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## of Science Frontier Research: F

# Mathematics and Decision Sciences


Highlights

Estimation of Crime Rate

Component Analysis Approach

Correlations of Dusty Fluid

Panel Data Analysis Approach

Discovering Thoughts, Inventing Future

VOLUME 15 ISSUE 2 VERSION 1.0

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Mathematics & Decision Sciences

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# On the Estimation of Crime Rate in the Southwest of Nigeria: Principal Component Analysis Approach

By Femi J. Ayoola, Adeyemi M. A. & Jabaru, S. O.
*University of Ibadan, Nigeria*

*Abstract-* Crime is at alarming rate in this part of world and there are many factors that are contributing to this antisocietal behavour both among the youths and old. In this work, principal component analysis (PCA) was used as a tool to reduce the dimensionality and to really know those variables that were crime prone in the study region. Data were collected on twenty-eight crime variables from National Bureau of Statistics (NBS) databank for a period of fifteen years, while retaining as much of the information as possible.

*Keywords:* *crime rates, data, southwest nigeria, principal component analysis, variables.*

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ONTHEESTIMATIONOFCRIMERATEINTHESOUTHWESTOFNIGERIAPRINCIPALCOMPONENTANALYSISAPPROACH

*Strictly as per the compliance and regulations of :*

# On the Estimation of Crime Rate in the Southwest of Nigeria: Principal Component Analysis Approach

Femi J. Ayoola [α], Adeyemi M. A. [σ] & Jabaru, S. O. [ρ]

*Abstract-* Crime is at alarming rate in this part of world and there are many factors that are contributing to this antisocietal behavour both among the youths and old. In this work, principal component analysis (PCA) was used as a tool to reduce the dimensionality and to really know those variables that were crime prone in the study region. Data were collected on twenty-eight crime variables from National Bureau of Statistics (NBS) databank for a period of fifteen years, while retaining as much of the information as possible.

We use PCA in this study to know the number of major variables and contributors to the crime in the Southwest Nigeria. The results of our analysis revealed that there were eight principal variables have been retained using the Scree plot and Loading plot which implies an eight-equation solution will be appropriate for the data.

The eight components explained 93.81% of the total variation in the data set. We also found that the highest and commonly committed crimes in the Southwestern Nigeria were: Assault, Grievous Harm and Wounding, theft/stealing, burglary, house breaking, false pretence, unlawful arms possession and breach of public peace.

*Keywords: crime rates, data, southwest nigeria, principal component analysis, variables.*

## I. Introduction

There is no universal definition of crime. This is as a result of changes in social, political, psychological and economic conditions. An act may be a crime in one society, but not in another (Danbazau, 2007). For example, prostitution, adultery and homosexuality between consenting adults have been wholly or partially removed from the criminal law in USA (Feldman, 1997) but are considered as crimes in Muslim communities such as Saudi Arabia. The constant changes in time also change the perception of society on crime. Today, it is becoming a crime to pollute the air and water. Therefore, the perception of an "act" to be a crime varies with time and space.

Crime is a universal phenomenon and differs only in degree among the various nations of the world. The Nigerian crime – problem is multidimensional and is capable of undermining its corporate existence as well as efforts towards sustainable development. The Nigeria corporate existence can be undermined by a number of factors among which is an escalating and uncontrolled crime problem Tanimu (2006). Security and crime have been deeply rooted in the political history of this country, particularly in recent time, which has emerged as a key concept in Nigeria's struggle for good governance, sustainable democracy and development.

Without mincing words the violence perpetrated in some part of countries in recent time constitutes public order crimes. Security is very importance for all human being regardless of one's status in the country. This will not be guarantee if the security sector

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is greedy and corrupt. More also, poverty reduction and development of democracy in this country will be better enhancing when the security of the citizen is guaranteed. It has been noted that the cost of crime and its control is equivalent to 5% GDP in the developed world, the figure rises to about 14% in developing nation (ICPC 1999).

The growth in urban crime rate in Nigeria is one of the major social problems facing the country in recent time. The dominance of crime in developing countries increases the volatility of the issue, for it pyramids one fear upon others. The concentration of violent crimes in major urban centers worldwide is therefore heralded as an indicator of the breakdown of urban systems. In many urban centers of Nigeria today, criminal activities and violence are assuming dangerous tendencies as they threaten lives and property, the national sense of well-being and coherence, peace, social order and security, thus, reducing the citizens' quality of life. (Agboola, 2000; Ahmed, 2010).

Notes

Over the years the rate of crime in Nigeria has been on the increase and these crimes are being carried out with more perfect and sophistication. This has led to the formation of various vigilante groups, to combat crimes in some parts of the country (Fajemirokun et al., 2006). One of the fundamental techniques to combat criminal activities is the better understanding of the dynamics of crime. Crime is often thought of as a moral threat and injurious to the society. However, it has been observed that the entire world is experiencing high criminal rate. The report of international crime victim survey (ICVS) has confirmed the situation. The report which was conducted on six major world region including Africa, Asia, central and eastern Europe, Latin America, and western Europe for the 1989 – 1996 period as shown that more than half of the urban respondents reported being victim at least once regardless of what part of the world they inhabit (Ackermen and Murray, 2004).

In this research work, we use PCA in determining the numbers of principal components (PC) to be used in explaining the crime data in southwestern Nigeria.

## II. Background of the Study

The major motivating factors of this study are centered on various socio-economic and political movements that transformed the country between 1995 and 2009. The country witnessed series of crime waves that transposed a new dispensation into the so-called modern democratic government". Hence, Nigeria witnessed different modes of governance from military to civilian regimes between these periods. The military, in the first instance, solely took advantage of its professional training by using violence to usurp power through coups and counter coups.

The politicians in their turn, and in their bids to absorb power, used hired-thugs, or paid assassins/hired killers to perpetrate violence and instill fears on their opponents. The frustrated masses took to armed robbery, formulation of militant groups as witnessed in the uproar of the youths from South-South and the Boko-Haram Sect from the North-Eastern path of the country disregarded the law. In view of the above, there is therefore, the need to look critically at the pattern and distribution of crime ascendant in some part of the country.

### *a) Research Questions*

In the light of the foregoing, some questions are raised and should be clarified to articulate the problem and objectives of this study:

1. What is the degree of relationship between the different crimes committed in the study area in the last fifteen years?
2. Which of the crimes committed accounted for higher percentage of total crimes in Southwestern Nigeria?
3. What are the important components present in the data?

Notes

4. Which crime (s) has high loadings on each of the rotated components?

5. What type of policy measures can be employed to reduce crime patterns at the regional or urban scale in Southwestern Nigeria?

### *b) Objectives of this Study*

1. To examine the degree of relationship between the different crimes committed in the study area in the last fifteen years.
2. To determine the crime that accounted for highest percentage of the total crimes in Southwestern Nigeria.
3. To conduct a principal component analysis to determine important components present in the data.
4. To examine the crime (s) with high loadings on each of the rotated components.
5. To recommend the policy measures that can be employed to reduce crime patterns at the regional or urban scale in Southwestern Nigeria.

### *c) Statistics of Crimes in Nigeria*

Nigeria has one of the highest crime rates in the world. Murder often accompanies minor burglaries. Rich Nigerians live in high – security compounds. Police in some states are empowered to "shoot on sight" violent criminals (Financial Times, 2009). There is no disagreement from both macro and micro level studies that the rate of crime in Nigeria has reached an unacceptable level. The fact file on losses between June 1999 and October 2001 painted a picture of robbery and murder victims akin to a declaration of war by hoodlums. Estimated properties cost is in billion of naira, while a total of 3680 people lost their lives. Assault related injuries, which include bruises, cuts black eyes and broken bones have severally been reported (CRSSYB, 2003). Some of these assault occurred as domestic violence, while other are inflicted by criminals on guards especially under volatile situation (Oshunkeye, 2004). These assault have resulted in damaged joint partial loss of hearing and vision, permanent disfigurement, scars from burns, knives and machete wound (Ikoh, 2002)

Aside from the human and sociological effect of violence crime, there is a significant economics cost to the country in which rate of crime and violence are high, such economic effect include increase absenteeism, decrease in labour market participation, reduced productivity that lower earning (Krug, Dahlberg, and Mercy, 2002). The growth in urban crime rate in Nigeria is one of the major social problems facing the country in recent time. The dominance of crime in developing countries increases the volatility of the issue for its pyramid one fear upon others. The concentration of violent crime in major urban centre world wide is therefore heralded as an indicator of the breakdown of urban system. In many urban centre of Nigeria today, criminals activities and violence are assuming dangerous tendencies as they threaten lives and properties, the national sense of well-being and coherence, peace, social order and security, thus reducing the citizen quality of life (Agboola, 2000; Ahmed, 2010)

## III. Methodology

The data required for this study was obtained from secondary source (National Bureau of Statistics) and it covered reported crime cases in Southwestern Nigeria that comprises - Oyo, Osun, Ondo, Ekiti, Ogun and Lagos States. There are twenty-eight crime variables identified in an attempt to identify the most salient variables to adapt in explaining the main distributional pattern of crimes in the study area using principal component analysis.

