Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 1 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Loubéré Magic Squares Semigroups and Groups

By Babayo A. M. & G. U. Garba
Federal University Kashere, Nigeria

Abstract- This work is a pioneer investigation of semigroups and groups over the Loubéré Magic Squares. By the Loubéré Magic Squares, we understand the magic squares formed by the De La Loubéré Procedure. The set of the Loubéré Magic Squares equipped with the matrix binary operation of addition forms a semigroup if the underlining set so considered is the multi set of natural numbers; and if we consider the multi set of integer numbers as the underlined set of entries of the square, the set of the squares enclosed with the aforementioned operation forms an abelian group. The Loubéré Magic Squares are always recognized with centre piece C and magic sum M(S). We showcase that the set of the centre pieces and the set of the magic sums form respective abelian groups if both are equipped with integer numbers operation of addition. We also explicate that the set of the eigen values of the squares enclosed with the integer addition (operation) forms an abelian group. We reveal that the subelement (a terminology we introduced) Magic Squares of the Loubéré Magic Squares forms a semigroup and the Subelement Magic Squares of the Loubéré Magic Squares Group forms a group, with respect to the matrix binary operation of addition.

Keywords: *semigroup, group, centre piece, eigen values, subelement, magic sum.*

GJSFR-F Classification : FOR Code : MSC 2010: 16W22

LOUBEREMAGICSQUARESSEMIGROUPSANDGROUPS

Strictly as per the compliance and regulations of :

Loubéré Magic Squares Semigroups and Groups

Babayo A. M. α & G. U. Garba σ

Abstract- This work is a pioneer investigation of semigroups and groups over the Loubéré Magic Squares. By the **Loubéré** Magic Squares, we understand the magic squares formed by the De La Loubéré Procedure. The set of the Loubéré Magic Squares equipped with the matrix binary operation of addition forms a semigroup if the underlining set so considered is the multi set of natural numbers; and if we consider the multi set of integer numbers as the underlined set of entries of the square, the set of the squares enclosed with the aforementioned operation forms an abelian group. The Loubéré Magic Squares are always recognized with centre piece C and magic sum M(S). We showcase that the set of the centre pieces and the set of the magic sums form respective abelian groups if both are equipped with integer numbers operation of addition. We also explicate that the set of the eigen values of the squares enclosed with the integer addition (operation) forms an abelian group. We reveal that the subelement (a terminology we introduced) Magic Squares of the Loubéré Magic Squares forms a semigroup and the Subelement Magic Squares of the Loubéré Magic Squares Group forms a group, with respect to the matrix binary operation of addition.

Keywords: semigroup, group, centre piece, eigen values, subelement, magic sum.

I. Introduction

This pioneering work disclosed a new realm of semigroup and group, the Loubéré Magic Squares Semigroup and Group. The set of the Loubéré Magic Squares of the arithmetic sequence of the set of the natural numbers or of its multi set form a semigroup which by analogy we refer to as the Loubéré Magic Squares Semigroup; and the set of the **Loubéré** Magic Squares of the arithmetic sequence of the set of integer numbers or of the multi set of the integer numbers form a group which by analogy we refer to as the **Loubéré** Magic Squares Group. The aforementioned semigroup [3] and group [4]are both with respect to the matrix binary operation of addition, thus they are both additive.

The collection of the centre pieces with formula $c_n = a_n + \left(\frac{m-1}{2}\right) j_n$ equipped with the integer addition forms an abelian group and the set of all the magic sums with formula $M(S_n) = \frac{m}{2}[2a_n + (m-1)j_n]$ equipped with the integer numbers binary operation of addition form an abelian group also, where $n = 1, 2, 3, \ldots$ and a_n , j_n are the corresponding first term and common difference along the main column respectively of $m \times m$ Loubéré Magic Squares.

We also showcase that the set of eigen values of the Loubéré Magic Squares enclosed with integer numbers operation of addition forms an abelian group. This is meaningful for the principal value of the eigen value corresponds to the magic sum [1].

Ref

1. Daryl Lynn Stephens, "Matrix Properties of Magic Squares," A Master of Science Professional Paper, College of Arts and Science, Denton, Texas, pp.32, 1993.

