Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 7 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Non-Local Solution of Mixed Integral Equation with Singular Kernel

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GJSFR-F Classification : *45B05, 45G10, 60R*

Strictly as per the compliance and regulations of :

6. N. I. Muskhelishvili, Singular Integral Equations, Noordhoff (1953).

Non-Local Solution of Mixed Integral Equation with Singular Kernel

M. A. Abdou [α], S. A. Raad [σ] & W. Wahied [ρ]

Abstract- In this paper, we consider a non-local mixed integral equation in position and time in the space $L_2[-1,1]\times C[0,T];T<1$. Then, using a quadratic numerical method, we have a system of Fredholm integral equations (SFIEs), where the existence of a unique solution is considered. Moreover, we consider Product Nystrom method (PNM), as a famous method to solve the singular integral equations, to obtain an algebraic system. Finally, some numerical results are considered, and the error estimate, in each case, is computed.

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I. Introduction

The integral equations have received considerable interest of many applications in different mathematical areas of sciences. Therefore, the authors established many analytic and numeric methods to obtain the solutions of the integral equations. For some works, the reader can forward to the following references [1-5]. For the analytical methods, one can use degenerate kernel method, Cauchy method (singular integral method), Laplace transformation method, Fourier transformation method, potential theory method, and Krien's method. More information for the analytic methods can be found in Muskhelishvili [6], Popov [7], Tricomi [8], Hochstad [9] and Green [10]. More recently, since analytical methods on practical problems often fail, numerical solutions of these equations are much studied subjected of numerous works. The interested reader should consult the fine exposition by Atkinson [11], Delves and Mohamed [12], Golberg [13] and Linz [14] for some different numerical methods. In [15] a mixed integral equation in one-dimensional is considered, under certain conditions, and the solution in a series form is obtained. In addition, a mixed integral equation of the second kind, when the Fredholm kernel takes a logarithmic form is discussed and solved in [16]. In [17] Kauthen used a collection method to solve the mixed integral equation with continuous kernel, numerically.

Consider the mixed integral equation of the second kind:

$$\mu\phi(x,t)=f(x,y)-H\left(x,t,\phi(x,t)\right)+\lambda\int_{-1}^{1}k\left(|x-y|\right)\phi(y,t)dy+\lambda\int_{0}^{t}F\left(|t-\tau|\right)\phi(x,\tau)d\tau. \quad (1)$$

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The given two continuous function $f(x,y)$ and $H(x,t,\phi(x,t))$ define in the Banach space $L_2[-1,1]\times C[0,T]$, $0\le t\le T<1$, where the first function is called the free term and the second function is known as the memory of the integral equation. The known functions $k(|x-y|)$ and $F(t,\tau)$ represent the kernels of Fredholm and Volterra integral terms, respectively. The unknown function $\phi(x,t)$ represents the solution of (1).The constant λ may be complex, has a physical meaning and μ, defines the kind of integral equation.

In order to guarantee the existence and uniqueness solution of Eq.(1), we assume the following conditions:

(i) The given function $H(x,t,\phi(x,t))$ with its partial derivatives with respect to position and time is continuous in the space $L_2[\Omega]\times C[0,T]$, for the constant $\mathrm{L}>\mathrm{L}_1$ and $\mathrm{L}>\mathrm{L}_2$ satisfies the following conditions:

$$(i.a)\ |H(x,t,\phi(x,t))|\le \mathrm{L}_1|\phi(x,t)| \qquad (i.b)\ |H(x,t,\phi(x,t))-H(x,t,\psi(x,t))|\le \mathrm{L}_2|\phi(x,t)-\psi(x,t)|$$

Where the norm is defined as $\|\phi(x,t)\|=\max_{0\le t\le T}\int_0^t\left[\int_{-1}^{1}|\phi(x,\tau)|^2dx\right]^{1/2}d\tau.$

(ii) The Fredholm kernel satisfies $\left[\int_{-1}^{1}\int_{-1}^{1}k(|x-y|)^2dydx\right]=M^2$, ($M$ is a constant).

(iii) The discontinuous function $F(|t-\tau|)$ is absolutely integrable with respect to τ for all

$0\le t\le T<1$, and satisfies $\int_0^t F(|t-\tau|)d\tau=N$, (N is constant).