### *a) The Study Area*

The South-Western part of Nigeria comprises Ekiti, Lagos, Ogun, Ondo, Osun and Oyo states and is mainly inhabited by the Yoruba, who are renowned for their strong industrial base, modern bureaucracy, accomplished academics and strong presence of a skilled labour force in various sectors. Thus the region attracts different categories of individuals and corporate bodies, including traders, professionals, businesspersons, administrators and students all of whom come from various parts of the country and beyond to explore the booming economic and educational opportunities. Due to the concentrated populations most especially in Lagos and Ibadan, the security of the region has always been undermined by criminal activities such as armed robbery, domestic violence etc.

*b) Principal Component Analysis*

Principal Components Analysis, is a data analysis tool that is usually used to reduce the dimensionality (number of variables) of a large number of interrelated variables, while retaining as much of the information (variation) as possible. PCA calculates an uncorrelated set of variables (factors or pc's). These factors are ordered so that the first few retain most of the variation present in all of the original variables. Unlike Factor Analysis, PCA always yields the same solution from the same data (apart from arbitrary differences in the sign).

Principal Component Analysis reduces multiple observed variables into fewer components that summarize their variance. Principal component analysis is a member of the general linear model (GLM) where *all* analyses are correlational term often used interchangeably with "factor analysis", however, there are slight differences. It is a method of reducing large data sets into more manageable "factors" or "components" method of identifying the most *useful* variables in a dataset and a method of identifying and classifying variables across common themes, or constructs that they represent.

Basically in principal component analysis, given p variables $X_1, X_2, \ldots, X_p$ measured on a sample of n subjects, then the ith principal component, $Z_i$ can be written as a linear combination of the original variables. Thus,

$$Z_i = a_{i1}X_1 + a_{i2}X_2 + \ldots + a_{ip}X_p$$

The principal components are chosen such that the first one, $Z_1 = a_{11}X_1 + a_{12}X_2 + \ldots + a_{1p}X_p$ accounts for as much of the variation in the data as (i. e. in the original variables) as possible subject to the constraint that

$$a_{11}^2 + a_{12}^2 + \ldots + a_{1p}^2 = 1$$

Then the second principal component $Z_2 = a_{21}X_1 + a_{22}X_2 + \ldots + a_{2p}X_p$ is chosen such that its variance is as high as possible. A similar constraint applies – namely, that

$$a_{21}^2 + a_{22}^2 + \ldots + a_{2p}^2 = 1$$

Another constraint is that the second component is chosen such that it is uncorrelated with the first component. The remaining principal components are chosen in the same way. When you do a principal component analysis, we get what is called eigenvalues which are the variances of the principal components. In other words, the first eigenvalue is the variance of the first principal component, the second eigenvalue is the variance of the second principal component and so on. Thus, because of the way principal components are selected, the first eigenvalue is the largest, followed by the second etc. There will be p eigenvalues altogether but some may be equal to zero.

## IV. Discussion of Findings and Conclusion

There are low pair-wise correlation in between many of the crimes and therefore cannot be used to explain one another, but however, at least moderate correlation exist in between sizeable number of the crimes. All the variables that are responsible for the causes of crimes committed in the study area are relatively important, though, there is

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variability in the contributions of each of these factors which could be obviously seen as the eigenvalues greater than one (table 1, appendix). From the varimax rotated factor loadings, factor I accounts for 31.415% of the total variance explained with its significant positive loading, Factor II exhibited a high positive loadings and accounts for 19.381% of the total variance, Factor III on the other hand, explained 10.217% of the total variance that has significant positive loadings on different crimes committed and reported. More importantly and again, Factor IV accounted for 8.307% of the total variance explained etc. The analysis and the scree plot (Figure 1, appendix) also revealed that only eight components are extracted. The eight components explain 93.806% (table 1, appendix) of the total variability of the data set. We also found that the highest and commonly committed crimes in the Southwestern part of Nigeria are Assault, Grievous Harm and Wounding, theft/stealing, burglary, house breaking, false pretence, unlawful arms possession and breach of public peace. The rotated factors give information about the extent to which the factors have been rotated. These rotated factors are just as good as the initial factors in explaining and reproducing the observed correlation matrix. Across each row, the highlighted are the factor that each variable loaded most strongly on.

-- The first12 crimes loaded strongly on Factor 1

-- The next 5 crimes loaded strongly on Factor 2

-- Forgery and arson loaded strongly on Factor 3

-- Murder, Armed Robbery and False Pretence loaded strongly on Factor 4

-- Perjury and slave dealing loaded strongly on Factor 5

-- Manslaughter and Forgery loaded strongly on Factor 6

-- Breach of public peace loaded strongly on Factor 7; while

-- Suicide loaded strongly on Factor 8

From table 2 (appendix), the equations of the principal components are:

$$Z_1 = 0.0993X_1 + 0.2455X_2 + \ldots - 0.0239X_{28}$$
$$Z_2 = 0.0995X_1 + 0.1469X_2 + \ldots + 0.3508X_{28}$$
$$Z_3 = -0.1685X_1 + 0.0775X_2 + \ldots + 0.0503X_{28}$$
$$Z_4 = 0.3868X_1 + 0.2711X_2 + \ldots - 0.1965X_{28}$$
$$Z_5 = -0.2984X_1 - 0.0331X_2 + \ldots + 0.0150X_{28}$$
$$Z_6 = 0.2011X_1 + 0.1469X_2 + \ldots - 0.3270X_{28}$$
$$Z_7 = 0.2711X_1 + 0.0625X_2 + \ldots + 0.0096X_{28}$$
$$Z_8 = 0.1496X_1 - 0.2092X_2 + \ldots - 0.1036X_{28}$$

The estimated principal components are displayed in Table 3 of the appendix.

## V. Recommendations

Government intervention in the provision of infrastructure and other basic amenities that would make life more meaningful should be encouraged. Government must also make it as matter of policy to shift its thinking about crime and punishment and turn its focus to crime prevention, addressing the root causes of crime such as lack of employment which is rampant among the youth, and devoting our resources to community building, education, and workforce development that provides jobs at a living wage because the future of Nigeria and our democracy depends on them. The family institution must also play its role by monitoring all the people in the family particularly the youth. The entire society must shun all values that encourage criminal activities.