Author α: Department of Mathematics and Computer Science, Faculty of Science, Federal University Kashere, P.M.B.0182, Gombe State, Nigeria. e-mail: baabaayo2014@gmail.com
Author σ: Department of Mathematics, Faculty of Science, Ahmadu Bello University Zaria, Kaduna State, Nigeria.
e-mail: gugarba@yahoo.com

Definition 1.1.

A magic square $n-1 \times n-1$ formed by removing the border cells of an $n \times n$ Loubéré Magic Square is called the subelement magic square of the $n \times n$ Loubéré Magic Square.

Remarks 1.1.

We have interest in the least subelement which is a subset of 3×3 Pancolumn Magic Squares. Purposefully, the 3×3 Loubéré Magic Square has no subelement for it is not a pancolumn. We explicate that the subelement magic squares of the Loubéré Magic Squares Semigroup forms a semi group and the subelement magic squares of the Loubéré Magic Squares Group forms a group with respect to the same underlining set and operation.

Notes

Global Journal of Science Frontier Research (F) Volume XV Issue I Version I Year 2015

II. Preliminaries

A basic magic square of order n is an arrangement of arithmetic sequence of common difference of 1 from 1 to n^2 in an $n \times n$ square grid of cells such that every row, column and diagonal add up to the same number, called the magic sum M(S) expressed as $M(S) = \frac{n^3+n}{2}$ and a centre piece C as $C = \frac{M(S)}{n}$.

a) Loubéré Procedure (NE-W-S or NW-E-S, the cardinal points)

Consider an empty $n \times n$ square of grids (or cells). Start, from the central column or row at a position $\left\lfloor \frac{n}{2} \right\rfloor$ where $\lfloor \square \rfloor$ is the greater natural number less than or equal to, with the number 1. The fundamental movement for filling the square is diagonally up, right (clock wise or NE or SE) or up left (anti clock wise or NW or SW) and one step at a time. If a filled cell (grid) is encountered, then the next consecutive number moves vertically down ward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S (depending on the position of the first term of the sequence) to the last row or first row or first column or last column.

Definition 2.1

Main Row or Column is the column or row of the Loubéré Magic Squares containing the first term and the last term of the arithmetic sequence in the square.

b) The Proof of the $\left\lfloor \frac{m^2}{2} \right\rfloor = a + \left(\frac{m-1}{2}\right) j$ and of the $M(S) = \frac{m}{2}[2a + (m-1)j]$, where $j = \frac{l-a}{m-1}$

Theorem 2.1.

Let the arithmetic sequence $a, a+d, \ldots, l = a + (n-1)d$ be arranged in an $m \times m$ Loubéré Magic Square. Then the magic sum of the square is expressed as $M(S) = \frac{m}{2}[\,2a + (m-1)j\,]$ and the middle term of the sequence (centre piece of the square) is expressed as $C = a + \left(\frac{m-1}{2}\right)j$ where j denotes the common difference of entries along the main column or row and is given as $j = \frac{l-a}{m-1}$.

Proof.

Consider any arbitrary General Loubéré Magic Square (here we consider 3×3) as follows:

$c+b$	$c-b-d$	$c+d$
$c-b+d$	c	$c+b-d$
$c-d$	$c+b+d$	$c-b$

Let $a = c - b - d$ and $l = c + b + d$. Then we have (from the square) an arithmetic sequence: $c - b - d, c - b, \ldots, c + b + d$ having the sums S as

$$S = (c - b - d) + (c - b) + \cdots + (c + b) + (c + b + d) \rightarrow (1)$$
$$+$$
$$S = (c + b + d) + (c + b) + \cdots + (c - b) + (c - b - d) \rightarrow (2)$$

Adding (1)and (2), $2s = 2c + 2c + \cdots$ n times

i.e. $2s = 2nc \Longrightarrow c = \frac{s}{n} \ldots (3)$ and $s = \frac{n}{2}(a + l) \ldots (4)$ from the Gaussian High School (Elementary) Method. Since our square is $m \times m$, m number of cells (terms) are on the main column whence $a = c - b - d$. Thus, (3) and (4) become $C = \frac{M(S)}{m}$ … (5) and $M(S) = \frac{m}{2}[a + l] \ldots (6)$ respectively. And, $l = a + (m - 1)j \ldots (7)$ where j is along the main column. Substituting (7) in (6), we have: $M(S) = \frac{m}{2}[2a + (m - 1)j]$ … (8). Substituting (8) in (5), we get: $C = a + \left(\frac{m-1}{2}\right)j \ldots (9)$ From (3) and (4), $C = \frac{1}{2}(a + l) = \left(a - \frac{a}{2}\right) + \frac{l}{2} = a + \frac{(l-a)}{2} = a + \frac{l-a}{m-1}\frac{m-1}{2}$, i.e. $C = a + \left(\frac{m-1}{2}\right)\frac{l-a}{m-1}$ … (10). Comparing (9) and (10), we have: $j = \frac{l-a}{m-1}$ … (11).

