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9×9 Composite Loubéré Magic Squares Infinite Abelian Group as a Miscellany Case of the 3×3 Loubéré Magic Squares Infinite Abelian Group

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9 × 9 Composite Loubéré Magic Squares Infinite Abelian Group as a Miscellany Case of the 3 × 3 Loubéré Magic Squares Infinite Abelian Group

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I. Introduction

It is remarkable that almost trivially the sets of eigen values, centre pieces and magic sums of the 3×3 Loubéré Magic Squares Infinite Abelian Group form Infinite Additive Abelian Groups. For Loubéŕe Magic Squares eigen values computations, see [2].

We highlighted consortium of miscellany effects of rotations and/or reflections [5] and/or enumerations of the 3×3 Loubéré Magic Squares to figure out the consortium of the 9×9 composites.

Establishing such a fact relationships set us conjecture that the 9×9 Composite Loubéré Magic Squares [1] Infinite Additive Abelian Group is a miscellany case of the 3×3 Loubéré Magic Squares Infinite Abelian Group.

II. Preliminaries

a) *Definition 2.1*

A basic magic square of order n can be defined as an arrangement of arithmetic sequence of common difference of 1 from 1 to n^2 in an $n \times n$ square grid of cells such that every row, column and diagonal add up to the same number, called the magic sum M(S) expressed as $M(S) = \frac{n^3+n}{2}$ and a centre piece C as $C = \frac{M(S)}{n}$.

b) *Definition 2.2*

A Composite Loubéŕe Magic Square is a magic square such that each of its cell (grid) is a Loubéré Magic Square. See also [1].

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Ref
1. Maya Ahmed (2004). Algebraic Combinatorics of Magic Squares, A Doctor of Philosophy Dissertation, University of California, Davis, pp. 85.

c) Definition 2.3

Main Row or Column is the column or row of the Loubéré Magic Squares containing the first term and the last term of the arithmetic sequence in the square.

d) Definition 2.4

A Loubéŕe Magic Square of type I is a magic square of arithmetic sequence entries such that the entries along the main column or row have a common difference and the main column or row is the central column or central row respectively.

e) Loubéré Procedure (NE-W-S or NW-E-S, the cardinal points)

Consider an empty $n \times n$ square of grids (or cells). Start, from the central column or row at a position $\lceil \frac{n}{2} \rceil$ where $\lceil \square \rceil$ is the greater integer number less than or equal to, with the number 1. The fundamental movement for filling the square is diagonally up, right (clock wise or NE or SE) or up left (anti clock wise or NW or SW) and one step at a time. If a filled cell (grid) is encountered, then the next consecutive number moves vertically down ward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S (depending on the position of the first term of the sequence) to the last row or first row or first column or last column.

The square grid of cells $[a_{ij}]_{n \times n}$ is said to be Loubéré Magic Square if the following conditions are satisfied.

i. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = k$;

ii. $\mathrm{trace}[a_{ij}]_{n \times n} = \mathrm{trace}[a_{ij}]_{n \times n}^{T} = k$; and

iii. $a_{1,\lceil \frac{n}{2} \rceil}$, $a_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$, $a_{n,\lceil \frac{n}{2} \rceil}$ are on the same main column or row and $a_{\lceil \frac{n}{2} \rceil, n}$, $a_{\lceil \frac{n}{2} \rceil \lceil \frac{n}{2} \rceil}$, $a_{\lceil \frac{n}{2} \rceil, 1}$ are on the same main column or row

,where $\lceil \square \rceil$ is the greater integer less or equal to, T is the transpose (of the square), k is the magic sum (magic product is defined analogously) usually expressed as $k = \frac{n}{2}[2a + (n-1)j]$ – from the sum of arithmetic sequence, where j is the common difference along the main column or row and a is the first term of the sequence– and $a_{\lceil \frac{n}{2} \rceil \lceil \frac{n}{2} \rceil} = \frac{k}{n}$.

f) Definition 2.6

Loubéré Magic Squares of type II are magic squares constructed with Loubéré Procedure with repeating – pattern - sequence.

g) Definition 2.7

A least subelement magic square of Loubéré Magic Square is a 3×3 Magic Square formed by removing boarder cells of the Loubéré Magic Squares.

h) Remark 2.8

The least subelements magic square of Loubéré Magic Squares are subsets of the semi pancolumn magic squares and the least subelement magic square of the composite Loubéré Magic Square is a 3×3 Loubéré Magic Square. If we use a repeating pattern sequence a, a, a, ... n times, b, b, b,...n times, c, c, c, ... n times, ... n number; we get Type II(a) Loubéré Magic Square and if instead we use a,b,c,... n number, a, b, c,... n number, ... n times; we get the Type II(b) Loubéré Magic Square.

i) Group

A non empty set G together with an operation * is known as a group if the following properties are satisfied.

i. G is closed with respect to *.i.e., $a * b \in G, \forall a, b \in G$.

Notes

ii. $*$ is associative in G. i.e., $a * (b * c) = (a * b) * c, \forall a, b, c \in G$.
iii. $\exists\, e \in G$, such that $e * a = a * e = a, \forall a \in G$. Here e is called the identity element in G with respect to $*$.
iv. $\forall a \in G, \exists b \in G$ such that $a * b = b * a = e$, where e is the identity element. Here b is called the inverse of a and similarly vise versa. The inverse of the element a is denoted as a^{-1}.