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Notes

## Appendix

*Table 1 :* Total Variance Explained

| Component | Initial Eigenvalues | | | Extraction Sums of Squared Loadings | | | Rotation Sums of Squared Loadings | | |
|---|---|---|---|---|---|---|---|---|---|
| | Total | % of Variance | Cumulative % | Total | % of Variance | Cumulative % | Total | %of Variance | Cumulative % |
| 1 | 9.813 | 35.045 | 35.045 | 9.813 | 35.045 | 35.045 | 8.796 | 31.415 | 31.415 |
| 2 | 5.202 | 18.578 | 53.623 | 5.202 | 18.578 | 53.623 | 5.427 | 19.381 | 50.796 |
| 3 | 3.497 | 12.489 | 66.111 | 3.497 | 12.489 | 66.111 | 2.861 | 10.217 | 61.013 |
| 4 | 2.670 | 9.537 | 75.648 | 2.670 | 9.537 | 75.648 | 2.326 | 8.307 | 69.320 |
| 5 | 1.613 | 5.761 | 81.409 | 1.613 | 5.761 | 81.409 | 2.024 | 7.227 | 76.547 |
| 6 | 1.343 | 4.796 | 86.205 | 1.343 | 4.796 | 86.205 | 1.838 | 6.565 | 83.112 |
| 7 | 1.082 | 3.864 | 90.069 | 1.082 | 3.864 | 90.069 | 1.689 | 6.033 | 89.145 |
| 8 | 1.046 | 3.737 | 93.806 | 1.046 | 3.737 | 93.806 | 1.305 | 4.661 | 93.806 |
| 9 | .551 | 1.969 | 95.775 | | | | | | |
| 10 | .382 | 1.363 | 97.138 | | | | | | |
| 11 | .298 | 1.065 | 98.203 | | | | | | |
| 12 | .237 | .845 | 99.049 | | | | | | |
| 13 | .182 | .649 | 99.698 | | | | | | |
| 14 | .085 | .302 | 100.000 | | | | | | |
| 15 | 5.366E-16 | 1.917E-15 | 100.000 | | | | | | |
| 16 | 4.163E-16 | 1.487E-15 | 100.000 | | | | | | |
| 17 | 3.230E-16 | 1.153E-15 | 100.000 | | | | | | |
| 18 | 2.668E-16 | 9.530E-16 | 100.000 | | | | | | |
| 19 | 1.447E-16 | 5.168E-16 | 100.000 | | | | | | |
| 20 | 4.408E-17 | 1.574E-16 | 100.000 | | | | | | |
| 21 | 2.843E-17 | 1.015E-16 | 100.000 | | | | | | |
| 22 | -6.922E-17 | -2.472E-16 | 100.000 | | | | | | |
| 23 | -1.082E-16 | -3.866E-16 | 100.000 | | | | | | |
| 24 | -1.720E-16 | -6.144E-16 | 100.000 | | | | | | |
| 25 | -2.920E-16 | -1.043E-15 | 100.000 | | | | | | |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 26 | -3.313E-16 | -1.183E-15 | 100.000 | | | | | | |
| 27 | -4.790E-16 | -1.711E-15 | 100.000 | | | | | | |
| 28 | -5.512E-16 | -1.968E-15 | 100.000 | | | | | | |
| Extraction Method: Principal Component Analysis. | | | | | | | | | |

Notes

*Table 2 :* Coefficient of the principal components (a's)

| Variables | Components | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 0.099279 | 0.099527 | -0.16845 | 0.386778 | -0.29842 | -0.20106 | 0.271104 | 0.149598 |
| 2 | 0.245485 | 0.146879 | 0.077539 | 0.271112 | -0.03307 | 0.038831 | 0.062488 | -0.20924 |
| 3 | -0.0348 | -0.14469 | -0.11765 | -0.37943 | -0.17873 | 0.306331 | -0.00288 | 0.466394 |
| 4 | 0.16919 | -0.22141 | -0.07005 | 0.148714 | -0.05905 | -0.07939 | -0.47683 | 0.113421 |
| 5 | 0.270704 | -0.08024 | 0.072726 | -0.18788 | -0.03386 | 0.111314 | 0.059604 | -0.16133 |
| 6 | 0.111729 | 0.392846 | -0.13262 | -0.01408 | -0.03779 | -0.02761 | 0.06345 | -0.03227 |
| 7 | 0.283792 | -0.00482 | 0.099464 | 0.149938 | -0.11338 | 0.12771 | 0.070179 | 0.071377 |
| 8 | 0.220586 | -0.0798 | 0.173794 | -0.04284 | 0.17401 | -0.20623 | -0.12402 | 0.427283 |
| 9 | 0.034157 | -0.15039 | -0.27647 | 0.253976 | 0.492111 | -0.01985 | -0.08268 | 0.019555 |
| 10 | 0.154186 | 0.316119 | -0.12406 | 0.080171 | 0.178735 | 0.018984 | -0.2288 | 0.052799 |
| 11 | 0.147483 | 0.292443 | -0.09412 | 0.048347 | -0.04173 | 0.349476 | -0.06441 | 0.175998 |
| 12 | -0.05491 | -0.20037 | -0.14492 | 0.401465 | -0.2992 | 0.090605 | -0.06633 | -0.10267 |
| 13 | 0.140779 | 0.378378 | -0.00267 | -0.10526 | 0.055116 | 0.063855 | 0.046145 | 0.030311 |
| 14 | 0.200474 | -0.19335 | 0.261494 | -0.03856 | 0.237001 | -0.04315 | 0.123054 | 0.098754 |
| 15 | 0.300073 | 0.074974 | 0.096255 | -0.00306 | -0.03071 | 0.035379 | -0.06057 | -0.01173 |
| 16 | 0.241335 | -0.04341 | 0.208553 | 0.053243 | -0.08897 | -0.0233 | -0.3134 | 0.087999 |
| 17 | -0.1494 | -0.11575 | 0.366305 | 0.029988 | 0.286605 | 0.046597 | -0.05287 | -0.28551 |
| 18 | -0.04661 | 0.148194 | 0.411224 | 0.080171 | 0.071651 | -0.15101 | 0.118247 | 0.379373 |
| 19 | 0.289539 | 0.037706 | -0.01925 | -0.03182 | -0.0622 | -0.21055 | 0.131706 | -0.05769 |
| 20 | 0.290496 | -0.1574 | 0.029946 | 0.017748 | -0.07008 | -0.03452 | -0.025 | -0.08213 |
| 21 | -0.11205 | -0.06533 | 0.327268 | 0.263768 | -0.27716 | 0.115629 | 0.177852 | 0.132976 |
| 22 | 0.17334 | -0.1631 | -0.23636 | -0.2295 | 0.007874 | 0.284758 | 0.267258 | 0.040088 |
| 23 | 0.258574 | -0.0855 | -0.22192 | -0.04712 | 0.09606 | -0.02071 | -0.10863 | -0.1496 |
| 24 | -0.06544 | 0.212646 | 0.140105 | 0.23378 | 0.171648 | 0.520331 | -0.24226 | 0.045955 |
| 25 | 0.073741 | -0.07804 | -0.14706 | 0.253976 | 0.421247 | 0.054363 | 0.481642 | 0.187731 |
| 26 | 0.186109 | -0.02762 | 0.281279 | -0.08078 | -0.01181 | 0.297702 | 0.197079 | -0.31582 |
| 27 | 0.282835 | -0.14206 | 0.041176 | -0.06671 | -0.07401 | -0.10873 | 0.034609 | -0.03031 |
| 28 | -0.02394 | 0.350756 | 0.050267 | -0.19645 | 0.01496 | -0.32704 | 0.009614 | -0.10364 |

*Table 3 :* The principal components

| PC1 | PC2 | PC3 | PC4 | PC5 | PC6 | PC7 | PC8 |
|---|---|---|---|---|---|---|---|
| 5.825186 | -0.68146 | 1.504577 | -1.19408 | 0.193327 | 0.948643 | -0.13663 | -0.89394 |
| 4.57694 | -0.59399 | 1.473616 | -0.49565 | -0.34718 | 0.864657 | 0.994138 | -1.14359 |
| 3.549147 | -0.39921 | 1.906892 | 0.343953 | 0.67018 | -1.71153 | -0.5453 | 2.049529 |
| 1.675142 | -0.69044 | 0.347119 | -0.29604 | -0.71392 | -0.44709 | -0.53793 | -0.01853 |
| 1.439421 | -0.45448 | -0.76772 | 0.065533 | -0.12329 | -1.48381 | -1.40536 | -0.2856 |
| -1.60097 | -1.97721 | -2.33998 | -3.89763 | -0.71447 | 1.137476 | 0.452038 | 1.567791 |
| 1.521179 | -1.44433 | -2.89953 | 2.282488 | 2.347398 | 0.104053 | 1.750423 | 0.75081 |
| -0.28285 | -0.07645 | -1.91988 | 1.520419 | -1.72622 | -0.27975 | 0.333432 | -0.98158 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -0.5335 | 0.155399 | -1.72468 | 1.684103 | -1.54625 | -0.14447 | 0.329784 | -0.56779 |
| -3.58183 | -0.24853 | 0.849903 | 0.150148 | 2.878182 | 0.938696 | -0.9338 | -1.06479 |
| -4.38745 | -0.059 | 2.201879 | -0.52608 | -0.25934 | -2.18826 | 1.325417 | -0.19313 |
| -3.97182 | -1.05605 | 0.532204 | -1.70555 | 0.369656 | -0.42268 | 0.350967 | -0.97929 |
| -2.98855 | 0.385129 | 3.269759 | 2.281612 | -1.1662 | 2.083303 | 0.201995 | 1.309468 |
| -1.67591 | -0.8069 | -1.46769 | 0.762995 | -0.24809 | 0.459766 | -2.2987 | 0.206917 |
| 0.435863 | 7.947505 | -0.96647 | -0.97623 | 0.38623 | 0.141 | 0.119528 | 0.243728 |