Definition 2.2.

A non empty set S equipped with a binary operation * is said to be a Semigroup (S,*) if it satisfies the following axioms:

i. $a, b \in S \Longrightarrow a * b \in S$; *and*

ii. $a, b, c \in S \Rightarrow a * (b * c) = (a * b) * c$.

If in addition to the 2 axioms above, the following axioms are satisfied; then we call the algebraic structure a group $(G,*)$.

iii. $\exists e \in S \ni a * e = e * a \; \forall a \in S$; and

iv. $\forall a \in S, \exists a^{-1} \in S \ni a * a^{-1} = a^{-1} * a = e \in S$.

If in addition to the above 4 axioms: I; ii; iii; and iv; the following axiom is satisfied; then we call (G,*) an abelian group.

v. $\forall a, b \in S, a * b = b * a$

Remark 2.1. The shift in notations from the use of S to G is intentional by the respective specialists.

III. The Loubéré Magic Squares Semigroup and Groups

We hereby present that the set of Loubéré Magic Squares L over the set of natural numbers equipped with the matrix binary operation of addition $\oplus$ forms a semigroup, and over the set of integer numbers forms a group-enclosed with the same operation.

a) Definition 3.1.

The square of grid of cells $[a_{ij}]_{n \times n}$ is said to be Loubéré Magic Square if the following conditions are satisfied.

i. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = k$

ii. $\mathrm{trace}[a_{ij}]_{n \times n} = \mathrm{trace}[a_{ij}]^{T}_{n \times n} = k$

iii. $a_{1,\left\lceil\frac{n}{2}\right\rceil}$, $a_{\left\lceil\frac{n}{2}\right\rceil,\left\lceil\frac{n}{2}\right\rceil}$, $a_{n,\left\lceil\frac{n}{2}\right\rceil}$ are on the same main column or row and $a_{\left\lceil\frac{n}{2}\right\rceil,n}$, $a_{\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n}{2}\right\rceil}$, $a_{\left\lceil\frac{n}{2}\right\rceil,1}$ are on the same main column or row,

Notes

where $[\square]$ is the greater integer less or equal to, T is the transpose (of the square), k is the magic sum (magic product is defined analogously) usually expressed as $k=\frac{n}{2}[2a+(n-1)j]$ – from the sum of arithmetic sequence, where j is the common difference along the main column or row and a is the first term of the sequence– and $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}=\frac{k}{n}$.

b) Theorem 3.2.

$(L,\oplus)$ forms an Infinite Commutative Semigroup if the underlining multi set is of natural numbers and it forms an Infinite Additive Abelian Group if the underlining multi set is of integer numbers.

Proof. Let $[a_{i,j}]_{n\times n}$ and $[b_{ij}]_{n\times n}\in L$. Then, by Definition 3.1, $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}=k$, $trace[a_{ij}]_{n\times n}=trace[a_{ij}]_{n\times n}^{T}=k$, and $a_{1,\lceil\frac{n}{2}\rceil}, a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}, a_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n}, a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}, a_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row, and $\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}=l$, $trace[b_{ij}]_{n\times n}=trace[b_{ij}]_{n\times n}^{T}=l$, and $b_{1,\lceil\frac{n}{2}\rceil}, b_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}, b_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $b_{\lceil\frac{n}{2}\rceil,n}, b_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}, b_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row.