The above definition of a group is given in [3]. If in addition to the above axioms, the following axiom is satisfied, we call $(G,*)$ an abelian group where $(G,*)$ is a denotation of a group.

v. $a * b = b * a, \forall a, b \in G$. That is all (not some of) the elements of G commutes.

j) The Proof of the General $\left|\frac{m^2}{2}\right| = a + \left(\frac{m-1}{2}\right) j$ *and of the General* $M(S) = \frac{m}{2}[2a + (m-1)j]$, *Where* $j = \frac{l-a}{m-1}$

i. *Theorem 2.11*

Let the arithmetic sequence $a, a+d, \dots, l = a + (n-1)d$ be arranged in an $m \times m$ Loubéré Magic Square. Then the magic sum of the square is expressed as $M(S) = \frac{m}{2}\,[\,2a + (m-1)j\,]$ and the middle term of the sequence (centre piece of the square) is expressed as $C = a + \left(\frac{m-1}{2}\right) j$ where j denotes the common difference of entries along the main column or row and is given as $j = \frac{l-a}{m-1}$.

Proof. Consider any arbitrary General Loubéré Magic Square [4] (here we consider 3×3) as follows:

$c+b$	$c-b-d$	$c+d$
$c-b+d$	c	$c+b-d$
$c-d$	$c+b+d$	$c-b$

Let $a = c - b - d$ and $l = c + b + d$. Then we have (from the square) an arithmetic sequence: $c-b-d, c-b, \dots, c+b+d$ having the sums S as

$$S = (c-b-d) + (c-b) + \cdots + (c+b) + (c+b+d) \rightarrow (1)$$
$$+$$
$$S = (c+b+d) + (c+b) + \cdots + (c-b) + (c-b-d) \rightarrow (2)$$
....
Adding (1)and (2), $\quad 2s = 2c + 2c + \cdots \qquad$ n times

i.e. $2s = 2nc \implies c = \frac{s}{n} \dots (3)$ and $s = \frac{n}{2}(a + l) \dots (4)$ from the Gaussian High School (Elementary) Method.

Since our square is $m \times m$, m number of cells (terms) are on the main column whence $a = c - b - d$. Thus, (3) and (4) become $C = \frac{M(S)}{m}$ … (5) and $M(S) = \frac{m}{2}[a + l] \dots (6)$ respectively. And, $l = a + (m-1)j \dots (7)$ where j is along the main column. Substituting (7) in (6), we have: $M(S) = \frac{m}{2}[2a + (m-1)j]$ … (8). Substituting (8) in (5), we get: $C = a + \left(\frac{m-1}{2}\right) j \dots (9)$ From (3) and (4), $C = \frac{1}{2}(a + l) = \left(a - \frac{a}{2}\right) + \frac{l}{2} = a + \frac{(l-a)}{2} = a + \frac{l-a}{m-1}\frac{m-1}{2}$, i.e. $C = a + \left(\frac{m-1}{2}\right)\frac{l-a}{m-1}$ … (10). Comparing (9) and (10), we have: $j = \frac{l-a}{m-1}$ … (11).

We consider $m \times m$ for the square is more general than the $n \times n$ considered initially.

Ref

3. Sreeranjini K.S, V.Madhukar Mallayya (2012). Semi Magic Squares as a Field, International Journal of Algebra, 6:1249-1256.

k) Centre Pieces and Magic Sums Abelian Groups

i. *Centre Pieces Abelian Group*

The set of the centre pieces $c_1, c_2, c_3, \dots$ of $m \times m$ Loubéré Magic Squares equipped with integer addition forms an infinite additive abelian group. Given the centre pieces $c_1, c_2, c_3, \dots$ of $m \times m$ Loubéré Magic Squares with corresponding formula $c_1 = a_1 + \left(\frac{m-1}{2}\right) j_1, c_2 = a_2 + \left(\frac{m-1}{2}\right) j_2,$

$$c_3 = a_3 + \left(\frac{m-1}{2}\right) j_3, \dots; \text{then}$$

- $c_1 + c_2 = (a_1 + a_2) + \left(\frac{m-1}{2}\right)(j_1 + j_2)$ is the centre piece of the $m \times m$ Loubéré Magic Square with first term $a_1 + a_2$ and common difference along the main column as $j_1 + j_2$. Hence, the set is *closed.*
- *Associativity.* This is an inherited property of the set of integer numbers:

$$c_1 + (c_2 + c_3) = (a_1 + a_2 + a_3) + \left(\frac{m-1}{2}\right)(j_1 + j_2 + j_3) = (c_1 + c_2) + c_3$$

- The *identity element* is the zero centre piece e.g.

$$\begin{bmatrix} \mathrm{c} & -\mathrm{d} & \mathrm{a} \\ -\mathrm{b} & 0 & \mathrm{b} \\ -\mathrm{a} & \mathrm{d} & -\mathrm{c} \end{bmatrix}$$

