,q^4,q^6,q^8;q^8)_\infty \tag{1.27}$$

$$(q^2;q^2)_\infty = (q^2;q^{12})_\infty(q^4;q^{12})_\infty(q^6;q^{12})_\infty(q^8;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty$$
$$= (q^2,q^4,q^6,q^8,q^{10},q^{12};q^{12})_\infty \tag{1.28}$$

$$(q^2;q^2)_\infty = (q^2;q^{16})_\infty(q^4;q^{16})_\infty(q^6;q^{16})_\infty(q^8;q^{16})_\infty(q^{10};q^{16})_\infty\times$$
$$\times(q^{12};q^{16})_\infty(q^{14};q^{16})_\infty(q^{16};q^{16})_\infty$$
$$= (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16};q^{16})_\infty \quad (1.29)$$

$$(q^2;q^2)_\infty = (q^2;q^{20})_\infty(q^4;q^{20})_\infty(q^6;q^{20})_\infty(q^8;q^{20})_\infty(q^{10};q^{20})_\infty(q^{12};q^{20})_\infty\times$$
$$\times(q^{14};q^{20})_\infty(q^{16};q^{20})_\infty(q^{18};q^{20})_\infty(q^{20};q^{20})_\infty$$
$$= (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16},q^{18},q^{20};q^{20})_\infty \quad (1.30)$$

$$(q^3;q^3)_\infty = (q^3;q^6)_\infty(q^6;q^6)_\infty = (q^3,q^6;q^6)_\infty \quad (1.31)$$

$$(q^4;q^4)_\infty = (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty = (q^4,q^8,q^{12};q^{12})_\infty \quad (1.32)$$

$$(q^4;q^4)_\infty = (q^4;q^{16})_\infty(q^8;q^{16})_\infty(q^{12};q^{16})_\infty(q^{16};q^{16})_\infty$$
$$= (q^4,q^8,q^{12},q^{16};q^{16})_\infty \quad (1.33)$$

$$(q^4;q^4)_\infty = (q^4;q^{20})_\infty(q^8;q^{20})_\infty(q^{12};q^{20})_\infty(q^{16};q^{20})_\infty(q^{20};q^{20})_\infty$$
$$= (q^4,q^8,q^{12},q^{16},q^{20};q^{20})_\infty \quad (1.34)$$

$$(q^4;q^4)_\infty = (q^4;q^{24})_\infty(q^8;q^{24})_\infty(q^{12};q^{24})_\infty(q^{16};q^{24})_\infty(q^{20};q^{24})_\infty(q^{24};q^{24})_\infty$$
$$= (q^4,q^8,q^{12},q^{16},q^{20},q^{24};q^{24})_\infty \quad (1.35)$$

$$(q^4;q^{12})_\infty = (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$
$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \quad (1.36)$$

$$(q^6;q^6)_\infty = (q^6;q^{12})_\infty(q^{12};q^{12})_\infty = (q^6,q^{12};q^{12})_\infty \quad (1.37)$$

$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty(q^{12};q^{24})_\infty(q^{18};q^{24})_\infty(q^{24};q^{24})_\infty$$
$$= (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \quad (1.38)$$

$$(q^6;q^{12})_\infty = (q^6;q^{60})_\infty(q^{18};q^{60})_\infty(q^{30};q^{60})_\infty(q^{42};q^{60})_\infty(q^{54};q^{60})_\infty$$
$$= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \quad (1.39)$$

$$(q^8;q^8)_\infty = (q^8;q^{24})_\infty(q^{16};q^{24})_\infty(q^{24};q^{24})_\infty = (q^8,q^{16},q^{24};q^{24})_\infty \quad (1.40)$$

$$(q^8;q^8)_\infty = (q^8;q^{48})_\infty(q^{16};q^{48})_\infty(q^{24};q^{48})_\infty(q^{32};q^{48})_\infty(q^{40};q^{48})_\infty(q^{48};q^{48})_\infty$$
$$= (q^8,q^{16},q^{24},q^{32},q^{40},q^{48};q^{48})_\infty \quad (1.41)$$

$$(q^8;q^{12})_\infty = (q^8;q^{60})_\infty(q^{20};q^{60})_\infty(q^{32};q^{60})_\infty(q^{44};q^{60})_\infty(q^{56};q^{60})_\infty$$
$$= (q^8,q^{20},q^{32},q^{44},q^{56};q^{60})_\infty \quad (1.42)$$

$$(q^8;q^{16})_\infty = (q^8;q^{48})_\infty(q^{24};q^{48})_\infty(q^{40};q^{48})_\infty = (q^8,q^{24},q^{40};q^{48})_\infty \quad (1.43)$$

$$(q^{10};q^{20})_\infty = (q^{10};q^{60})_\infty(q^{30};q^{60})_\infty(q^{50};q^{60})_\infty = (q^{10},q^{30},q^{50};q^{60})_\infty \quad (1.44)$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{24})_\infty(q^{24};q^{24})_\infty = (q^{12},q^{24};q^{24})_\infty \quad (1.45)$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{60})_\infty(q^{24};q^{60})_\infty(q^{36};q^{60})_\infty(q^{48};q^{60})_\infty(q^{60};q^{60})_\infty$$
$$= (q^{12},q^{24},q^{36},q^{48},q^{60};q^{60})_\infty \quad (1.46)$$