Then,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}+\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}=k+l=\mathrm{trace}[a_{ij}]_{n\times n}+\mathrm{trace}[b_{ij}]_{n\times n}=\mathrm{trace}[a_{ij}+b_{ij}]_{n\times n}$$

$$\mathrm{trace}[a_{ij}]_{n\times n}^{T}+\mathrm{trace}[b_{ij}]_{n\times n}^{T}=\mathrm{trace}[a_{ij}+b_{ij}]_{n\times n}^{T}=k+l,$$ and $a_{1,\lceil\frac{n}{2}\rceil}+b_{1,\lceil\frac{n}{2}\rceil}$ $a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}+b_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}$, $a_{n,\lceil\frac{n}{2}\rceil}+b_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n}+b_{\lceil\frac{n}{2}\rceil,n}$, $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}+b_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $a_{\lceil\frac{n}{2}\rceil,1}+b_{\lceil\frac{n}{2}\rceil,1}$ since $a_{1,\lceil\frac{n}{2}\rceil}, a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}, a_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n}, a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}, a_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row, and $b_{1,\lceil\frac{n}{2}\rceil}, b_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}, b_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $b_{\lceil\frac{n}{2}\rceil,n}, b_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}, b_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row.

i. *Associativity*

Let $[a_{i,j}]_{n\times n}, [b_{i,j}]_{n\times n}$ *and* $[c_{i,j}]_{n\times n}\in L$. Then, by Definition 3.1, $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}=k$, $\mathrm{trace}[a_{ij}]_{n\times n}=\mathrm{trace}[a_{ij}]_{n\times n}^{T}=k$, and $a_{1,\lceil\frac{n}{2}\rceil}, a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}, a_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n}, a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}, a_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row, $\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}=l$, $\mathrm{trace}[b_{ij}]_{n\times n}=\mathrm{trace}[b_{ij}]_{n\times n}^{T}=l$ and $b_{1,\lceil\frac{n}{2}\rceil}, b_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}, b_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $b_{\lceil\frac{n}{2}\rceil,n}, b_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}, b_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row.

and $\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}=m$, $\mathrm{trace}[c_{ij}]_{n\times n}=\mathrm{trace}[c_{ij}]_{n\times n}^{T}=m$ and $c_{1,\lceil\frac{n}{2}\rceil}, c_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}, c_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $c_{\lceil\frac{n}{2}\rceil,n}, c_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}, c_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row

Then,

Notes

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} + \left(\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\right) = \mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n\times n}} + \left(\mathrm{trace}[\mathrm{b_{ij}}]_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{c_{ij}}]_{\mathrm{n\times n}}\right)$$
$$= \mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{b_{ij}}]_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{c_{ij}}]_{\mathrm{n\times n}}$$
$$= \left(\mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{b_{ij}}]_{\mathrm{n\times n}}\right) + \mathrm{trace}[\mathrm{c_{ij}}]_{\mathrm{n\times n}} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}$$

Notes

$$\mathrm{trace}[\mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} + \left(\mathrm{trace}[\mathrm{b_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{c_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}}\right) = \mathrm{trace}[\mathrm{a_{ij}} + \mathrm{b_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} =$$

$\mathrm{trace}[\mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} + \left(\mathrm{trace}[\mathrm{b_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{c_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}}\right) = \mathrm{k} + \mathrm{l} + \mathrm{m}$, and $a_{1,\lceil\frac{n}{2}\rceil} + \left(b_{1,\lceil\frac{n}{2}\rceil} + c_{1,\lceil\frac{n}{2}\rceil}\right)$, $a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil} + \left(b_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil} + c_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}\right)$, $a_{n,\lceil\frac{n}{2}\rceil} + \left(b_{n,\lceil\frac{n}{2}\rceil} + c_{n,\lceil\frac{n}{2}\rceil}\right)$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n} + \left(b_{\lceil\frac{n}{2}\rceil,n} + c_{\lceil\frac{n}{2}\rceil,n}\right)$, $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil} + \left(b_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil} + c_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}\right)$, $a_{\lceil\frac{n}{2}\rceil,1} + \left(b_{\lceil\frac{n}{2}\rceil,1} + c_{\lceil\frac{n}{2}\rceil,1}\right)$, then $\left(a_{\lceil\frac{n}{2}\rceil,n} + b_{\lceil\frac{n}{2}\rceil,n}\right) + c_{\lceil\frac{n}{2}\rceil,n}$, $\left(a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil} + b_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}\right) + c_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $\left(a_{\lceil\frac{n}{2}\rceil,1} + b_{\lceil\frac{n}{2}\rceil,1}\right) + c_{\lceil\frac{n}{2}\rceil,1}$ *and* $a_{1,\lceil\frac{n}{2}\rceil}$, $a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}$, $a_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n}$, $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $a_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row , and $b_{1,\lceil\frac{n}{2}\rceil}$, $b_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}$, $b_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $b_{\lceil\frac{n}{2}\rceil,n}$, $b_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $b_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row, and $c_{1,\lceil\frac{n}{2}\rceil}$, $c_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}$, $c_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $c_{\lceil\frac{n}{2}\rceil,n}$, $c_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $c_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row.