- Given an arbitrary centre piece $c_n = a_n + \left(\frac{m-1}{2}\right) j_n$ of the $m \times m$ Loubéré Magic Square, there exists another centre piece c_{-n} of another $m \times m$ Loubéré Magic Square having first term as $-a_n$ and common difference along the main column or row as $-j_n$, thus its formulae is $c_{-n} = -a_n + \left(\frac{m-1}{2}\right)(-j_n)$ and is such that $c_n + c_{-n} = c_{-n} + c_n = (a_n - a_n) + \left(\frac{m-1}{2}\right)[\, j_n - j_n] = 0 = c_i$, the identity centre piece. c_n and $\mathrm{c}_{-\mathrm{n}}$ are *inverses* of each other.
- Clearly $c_1 + c_2 = a_1 + a_2 + \left(\frac{m-1}{2}\right)(j_1 + j_2) = a_2 + a_1 + \left(\frac{m-1}{2}\right)(j_2 + j_1) = c_2 + c_1$

The set equipped with the operation is an abelian group.

ii. *Magic Sum Abelian Groups*

The set of the magic sums $M(s_1), M(s_2), M(s_3), \dots$of $m \times m$ Loubéré Magic Squares equipped with the integer number binary operation of addition forms an infinite additive abelian group. Given the magic sums $M(s_1), M(s_2), M(s_3), \dots$of $m \times m$ Loubéré Magic Squares with corresponding formula

$$M(s_1) = \frac{m}{2}\left[2a_1 + (m-1)j_1, M(s_2) = \frac{m}{2}[2a_2 + (m-1)j_2], M(s_3\right) = \frac{m}{2}[2a_3 + (m-1)j_3, \dots;$$

then (as in the above).

- $M(s_1) + M(s_2) = M(s_?)$ where $\mathrm{M}(s_?)$ is a magic sum of another $\mathrm{m} \times \mathrm{m}$ Loubéré Magic Square with first term $a_1 + a_2$ and common difference along the main column as $j_1 + j_2$.

The axioms: ii, iii, iv and v follow, by analogy to the centre piece infinite additive abelian group properties, immediately.

iii. *Eigen Values Abelian Group*

The Eigen values computation in the magic squares is what is zealotly prophesized that magic squares are special type of matrices, hence the definition of the magic squares, we do not love to like such a sudden conclusion if loving to liking forces choosing the definitions in terms of just the square grids (or cells).

Notes

We want to show through concrete examples that the set of Eigen Values of the Loubéré Magic Squares with usual integer numbers binary operation of addition forms a group. Consider the following arbitrary two 3 × 3 Loubéré Magic Squares −which we let

$$\mathrm{a} = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 1 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \mathrm{b} = \begin{bmatrix} 2 & -5 & 0 \\ -3 & -1 & 1 \\ -2 & 3 & -4 \end{bmatrix} \text{ and their sum } \mathrm{c} = \begin{bmatrix} 6 & -8 & 2 \\ -4 & 0 & 4 \\ -2 & 8 & -6 \end{bmatrix}$$

We compute the eigen values for a as follows: The corresponding matrix of a is $(a) = \begin{pmatrix} 4 & -3 & 2 \\ -1 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix}$, its eigen vector is $|a - \lambda I| = \begin{vmatrix} 4-\lambda & -3 & 2 \\ -1 & 1-\lambda & 3 \\ 0 & 5 & -2-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 3\lambda^2 - 24\lambda - 72 = (\lambda - 3)(\lambda^2 - 24) = 0$ having eigen values $\lambda_{a_1} = 3, \lambda_{a_2} = 4.9 \text{ and } \lambda_{a_3} = -4.9$.

We compute the eigen values for b as follows: The corresponding matrix of b is $(b) = \begin{pmatrix} 2 & -5 & 0 \\ -3 & -1 & 1 \\ -2 & 3 & -4 \end{pmatrix}$, its eigen vector is $|b - \lambda I| = \begin{vmatrix} 2-\lambda & -5 & 0 \\ -3 & -1-\lambda & 1 \\ -2 & 3 & -4-\lambda \end{vmatrix} = 0$ i.e. $\lambda^3 + 3\lambda^2 - 24\lambda - 72 = (\lambda + 3)(\lambda^2 - 24) = 0$ with eigen values $\lambda_{b_1} = -3, \lambda_{b_2} = 4.9 \text{ and } \lambda_{b_3} = -4.9$.

We compute the eigen values for c as follows: The corresponding matrix of c is $(c) = \begin{pmatrix} 6 & -8 & 2 \\ -4 & 0 & 4 \\ -2 & 8 & -6 \end{pmatrix}$, its eigen vector is $|c - \lambda I| = \begin{vmatrix} 6-\lambda & -8 & 2 \\ -4 & -\lambda & 4 \\ -2 & 8 & -6-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 96\lambda = 0$ with corresponding eigen values $\lambda_{c_1} = 0, \lambda_{c_2} = 9.8$ and $\lambda_{c_3} = -9.8$.

We now conclude this session by showing that the set of eigen values satisfies *The Properties of an Additive Abelian Group* as follows:

Closure Property. Consider any 3 arbitrary Loubéré Magic Squares a, b, c; such that $a + b = c$; then from the example above, the corresponding eigen values of a; $\lambda_{a_1}, \lambda_{a_2}\lambda_{a_3}$; the corresponding eigen values of b; $\lambda_{b_1}, \lambda_{b_2}, \lambda_{b_3}$; are such that $\lambda_{a_1} + \lambda_{b_1} = \lambda_{c_1}$, $\lambda_{a_2} + \lambda_{b_2} = \lambda_{c_2}$, and $\lambda_{a_3} + \lambda_{b_3} = \lambda_{c_3}$ where $\lambda_{c_1}, \lambda_{c_2}, \lambda_{c_3}$ are the corresponding eigen values of c.