Notes

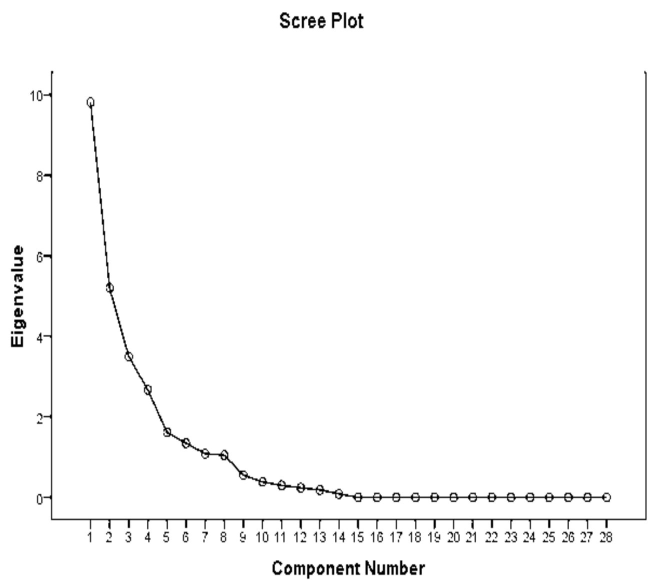


*Figure 1 :* The Scree Plot

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# On the Investigation of Determinant Variables on Economic Growth Rate in some African Countries using Panel Data Analysis Approach

By Femi J. Ayoola & Femi Adepegba
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*GJSFR-F Classification : FOR Code : MSC 2010: 11D72, 62H25*

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*Strictly as per the compliance and regulations of :*

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# On the Investigation of Determinant Variables on Economic Growth Rate in some African Countries using Panel Data Analysis Approach

Femi J. Ayoola [α] & Femi Adepegba [σ]

*Abstract-* In most African Countries, increase in Gross Domestic Products (GDP) has not translated to economic growth and development. For some decades had a lot of contestson economic growth and development has been a serious issues.The focus of this study is to analysing the effects of economic determinants on economic growth rate in some African Countries by employing panel data analysis. Yearly data were used from 1990 to 2013 time period. The data was obtained from the world economic outlook database of the International Monetary Fund (IMF), for probing the effects of these variables on growth rate in some selected African countries which include: Nigeria, Algeria, Angola, Benin, Botswana, Burundi, Cape-Verde, Cameroun, Central African Republic, Chad, Republic Of Congo, Cote di' Voire, Egypt, Equatorial-Guinea, Ethiopia, Gabon, Ghana, Guinea Bissau, Kenya, Lesotho, Madagascar, Mali, Mauritius, Morocco, Mozambique, Niger, Rwanda, Senegal, Seychelles, Sierra Leone, South Africa, Sudan, Swaziland, Tanzania, Togo, Tunisia, and Uganda. The effects of 6 macroeconomic variables on GDP were critically examined.

We used 37 Countries GDP as our dependent variable and 6 independent variables used in this study include: Total Investment (totinv), Inflation (inf),Population (popl), current account balance (cab), volume of imports of goods and services (vimgs), and volume of exports of goods and services (vexgs). The results of our analysis shows that total investment, population and volume of exports of goods and services strongly affect the economic growth. We noticed that population of these selected countries positively affect the GDP while total investment and volume of exports negatively affect GDP. On the contrary, inflation, current account balance and volume of imports of goods and services' contribution to the GDP are insignificant.

The results of this study would be useful for individual African governments for developing a suitable and appropriate economic policies and strategies. It will also help investors to understand the economic nature and viability of Africa as a continent as well as its individual countries.

*Keywords: african countries, gross domestic products, static panel data models, economic growth and development, macroeconomic variables.*

## I. Introduction

Literature has shown that in the last three decades, African countries had many situations which have adverse effects on economic growth, these situations as resulted to the continent's economic unsteadiness. Their challenges include economic under development, poverty, youth's unemployment, over-population, political instability, and terrorism among the idle hands in some African countries.

NihatTaş et al (2013)used static linear panel data models to determine the effects of 11 independent macro-economic variables on GDP of 31 EU member, acceding and candidate countries for the period 2002-2012. He opined that level of population affects economic growth positively. While the level of unemployment and total expenditure negatively affects economic growth. And that the research results were especially

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useful for the EU candidate countries like Iceland, Serbia and Turkey for developing convenient economic strategies.

Tsoukas S. (2011) performed his research on five Asian countries using panel. The Asian countries are Indonesia, Korea, Malaysia, Singapore and Thailand over the period 1995–2007. He analysed the connections between firm survival and financial development. He discovered country-level indicators of financial development plays an essential role in influencing firm survival and large firms would benefit the most fromdevelopments in the stock market, while small firms are most harshly dealt with for high levels of financial intermediation.

Notes

Beine M., et al (2011) introduceda new panel data approach for investigating the impact of skilled emigration on human capital accumulation. The data covers 147 countries over the1975-2000 period using dynamic regression models. They concluded that skilled migration prospects foster human capital accumulation in low income countries using dynamic regression models to test predictions.Lee C.C. and Chang C.P. (2008) used the new heterogeneous panel co-integration technique to re- examine the long run co-movements and casual relationship between tourism development and economic growth for OECD and non-OECD nations for the 1990-2002 period. They found that tourism development has a greater influence on GDP in the non-OECD countries than in OECD countries.

Sukiassyan G. (2007) empirically weighs that relationship with data from the transition economies of Central and Eastern Europe and the Commonwealth of Independent States. He studiedseveralscopes of the growth-inequality argument. His outcomes for transition countries show a strong but negative contemporaneous growth-inequality association.Lee C.C. and Chang C.P. (2007) engaged a new panel data stationary testing technique with a view to re-examining the dynamic connections between energy consumption per capita and real GDP per capita in 22 developed countries and 18 developing countries. It was discovered that in individual countries, structural breaks occurs near other variables in both developed and the developing countries due to tight association between energy consumption and the GDP.

Bortolotti B., et al (2003) discovers the reasons why governments implement privatization, and the magnitude, degree of privatization processes around the world using panel of34 countries over 1977-1999 time period. They discovered market, budget and institutional constraints which influences privatization.De Haas R. and Van Lelyveld I. (2006)investigated whether indigenous and non-indigenous banks in Central and Eastern Europe respond differently to business cycles and banking disasters. They used a panel database with over 250 banks between 1993 and 2000. They proved that during crises periods, local banks contract their credit. In contrast, foreign banks play astabilizing role by keeping their credit base stable. They also discovered a significant negative affiliation between home country economic growth and host country credit by foreign bank subsidiaries.

## II. Macro-Economic Determinants

The model used in this work is made up of six independent variables which are total investment, inflation (average consumer price), current account balance, population, volume of imports of goods and services and volume of exports of goods and services, while the dependent variable of interest is the gross domestic product (GDP). Gross Domestic Product by definition is the value of all goods and services produced in a country over time.Gross Domestic Product can be seen as the economic health of goods and services produced by a country and services used by individuals, firms, foreigners and the governing bodies. GDP entailsgovernment spending,consumer spending, investment expenditure and net exports hence it portrays comprehensive image of an economy. GDP is not only used as a determinant for most government and economic decision-makers for planning and policy design, but also it helps the investors to accomplish their folders by providing them with regulation about the condition of the economy,NihatTaş et al (2013).

Notes

Economic determinants can be described as pointers which are capable to explain important behaviour, characteristic and attribute of economic variable of interest. Balance ofPayments Manual released by International Monetary Fund (IMF) on international standards regarding the compilation of balance of payments statistics in order to provide guidance to member countries, in a more explicitly explained balance of payments as a statistical statement that systematically records all the economic transactions between residents of a country and non-residents for a specific time period.