ii. *Identity Element*

$\exists[a_{ij}]_{n\times n} \in L$ *and* is said to be Loubéré Magic Square if the following conditions are satisfied.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = 0$$

$$\mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n\times n}} = \mathrm{trace}[\mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} = 0$$

$a_{1,\lceil\frac{n}{2}\rceil} = 0$, $a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil} = 0$, $a_{n,\lceil\frac{n}{2}\rceil} = 0$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n} = 0$, $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil} = 0$, $a_{\lceil\frac{n}{2}\rceil,1} = 0$ are on the same main column or row, whence $\mathrm{a_{ij}} = 0, \forall \mathrm{i,j} \Rightarrow$ the identity is$[0]_{\mathrm{n\times n}} \in \mathrm{L}$

iii. *Inverse Element Property*

Given $[a_{i,j}]_{n\times n} \in L \ni \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = k$,

$\mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n\times n}} = \mathrm{trace}[\mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} = \mathrm{k}$, and

$a_{1,\lceil\frac{n}{2}\rceil}$, $a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}$, $a_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $a_{\lceil\frac{n}{2}\rceil,n}$, $a_{\lceil\frac{n}{2}\rceil\lceil\frac{n}{2}\rceil}$, $a_{\lceil\frac{n}{2}\rceil,1}$ are on the same main column or row, there exists $[-a_{i,j}]_{n\times n} \in L$ such that $\sum_{i=1}^{n}\sum_{j=1}^{n} -a_{ij} = -k$, $\mathrm{trace}[-\mathrm{a_{ij}}]_{\mathrm{n\times n}} = \mathrm{trace}[-\mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} = -\mathrm{k}$, and $-a_{1,\lceil\frac{n}{2}\rceil}$, $-a_{\lceil\frac{n}{2}\rceil,\lceil\frac{n}{2}\rceil}$, $-a_{n,\lceil\frac{n}{2}\rceil}$ are on the same main column or row and $-a_{[\frac{n}{2}],n}$, $-a_{[\frac{n}{2}][\frac{n}{2}]}$, $-a_{[\frac{n}{2}],1}$ are on the same main column or row. Thus, $[a_{ij}]_{n\times n} + [-a_{ij}]_{n\times n} = [-a_{ij}]_{n\times n} + [a_{ij}]_{n\times n} = [0]_{\mathrm{n\times n}}$.

iv. *Commutativity*

$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} = k + l = l + k = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$ and $\mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{b_{ij}}]_{\mathrm{n\times n}} = \mathrm{trace}[\mathrm{a_{ij}} + \mathrm{b_{ij}}]_{\mathrm{n\times n}} = k + l = l + k = \mathrm{trace}[\mathrm{b_{ij}} + \mathrm{a_{ij}}]_{\mathrm{n\times n}} = \mathrm{trace}[\mathrm{b_{ij}}]_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n\times n}}$

$\mathrm{trace}[\mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{b_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} = \mathrm{trace}[\mathrm{a_{ij}} + \mathrm{b_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} = \mathrm{k} + \mathrm{l} = \mathrm{l} + \mathrm{k} = \mathrm{trace}[\mathrm{b_{ij}} + \mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} = \mathrm{trace}[\mathrm{b_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}} + \mathrm{trace}[\mathrm{a_{ij}}]^{\mathrm{T}}_{\mathrm{n\times n}}$, and $a_{1,[\frac{n}{2}]} + b_{1,[\frac{n}{2}]}$, $a_{[\frac{n}{2}],[\frac{n}{2}]} + b_{[\frac{n}{2}],[\frac{n}{2}]}$, $a_{n,[\frac{n}{2}]} + b_{n,[\frac{n}{2}]}$ are on the same main column or row and $a_{[\frac{n}{2}],n} + b_{[\frac{n}{2}],n}$, $a_{[\frac{n}{2}][\frac{n}{2}]} + b_{[\frac{n}{2}][\frac{n}{2}]}$, $a_{[\frac{n}{2}],1} + b_{[\frac{n}{2}],1}$ as well as $b_{1,[\frac{n}{2}]} + a_{1,[\frac{n}{2}]}$, $b_{[\frac{n}{2}],[\frac{n}{2}]} + a_{[\frac{n}{2}],[\frac{n}{2}]}$, $b_{n,[\frac{n}{2}]} + a_{n,[\frac{n}{2}]}$ are on the same main column or row and $b_{[\frac{n}{2}],n} + a_{[\frac{n}{2}],n}$, $b_{[\frac{n}{2}][\frac{n}{2}]} + a_{[\frac{n}{2}][\frac{n}{2}]}$, $b_{[\frac{n}{2}],1} + a_{[\frac{n}{2}],1}$.