Associativity Property. Since Loubéré Magic Squares are a semi group (which is easy to observe), the eigen values are associative.

Identity Element Property. The eigen value 0 is the identity element that corresponds to the sum of the Loubéré Magic Squares of opposite eigen values as in the above.

Inverse Elements Property. For any arbitrary eigen value λ_{m} corresponding to a Loubéré Magic Square m, there exist a $-\lambda_{m}$ eigen value corresponding to another Loubéré Magic Square such that $\lambda_{m} + (-\lambda_{m})$ gives the identity element which is formed as a result of matrix addition of the aforementioned Loubéré Magic Squares.

Commutativity. Integer numbers binary operation of addition is commutative.

This completes the proof. The idea of eigen values computation of a magic square is conceived from the work of [2].

III. 9 × 9 Composite Loubéré Magic Squares Infinite Abelian Group as a Miscellany Case of the 3 × 3 Loubéré Magic Squares Infinite Abelian Group

Let n stands for number of columns, d stands for common difference of entries and f stands for first term of the aforementioned square. Then $\mathcal{S}_{(n,\ d,\ f)}$ denotes the

Ref

2. Daryl Lynn Stephens (1993). Matrix Properties of Magic Squares, Master of Science Professional Paper, College of Arts and Sciences, Denton, Texas, pp. 32.

sequences of $n \times n$ Loubére Magic Squares of type I of d common difference of its entries and of first terms f, ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ denotes the sequences of $n \times n$ Loubéré Magic Squares of type II(a) of d common difference of its entries and of first terms f, ${}^{2}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ denotes the sequence of type II(b), $\mathcal{CS}_{(n,\ d,\ f)}$ denotes the sequence of the composites of $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{CT}_{(n,\ d,\ f)}$ denotes the sequence of composites of ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$, $\mathcal{CST}_{(n,\ d,\ f)}$ denotes the sequence of the composites of $\mathcal{S}_{(n,\ d,\ f)}$ having entries ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ or simply T(f), and $\mathcal{CTS}_{(n,\ d,\ f)}$ denotes the sequence of the composites of ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ having entries $\mathcal{S}_{(n,\ d,\ f)}$. Then, the sequences are as follows:

$$\mathcal{S}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} 5 & -2 & 3 \\ 0 & 2 & 4 \\ 1 & 6 & -1 \end{bmatrix}, \begin{bmatrix} 6 & -1 & 4 \\ 1 & 3 & 5 \\ 2 & 7 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 5 \\ 2 & 4 & 6 \\ 3 & 8 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 9 & 2 & 7 \\ 4 & 6 & 8 \\ 5 & 10 & 3 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} 12 & -2 & 8 \\ 2 & 6 & 10 \\ 4 & 14 & 0 \end{bmatrix}, \begin{bmatrix} 13 & -1 & 9 \\ 3 & 7 & 11 \\ 5 & 15 & 1 \end{bmatrix}, \begin{bmatrix} 14 & 0 & 10 \\ 4 & 8 & 12 \\ 6 & 16 & 2 \end{bmatrix}, \begin{bmatrix} 15 & 1 & 11 \\ 5 & 9 & 13 \\ 7 & 17 & 3 \end{bmatrix}, \begin{bmatrix} 16 & 2 & 12 \\ 6 & 10 & 14 \\ 8 & 18 & 4 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} 19 & -2 & 13 \\ 4 & 10 & 16 \\ 7 & 22 & 1 \end{bmatrix}, \begin{bmatrix} 20 & -1 & 14 \\ 5 & 11 & 17 \\ 8 & 23 & 2 \end{bmatrix}, \begin{bmatrix} 21 & 0 & 15 \\ 6 & 12 & 18 \\ 9 & 24 & 3 \end{bmatrix}, \begin{bmatrix} 22 & 1 & 16 \\ 7 & 13 & 19 \\ 10 & 25 & 4 \end{bmatrix}, \begin{bmatrix} 23 & 2 & 17 \\ 8 & 14 & 20 \\ 11 & 26 & 5 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{S}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} -9 & -2 & -7 \\ -4 & -6 & -8 \\ -5 & -10 & -3 \end{bmatrix}, \begin{bmatrix} -8 & -1 & -6 \\ -3 & -5 & -7 \\ -4 & -9 & -2 \end{bmatrix}, \begin{bmatrix} -7 & 0 & -5 \\ -2 & -4 & -6 \\ -3 & -8 & -1 \end{bmatrix}, \begin{bmatrix} -6 & 1 & -4 \\ -1 & -3 & -5 \\ -2 & -7 & 0 \end{bmatrix}, \begin{bmatrix} -5 & 2 & -3 \\ 0 & -2 & -4 \\ -1 & -6 & 1 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} -16 & -2 & -12 \\ -6 & -10 & -14 \\ -8 & -18 & -4 \end{bmatrix}, \begin{bmatrix} -15 & -1 & -11 \\ -5 & -9 & -13 \\ -7 & -17 & -3 \end{bmatrix}, \begin{bmatrix} -14 & 0 & -10 \\ -4 & -8 & -12 \\ -6 & -16 & -2 \end{bmatrix}, \begin{bmatrix} -13 & 1 & -9 \\ -3 & -7 & -11 \\ -5 & -15 & -1 \end{bmatrix}, \begin{bmatrix} -12 & 2 & -8 \\ -2 & -6 & -10 \\ -4 & -14 & 0 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} -23 & -2 & -17 \\ -8 & -14 & -20 \\ -11 & -26 & -5 \end{bmatrix}, \begin{bmatrix} -22 & -1 & -16 \\ -7 & -13 & -19 \\ -10 & -25 & -4 \end{bmatrix}, \begin{bmatrix} -21 & 0 & -15 \\ -6 & -12 & -18 \\ -9 & -24 & -3 \end{bmatrix}, \begin{bmatrix} -20 & 1 & -14 \\ -5 & -11 & -17 \\ -8 & -23 & -2 \end{bmatrix}, \begin{bmatrix} -19 & 2 & -13 \\ -4 & -10 & -16 \\ -7 & -22 & -1 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{T}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & -2 \\ -2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 4 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 5 \\ 5 & 3 & 1 \\ 1 & 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 6 \\ 6 & 4 & 2 \\ 2 & 6 & 4 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 5 \\ 5 & 2 & -1 \\ -1 & 5 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 0 & 6 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 7 \\ 7 & 4 & 1 \\ 1 & 7 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 2 & 8 \\ 8 & 5 & 2 \\ 2 & 8 & 5 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{T}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} -3 & -2 & -4 \\ -4 & -3 & -2 \\ -2 & -4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & -1 & -3 \\ -3 & -2 & -1 \\ -1 & -3 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -2 \\ -2 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 2 & 8 \\ 8 & 5 & 2 \\ 2 & 8 & 5 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} -4 & -2 & -6 \\ -6 & -4 & -2 \\ -2 & -6 & -4 \end{bmatrix}, \begin{bmatrix} -3 & -1 & -5 \\ -5 & -3 & -1 \\ -1 & -5 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 0 & -4 \\ -4 & -2 & 0 \\ 0 & -4 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -3 \\ -3 & -1 & 1 \\ 1 & -3 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}, \ldots$$