The balance of payments statistics is grouped into two major categories Current Account and Capital and Financial Account. The current account contains all transactions that involve real sources (including volume of imports and exports of goods and services) and current transfers while the capital and financial accounts show how these transactions are financed. Deficits and Surpluses are natural consequence economic dealings between countries. They show the degree of a country dependence on borrowing from the rest of the world or the amount of its resources it lend abroad. A country that recorded surplus current account transfers consumption from today to tomorrow by investing abroad and a country with a deficit can increase its investments but must transfer future income abroad to redeem its external debt. Both surpluses and deficits can simply be the result of an appropriate allocation of savings, taking to account different investment opportunities across countries.In particular, countries with a rapidly ageing population may find it opportune to save today to smooth consumption over time.

On the other hand, current account deficits and surpluses are part of the adjustment process in a monetary union. They absorb asymmetric shocks in the absence of independent monetary policy and nominal exchange rate adjustment. To determine the state of economy of a country is via the comparison of general government gross debt, revenue, total investment, total expenditure and national savings. For example, if the government gross debt is low to GDP percentage, itpoint towardsa robust economy, whereas, high government debt with respect to GDP means financial distress for a nation.

## III. Methodology

### a) The Models

The static random panel data model takes the form:

$$y_{it} = \beta_0 + X_{it}\beta + \mu_i + v_{it}$$
$$u_{it} = \mu_i + v_{it} \qquad i = 1,2,.....,37, \qquad t = 1,2,.....,24 \tag{3.1}$$

where

$y_{it}$ is the dependent variable (GDP),

$X_{it}$ is the matrix of explanatory variables with coefficients $\beta$,

$\beta_0$ is the constant term,

$\mu_i$ represents unobserved individual effects for N cross sections,

$v_{it}$ represents random or idiosyncratic disturbances.

In an "ideal" model, the majority of the overall variation should be captured in the crosssectional effect. These "effect" are often referred to in the literature as errorcomponents, because in essence, the error term is being broken down into two components: cross-sectional, and idiosyncratic.

### b) Fixed Effects Models

These models do not make any assumptions regarding the joint distribution of the $X_{it}$, and terms. In theory, separate coefficients can be estimated for each individual crosssection or time period using ordinary least squares (OLS), but in practice, some type of transformation must be performed. Consider the one-way fixed effects model:

$$y_{it} = \beta_0 + X_{it}\beta + \mu_i + v_{it}$$

In matrix form, we have

$$\begin{bmatrix} \underset{(TX1)}{y_1} \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(NTX1)}{y_N} \end{bmatrix} = \begin{bmatrix} \underset{(TXk)}{X_1} \\ X_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(NTXk)}{X_N} \end{bmatrix} \begin{bmatrix} \underset{(TX1)}{\beta_1} \\ \beta_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(kX1)}{\beta_k} \end{bmatrix} + \begin{bmatrix} d_1 & \underset{(NTXN)}{d_2} & .. & .. & .. & d_N \end{bmatrix} \begin{bmatrix} \underset{(TX1)}{i_1\mu_1} \\ i_2\mu_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(NTX1)}{i_N\mu_N} \end{bmatrix} + \begin{bmatrix} \underset{(TX1)}{v_1} \\ v_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(NTX1)}{v_N} \end{bmatrix} \qquad (3.2)$$

Notes

where k represents the number of parameters in the model and $i_1$ represents a matrix of ones with dimension T. Rewriting, we have

$$\begin{bmatrix} \underset{(TX1)}{y_1} \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(888X1)}{y_N} \end{bmatrix} = \begin{bmatrix} \underset{(TXk)}{X_1} \\ X_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(888Xk)}{X_N} \end{bmatrix} \begin{bmatrix} \underset{(TX1)}{\beta_1} \\ \beta_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(kX1)}{\beta_k} \end{bmatrix} + \begin{bmatrix} \underset{(TX1)}{i_1\mu_1} \\ i_2\mu_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(888X1)}{i_N\mu_N} \end{bmatrix} + \begin{bmatrix} \underset{(TX1)}{v_1} \\ v_2 \\ \vdots \\ \vdots \\ \vdots \\ \underset{(888X1)}{v_N} \end{bmatrix} \qquad (3.3)$$

$$d_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & \underset{(888XN)}{d_2} & .. & .. & .. & d_{37} \end{bmatrix}$$

The parameter vector is now $\begin{bmatrix} \beta \\ \mu \end{bmatrix}$ as opposed to simply $\beta$ as in OLS.

## IV. Analysis

### a) *Variable Declaration and Descriptive Statistics*

The data used in this study is a panel data set of 37 African countriesfor the 1990-2013 time periods. It is a balanced, macro panel database with N*T*(K+1) = 37x24x7 = 6216 observations. Each variable has N*T = 37x24 = 888 observations. Regressand is GDP (billion dollars) and there are six regressors.

Table 1 presents the independent variables, measuring units and their abbreviations used in the analysis to represent them.

Notes

*Table 1 :* Predictor variables and their measuring units

| Codes | Variables | Units |
|---|---|---|
| totinv. | Total investment | % of GDP |
| Inf | Inflation, average consumer prices | % change |
| Popl | Population (10,000,000) | Persons |
| vimgs | Volume of imports of goods and services | % change |
| vexgs | Volume of exports of goods and services | % change |
| Cab | Current account balance | % of GDP |

*Source: International Monetary Fund world economic outlook database.*

The descriptive statistics of the variables used in this research are displayed in Table 2. Descriptive statistics values are ordinary and there are no exceptional values in the dataset.The mean value of GDP for 37 countries is $17.84 billion as observed.

*Table 2 :* Summary of Statistics

| Variable | Obs | Mean | Std. Dev. | Min | Max |
|---|---|---|---|---|---|
| Gdp | 888 | 1784.489 | 2763.974 | 94.93 | 23432.39 |
| totinv | 888 | 23.7008 | 18.19449 | 2.48 | 227.479 |
| Inf | 888 | 21.8019 | 175.6216 | -10.874 | 4146.01 |
| vimgs | 888 | 7.20244 | 19.1821 | -61.368 | 163.557 |
| vexgs | 888 | 8.383158 | 28.57126 | -70.657 | 560.871 |
| Popl | 888 | 18.87216 | 25.15105 | .07 | 169.282 |
| Cab | 888 | -5.449375 | 12.53256 | -147.997 | 34.449 |

Table 3 shows the correlation coefficients between the economic indicators used. The highest correlations among the explanatory variables are coefficient between totinv and cab which is -0.53, though they have negative association. Relationship exists among the predictor variables but its magnitude poses no threat on the analysis.

*Table 3 :* Correlation Coefficients between the Macro-economic Indicators

| | gdp | totinv | inf | vimgs | vexgs | Popl | cab |
|---|---|---|---|---|---|---|---|
| gdp | 1.0000 | | | | | | |
| totinv | 0.2028 | 1.0000 | | | | | |
| inf | -0.0457 | 0.0245 | 1.0000 | | | | |
| vimgs | -0.0515 | 0.1978 | 0.0235 | 1.0000 | | | |
| vexgs | -0.0736 | 0.2274 | -0.0015 | 0.1889 | 1.0000 | | |
| popl | -0.1519 | -0.0974 | -0.0070 | 0.0005 | -0.0454 | 1.0000 | |
| cab | 0.1348 | -0.5311 | -0.0264 | -0.2007 | -0.1292 | 0.1535 | 1.0000 |

*b) Static Linear Panel Data Models*

To obtain the association between macro-economic explanatory variables and the dependent variable, the random effects model and the fixed effects model, the most prominent static linear panel data analysis models, are used. The dependent variable is modelled as a function of 6 determinants.

The fixed effects model is

$$gdp_{it} = \alpha_i + \beta_1 totinv_{it} + \beta_2 \inf_{it} + \beta_3 popl_{it} + \beta_4 vimgs_{it} + \beta_5 vexgs_{it} + \beta_6 cab_{it} + U_{it} \quad (4.1)$$

and the random effects model:

$$gdp_{it} = \beta_1 totinv_{it} + \beta_2 \inf_{it} + \beta_3 popl_{it} + \beta_4 vimgs_{it} + \beta_5 vexgs_{it} + \beta_6 cab_{it} + (\alpha_i + u_{it}) \quad (4.2)$$

i represent the country number, t stands for the year; $U_{it}$ is the error term for the fixed effects estimators and $(\alpha_i + U_{it})$ is the composite error term for the random effects estimator. When the individual (country) effects are not correlated with the predictors, they are called random effects. Since the country specific effects is uncorrelated with the regressors, then the country specific effects is classified as additional random disturbances. They are known as fixed effects if the country specific effects are correlated with the predictors. But if there is no country specific effect in the model, then, the model assumes the pooled ordinary least squares

$$gdp_{it} = \mu + \beta_1 totinv_{it} + \beta_2 \inf_{it} + \beta_3 popl_{it} + \beta_4 vimgs_{it} + \beta_5 vexgs_{it} + \beta_6 cab_{it} + U_{it} \quad (4.3)$$

*Table 4 :* Testing for the Country Specific Effects

| $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n$ | |
|---|---|
| $F(36,845) = 43.60$ | prob > F = 0.0000 |

The null hypothesis states that the constant term is equal across countries and this is tested to determine if the pooled estimator would produce consistent estimates. It is also referred to as heterogeneity test using F test. Since the p-value =0.000 from table 4, $H_0$ is rejected, giving us the importance of retaining country specific effects in our analysis. Hence, OLS is inconsistent and inappropriate. Individual countries have different intercept which authenticated the adoption of other estimators rather than OLS.