We can now consider the general multi set. The Loubéré Magic Squares over the multi set of integer numbers, since multi set of natural numbers is its subset, is a semi pandiagonal. By semi pandiagonal, we mean in $n \times n$ square, n elements repeats on every row, column and on a diagonal. Though the sum of the numbers on the rows, the columns and the diagonals add up to the magic sum; yet one diagonal has an n repetition of one element. To change the orientation (from left to right or the reverse) of the pandiagonal of the 3×3, use the sequence $a, a, a, b, b, b, c, c, c$ rather than $a, b, c, a, b, c, a, b, c$.

We can now show that they form a group as in the above. Consider 3 arbitrary elements of the set of Lefty Semi Pandiagonal Loubéré Magic Squares,

b	a	c
c	b	a
a	c	b

,

e	d	f
f	e	d
d	f	e

and

h	g	i
i	h	g
g	i	h

Then,

b	a	c
c	b	a
a	c	b

+

e	d	f
f	e	d
d	f	e

=

b+e	a+d	c+f
c+f	b+e	a+d
a+d	c+f	b+e

Notes

Notes

i. is also a Lefty Semi Pandiagonal 3×3 Loubéré Magic Squares, hence *closure property* is satisfied.

ii. *Associativity.* It is clear (even from inherited property of the underlining set) that

$$\begin{bmatrix} b & a & c \\ c & b & a \\ a & c & b \end{bmatrix} + \left(\begin{bmatrix} e & d & f \\ f & e & d \\ d & f & e \end{bmatrix} + \begin{bmatrix} h & g & i \\ i & h & g \\ g & i & h \end{bmatrix} \right) = \left(\begin{bmatrix} b & a & c \\ c & b & a \\ a & c & b \end{bmatrix} + \begin{bmatrix} e & d & f \\ f & e & d \\ d & f & e \end{bmatrix} \right) + \begin{bmatrix} h & g & i \\ i & h & g \\ g & i & h \end{bmatrix}$$

iii. The *identity* is −as in the above−

0	0	0
0	0	0
0	0	0

iv. Let the following be an arbitrary Lefty Semi Pandiagonal Magic Square.

V	U	W
W	V	U
U	W	V

Clearly, its *inverse* is

-V	-U	-W
-W	-V	-U
-U	-W	-V

v. Every 2 Loubéré Magic Squares (whether semi pancolumn or not) over multi set of naturals or over multi set of integer numbers *commute* since natural and integer numbers commutes.

Thus the group and the semigroups of the Loubéré Magic Squares are commutative.

IV. Centre Pieces and Magic Sums Abelian Groups

a) *Centre Pieces Abelian Group*

The set of the centre pieces $c_1, c_2, c_3, \dots$ of $m \times m$ Loubéré Magic Squares equipped with the integer number binary operation of addition forms an infinite abelian group. Given the centre pieces $c_1, c_2, c_3, \dots$ of $m \times m$ Loubéré Magic Squares with corresponding formula

$$c_1 = a_1 + \left(\frac{m-1}{2}\right) j_1, c_2 = a_2 + \left(\frac{m-1}{2}\right) j_2,\ c_3 = a_3 + \left(\frac{m-1}{2}\right) j_3, \dots; \text{then}$$

i. $c_1 + c_2 = (a_1 + a_2) + \left(\frac{m-1}{2}\right)(j_1 + j_2)$ is the centre piece of the $m \times m$ Loubéré Magic Square with first term $a_1 + a_2$ and common difference along the main column $j_1 + j_2$. Hence, the set is closed.

ii. This is an inherited property of the set of integer numbers:

$$c_1 + (c_2 + c_3) = (a_1 + a_2 + a_3) + \left(\frac{m-1}{2}\right)(j_1 + j_2 + j_3) = (c_1 + c_2) + c_3$$

iii. The identity element is the zero centre piece e.g.

