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A Class of Multivalent Harmonic Functions Involving Salagean Operator

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Introduction- A continuous complex valued function f=u+iv defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. Let F and G be analytic in D so that F(0)=G(0)=0, ReF = Ref=u, ReG = Imf=v by writing (F+iG)/2 = h, (F-iG)/2 = g, The function f admits the representation f = h + g , where h and g are analytic in D. h is called the analytic part of f and g, the co-analytic part of f.

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A Class of Multivalent Harmonic Functions Involving Salagean Operator

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I. Introduction

A continuous complex valued function f=u+iv defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. Let F and G be analytic in D so that F(0)=G(0)=0, ReF = Ref=u, ReG = Imf=v by writing (F+iG)/2 = h, (F-iG)/2 = g, The function f admits the representation $f = h + \overline{g}$ where h and g are analytic in D. h is called the analytic part of f and g, the co-analytic part of f.

Ahuja and Jahangiri [1], [2] introduce and studied certain subclasses of the family SH(m), m ≥ 1 of all multivalent harmonic and orientation preserving functions in $\Delta = \{z : |z| < 1\}$. A function f in SH(m) can be expressed as $f = h + \overline{g}$, where h and g are analytic functions of the form

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}$$

$$g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1. \tag{1}$$

For analytic function $h(z) \in S(m)$ Salagean [3] introduced an operator D_m^{ν} defined as follows:

$$D_m^0 h(z) = h(z), \; D_m^1 h(z) = D_m(h(z)) = \frac{z}{m} h'(z) \text{ and}$$

$$D_m^{\nu} h(z) = D_m(D_m^{\nu-1} h(z)) = \frac{z(D_m^{\nu-1} h(z))'}{m} \tag{2}$$

$$= z + \sum_{n=2}^{\infty} \left(\frac{n+m-1}{m} \right)^{\nu} a_{n+m-1} z^{n+m-1}, \quad \nu \in N.$$

Whereas, Jahangiri et al. [4] defined the Salagean operator $D_m^{\nu} f(z)$ for multivalent harmonic function as follows:

$$D_m^{\nu} f(z) = D_m^{\nu} h(z) + (-1)^{\nu} D_m^{\nu} g(z) \tag{3}$$

where,

$$D_m^{\nu} h(z) = z^m + \sum_{n=2}^{\infty} \left(\frac{n+m-1}{m} \right)^{\nu} a_{n+m-1} z^{n+m-1}$$

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1. Ahuja,O.P. and Jahangiri,J.M., Multivalent Harmonic starlike Functions, Ann. Univ Mariae Curie – Sklodowska, Section A, 55 (1) (2001), 1-13.

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$$D_m^{\nu} g(z) = \sum_{n=1}^{\infty} \left(\frac{n+m-1}{m} \right)^{\nu} b_{n+m-1} z^{n+m-1}.$$

In this paper we define a sub class $H_m(\lambda, \nu, \alpha)$ of m-valent harmonic functions involving Salagean operator $D_m^{\nu} f(z)$ as follows:

Definition 1

Let $f(z) = h(z) + \overline{g(z)}$ be the harmonic multivalent function of the form (1), then $f \in H_m(\lambda, \nu, \alpha)$ if and only if

$$\operatorname{Re}\left\{ (1-\lambda)\frac{D_m^{\nu} f(z)}{z^m} + \lambda \frac{\frac{\partial}{\partial\theta} D_m^{\nu} f(z)}{\frac{\partial}{\partial\theta} z^m} \right\} > \alpha \tag{4}$$

where $0 \le \alpha < 1, \lambda \ge 0$, $z = re^{i\theta} \in \Delta$ and $D_m^{\nu} f(z)$ is defined by (3) and

$$\frac{\partial}{\partial\theta} D_m^{\nu} f(z) = i\left[z(D_m^{\nu} h(z))' - (-1)^{\nu} \overline{z(D_m^{\nu} g(z))'} \right], \quad \frac{\partial}{\partial\theta} z^m = imz^m.$$

We denote the subclass $TH_m(\lambda, \nu, \alpha)$ consist of harmonic functions $f_{\nu} = h + \overline{g}_{\nu}$ in $H_m(\lambda, \nu, \alpha)$ so that h and g_{ν} are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1},$$

$$g_{\nu}(z) = (-1)^{\nu} \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \; |b_m| < 1.$$

Also note that $TH_m(\lambda, \nu, o) \equiv TH_m(\lambda, \nu)$.

The class $H_m(\lambda, \nu, \alpha)$ provides a transition between two classes:

$$\operatorname{Re}\left\{ \frac{D_m^{\nu} f(z)}{z^m} \right\} > \alpha \text{ and } \operatorname{Re}\left\{ \frac{\frac{\partial}{\partial\theta} D_m^{\nu} f(z)}{\frac{\partial}{\partial\theta} z^m} \right\} > \alpha \text{ as } \lambda \text{ moves between } 0 \text{ and } 1.$$

Denote $H_m(0, \nu, \alpha)$ by $P_m(\nu, \alpha)$ and $H_m(1, \nu, \alpha)$ by $Q_m(\nu, \alpha)$.