(iv) The given function $f(x,t)$ with its partial derivatives with respect to position x and time t are continuous in the space $L_2[-1,1]\times C[0,T]$ and its norm is defined as

$$\|f(x,t)\|=\max_{0\le t\le T}\left|\int_0^t\left\{\int_{-1}^{1}f^2(x,\tau)dx\right\}^{\frac{1}{2}}d\tau\right|=V=\text{costant}.$$

a) *Theorem 1 (without proof):*

If the conditions (i) –(iii) are satisfied, then Eq. (1) has a unique solution $\phi(x,t)$ in the space $L_2[-1,1]\times C[0,T]$ inside the sphere of radius ρ such that:

$$\rho=V/[\mu-(\mathrm{L}+|\lambda|M+|\lambda|TN)],\quad (\mathrm{L}+|\lambda|M+|\lambda|TN)<\mu. \blacksquare \tag{2}$$

In the remainder, of this paper a suitable quadratic numerical method is used to reduce the mixed integral equation into **SFIEs** of the second kind. Then using PNM, as a suitable numerical method to solve the singular integral equations, the **SFIEs** will reduce to an algebraic system. Finally, many numerical results are calculated when the kernel takes a logarithmic form and Carleman function forms. Moreover, the error estimate, in each case, is computed.

II. System of Fredholm Integral Equations

In this section, a quadratic numerical method is used; see Atkinson [11], Delves and Mohamed [12], to obtain **SFIEs** of the second kind, where the existence and uniqueness of the integral

Ref

11. K. E. Atkinson, A Survey of Numerical Method for the Solution of Fredholm Integral Equation of the Second Kind, SIAM, Philadelphia, 1976.

system are considered. Moreover, the equivalence between the **F-VIE** and the **SFIEs** is obtained.

For this, we divide the interval $[0,T]$ into m subintervals, by means of the points $0=t_0<t_1<...<t_m=T$, where $t=t_i,\tau=t_j, i,j=0,1,2,...,m$, then using the quadrature formula. The formula (1) can be adapted in the following form

Notes

$$\mu_i\phi_i(x)=f_i(x)-H_i(x,\phi_i(x))+\lambda\sum_{j=0}^{i-1}\omega_j F_{i,j}\phi_j(x)+\lambda\int_{-1}^{1}k(|x-y|)\phi_i(y)dy,\qquad i=0,1,2,...,m. \quad (3)$$

Here, we used the following nations:

$$\phi(x,t_i)=\phi_i(x),\ H(x,t_i,\phi(x,t_i))=H_i(x,\phi_i(x));\ F(|t_i-t_j|)=F_{i,j};\ \mu_i=(\mu-\omega_i F_{i,i}),$$

$$f(x,t_i)=f_i(x);\qquad \omega_j=\begin{cases}h_j/2 & j=0,\ \ j=i\\ h_j & 0<j<i\end{cases}.$$

Here, $h_i=\max\limits_{0\le j\le m}(t_{j+1}-t_j), h_i$ is the step size of integration, and ω_j are the weights,

The value of i and p depend on the number of derivative of $v(t,\tau)$ with respect to t for all $\tau\in[0,T]$. Here, we neglect the term of the error of the quadratic numerical method $0(h_i^{p+1})$.

If in (3) $\mu=\omega_i F_{i,i}$, we have homogeneous **SFIEs**. While, the system is nonhomogeneous if $\mu_i=(\mu-\omega_i F_{i,i})\neq 0$.

Definition 1: The estimate error $R_{m,i}$, of the quadratic method, is determined by the relation

$$R_{m,i}=\left|\int_0^t F(|t_i-t_j|)\phi(x,\tau)d\tau-\sum_{j=0}^{i}\omega_j F_{i,j}\phi_j(x)\right|,\qquad i=1,2,...,m\ .\blacksquare \quad (4)$$

Remark1: Consider $\Phi(x)=\{\phi_0(x),\phi_1(x),...,\phi_i(x),...\}$ be the set of all continuous functions in E, where $\phi_i(x)\in L_2[-1,1]$ for all i, and define on E the norm by

$$\|\phi\|_E=\max_i\left[\int_{-1}^{1}|\phi_i(x)|^2dx\right]^{\frac{1}{2}}=\max_i\|\phi_i\|_{L_2[\Omega]}\qquad,\forall i. \quad (5)$$

Then, E is a Banach space.$\blacksquare$

In order to guarantee the existence of a unique solution of (3) in the Banach space E, we assume the following:

$$(1).\ \max_j\sum_{j=0}^{i-1}|\omega_j F_{i,j}|\le N^*,\ \ (2).\ \left[\int_{-1}^{1}\int_{-1}^{1}|k(|x-y|)|^2dydx\right]=M^2,\ (3).\ \|f\|_E=\max_i\|f_i\|=V^*,\ \forall i.$$

(4). The functions $H_i(x,\phi_i(x))$, for the constants $L^*>L_1^*$ and $L^*>L_2^*$ satisfies the following conditions:

$$(4.a).\ |H_i(x,\phi_i(x))|\le L_1^*|\phi_i(x)|,\quad (4b).\ |H_i(x,\phi_i(x))-H_i(x,\psi_i(x))|\le L_2^*|\phi_i(x)-\psi_i(x)|.$$

b) Theorem 2 (without proof): The **SFIEs** (4) have a unique solution $\phi_i(x)$ in the Banach space E under the conditions**:**

$$|\mu^*| \le V^*/[1-(L^*+|\lambda|N^*+|\lambda|M)]; \quad \mu^*=\min_i|\mu_i|. \blacksquare \tag{6}$$

III. Product Nystrom Method

In this section, we discuss the numerical solution of Eq. (3), when the kernel of position has a singular term, using **PNM**, see [11, 12].