C	-D	A
-B	0	B
-A	D	-C

iv. Given an arbitrary centre piece $c_n = a_n + \left(\frac{m-1}{2}\right) j_n$ of the $m \times m$ Loubéré Magic Square, there exists another centre piece c_{-n} of another $m \times m$ Loubéré Magic Square having first term as $-a_n$ and common difference along the main column or row as $-j_n$, thus its formula is $c_{-n} = -a_n + \left(\frac{m-1}{2}\right)(-j_n)$ such that $c_n + c_{-n} = c_{-n} + c_n = (a_n - a_n) + \left(\frac{m-1}{2}\right)[j_n - j_n] = 0 = c_i$, the identity centre piece.

v. Clearly $c_1 + c_2 = a_1 + a_2 + \left(\frac{m-1}{2}\right)(j_1 + j_2) = a_2 + a_1 + \left(\frac{m-1}{2}\right)(j_2 + j_1) = c_2 + c_1$

The set equipped with the operation is an abelian group.

b) *Magic Sum Abelian Groups*

The set of the magic sums $M(s_1), M(s_2), M(s_3), \dots$ of $m \times m$ Loubéré Magic Squares equipped with the integer binary operation of addition form an infinite abelian group. Given the magic sums $M(s_1), M(s_2), M(s_3), \dots$ of $m \times m$ Loubéré Magic Squares with corresponding formula

$$M(s_1) = \frac{m}{2}[2a_1 + (m-1)j_1, M(s_2) = \frac{m}{2}[2a_2 + (m-1)j_2], M(s_3)) = \frac{m}{2}[2a_3 + (m-1)j_3, \dots;$$

then (as in the above):

i. $M(s_1) + M(s_2) = M(s_?)$ where $M(s_?)$ is a magic sum of another $m \times m$ Loubéré Magic Square with first term $a_1 + a_2$ and common difference along the main column as $j_1 + j_2$.

The axioms: ii, iii, iv and v follow, by analogy to the centre piece abelian group, immediately.

Notes

Notes

V. Eigen Values Abelian Group

The Eigen values computation in the magic squares is what is zealotly prophesized that magic squares are special type of matrices, hence the definition of the magic squares, we do not love to like such a sudden conclusion if loving to liking forces choosing the definitions in terms of just the square grids (or cells).

We want to show through concrete examples that the set of Eigen Values of the Loubéré Magic Squares with the usual integer numbers binary operation of addition forms a group. Consider the following arbitrary two 3×3 Loubéré Magic Squares – which we let

a =

4	3	2
-1	1	3
0	5	-2

, b =

2	-5	0
-3	-1	1
-2	3	-4

and their sum c =

6	-8	2
-4	0	4
-2	8	-6

We compute the eigen values for a as follows: The corresponding matrix of a is $(a) = \begin{pmatrix} 4 & -3 & 2 \\ -1 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix}$, its eigen vector is $|a - \lambda I| = \begin{vmatrix} 4-\lambda & -3 & 2 \\ -1 & 1-\lambda & 3 \\ 0 & 5 & -2-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 3\lambda^2 - 24\lambda - 72 = (\lambda - 3)(\lambda^2 - 24) = 0$ having characteristic equation as $\lambda_{a_1} = 3$, $\lambda_{a_2} = 4.9$ and $\lambda_{a_3} = -4.9$.

We compute the eigen values for b as follows: The corresponding matrix of b is $(b) = \begin{pmatrix} 2 & -5 & 0 \\ -3 & -1 & 1 \\ -2 & 3 & -4 \end{pmatrix}$, its characteristic equation is $|b - \lambda I| = \begin{vmatrix} 2-\lambda & -5 & 0 \\ -3 & -1-\lambda & 1 \\ -2 & 3 & -4-\lambda \end{vmatrix} = 0$ i.e. $\lambda^3 + 3\lambda^2 - 24\lambda - 72 = (\lambda + 3)(\lambda^2 - 24) = 0$ with eigen values $\lambda_{b_1} = -3, \lambda_{b_2} = 4.9$ and $\lambda_{b_3} = -4.9$.