Notes

$$(q^{16};q^{16})_\infty=(q^{16};q^{48})_\infty(q^{32};q^{48})_\infty(q^{48};q^{48})_\infty=(q^{16},q^{32},q^{48};q^{48})_\infty \tag{1.47}$$

$$(q^{20};q^{20})_\infty=(q^{20};q^{60})_\infty(q^{40};q^{60})_\infty(q^{60};q^{60})_\infty=(q^{20},q^{40},q^{60};q^{60})_\infty \tag{1.48}$$

Notes

The outline of this paper is as follows. In sections 2, we record a set of known results which are found to be useful in the paper. In section 3, we state and prove our main results, associated with the families given in (1.15)-(1.17) and (1.22), which depict the inter-relationships between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities.

II. Preliminaries

In [3], following identities are given

$$\sum_{n=-\infty}^{\infty} q^{n^2}+\sum_{n=-\infty}^{\infty} q^{2n^2}=2\frac{(q^3,q^5,q^8;q^8)_\infty}{(q,q^4,q^7;q^8)_\infty} \tag{2.1}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2}+\sum_{n=-\infty}^{\infty} q^{2n^2}=2\left(\begin{matrix} q^3,q^5,q^8 \\ \\ q,q^4,q^7\end{matrix}\;;q^8\right) \tag{2.2}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2}-\sum_{n=-\infty}^{\infty} q^{2n^2}=2q\frac{(q,q^7,q^8;q^8)_\infty}{(q^3,q^4,q^5;q^8)_\infty} \tag{2.3}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2}-\sum_{n=-\infty}^{\infty} q^{2n^2}=2q\left(\begin{matrix} q,q^7,q^8 \\ \\ q^3,q^4,q^5\end{matrix}\;;q^8\right) \tag{2.4}$$

In [4; Theorem 3], following identities are given

$$\sum_{n=-\infty}^{\infty} q^{n^2}+\sum_{n=-\infty}^{\infty} q^{5n^2}=2\left(\begin{matrix} q^2,q^8,q^{10},q^{12},q^{18},q^{20} \\ \\ q,q^4,q^9,q^{11},q^{16},q^{19}\end{matrix}\;;q^{20}\right) \tag{2.5}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2}-\sum_{n=-\infty}^{\infty} q^{5n^2}=2q\left(\begin{matrix} q^4,q^6,q^{10},q^{14},q^{16},q^{20} \\ \\ q^3,q^7,q^8,q^{12},q^{13},q^{17}\end{matrix}\;;q^{20}\right) \tag{2.6}$$

III. Main Results

We have the following identities

$$\sum_{n=-\infty}^{\infty} q^{n^2}+\sum_{n=-\infty}^{\infty} q^{2n^2}=2\frac{(q^3,q^5,q^8;q^8)_\infty}{(q,q^4,q^7;q^8)_\infty}=2\left(\begin{matrix} q^3,q^5,q^8 \\ \\ q,q^4,q^7\end{matrix}\;;q^8\right)$$

$$=2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}.\left(\frac{1}{1-}\;\frac{q^4}{1+}\;\frac{q^4(1-q^4)}{1-}\;\frac{q^{12}}{1+}\;\frac{q^8(1-q^8)}{1-}\;\frac{q^{20}}{1+}\;\frac{q^{12}(1-q^{12})}{1-\cdots}\right);(|q|<1)$$

$$=2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}R(4,4,1,1,1,2) \tag{3.1}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q,q^7,q^8;q^8)_\infty}{(q^3,q^4,q^5;q^8)_\infty} = 2q\begin{pmatrix} q,q^7,q^8 & \\ & ;q^8 \\ q^3,q^4,q^5 & \end{pmatrix}$$

$$= 2q\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}.\left(\frac{1}{1-}\frac{q^4}{1+}\frac{q^4(1-q^4)}{1-}\frac{q^{12}}{1+}\frac{q^8(1-q^8)}{1-}\frac{q^{20}}{1+}\frac{q^{12}(1-q^{12})}{1-\cdots}\right);(|q|<1)$$

$$= 2q\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}R(4,4,1,1,1,2) \quad (3.2)$$

Notes

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{5n^2} = 2\begin{pmatrix} q^2,q^{10},q^{18},q^{20} & \\ & ;q^{20} \\ q,q^9,q^{11},q^{19} & \end{pmatrix}\times$$

$$\times\left(1+\frac{q^4}{1+}\frac{q^8}{1+}\frac{q^{12}}{1+}\frac{q^{16}}{1+}\frac{q^{20}}{1+}\frac{q^{24}}{1+\cdots}\right);(|q|<1) \quad (3.3)$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q\begin{pmatrix} q^6,q^{10},q^{14},q^{20} & \\ & ;q^{20} \\ q^3,q^7,q^{13},q^{17} & \end{pmatrix}\times$$

$$\times\left(\frac{1}{1+}\frac{q^4}{1+}\frac{q^8}{1+}\frac{q^{12}}{1+}\frac{q^{16}}{1+}\frac{q^{20}}{1+}\frac{q^{24}}{1+\cdots}\right);(|q|<1) \quad (3.4)$$