For this, write the singular kernel of (3) in the form

$$k(|x-y|)=\overline{k(|x-y|)}\,\ell(x,y) \tag{7}$$

Where, $\overline{k(|x-y|)}$ is a badly behavior, while $\ell(x,y)$ well behavior.

In view of (7), the **SFIEs** (3), can be adapted in the form

$$\mu_i\phi_{i,p}=f_{i,p}-H_{i,p}(\phi_{i,p})+\lambda\sum_{j=0}^{i-1}\omega_j F_{i,j}\phi_{j,p}+\lambda\sum_{q=0}^{N}w_{p,q}\ell_{p,q}\phi_{i,q}, \; i=1,2,\dots,N. \tag{8}$$

Where, we use the following definitions $\phi_i(x_p)=\phi_{i,p}; H_i(x_p,\phi_i(x_p))=H_{i,p}(\phi_{i,p})$:

$\ell(x_p,x_q)=\ell_{p,q}\; x_p=y_p=-1+ph_1, p=0,1,\dots,N$ with $h_1=\frac{2}{N}, N$ even.

In addition, the weight terms $w_{p,q}$ are given by, see [12],

$$w_{p,0}=\beta_{p,1}, \quad w_{p,2q+1}=-2\gamma_{p,q+1}, \quad w_{p,2q}=\alpha_{pq}+\beta_{p,q+1}, \qquad w_{p,N}=\alpha_{p,N/2}. \tag{9}$$

$$\alpha_{p,q}=\frac{h_1}{2}\int_0^2\xi(\xi-1)\overline{k(|y_p-(y_{2q-2}+\xi h_1)|)}d\xi, \quad \beta_{p,q}=\frac{h_1}{2}\int_0^2(\xi-1)(\xi-2)\overline{k(|y_p-(y_{2q-2}+\xi h_1)|)}d\xi,$$

$$\gamma_{p,q}=\frac{h_1}{2}\int_0^2\xi(2-\xi)\overline{k(|y_p-(y_{2q-2}+\xi h_1)|)}d\xi.$$

Here, in (9), we introduce the change of variable $y=y_{2q-2}+\xi h_1, \quad 0\le\xi\le 2$

Definition 2: The relation between the estimate local error $R_{N,q}$ and Eq. (8) is

$$R_{N,q}=\left|\int_{-1}^{1}k(x,y)\phi_i(y)dy-\sum_{q=0}^{N}w_{pq}\ell_{p,q}\phi_{i,q}\right|. \blacksquare \tag{10}$$

Definition 3: The **PNM** is said to be convergent of order r in the interval $[-1,1]$, if and only if for sufficiently large N, there exists a constant $s>0$ independent of N such that

$$\left\|\phi_i(x)-(\phi_i(x))_N\right\|_\infty \le sN^{-r}. \blacksquare \tag{11}$$

In order to guarantee the existence of unique solution of the **NAS** (8) in the Banach space ℓ_∞, we assume the following conditions:

Ref

12. L. M. Delves and J.L. Mohamed, Computational Methods for Integral Equations, Phila-delphia, 1985.

$(a)\max_{j}\sum_{j=0}^{i-1}\left|\omega_j F_{i,j}\right|\le N^*$; $\quad (b)\sup_{q}\sum_{q=0}^{N}\left|w_{p,q}\ell_{p,q}\right|\le c$; $\quad (c)\ \|f\|_{\ell_\infty}=\sup_{i,p}\left|f_{i,p}\right|=V_1^*$ (N^*, V_1^*, c constants).

(d) For the constants $L'>L_1'$ and $L'>L_2'$, $H_{i,p}\left(\phi_{i,p}\right)$ satisfies the conditions:

$$(d.1)\ \left|H_{i,p}\left(\phi_{i,p}\right)\right|\le L_1'\left|\phi_{i,p}\right|,\quad (d.2)\ \left|H_{i,p}\left(\phi_{i,p}\right)-H_{i,p}\left(\psi_{i,p}\right)\right|\le L_2'\left|\phi_{i,p}-\psi_{i,p}\right|.$$