Notes

$$\mathcal{T}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} -5 & -2 & -8 \\ -8 & -5 & -2 \\ -2 & -8 & -5 \end{bmatrix}, \begin{bmatrix} -4 & -1 & -7 \\ -7 & -4 & -1 \\ -1 & -7 & -4 \end{bmatrix}, \begin{bmatrix} -3 & 0 & -6 \\ -6 & -3 & 0 \\ 0 & -6 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 1 & -5 \\ -5 & -2 & 1 \\ 1 & -5 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -4 \\ -4 & -1 & 2 \\ 2 & -4 & -1 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CS}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} \\ \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} \\ \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} \\ \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} \\ \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} \\ \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} \\ \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 1,\ 5)} & \mathcal{S}_{(3,1,-2)} & \mathcal{S}_{(3,\ 1,3)} \\ \mathcal{S}_{(3,\ 1,0)} & \mathcal{S}_{(3,\ 1,2)} & \mathcal{S}_{(3,\ 1,4)} \\ \mathcal{S}_{(3,1,1)} & \mathcal{S}_{(3,\ 1,6)} & \mathcal{S}_{(3,1,-1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 1,6)} & \mathcal{S}_{(3,\ 1,-1)} & \mathcal{S}_{(3,1,4)} \\ \mathcal{S}_{(3,\ 1,1)} & \mathcal{S}_{(3,\ 1,3)} & \mathcal{S}_{(3,\ 1,5)} \\ \mathcal{S}_{(3,\ 1,2)} & \mathcal{S}_{(3,\ 1,7)} & \mathcal{S}_{(3,\ 1,0)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 1,7)} & \mathcal{S}_{(3,\ 1,0)} & \mathcal{S}_{(3,1,5)} \\ \mathcal{S}_{(3,\ 1,2)} & \mathcal{S}_{(3,\ 1,4)} & \mathcal{S}_{(3,\ 1,6)} \\ \mathcal{S}_{(3,\ 1,3)} & \mathcal{S}_{(3,\ 1,8)} & \mathcal{S}_{(3,\ 1,1)} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 2,12)} & \mathcal{S}_{(3,\ 2,-2)} & \mathcal{S}_{(3,\ 2,8)} \\ \mathcal{S}_{(3,\ 2,2)} & \mathcal{S}_{(3,2,6)} & \mathcal{S}_{(3,\ 2,10)} \\ \mathcal{S}_{(3,\ 2,4)} & \mathcal{S}_{(3,\ 2,14)} & \mathcal{S}_{(3,\ 2,0)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 2,13)} & \mathcal{S}_{(3,\ 2,-1)} & \mathcal{S}_{(3,\ 2,9)} \\ \mathcal{S}_{(3,\ 2,3)} & \mathcal{S}_{(3,\ 2,7)} & \mathcal{S}_{(3,\ 2,11)} \\ \mathcal{S}_{(3,\ 2,5)} & \mathcal{S}_{(3,\ 2,15)} & \mathcal{S}_{(3,\ 2,1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 2,14)} & \mathcal{S}_{(3,\ 2,0)} & \mathcal{S}_{(3,\ 2,10)} \\ \mathcal{S}_{(3,\ 2,4)} & \mathcal{S}_{(3,\ 2,8)} & \mathcal{S}_{(3,\ 2,12)} \\ \mathcal{S}_{(3,\ 2,6)} & \mathcal{S}_{(3,\ 2,16)} & \mathcal{S}_{(3,2,2} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 3,19)} & \mathcal{S}_{(3,\ 3,-2)} & \mathcal{S}_{(3,\ 3,13)} \\ \mathcal{S}_{(3,\ 3,4)} & \mathcal{S}_{(3,\ 3,10)} & \mathcal{S}_{(3,\ 3,16)} \\ \mathcal{S}_{(3,\ 3,7)} & \mathcal{S}_{(3,\ 3,22)} & \mathcal{S}_{(3,3,1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 3,20)} & \mathcal{S}_{(3,\ 3,-1)} & \mathcal{S}_{(3,\ 3,14)} \\ \mathcal{S}_{(3,3,5)} & \mathcal{S}_{(3,\ 3,11)} & \mathcal{S}_{(3,\ 3,17)} \\ \mathcal{S}_{(3,\ 3,8)} & \mathcal{S}_{(3,3,23)} & \mathcal{S}_{(3,\ 3,2)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 3,21)} & \mathcal{S}_{(3,3,0)} & \mathcal{S}_{(3,3,15)} \\ \mathcal{S}_{(3,\ 3,6)} & \mathcal{S}_{(3,3,122)} & \mathcal{S}_{(3,3,18)} \\ \mathcal{S}_{(3,\ 3,9)} & \mathcal{S}_{(3,\ 