Table 5 displays the pooled OLS, fixed effects and random effects models results. Since the country specific effects has been confirmed to be retained, then the OLS estimates is unreliable to make conclusions.

*Table 5 :* Pooled OLS, Fixed Effects and Random Effects Models

| Variables | OLS | FE | RE |
|---|---|---|---|
| totinv | 62.40823 | -71.31068 | -51.65046 |
| | 5.622320 | 6.920889 | 6.833444 |
| | 0.0000 | 0.0000 | 0.0000 |
| inf | -0.738179 | -0.2700644 | -0.407897 |
| | 0.483264 | 0.3227281 | 0.334968 |
| | 0.1270 | 0.403 | 0.2230 |
| popl | -18.92739 | 61.44893 | 24.68474 |
| | 3.416879 | 9.84715 | 7.957297 |
| | 0.0000 | 0.0000 | 0.0020 |
| vimgs | -5.364951 | 3.831325 | 3.451583 |
| | 4.597708 | 2.85451 | 2.970446 |

Notes

Notes

| | | | |
|---|---|---|---|
| | 0.2436 | 1.34 | 0.2450 |
| vexgs | -11.80921 | -5.105807 | - 6.721481 |
| | 3.085253 | 1.914438 | 1.991925 |
| | 0.0001 | 0.008 | 0.0020 |
| cab | 78.26949 | -8.539097 | 4.024485 |
| | 8.104455 | 6.611919 | 6.721481 |
| | 0.0000 | 0.197 | 0.5490 |
| cons. | 1242.818 | 2289.499 | 2599.546 |
| | 159.5281 | 214.1394 | 363.4093 |
| | 0.000 | 0.000 | 0.000 |

The Lagrange Multiplier Test helps to decide between a random effects regression and a simple OLS regression. The null hypothesis is that the variances if the country specific effects equals zero. Deducing from Table 6, LM test shows that there is country specific effects.

*Table 6 :* The Breusch-Pagan Lagrange Multiplier Test Results

| Lagrange Multiplier Test | |
|---|---|
| Null Hypothesis: $\text{var}(u) = 0$ –Pooled ols regression is appropriate.″ | |
| LM $\chi^2_1 = 2211.4$ | prob. $> \chi^2 = 0.0000$ |

In view of this, pooled OLS model presented in the first column is unreliable. Although 4 of the independent variables are estimated to be statistically significant, while the last two columns estimated only 3factors to be statistically significant. These 3 significant variables which are totinv, popl and vexgs were further estimated with the fixed and the random effects models and their output are shown in the first two columns of Table 7 below

*Table 7 :* Static Linear Panel Data Models with Contemporaneous Correlation

| **Variables** | **FE** | **RE** | **FE-RB** | **FE-PCSE** |
|---|---|---|---|---|
| Totinv | -64.12924 | -53.57096 | -64.12924 | -64.12924 |
| | 4.966806 | 5.005964 | 11.77830 | 16.39941 |
| | 0.000 | 0.000 | 0.000 | 0.0001 |
| Popl | 60.45303 | 26.2126 | 60.45303 | 60.45303 |
| | 9.655807 | 7.855872 | 6.981408 | 6.746504 |
| | 0.000 | 0.001 | 0.000 | 0.0000 |
| Vexgs | -4.995793 | -5.613376 | -4.995793 | -4.995793 |
| | 1.889213 | 1.966326 | 1.960788 | 5.509792 |
| | 0.008 | 0.004 | 0.0110 | 0.0008 |
| cons | 2205.405 | 2606.534 | 2205.405 | 2205.405 |
| | 209.5945 | 359.6452 | 303.7689 | 351.8542 |
| | 0.000 | 0.000 | 0.000 | 0.0000 |

Hausman test is used to validate the assumptions of the random effects estimator that the country specific effects are uncorrelated with the explanatory variables and the extra orthogonality conditions are satisfied. The random effects model assumes the country specific effects as a random draw that is uncorrelated with the predictors and the overall error term.

Notes

*Table 8 :* Hausman Specification Test Result

| Variables | Fixed Effects (b) | Random Effects (B) | Difference (b-B) |
|---|---|---|---|
| Totinv | -64.12924 | -53.57096 | -10.55827 |
| Popl | 60.45303 | 26.2126 | 34.24043 |
| Vexgs | -4.995793 | -5.613376 | 0.6175828 |

$H_o$: difference in coefficients not systematic (RE is consistent).
$\chi^2_3 =$ (b-B)'[(V_b-V_B)^(-1)](b-B) = 78.846
prob. $>\chi^2 = 0.0000$

The null hypothesis of the Hausman test is rejected.Therefore, country specific effects are correlated with the predictor variables. Since the random effects estimator is found inconsistent, it gives way for the fixed effects estimator as the only appropriate estimator.

Despite this, all the necessary and in fact important assumptions of the fixed effects estimator must be met, such as homoscedasticity, no serial correlation and no contemporaneous correlation. These entire diagnostic tests must be done before using FE estimator. Modified Wald test is used for testing homoscedasticity (null hypothesis = homoscedasticity

### *c) Diagnostic Tests*

#### i. *Heteroscedasticity*

This is tested using the Modified Wald test for group-wise heteroscedasticity. The null hypothesis is that the cross sectional variances are equal against the alternative hypothesis that state otherwise. It is Chi-square tested.

$$H_O : \sigma_i^2 = \sigma^2, \chi^2_{37} = 2.8 * 10^5, p > \chi^2_{37} = 0.0000$$

Since the test is significant, we reject the null hypothesis and conclude that the cross sectional variances are not equal, thus, the model has heteroscedasticity.

Serial Correlation: Using the Durbin-Watson statistic (0.120632), it is concluded that there is evidence of positive serial correlation in the residuals since the DW statistic is less than 2.

Table 7 shows the fixed effects model withFE-RB the Huber-White standard errors that is robust to heteroscedasticity and serial correlation, FE-PCSE with panel corrected standard errors that is robust to heteroscedasticity and the cross sectional correlation (contemporaneous correlation) The three models have the same coefficient estimates but with different standard errors. Finally, because of the violations of the assumptions and the nature of the model estimators, the last is used to deduce the relationship between the regressand and the regressors.

$$gdp_{it} = 2205.41 - 64.13 totinv_{it} + 60.45 popl_{it} - 5.0 vexgs_{it} \qquad 4.4$$

The above model (4.4) can be explain thus; the three economic determinants (i.e. totinv, popl and vexgs.), are significant to the GDP given their p-value to be 0.0001,0.0000 and 0.0008 respectively. The coefficient of totinv (-64.13) implies if the total investment rate increases by 1%, the gross domestic product decreases about \$0.6413 billion. The estimated coefficient of popl (60.45) indicates that if the population increases by 10million, the gross domestic product increases by about \$0.605billion. And the dependent variable (GDP) decreases about \$0.05billion if the volume of exports of goods and services increases 1%, because the coefficient of vexgs (-5.0 approximately).

## V. Conclusion and Suggestions

In this research work, the authors employed the linear static panel data procedures to analyse the cross sectional effects of some crucial macroeconomic determinants (total investment, inflation, population, current account balance, volume of imports of goods and services and volume of exports of goods and services) of African countries during the period 1990-2013. The major deductions include; total investment and volume of exports of goods and services affects economic growth negatively. That is 1% increase in total investment and volume of exports of goods and services yield a decrease of about \$0.6413billion and \$0.05billion on GDP respectively. Also, level of population has positive effects on economic growth. Because 10million increase in population leads to increase in GDP by over \$0.6billion.