Notes

a) Theorem3 (without proof): The **NAS** (8) has a unique solution $\phi_{i,p}$ in the space ℓ_∞; $\|\phi\|_{\ell_\infty}=\sup_{i,p}\left|\phi_{i,p}\right|$, under the following conditions

$$\left\|\phi_{i,p}\right\|\le V^*/[\mu^*-\left(L'+|\lambda|N^*+|\lambda|c\right)];\quad \mu^*=\min_i|\mu_i|.\ \blacksquare \tag{12}$$

The equivalence between the **NAS** (8) and the **SFIEs** (3) is satisfies if:

$$\sum_{q=0}^{N}w_{p,q}\ell_{p,q}\phi_i\left(qh_1\right)\to\int_{-1}^{1}k\left(x,y\right)\phi_i\left(y\right)dy\ (N\to\infty).$$

Theorem 4: Under the conditions of theorem3, the sequence of functions, $\{\Phi_N\}=\left\{\left(\phi_{i,p}\right)_N\right\}$, of (10) convergence uniformly to the solution $\Phi=\left\{\phi_{i,p}\right\}$ of (3) in the space ℓ_∞.

Proof: From Eq. (8), we write

$$\left|\phi_{i,p}-\left(\phi_{i,p}\right)_N\right|\le\frac{1}{|\mu_i|}\left\{\left|H_{i,p}\left(\phi_{i,p}\right)-H_{i,p}\left(\left(\phi_{i,p}\right)_N\right)\right|+|\lambda|\sum_{j=0}^{i-1}\left|\omega_j F_{i,j}\right|\left|\phi_{j,p}-\left(\phi_{j,p}\right)_N\right|+\right.$$
$$\left.|\lambda|\sum_{q=0}^{N}\left|w_{p,q}\ell_{p,q}\right|\left|\phi_{i,q}-\left(\phi_{i,q}\right)_N\right|\right\}. \tag{13}$$

Using the conditions (a-d) of theorem 3, and using condition (3) of theorem 2, the above inequality can be adapted in the form

$$\sup_{i,p}\left|\phi_{i,p}-\left(\phi_{i,p}\right)_N\right|\le\sigma\left\|\Phi-\Phi_N\right\|_{\ell_\infty},\quad \sigma=\frac{\left(L'+|\lambda|N^*+|\lambda|c\right)}{\mu^*}<1.$$

Hence, we have $\left\|\Phi-\Phi_N\right\|_{\ell_\infty}\to 0$ as $N\to\infty.\ \blacksquare$

Definition 4: The estimate total error of **PNM**, $R_{m,N}=R_{m,i}+R_{N,q}$ of Eq.(1) is determined by the following relation

$$R_{m,N}=\left|\int_{-1}^{1}k\left(x,y\right)\phi\left(y,t\right)dy+\int_{0}^{t}F\left(t,\tau\right)\phi\left(x,\tau\right)d\tau-\sum_{q=0}^{N}w_{p,q}\ell_{p,q}\phi_{i,q}-\sum_{j=0}^{i}\omega_j F_{i,j}\phi_{j,p}\right|. \tag{14}$$

•The equivalence between the NAS and F-VIE:

When $m,N\to\infty$, the sum

$$\sum_{q=0}^{N}w_{pq}\ell_{p,q}\phi_{i,q}+\sum_{j=0}^{i}\omega_j v_{ij}\phi_{j,p}\to\int_{-1}^{1}k\left(|x-y|\right)\phi\left(y,t\right)dy+\int_{0}^{t}F\left(t,\tau\right)\phi\left(x,\tau\right)d\tau.$$

Then the solution of the **NAS** (8) becomes the solution of **F-VIE** (1).

Theorem 5: **if** the conditions (i) and (ii) of theorem1 are satisfied, then the sequence of functions $\{\Phi_{m,N}\} = \{\phi_{m,N}(x,t)\}$ convergence uniformly to the exact solution $\Phi = \phi(x,t)$ of Eq. (1)) in the space $L_2[-1,1]\times C[0,T]$.

Proof: the formula (1) with its approximation solution gives

$$\mu\|\Phi-\Phi_{m,N}\| \le \|H(x,t,\phi(x,t)) - H(x,t,\phi_{m,N}(x,t))\| + |\lambda| \left\| \int_{-1}^{1} |k(|x-y|)| |\phi(y,t)-\phi_{m,N}(y,t)| dy \right\| + \qquad (19)$$

$$|\lambda| \left\| \int_0^t |F(t,\tau)| |\phi(x,\tau)-\phi_{m,N}(x,\tau)| d\tau \right\|.$$

Using $|F(t,\tau)| \le N$, $F(t,\tau)$ is continuous, then with the aid of conditions (i-2), (ii) then, applying Cauchy Schwarz inequality, the above inequality becomes

$$\mu\|\Phi-\Phi_{m,N}\| \le \xi\|\Phi-\Phi_{m,N}\|, \quad \xi = (L_2 + |\lambda| NT + |\lambda| M) < 1.$$

Hence, we have $\|\Phi-\Phi_{m,N}\|_{L_2[\Omega]\times C[0,T]} \to 0$ as $m, N \to \infty$.