3,24)} & \mathcal{S}_{(3,3,3)} \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CS}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} -\mathcal{S}_{(3,\ 1,\ 5)} & -\mathcal{S}_{(3,1,-2)} & -\mathcal{S}_{(3,\ 1,3)} \\ -\mathcal{S}_{(3,\ 1,0)} & -\mathcal{S}_{(3,\ 1,2)} & -\mathcal{S}_{(3,\ 1,4)} \\ -\mathcal{S}_{(3,1,1)} & -\mathcal{S}_{(3,\ 1,6)} & -\mathcal{S}_{(3,1,-1)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 1,6)} & -\mathcal{S}_{(3,\ 1,-1)} & -\mathcal{S}_{(3,1,4)} \\ -\ \mathcal{S}_{(3,\ 1,1)} & -\mathcal{S}_{(3,\ 1,3)} & -\mathcal{S}_{(3,\ 1,5)} \\ \mathcal{S}_{(3,\ 1,2)} & -\mathcal{S}_{(3,\ 1,7)} & -\mathcal{S}_{(3,\ 1,0)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 1,7)} & -\mathcal{S}_{(3,\ 1,0)} & -\mathcal{S}_{(3,1,5)} \\ -\ \mathcal{S}_{(3,\ 1,2)} & -\mathcal{S}_{(3,\ 1,4)} & -\mathcal{S}_{(3,\ 1,6)} \\ -\mathcal{S}_{(3,\ 1,3)} & -\mathcal{S}_{(3,\ 1,8)} & -\mathcal{S}_{(3,\ 1,1)} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} -\mathcal{S}_{(3,\ 2,12)} & -\mathcal{S}_{(3,\ 2,-2)} & -\mathcal{S}_{(3,\ 2,8)} \\ -\mathcal{S}_{(3,\ 2,2)} & -\mathcal{S}_{(3,2,6)} & -\mathcal{S}_{(3,\ 2,10)} \\ -\mathcal{S}_{(3,\ 2,4)} & -\mathcal{S}_{(3,\ 2,14)} & -\mathcal{S}_{(3,\ 2,0)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 2,13)} & -\mathcal{S}_{(3,\ 2,-1)} & -\mathcal{S}_{(3,\ 2,9)} \\ -\mathcal{S}_{(3,\ 2,3)} & -\mathcal{S}_{(3,\ 2,7)} & -\mathcal{S}_{(3,\ 2,11)} \\ -\mathcal{S}_{(3,\ 2,5)} & -\mathcal{S}_{(3,\ 2,15)} & -\mathcal{S}_{(3,\ 2,1)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 2,14)} & -\mathcal{S}_{(3,\ 2,0)} & -\mathcal{S}_{(3,\ 2,10)} \\ -\mathcal{S}_{(3,\ 2,4)} & -\mathcal{S}_{(3,\ 2,8)} & -\mathcal{S}_{(3,\ 2,12)} \\ -\mathcal{S}_{(3,\ 2,6)} & -\mathcal{S}_{(3,\ 2,16)} & -\mathcal{S}_{(3,2,2} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} -\mathcal{S}_{(3,\ 3,19)} & -\mathcal{S}_{(3,\ 3,-2)} & -\mathcal{S}_{(3,\ 3,13)} \\ -\mathcal{S}_{(3,\ 3,4)} & -\mathcal{S}_{(3,\ 3,10)} & -\mathcal{S}_{(3,\ 3,16)} \\ -\mathcal{S}_{(3,\ 3,7)} & -\mathcal{S}_{(3,\ 3,22)} & -\mathcal{S}_{(3,3,1)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 3,20)} & -\mathcal{S}_{(3,\ 3,-1)} & -\mathcal{S}_{(3,\ 3,14)} \\ -\mathcal{S}_{(3,3,5)} & -\mathcal{S}_{(3,\ 3,11)} & -\mathcal{S}_{(3,\ 3,17)} \\ -\mathcal{S}_{(3,\ 3,8)} & -\mathcal{S}_{(3,3,23)} & -\mathcal{S}_{(3,\ 3,2)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 3,21)} & -\mathcal{S}_{(3,3,0)} & -\mathcal{S}_{(3,3,15)} \\ \mathcal{S}_{(3,\ 3,6)} & -\mathcal{S}_{(3,3,122)} & -\mathcal{S}_{(3,3,18)} \\ -\mathcal{S}_{(3,\ 3,9)} & -\mathcal{S}_{(3,\ 3,24)} & -\mathcal{S}_{(3,3,3)} \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CT}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} T(-2) & T(-2) & T(-2) \\ T(-2) & T(-2) & T(-2) \\ T(-2) & T(-2) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(-1) & T(-1) \\ T(-1) & T(-1) & T(-1) \\ T(-1) & T(-1) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(0) & T(0) \\ T(0) & T(0) & T(0) \\ T(0) & T(0) & T(0) \end{bmatrix}, \begin{bmatrix} T(1) & T(1) & T(1) \\ T(1) & T(1) & T(1) \\ T(1) & T(1) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(2) & T(2) \\ T(2) & T(2) & T(2) \\ T(2) & T(2) & T92) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} T(-1) & T(-2) & T(0) \\ T(0) & T(-1) & T(-2) \\ T(-2) & T(0) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(-1) & T(1) \\ T(1) & T(0) & T(-1) \\ T(-1) & T(1) & T(0) \end{bmatrix}, \begin{bmatrix} T(1) & T(0) & T(2) \\ T(2) & T(1) & T(0) \\ T(0) & T(2) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(1) & T(3) \\ T(3) & T(2) & T(1) \\ T(1) & T(3) & T(2) \end{bmatrix}, \begin{bmatrix} T(3) & T(2) & T(4) \\ T(4) & T(3) & T(2) \\ T(2) & T(4) & T(3) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} T(0) & T(-2) & T(2) \\ T(2) & T(0) & T(-2) \\ T(-2) & T(2) & T(0) \end{bmatrix}, \begin{bmatrix} T(1) & T(-1) & T(3) \\ T(3) & T(1) & T(-1) \\ T(-1) & T(3) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(0) & T(4) \\ T(4) & T(2) & T(0) \\ T(0) & T(4) & T(2) \end{bmatrix}, \begin{bmatrix} T(3) & T(1) & T(5) \\ T(5) & T(3) & T(1) \\ T(1) & T(5) & T(3) \end{bmatrix}, \begin{bmatrix} T(4) & T(2) & T(6) \\ T(6) & T(4) & T(2) \\ T(2) & T(6) & T(4) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} T(1) & T(-2) & T(4) \\ T(4) & T(1) & T(-2) \\ T(-2) & T(4) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(-1) & T(5) \\ T(5) & T(2) & T(-1) \\ T(-1) & T(5) & T(2) \end{bmatrix}, \begin{bmatrix} T(3) & T(0) & T(6) \\ T(6) & T(3) & T(0) \\ T(0) & T(6) & T(3) \end{bmatrix}, \begin{bmatrix} T(4) & T(1) & T(7) \\ T(7) & T(4) & T(1) \\ T(1) & T(7) & T(4) \end{bmatrix}, \begin{bmatrix} T(5) & T(2) & T(8) \\ T(8) & T(5) & T92) \\ T(2) & T(8) & T(5) \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CT}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} T(-3) & T(-2) & T(-4) \\ T(-4) & T(-3) & T(-2) \\ T(-2) & T(-4) & T(-3) \end{bmatrix}, \begin{bmatrix} T(-2) & T(-1) & T(-3) \\ T(-3) & T(-2) & T(-1) \\ T(-1) & T(-3) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(0) & T(-2) \\ T(-2) & T(-1) & T(0) \\ T(0) & T(-2) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(1) & T(-1) \\ T(-1) & T(0) & T(1) \\ T(1) & T(-1) & T(0) \end{bmatrix}, \begin{bmatrix} T(5) & T(2) & T(8) \\ T(8) & T(5) & T(2) \\ T(2) & T(8) & T(5) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} T(-4) & T(-2) & T(-6) \\ T(-6) & T(-4) & T(-2) \\ T(-2) & T(-6) & T(-4) \end{bmatrix}, \begin{bmatrix} T(-3) & T(-1) & T(-5) \\ T(-5) & T(-3) & T(-1) \\ T(-1) & T(-5) & T(-3) \end{bmatrix}, \begin{bmatrix} T(-2) & T(0) & T(-4) \\ T(-4) & T(-2) & T(0) \\ T(0) & T(-4) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(1) & T(-3) \\ T(-3) & T(-1) & T(1) \\ T(1) & T(-3) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(2) & T(-2) \\ T(-2) & T(0) & T(2) \\ T(2) & T(-2) & T(0) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} T(-5) & T(-2) & T(-8) \\ T(-8) & T(-5) & T(-2) \\ T(-2) & T(-8) & T(-5) \end{bmatrix}, \begin{bmatrix} T(-4) & T(-1) & T(-7) \\ T(-7) & T(-4) & T(-1) \\ T(-1) & T(-7) & T(-4) \end{bmatrix}, \begin{bmatrix} T(-3) & T(0) & T(-6) \\ T(-6) & T(-3) & T(0) \\ T(0) & T(-6) & T(-3) \end{bmatrix}, \begin{bmatrix} T(-2) & T(1) & T(-5) \\ T(-5) & T(-2) & T(1) \\ T(1) & T(-5) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(2) & T(-4) \\ T(-4) & T(-1) & T(2) \\ T(2) & T(-4) & T(-1) \end{bmatrix}, \ldots$$