We compute the eigen values for c as follows: The corresponding matrix of c is $(c) = \begin{pmatrix} 6 & -8 & 2 \\ -4 & 0 & 4 \\ -2 & 8 & -6 \end{pmatrix}$, its characteristic equation is $|c - \lambda I| = \begin{vmatrix} 6-\lambda & -8 & 2 \\ -4 & -\lambda & 4 \\ -2 & 8 & -6-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 96\lambda = 0$ with corresponding eigen values $\lambda_{c_1} = 0, \lambda_{c_2} = 9.8$ and $\lambda_{c_3} = -9.8$.

We now conclude this session by showing that the set of eigen values satisfies *The Properties of a Group* as follows:

Closure Property. Consider any 3 arbitrary Loubéré Magic Squares a, b, c; such that $a + b = c$; then from the example above, the corresponding eigen values of a; $\lambda_{a_1}, \lambda_{a_2}\lambda_{a_3}$; the corresponding eigen values of b; $\lambda_{b_1}, \lambda_{b_2}, \lambda_{b_3}$; are such that $\lambda_{a_1} + \lambda_{b_1} = \lambda_{c_1}$, $\lambda_{a_2} + \lambda_{b_2} = \lambda_{c_2}$, and $\lambda_{a_3} + \lambda_{b_3} = \lambda_{c_3}$ where $\lambda_{c_1}, \lambda_{c_2}, \lambda_{c_3}$ are the corresponding eigen values of c.

Associativity Property. Since Loubéré Magic Squares are a semigroup (which is easy to observe), the eigen values are associative.

Identity Element Property. The eigen value 0 is the identity element that corresponds to the sum of the Loubéré Magic Squares of opposite eigen values as in the above.

Inverse Elements Property. For any arbitrary eigen value λ_m corresponding to a Loubéré Magic Square m, there exist a $-\lambda_m$ eigen value corresponding to another Loubéré Magic Square such that $\lambda_m + (-\lambda_m)$ gives the identity element which is formed as a result of matrix addition of the aforementioned Loubéré Magic Squares.

Commutativity. Consider any 2 arbitrary Loubéré Magic Squares a, b, ; such that $a + b = b + a$; then from the example above, the corresponding eigen values of a; $\lambda_{a_1}, \lambda_{a_2} \lambda_{a_3}$; the corresponding eigen values of b; $\lambda_{b_1}, \lambda_{b_2}, \lambda_{b_3}$; are such that $\lambda_{a_1} + \lambda_{b_1} = \lambda_{b_1} + \lambda_{a_1}$, $\lambda_{a_2} + \lambda_{b_2} = \lambda_{b_2} + \lambda_{a_2}$, and $\lambda_{a_3} + \lambda_{b_3} = \lambda_{b_3} + \lambda_{a_3}$.

The idea of eigen values computation of a magic square is conceived from the work of [1].

VI. The Subelement Magic Squares Semigroup and Group

The set of least subelement of Loubéré Magic Squares is a subset of pancolumn 3×3 Magic Squares. By convention, the 3×3 Loubéré Magic Square(since not pancolumn) is not a self subelement. The sum of two arbitrary subelements of $m \times m$ Loubéré Magic Squares is a subelement of $m \times m$ Loubéré Magic Square, hence closure property is exhibited. Associativity , Identity, Inverse and Commutativity Properties are inherited from the super elements, the Loubéré Magic Squares. Both the binary and the unary operations of the super elements and of the subelements are equal.

References Références Referencias

1. Daryl Lynn Stephens, "Matrix Properties of Magic Squares," A Master of Science Professional Paper, College of Arts and Science, Denton, Texas, pp.32, 1993.
2. Lee C.F. Sallows (1986). Adventures with Turtle Shell and Yew between the Mountains of Mathematics and the Lowlands of Logology, ABACUS, Spriger-Verlag, New York, Inc,vol. 4, pp.1, 1986.
3. John M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, New York, United States, vol.1,pp.1, 2003.
4. Joseph J. Rotman, *A First Course in Abstract Algebra*, Prentice Hall, Upper Saddle River, New Jersey, 4: 134, 20--.

Notes

Global Journals Inc. (US) Guidelines Handbook 2015

www.GlobalJournals.org