Corollary 2: the total error $R_{m,N}$ satisfies $\lim_{m,N\to\infty} R_{m,N} = 0$.

IV. Applications

Assume the **NF-VIE** of the second kind:

$$\phi(x,t) = f(x,t) - H(x,t,\phi(x,t)) + \lambda \int_{-1}^{1} k(|x-y|)\,\phi(y,t)\,dy + \lambda \int_0^t F(t,\tau)\,\phi(x,\tau)\,d\tau$$

Where the kernel of Fredholm term has Carleman $k(|x-y|) = |x-y|^{-\upsilon}$, $0 < \upsilon < \frac{1}{2}$, and Logarithmic $k(|x-y|) = \log|x-y|$ kernel, the historical function $H(x,t,\phi(x,t))$ take a linear form $(x,t)\,t^2$, and a nonlinear form $\phi^2(x,t)$, the kernel of Volterra term $F(t,\tau) = t\,\tau^2$, $\lambda = 0.01$ and the exact solution : $\phi(x,t) = x^2\,t^2$. Applying **PNM**, the results are obtained numerically by Maple 12 software, for $t = 0.0008, 0.05, 0.8,$ with $\alpha = 0.2, 0.3$ and 0.4. The interval $[-1,1]$ is divided into $N = 21$ unites.

Application 1: When the singular kernel takes the Carleman function form

$$k(|x-y|) = |x-y|^{-\upsilon}\ , \ \ 0 < \upsilon < \frac{1}{2}$$

Case 1: $H(x,t,\phi(x,t)) = t\,\phi(x,t)$,

Here, the integral equation (1) takes the linear form

$$\delta(t)\phi(x,t) = f(x,t) + \lambda \int_{-1}^{1} k(|x-y|)\,\phi(y,t)\,dy + \lambda \int_0^t F(t,\tau)\,\phi(x,\tau)\,d\tau; \quad \delta = (1-t)$$

Notes

Notes

Table (1)

(1-i) T=0.0008

X	EX.	$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
		APP	ERR	APP	ERR	APP	ERR
-1	6.400E-07	6.400E-07	0.000E+00	6.400E-07	4.000E-16	6.400E-07	5.000E-16
-0.8	4.096E-07	4.096E-07	1.000E-16	4.096E-07	3.000E-16	4.096E-07	2.000E-16
-0.6	2.304E-07	2.304E-07	3.000E-16	2.304E-07	2.000E-16	2.304E-07	1.000E-16
-0.4	1.024E-07	1.024E-07	1.000E-16	1.024E-07	0.000E-00	1.024E-07	1.000E-16
-0.2	2.560E-08	2.560E-08	5.000E-17	2.560E-08	4.000E-17	2.560E-08	1.000E-17
0	0.000E-00	5.200E-18	5.200E-18	2.200E-18	2.200E-18	-7.400E-18	7.400E-18
0.2	2.560E-08	2.560E-08	3.000E-17	2.560E-08	2.000E-17	2.560E-08	2.000E-17
0.4	1.024E-07	1.024E-07	1.000E-16	1.024E-07	0.000E-00	1.024E-07	1.000E-16
0.6	2.304E-07	2.304E-07	3.000E-16	2.304E-07	2.000E-16	2.304E-07	1.000E-16
0.8	4.096E-07	4.096E-07	0.000E+00	4.096E-07	0.000E-00	4.096E-07	0.000E+00
1	6.400E-07	6.400E-07	3.000E-16	6.400E-07	3.000E-16	6.400E-07	5.000E-16

Table (2)

(1-ii) T=0.05

X	EX.	$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
		APP	ERR	APP	ERR	APP	ERR
-1	2.500E-03	2.500E-03	5.000E-12	2.500E-03	6.000E-12	2.500E-03	4.000E-12
-0.8	1.600E-03	1.600E-03	2.000E-12	1.600E-03	4.000E-12	1.600E-03	2.000E-12
-0.6	9.000E-04	9.000E-04	1.800E-12	9.000E-04	1.600E-12	9.000E-06	8.910E-12
-0.4	4.000E-04	4.000E-04	9.000E-13	4.000E-04	8.000E-13	4.000E-04	8.000E-13
-0.2	1.000E-04	1.000E-04	2.000E-13	1.000E-04	2.000E-13	1.000E-04	3.000E-13
0	0.000E+00	3.060E-14	3.060E-14	3.160E-14	3.160E-14	3.450E-14	3.450E-14
0.2	1.000E-04	1.000E-04	2.000E-13	1.000E-04	2.000E-13	1.000E-04	3.000E-13
0.4	4.000E-04	4.000E-04	9.000E-13	4.000E-04	1.000E-12	4.000E-04	7.000E-13
0.6	9.000E-04	9.000E-04	1.900E-12	9.000E-04	1.700E-12	9.000E-04	1.500E-12
0.8	1.600E-03	1.600E-03	3.000E-12	1.600E-03	3.000E-12	1.600E-03	2.000E-12
1	2.500E-03	2.500E-03	5.000E-12	2.500E-03	6.000E-12	2.500E-03	4.000E-12