$\vdots$

a) Remarks 3.1

$\mathcal{CTS}_{(\mathrm{n},\ \mathrm{d},\ \mathrm{f})}$ and $\mathcal{CST}_{(\mathrm{n},\ \mathrm{d},\ \mathrm{f})}$ are enumerated analogously.

Notes

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b) The Generalized 3×3 *Loubére Magic Square*

Let $\mathbb{Z}$ denotes the set of integer numbers, $\overline{\vee}$ denotes exclusive 'or' and $\vee$ denotes inclusive 'or'. Then the general 3×3 Loubéré Magic Square is given by

$$G_{3\times3\,L} := \left\{ \left(\begin{bmatrix} f+7d & f & f+5d \\ f+2d & f+4d & f+6d \\ f+3d & f+8d & f+d \end{bmatrix} \vee \begin{bmatrix} c+b & c-(b+d) & c+d \\ c-b+d & c & c+b-d \\ c-d & c+(b+d) & c-b \end{bmatrix} \right)^{1 \text{ or } C} \overline{\vee} \right.$$

$$\left. \vee \left(\begin{bmatrix} d & f & c \\ c & d & f \\ f & c & d \end{bmatrix} \vee \begin{bmatrix} c & f & d \\ f & d & c \\ d & c & f \end{bmatrix} \right)^{1 \text{ or } M \text{ or } C} : c, d, f \in \mathbb{Z} \right\}$$

Where S^M denotes the miscellany effects of rotations and"or reflections of S and S^C denotes composition of S.

The advantage of this generalization is that it has covered both miscellany effects and composites. It also consider 9×9 Composite Loubéré a special case of the 3×3 Loubéré .

c) Theorem 3.3.

The set of $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$,$\mathcal{CS}_{(n,\ d,\ f)}$,$\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ form Infinite Additive Abelian Groups.

Proof. The sum of two sequences of the types $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$,$\mathcal{CS}_{(n,\ d,\ f)}$,$\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ is a sequence of their type. Thus, *Closure Property* is exhibited.

Associativity Property. This is an inherited property of closure above whence we have integer number entries in the sequences.

Identity Property. The identities are the sequence $\mathcal{S}_{(n,\ 0,\ f)}$, $\mathcal{T}_{(n,\ 0,\ f)}$, $\mathcal{CS}_{(n,\ 0,\ f)}$,$\mathcal{CT}_{(n,\ 0,\ f)}$, $\mathcal{CTS}_{(n,\ 0,\ f)}$ and $\mathcal{CST}_{(n,\ 0,\ f)}$ respectively.

Inverse Property. Each element in $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$,$\mathcal{CS}_{(n,\ d,\ f)}$,$\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ has an inverse. Examples. $\mathcal{S}_{(n,\ z,\ y)}$, $\mathcal{T}_{(n,\ ,z,\ y)}$, $\mathcal{CS}_{(n,\ z,\ y)}$,$\mathcal{CT}_{(n,\ z,\ y)}$, $\mathcal{CTS}_{(n,\ z,\ y)}$ and $\mathcal{CST}_{(n,\ z,\ y)}$ have inverses $\mathcal{S}_{(n,-z,\ y)}$, $\mathcal{T}_{(n,\ ,-z,\ y)}$, $\mathcal{CS}_{(n,-z,\ y)}$,$\mathcal{CT}_{(n,-z,\ y)}$, $\mathcal{CTS}_{(n,-z,\ y)}$ and $\mathcal{CST}_{(n,-z,\ y)}$ respectively.

Commutativity. Integer numbers binary operation of addition is commutative.
This completes the proof.

d) Conjecture 3.4

The 3 Eigen Values, Magic Sums and Centre Pieces of the 9×9 Composite Loubéré Magic Square are 3 times that of the 3×3 Loubéré Magic Square.

Proof. This is manifested clearly in the enumeration of $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$, $\mathcal{CS}_{(n,\ d,\ f)}$,$\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ above.

e) Theorem 3.5

The 3 Eigen Values, Magic Sums and Centre Pieces of the 9×9 Composite Loubéré Magic Square that are multiples of that of the 3×3 Loubéré Magic Square form Infinite Additive Abelian Groups.

Proof. If a set of integer numbers equipped with an operation is a group, then 3 times the set of corresponding elements of the set equipped with the same operation is also a group.

Notes

Notes

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