Proof of (3.1): From (2.1) and (2.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5,q^8;q^8)_\infty}{(q,q^4,q^7;q^8)_\infty} = 2\begin{pmatrix} q^3,q^5,q^8 & \\ & ;q^8 \\ q,q^4,q^7 & \end{pmatrix} \quad (3.1.1)$$

Which can be represented as

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}\times\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \quad (3.1.2)$$

Applying (1.15) for $q = q^4$, in (3.1.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2}$$

$$= 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}.\left(\frac{1}{1-}\frac{q^4}{1+}\frac{q^4(1-q^4)}{1-}\frac{q^{12}}{1+}\frac{q^8(1-q^8)}{1-}\frac{q^{20}}{1+}\frac{q^{12}(1-q^{12})}{1-\cdots}\right);(|q|<1) \quad (3.1.3)$$

Applying (1.18) for $m = 4$, in (3.1.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}R(4,4,1,1,1, \quad (3.1.4)$$

Joining (3.1.1), (3.1.3) and (3.1.4) togather, we get the required result (3.1). Proof of (3.2): From (2.3) and (2.4), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q,q^7,q^8;q^8)_\infty}{(q^3,q^4,q^5;q^8)_\infty} = 2q\left(\begin{matrix} q,q^7,q^8 \\ q^3,q^4,q^5 \end{matrix} ;q^8\right) \tag{3.2.1}$$

Which can be represented as

Notes

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty} \times \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \tag{3.2.2}$$

Applying (1.15) for $q = q^4$, in (3.2.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q,q^7,q^8;q^8)_\infty}{(q^3,q^4,q^5;q^8)_\infty} = 2q\left(\begin{matrix} q,q^7,q^8 \\ q^3,q^4,q^5 \end{matrix} ;q^8\right)$$

$$= 2q\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}.\left(\frac{1}{1-}\,\frac{q^4}{1+}\,\frac{q^4(1-q^4)}{1-}\,\frac{q^{12}}{1+}\,\frac{q^8(1-q^8)}{1-}\,\frac{q^{20}}{1+}\,\frac{q^{12}(1-q^{12})}{1-\cdots}\right)\cdots\ (|q|<1) \tag{3.2.3}$$

Applying (1.18) for $m = 4$, in (3.2.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} == 2q\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}R(4,4,1,1,1,2) \tag{3.2.4}$$

Joining (3.2.1), (3.2.3) and (3.2.4) togather, we get required result (3.2). Proof of (3.3): From (2.5), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{5n^2} = 2\left(\begin{matrix} q^2,q^{10},q^{18},q^{20} \\ q,q^9,q^{11},q^{19} \end{matrix} ;q^{20}\right) \times \frac{(q^8;q^{20})_\infty(q^{12};q^{20})_\infty}{(q^4;q^{20})_\infty(q^{16};q^{20})_\infty} \tag{3.3.1}$$

Applying (1.17) for $q = q^4$, in (3.3.1), we get required result (3.3). Proof of (3.4): From (2.6), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q\left(\begin{matrix} q^6,q^{10},q^{14},q^{20} \\ q^3,q^7,q^{13},q^{17} \end{matrix} ;q^{20}\right) \times \frac{(q^4;q^{20})_\infty(q^{16};q^{20})_\infty}{(q^8;q^{20})_\infty(q^{12};q^{20})_\infty} \tag{3.4.1}$$

Applying (1.16) for $q = q^4$, in (3.4.1), we get required result (3.4).

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References Références Referencias

1. M. P. Chaudhary : *Generalization for character formulas in terms of continued fraction identities,* Malay J. Mat. 1(1)(2014) 24-34.
2. George Andrews, Kathrin Bringman, and Karl Mahlburg : *Double series Representations for Schur's partition function and related identities*, pre-print.
3. Shaun Cooper and Micheal Hirschhorn : *Om some infinite product identities*, Rocky Mountain Journal, 31(2001), 131-139.

4. Richard Becksmith, John Brillhart and Irving Gerst : *Some infinite product identities*, Math. Comp. 51(1988), 301-314.
5. M. P. Chaudhary : *On q-product identities*, Pacific J. Appl. Math.(5)(2)(2013) 123-129.
6. M. P. Chaudhary : *Applications of continued fraction identities*, Global J. Sci. Frontier Res. Math. Decision Sci. 12(4)(2012) 31-37.
7. M. P. Chaudhary and Sangeeta Chaudhary : *On 3-dissection property, continued fraction identities and combinatorial partition identities*, Preprint 2015.

Notes