Table (3)

(1-iii) T=0.8

X	EX.	$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
		APP	ERR	APP	ERR	APP	ERR
-1	0.64	0.64006387	6.38739E-05	0.640063922	6.39213E-05	0.640063992	6.39914E-05
-0.8	0.4096	0.409641023	4.10232E-05	0.409641088	4.10877E-05	0.409641183	4.11831E-05
-0.6	0.2304	0.230423216	2.32163E-05	0.230423263	2.32632E-05	0.230423329	2.33291E-05
-0.4	0.1024	0.10241049	1.04902E-05	0.102410519	1.05193E-05	0.102410558	1.05578E-05
-0.2	2.56E-02	2.56E-02	2.85211E-06	2.56E-02	2.86893E-06	2.56E-02	2.88919E-06
0	0	3.06E-07	3.05785E-07	3.18E-07	3.18284E-07	3.32E-07	3.32246E-07
0.2	2.56E-02	2.56E-02	2.85211E-06	2.56E-02	2.86892E-06	2.56E-02	2.88918E-06
0.4	0.1024	0.10241049	1.04901E-05	0.102410519	1.05191E-05	0.102410558	1.05577E-05
0.6	0.2304	0.230423217	2.32164E-05	0.230423263	2.32633E-05	0.230423329	2.33293E-05
0.8	0.4096	0.409641023	4.10228E-05	0.409641088	4.10877E-05	0.409641183	4.11832E-05
1	0.64	0.640063874	6.38738E-05	0.640063922	6.39213E-05	0.640063992	6.39915E-05

Case 2: When the non-local term is nonlinear $H(x,t,\phi(x,t)) = \phi^2(x,t)$

Table (4)

(2-i) T=0.0008

X	EX.	$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
		APP	ERR	APP	ERR	APP	ERR
-1	6.400E-07	6.351E-07	4.901E-09	6.344E-07	5.620E-09	6.333E-07	6.680E-09
-0.8	4.096E-07	4.043E-07	5.290E-09	4.033E-07	6.263E-09	4.019E-07	7.688E-09
-0.6	2.304E-07	2.253E-07	5.085E-09	2.246E-07	5.781E-09	2.236E-07	6.753E-09
-0.4	1.024E-07	9.757E-08	4.828E-09	9.715E-08	5.249E-09	9.660E-08	5.804E-09
-0.2	2.560E-08	2.096E-08	4.640E-09	2.073E-08	4.875E-09	2.045E-08	5.153E-09
0	0.000E+00	4.573E-09	4.573E-09	4.742E-09	4.742E-09	4.925E-09	4.925E-09
0.2	2.560E-08	2.096E-08	4.640E-09	2.073E-08	4.874E-09	2.045E-08	5.152E-09
0.4	1.024E-07	9.758E-08	4.823E-09	9.716E-08	5.241E-09	9.661E-08	5.792E-09
0.6	2.304E-07	2.253E-07	5.064E-09	2.247E-07	5.748E-09	2.237E-07	6.708E-09
0.8	4.096E-07	4.044E-07	5.233E-09	4.034E-07	6.176E-09	4.020E-07	7.573E-09
1	6.400E-07	6.354E-07	4.563E-09	6.350E-07	4.959E-09	6.345E-07	5.485E-09

Table (5)

(2-ii) T=0.05

X	EX.	$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
		APP	ERR	APP	ERR	APP	ERR
-1	2.5E-03	2.481E-03	1.905E-05	2.478E-03	2.185E-05	2.474E-03	2.597E-05
-0.8	1.6E-03	1.579E-03	2.060E-05	1.576E-03	2.439E-05	1.570E-03	2.994E-05
-0.6	9.0E-04	8.802E-04	1.983E-05	8.775E-04	2.254E-05	8.737E-04	2.633E-05
-0.4	4.0E-04	3.812E-04	1.885E-05	3.795E-04	2.049E-05	3.773E-04	2.265E-05
-0.2	1.0E-04	8.188E-05	1.812E-05	8.096E-05	1.904E-05	7.988E-05	2.012E-05
0	0.0E+00	1.786E-05	1.786E-05	1.853E-05	1.853E-05	1.924E-05	1.924E-05
0.2	1.0E-04	8.188E-05	1.812E-05	8.096E-05	1.904E-05	7.988E-05	2.012E-05
0.4	4.0E-04	3.812E-04	1.882E-05	3.795E-04	2.046E-05	3.774E-04	2.261E-05
0.6	9.0E-04	8.803E-04	1.975E-05	8.776E-04	2.241E-05	8.738E-04	2.616E-05
0.8	1.6E-03	1.580E-03	2.038E-05	1.576E-03	2.405E-05	1.571E-03	2.949E-05
1	2.5E-03	2.482E-03	1.774E-05	2.481E-03	1.928E-05	2.479E-03	2.132E-05