In this paper first we obtained the sufficient coefficient condition for $f(z) \in H_m(\lambda, \nu, \alpha)$ and then it is shown that this coefficient condition is also necessary for $f(z) \in TH_m(\lambda, \nu, \alpha)$. Also distortion bounds, extreme points, convex combination, integral operator, convolution condition, radius of convexity, radius of starlikeness for the functions $f(z) \in TH_m(\lambda, \nu, \alpha)$ are obtained.

II. Main Results

a) Theorem 1 (Sufficient coefficient condition for $H_m(\lambda, \nu, \alpha)$*)*

Assume that $f = h + \overline{g}$, h and g be given by (1) and $\lambda \ge 0$, if

$$\sum_{n=2}^{\infty} \left(\frac{n+m-1}{m} \right)^{\nu} \left[\left(\frac{n+m-1}{m} \right) \lambda + (1-\lambda) \right] |a_{n+m-1}| +$$

$$\sum_{n=1}^{\infty} \left(\frac{n+m-1}{m} \right)^{\nu} \left| \left(\frac{n+m-1}{m} \right) \lambda - (1-\lambda) \right| |b_{n+m-1}| \le 1-\alpha, 0 \le \alpha < 1 \tag{6}$$

then, $f(z) \in H_m(\lambda, \nu, \alpha)$.

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b) Remark 2

The coefficient bound (6) in above theorem is sharp for the function

$$f(z) = z^m + \sum_{n=2}^{\infty} \frac{x_n}{\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda + (1-\lambda)\right]} z^{n+m-1} \tag{7}$$

$$+\sum_{n=1}^{\infty} \frac{\overline{y_n}}{\left(\frac{n+m-1}{m}\right)^{\nu}\left\|\left[\left(\frac{n+m-1}{m}\right)\lambda - (1-\lambda)\right]\right\|} \overline{z^{n+m-1}}$$

where

$$\frac{1}{1-\alpha}\left(\sum_{n=2}^{\infty} | x_n | + \sum_{n=1}^{\infty} | y_n |\right) = 1 .$$

c) Remark 3

For $\lambda \geq 1$,

$$1 \leq \left(\frac{n+m-1}{m}\right) \leq \left[\left(\frac{n+m-1}{m}\right)\lambda + (1-\lambda)\right] \leq \left[\left(\frac{n+m-1}{m}\right)\lambda - (1-\lambda)\right]. \tag{8}$$

d) Corollary 4

Let $f = h + \overline{g}$ be such that h and g are given by (1) and let

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda + (1-\lambda)\right] | a_{n+m-1} | + \tag{9}$$

$$\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda - (1-\lambda)\right] | b_{n+m-1} | \leq 1-\alpha$$

for $\lambda \geq 1$ and $0 \leq \alpha < 1$, then $f \in H\ (\lambda, \nu, \alpha)$.

Putting $\lambda = 0$ in Theorem 1 the following Corollary is obtained.

e) Corollary 5

Let $f = h + \overline{g}$ be such that h and g are given by (1) and let

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu} | a_{n+m-1} | + \sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu} | b_{n+m-1} | \leq 1-\alpha$$

for $0 \leq \alpha < 1$, then $f \in P_m(\nu, \alpha)$.

Putting λ=1 in Theorem 1 the following Corollary is obtained.

f) Corollary 6

Let $f = h + \overline{g}$ be such that h and g are given by (1) and let

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu+1} | a_{n+m-1} | + \sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu+1} | b_{n+m-1} | \leq 1-\alpha$$

for $0 \leq \alpha < 1$, then $f \in Q_m(\nu, \alpha)$.

g) Remark 7

$H_m(\lambda, \nu, \alpha_2) \subseteq H_m(\lambda, \nu, \alpha_1)$ for $\alpha_1 \leq \alpha_2$. Also, $Q_m(\nu, \alpha) \subset P_m(\nu, \alpha)$.

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h) Theorem 8 (Coefficient inequality for $TH_m(\lambda,\nu,\alpha)$ *)*

Let $f_\nu = h + \overline{g}_\nu$ be so that h and g_ν are given by (5). Then,
$f_\nu \in TH_m(\lambda,\nu,\alpha)$

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]|a_{n+m-1}|+$$

$$\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right]|b_{n+m-1}|\le 1-\alpha \qquad (10)$$

where $0\le\alpha<1, \lambda\ge 1 \text{ and } |a_m|=1$.

i) Theorem 9 (Distortion Bounds)