Having known the effects of these determinants, African individual state governments should critically look into the significance of the estimated macroeconomic determinants for re-strategizing economic policies as well as using them to improve their decision making. Private investors were advised to study the impact of these economic determinants with a view to maximizing their profit.

Notes

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## Appendix

*Figure 1 :* Presents the Panel Line Graph of the GDP for the Individual Countries

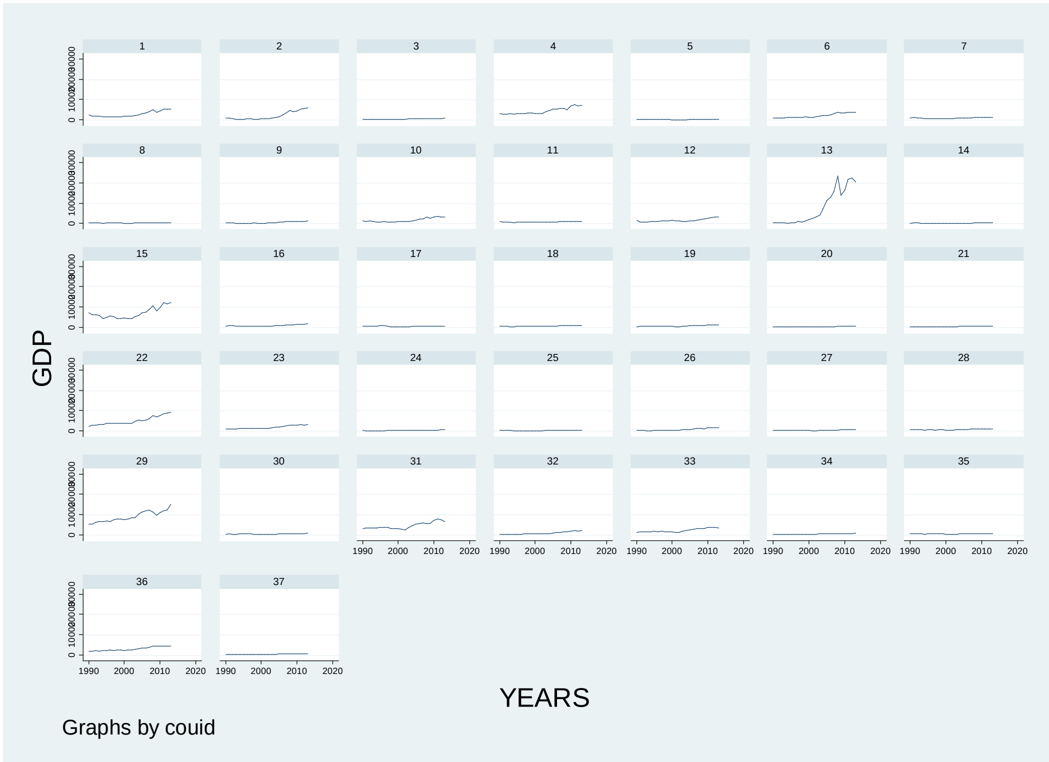
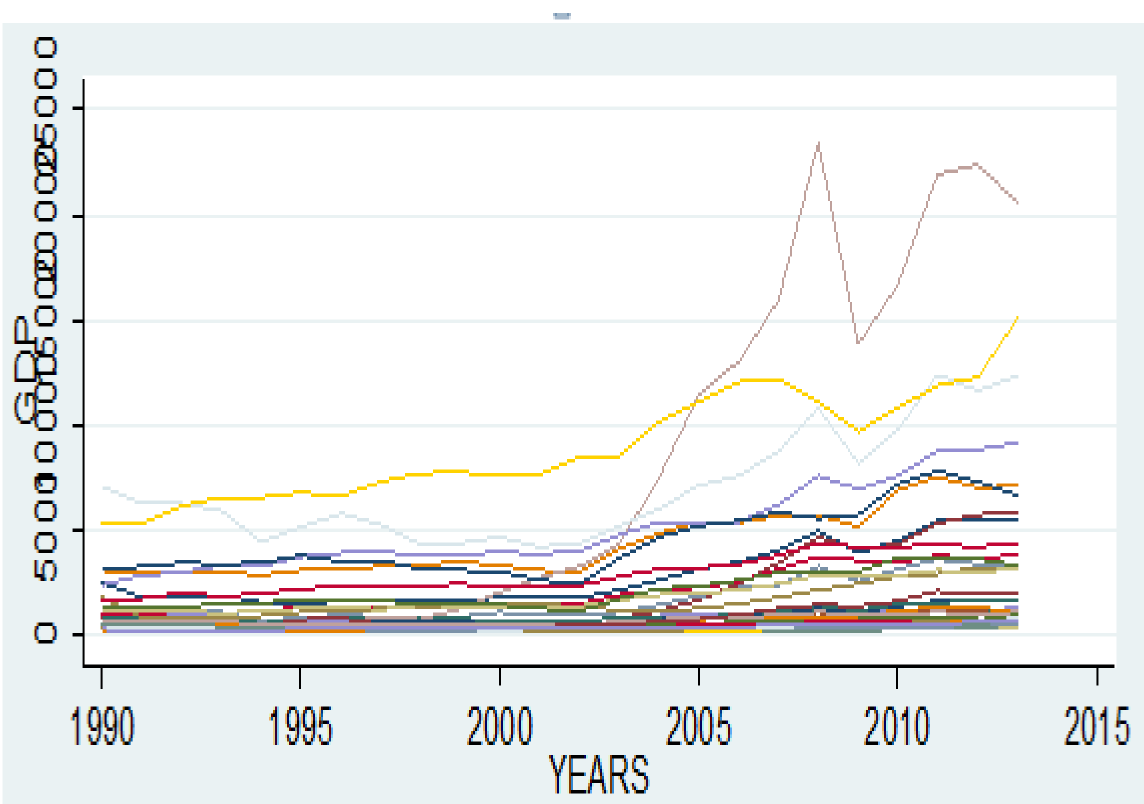



*Figure 2 :*Represent the Joint Graph of GDP for all the Countries under Investigation

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# Chebychev Polynomials of the first Kind and Whittaker's Constant

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*Abstract-* In this paper, we provide the conditions required for the Chebychev polynomials of the first kind $\{T_n(z)\}_{n\geq 0}$ to be a basic set. Then, we prove that the domain of effectiveness is a unit disc $D(R) = (0,1)$ related to the radius R of convergence of the associated basic series of $\{T_n(z)\}_{n\geq 0}$. We then give the Cannon condition satisfied by $\{T_n(z)\}_{n\geq 0}$ and the corresponding Whittaker's constant which is better than that obtained in the previous works using the Goncharov's polynomials. The order and type of the polynomials are also given.

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*Strictly as per the compliance and regulations of :*

Ref

1. Z.M.G. Kishka and A. El-Sayed Ahmed, On the order and type of basic and composite sets of polynomials in complete Reinhardt domains, period. Math. Hung., 46(1), 67-76 (2003).

# Chebychev Polynomials of the first Kind and Whittaker's Constant

A. Anjorin [α] & S. Akinbode [σ]

*Abstract-* In this paper, we provide the conditions required for the Chebychev polynomials of the first kind $\{T_n(z)\}_{n\geq 0}$ to be a basic set. Then, we prove that the domain of effectiveness is a unit disc $D(R) = (0,1)$ related to the radius R of convergence of the associated basic series of $\{T_n(z)\}_{n\geq 0}$. We then give the Cannon condition satisfied by $\{T_n(z)\}_{n\geq 0}$ and the corresponding Whittaker's constant which is better than that obtained in the previous works using the Goncharov's polynomials. The order and type of the polynomials are also given.

## I. Introduction

The basic sets of polynomials continue to be at the core of many investigations [1]-[19] since the work of Whittaker [12]. The properties of series of the form $C_0P_0(z) + C_1P_1(z) + \cdots +$ where $P_0(z), P_1(z) \cdots$ are prescribed polynomials differ widely according to the particular polynomials choosen. For example, the region of convergence may be a circle (Taylor Series), an ellipse (series of Legendre polynomials), a half-plane (Newton's interpolation series) etc. Whittaker [12], in his attempt to find common properties exhibited by all these polynomials, introduced the subject of basic sets of polynomials. In his work, he gave the definition of basic set, basic series and effectiveness of basic sets. Cannon [13] obtained the necessary and sufficient conditions for the effectiveness of basic sets for classes of functions of finite radii of regularity and of entire functions. Nassif and Adepoju [18] investigated the zeros of polynomials belonging to simple sets. Wakid and Maker [2] also contributed to the investigations of the zeros of polynomials belonging to simple sets. Initially, the subject has been approached through the classical treatment. Then Newns [19] laid down the treatment of the subject based on functional analysis consideration. Over the years, this approach has received further advancement through the works of Falgas [16], Adepoju [3] and Kishka and El-Sayed Ahmed [1].