Table (6)

(2-iii) T=0.8

X	EX.	$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
		APP	ERR	APP	ERR	APP	ERR
-1	0.64	0.63788999	0.00211002	0.637573604	0.002426397	0.637107531	0.00289247
-0.8	0.4096	0.40671993	0.00288007	0.406183063	0.003416937	0.405396058	0.004203942
-0.6	0.2304	0.22693220	0.00346780	0.226453153	0.003946847	0.225783412	0.004616588
-0.4	0.1024	0.09838890	0.00401110	0.098036574	0.004363426	0.097572594	0.004827406
-0.2	2.56E-02	2.12E-02	0.00443337	2.09E-02	0.004658686	2.07E-02	0.004925819
0	0	4.60E-03	0.00459856	4.77E-03	0.004769697	4.95E-03	0.004954065
0.2	2.56E-02	2.12E-02	0.00443282	2.09E-02	0.004657871	2.07E-02	0.004924744
0.4	0.1024	0.09839359	0.00400641	0.098043649	0.004356351	0.097582076	0.004817924
0.6	0.2304	0.22694706	0.00345294	0.226475929	0.003924072	0.225814449	0.004585551
0.8	0.4096	0.40675174	0.00284827	0.406231277	0.003368724	0.405460083	0.004139917
1	0.64	0.63803898	0.00196103	0.637865533	0.002134467	0.637635663	0.002364338

Notes

Application 2: When the singular kernel takes the logarithmic function

$$k(|x-y|) = \ln|x-y|$$

Table (7)

Case 3: $H(x,t,\phi(x,t)) = t^2\,\phi(x,t)$

X	T=0.0008			T=0.05			T=0.8		
	EX	APP	ERR	EX	APP	ERR	EX	APP	ERR
-1	6.400E-07	6.400E-07	6.000E-16	2.5E-03	2.500E-03	4.000E-12	6.400E-01	0.64006336	6.33562E-05
-0.8	4.096E-07	4.096E-07	2.000E-16	1.6E-03	1.600E-03	2.000E-12	4.096E-01	0.40964041	4.04056E-05
-0.6	2.304E-07	2.304E-07	3.000E-16	9.0E-04	9.000E-04	1.000E-12	2.304E-01	0.23042266	2.26594E-05
-0.4	1.024E-07	1.024E-07	1.000E-16	4.0E-04	4.000E-04	7.000E-13	1.024E-01	0.10241001	1.00122E-05
-0.2	2.560E-08	2.560E-08	0.000E-00	1.0E-04	1.000E-04	0.000E+00	2.560E-02	2.56E-02	2.43323E-06
0	0.000E-00	3.605E-18	3.605E-18	0.0E+00	2.703E-15	2.703E-15	0.000E-00	9.17E-08	9.17299E-08
0.2	2.560E-08	2.560E-08	1.000E-17	1.0E-04	1.000E-04	7.000E-14	2.560E-02	2.56E-02	2.43325E-06
0.4	1.024E-07	1.024E-07	2.000E-16	4.0E-04	4.000E-04	5.000E-13	1.024E-01	0.10241001	1.00122E-05
0.6	2.304E-07	2.304E-07	0.00E+00	9.0E-04	9.000E-04	1.000E-12	2.304E-01	0.23042266	2.26595E-05
0.8	4.096E-07	4.096E-07	1.000E-16	1.6E-03	1.600E-03	3.000E-12	4.096E-01	0.40964041	4.04057E-05
1	6.400E-07	6.400E-07	5.000E-16	2.5E-03	2.500E-03	4.000E-12	6.400E-01	0.64006336	6.33562E-05

Table (8)