If $f_\nu \in TH_m(\lambda,\nu,\alpha)$ and $\lambda\ge 1, |z|=r<1$, then

$$|f_\nu(z)|\le(1+|b_m|)r^m+\frac{r^{m+1}}{\left(\frac{m+1}{m}\right)^{\nu}}\left[\frac{m(1-\alpha)}{(m+\lambda)}-\frac{m(2\lambda-1)}{(m+\lambda)}|b_m|\right] \qquad (11)$$

and

$$|f_\nu(z)|\ge(1-|b_m|)r^m-\frac{r^{m+1}}{\left(\frac{m+1}{m}\right)^{\nu}}\left[\frac{m(1-\alpha)}{(m+\lambda)}-\frac{m(2\lambda-1)}{(m+\lambda)}|b_m|\right]. \qquad (12)$$

j) Corollary 10

Let $f_\nu \in TH_m(\lambda,\nu,\alpha)$ then for $|z|=r<1$ and $\lambda\ge 1$

$$\left[w:|w|<\left\{\frac{(m+1)^{\nu}(m+\lambda)-m^{\nu+1}(1-\alpha)}{(m+1)^{\nu}(m+\lambda)}\right\}+\left\{\frac{(2\lambda-1)-(m+1)^{\nu}(m+\lambda)}{(m+1)^{\nu}(m+\lambda)}\right\}|b_m|\right]\subset f_\nu(\Delta). \qquad (13)$$

k) Theorem 11 (Extreme Points)

Let f_ν be given by (5) then $f_\nu \in TH_m(\lambda,\nu,\alpha)$; $\lambda\ge 1$ if and only if.

$$f_\nu(z)=\sum_{n=1}^{\infty}\left[x_{n+m-1}h_{n+m-1}(z)+y_{n+m-1}g_{n+m-1,\nu}(z)\right], \qquad (14)$$

where

$$h_m(z)=z^m,\ h_{n+m-1}(z)=z^m-\frac{1}{\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]}z^{n+m-1},\ (n=2,3\ldots)$$

and

$$g_{n+m-1,\nu}(z)=z^m+(-1)^{\nu}\frac{1}{\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right]}\overline{z}^{n+m-1},\ (n=1,2,3\ldots.)$$

$$x_{n+m-1}\ge 0,\ y_{n+m-1}\ge 0,\ x_m=1-\sum_{n=2}^{\infty}x_{n+m-1}-\sum_{n=1}^{\infty}y_{n+m-1}.$$

In particular, the extreme points of $TH_m(\lambda,\nu,\alpha)$ are $\{h_{n+m-1}\}$ and $\{g_{n+m-1,\nu}\}$.

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l) Theorem 12 (Convex Combination)

If $f_{i,v}(i=1,2\ldots)$ belongs to $TH_m(\lambda,\nu,\alpha)$; $\lambda \geq 1$ then the function $\sum_{i=1}^{\infty} t_i f_{i,v}(z)$ is also in $TH_m(\lambda,\nu,\alpha)$ where $f_{i,v}$ is defined by

$$f_{i,v} = z^m - \sum_{n=2}^{\infty} |a_{n+m-1,i}| z^{n+m-1} + (-1)^{\nu} \sum_{n=1}^{\infty} |b_{n+m-1,i}| \overline{z}^{n+m-1} (i=1,2\ldots) \tag{15}$$

and $0 \leq t_i < 1, \sum_{i=1}^{\infty} t_i = 1$.

m) Definition 2

The harmonic generalized Bernardi-Libera-Livingston integral operator $L_c(f(z))$ for m-valent functions is defined by

$$L_c(f(z)) = \frac{c+m}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+m}{z^c}\int_0^z t^{c-1}g(t)dt}, \qquad c > -1.$$

n) Theorem 13 (Integral Operator)

Let $f \in TH_m(\lambda,\nu,\alpha)$; $\lambda \geq 1$. Thus $L_c(D_m^{\nu} f_v(z))$ belongs to the class $TH_m(\lambda,\nu,\alpha)$.

o) Theorem 14 (Convolution Condition)

Let $f_v \in TH_m(\lambda,\nu,\alpha)$ and $F_v \in TH_m(\lambda,\nu,\alpha)$; $\lambda \geq 1$ then the convolution

$$(f_v * F_v)(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}A_{n+m-1}| z^{n+m-1} +$$

$$+(-1)^{\nu} \sum_{n=1}^{\infty} |b_{n+m-1}B_{n+m-1}| \overline{z}^{n+m-1} \in TH_m(\lambda,\nu,\alpha).$$

p) Theorem 15 (Radius of Convexity)

The radius of convexity for the function $f_v \in TH_m(\lambda,\nu)$ is given by

$$r_0 = \frac{\left(\frac{m+1}{m}\right)^{\nu-2}}{1-(2\lambda-1)|b_m|}, \text{ for } \lambda \geq 1.$$

q) Theorem 16 (Radius of Starlikeness)

The radius of starlikeness for the function $f_v \in TH_m(\lambda,\nu)$ is given by

$$r_0 = \frac{\left(\frac{m+1}{m}\right)^{\nu-1}}{1-(2\lambda-1)|b_m|}, \text{ for } \lambda \geq 1.$$

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