Case 4: $H(x,t,\phi(x,t)) = \phi^2(x,t)$

X	T=0.0008			T=0.05			T=0.8		
	EX	APP	ERR	EX	APP	ERR	EX	APP	ERR
-1	6.400E-07	6.388E-07	1.189E-09	2.500E-03	2.495E-03	4.621E-06	6.400E-01	6.395E-01	4.816E-04
-0.8	4.096E-07	4.084E-07	1.229E-09	1.600E-03	1.595E-03	4.785E-06	4.096E-01	4.090E-01	6.464E-04
-0.6	2.304E-07	2.300E-07	4.412E-10	9.000E-04	8.983E-04	1.720E-06	2.304E-01	2.301E-01	2.804E-04
-0.4	1.024E-07	1.029E-07	4.730E-10	4.000E-04	4.018E-04	1.846E-06	1.024E-01	1.028E-01	4.066E-04
-0.2	2.560E-08	2.675E-08	1.149E-09	1.000E-04	1.045E-04	4.487E-06	2.560E-02	2.670E-02	1.097E-03
0	0.000E+00	1.394E-09	1.394E-09	0.000E+00	5.447E-06	5.447E-06	0.00E+00	1.394E-03	1.394E-03
0.2	2.560E-08	2.675E-08	1.149E-09	1.000E-04	1.045E-04	4.487E-06	2.560E-02	2.670E-02	1.097E-03
0.4	1.024E-07	1.029E-07	4.730E-10	4.000E-04	4.018E-04	1.846E-06	1.024E-01	1.028E-01	4.066E-04
0.6	2.304E-07	2.300E-07	4.412E-10	9.000E-04	8.983E-04	1.720E-06	2.304E-01	2.301E-01	2.804E-04
0.8	4.096E-07	4.084E-07	1.229E-09	1.600E-03	1.595E-03	4.785E-06	4.096E-01	4.090E-01	6.464E-04
1	6.400E-07	6.388E-07	1.189E-09	2.500E-03	2.495E-03	4.621E-06	6.400E-01	6.395E-01	4.816E-04

V. Conclusions

●The non-local term is called the histories of the problem and is considered with negative sign

I- For the Carleman kernel $k(|x-y|) = |x-y|^{-v}$ and for the linear non- local term $H(x,t,\phi(x,t)) = t\,\phi(x,t)$, *we* have E. $_{\text{Max.}}$ and E.Min. respectively, the following:

(i) In Table (1) at T=0.0008: for $\alpha = 0.2$ are ,respectively3.000E-16 , 0.000E-00. While *for* $\alpha = 0.3$ are 4.000E-16 and 0.000E-00. Finally at $\alpha = 0.4$ are 5.000E-16 and 0.000E-00.

(ii) In Table (2) at T=0.005: for $\alpha = 0.2$; 5.000E-12 and 2.000E-13. For $\alpha = 0.3$ are 6.000E-12 and 1.000E-12. While $\propto = 0.4$ are 8.910E-12 and 3.450E-14.

(iii) In Table (3) at T=0.8: for $\alpha = 0.2$; 6.38738E-05 and 3.05785E-07; for $\alpha = 0.3$ are 6.39213E-05 and 1.05193E-05. Finally for $\propto = 0.4$ are 6.39914E-05 and 1.05577E-05.

II- For the Carleman kernel and nonlinear non- local term $H(x,t,\phi(x,t)) = \phi^2(x,t)$

(iv) In Table (4) at T=0.0008: for $\alpha = 0.2$ we obtain 5.290E-09, 4.563E-09. Also, $for\ \alpha = 0.3$: we get 6.263E-09and 4.742E-09. Finally for $\alpha = 0.4$ we have 7.688E-09and 4.925E-09.

(v) In Table (5) at T=0.005: for $\alpha = 0.2$; 2.060E-05and 1.774E-05. For $\alpha = 0.3$ are 2.439E-05and 1.853E-05. For $\propto = 0.4$ are 2.994E-05 and 1.924E-05.

(vi) In Table (6) at T=0.8: for $\alpha = 0.2$; 0.00459856 and 0.00196103; for $\alpha = 0.3$ are 0.004769697and 0.002134467. For $\propto = 0.4$ are 0.004954065and 0.002364338.

Notes

III- For a logarithmic kernel $k(|x-y|) = \ln|x-y|$ and linear non- local term $H(x,t,\phi(x,t)) = t^2\,\phi(x,t)$ *the* E. $_{Max.}$ and E. Min are given respectively, as the following:

(vii) In Table (7) at T=0.0008:, we have 6.000E-16 and 0.000E-00. At T=0.005 we have 4.000E-12; 0.000E-00. At T=0. 8, 6.33562E-05, 9.17299E-08.

IV- For a logarithmic and non linear non- local term $H(x,t,\phi(x,t)) = \phi^2(x,t)$ the E. $_{Max.}$ and E.Min are respectively,

(viii) In Table (8) at T=0.0008:, we have 1.229E-09 and 4.412E-10. At T=0.005 5.447E-06; 1.720E-06. At T=0. 8, 4.816E-04, 1.097E-03.

From the above results, we deduce that the error in the linear non- local function is less than the error in the nonlinear case. This result is true, where the integral equation without non-local term in the linear case.

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