*Definition 1.1* *[1] A sequence $\{P_n(z)\}_{n\geq 0}$ of polynomials is said to form a basic set if and only if polynomial $P_i(z), i = 0, 1, \cdots$ admits a unique finite linear combination of the polynomials of the set:*

$$P(z) = \sum_{k=0}^{n} C_k P_k(z), \quad where \;\; n < \infty. \tag{1.1}$$

Indeed, the polynomials $\{P_n(z)\}_{n\geq 0}$ are linearly independent, i.e.,

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$$\sum_{k=0}^{n} C_k P_k(z) = 0$$

implies

$$C_0 = C_1 = C_2 = \cdots = C_n = 0.$$

The set of Polynomials $\{z^n\}_{n\geq 0}$ has a unique representation of the form

$$z^n = \sum_{k=0}^{n} \pi_{n,k} P_k(z), \quad \text{where} \ \ \pi_{n,k} = C_k^n$$

so that

$$1 = \sum_{k=0}^{0} \pi_{0,k} P_k(z), \quad z = \sum_{k=0}^{1} \pi_{1,k} P_k(z), \ \ z^2 = \sum_{k=0}^{2} \pi_{2,k} P_k(z), \ \ \cdots, z^n = \sum_{k=0}^{n} \pi_{n,k} P_k(z). \quad (1.2)$$

In general, any polynomial of the form

$$P(z) = \sum_{i=0}^{k} P_i z^i$$

can be written

$$\begin{aligned}
P(z) &= \sum_{i=0}^{k} P_i z^i = \sum_{i=0}^{k} \left( \sum_{j=0}^{i} \pi_{i,j} P_j(z) \right) && (1.3)\\
&= \sum_{i=0}^{k} P_i \left( \pi_{i,0} P_0(z) + \pi_{i,1} P_1(z) + \cdots \right) && (1.4)\\
&= \sum_{i=0}^{k} P_i \pi_{i,0} P_0(z) + \sum_{i=0}^{k} P_i \pi_{i,1} P_1(z) + \cdots && (1.5)\\
&= \sum_{n} C_n P_n(z), && (1.6)
\end{aligned}$$

so that the representation is unique.
To every basic set there corresponds an associated basic series. So, if $\{P_n(z)\}_{n\geq 0}$ forms a basic set, then the corresponding basic series can be written as

$$\begin{aligned}
f(z) = \sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} a_n \left\{ \left( \sum_{k=0}^{n} \pi_{n,k} P_k(z) \right) \right. \\
&= \sum_{n=0}^{\infty} a_n \left( \pi_{n,0} P_0(z) + \pi_{n,1} P_1(z) + \cdots \right) \\
&= \sum_{n=0}^{\infty} \pi_n f(0) P_n(z), && (1.7)
\end{aligned}$$

where

$$\pi_n f(0) = a_0 \pi_{0,n} + a_1 \pi_{1,n} + a_2 \pi_{2,n} + \cdots = \sum_{k=0}^{\infty} a_k \pi_{k,n}.$$

Notes

Substituting $a_k \equiv f^k(0)/k!$, we have

$$\pi_n f(0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \pi_{k,n} = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_{k,n} \frac{d^k f(0)}{dz^k} \tag{1.8}$$

so that the corresponding basic series takes the form

$$f(z) = \sum_{n=0}^{\infty} \pi_n f(0) P_n(z), \tag{1.9}$$

where $\pi_j, j = 1, 2, \cdots, n$ can be regarded as the elements of the set of operators of the basic set, $\{\pi_n\}_{n\geq 0}$, given by

$$\pi_n = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_{n,k} \frac{d^k}{dz^k}. \tag{1.10}$$

This set is called a basic set of operators if these operators are associated with a basic set of polynomials $\{P_n(z)\}_{n\geq 0}$. Prior to the definition of the effectiveness of basic sets, we recall that in the complex plane, a domain is an open connected set. A regular closed curve is usually denoted by $D(C)$, the domain interior to $C$; its closure is denoted by $\bar{D}(C)$ and the class of functions regular in $D(C)$ is written as $H(C)$. When $C$ is a circle of radius $|z| = r$, the above entities are referred to as $D(r), \bar{D}(r)$ and $H(r)$, respectively.

*Definition 1.2* *Let $f(z)$ be a function regular in the domain $D(R)$. The basic series* $\sum_{n=0}^{\infty} \pi_n f(0) P_n(z)$ *of (1.7) is said to represent $f(z)$ in the domain if it converges uniformly to $f(z)$ in $D(R)$. If the domain is a circle, we simply state that the basic set $\{P_n(z)\}_{n\geq 0}$ represents $f(z)$ in $|z| \leq R$. We write this as*

$$f(z) \sim \sum_{n=0}^{\infty} \pi_n f(0) P_n(z) \tag{1.11}$$

*Definition 1.3* *[3] A basic series (or a basic set) $\{P_n(z)\}_{n\geq 0}$ is effective in a domain $D(R)$ if every function $f(z)$ regular in $D(R)$ is represented by the basic series.*

The following is relevant for the effectiveness in closed circle. Suppose the set $\{P_n(z)\}_{n\geq 0}$ is basic. We denote by

$$M_k(R) = \max_{|z|=r} |P_k(z)| \tag{1.12}$$

the maximum value of $P_k(z) \in \{P_n(z)\}_{n\geq 0}$ over $|z| = R$. The Cannon sum [13] $W_n(R)$ of the basic set $\{P_n(z)\}_{n\geq 0}$ is defined by

$$W_n(R) = \sum_{k=0}^{\infty} |\pi_{n,k}| M_k(R), \tag{1.13}$$

and the Cannon function is denoted by

$$\lambda(R) = \overline{\lim}_{n\to\infty} \{W_n(R)\}^{\frac{1}{n}} = \lim_{n\to\infty} \sup \{W_n(R)\}^{\frac{1}{n}}. \tag{1.14}$$

Ref

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This definition is satisfied in both open and closed circles with the notation $<$ and $\leq$ respectively. Let us now define the Cannon condition.

*Definition 1.4* *[3] A set $\{P_n(z)\}_{n\geq 0}$ of polynomials in which the polynomial $P_n(z)$ is of degree $n$, is necessarly basic and is called a simple set. Let $\{P_n(z)\}_{n\geq 0}$ be a basic set of polynomials. Then the number $V_{(n)}$ is defined as the number of polynomials of the set whose degree is less than $n$. The polynomials of the set $\{P_n(z)\}_{n\geq 0}$ are linearly independent if $V_{(n)} \leq n$ for $n \geq 1$. The number $D_n$ is also defined as the degree of polynomials of the highest degree in the representation (1.1).*

For

$$z^n = \sum_{k=0}^{n} \pi_{n,k} P_k(z), \quad \text{we have} \quad D_n \geq n.$$

*Definition 1.5* *[1] [3] Let $N_n$ be the number of non-zero terms in the representation. Then the basic set $\{P_n(z)\}_{n\geq 0}$ is called the Cannon set if $N_n^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$. This condition is called the Cannon condition.*

In this paper, we are interested in the investigation of properties of the Chebychev basic set. In Section 2, we prove that the set of Chebychev polynomials $\{T_n(z)\}_{n\geq 0}$ is a basic set. In Section 3, we provide the corresponding associated basic series. In section 4, we show the effectiveness of the basic set. In section 5, we prove that $\{T_n(z)\}_{n\geq 0}$ forms a Cannon set. Finally, in section 6, we infer that the Chebychev basic set gives better improvement for the Whittaker constant than the other sets of classical polynomials.

## II. Basic Set of Chebychev Polynomials

Let us prove the following

*Theorem 2.1* *The set of Chebychev polynomials $\{T_n(z)\}_{n\geq 0}$ is a basic set.*

(i) We first show that the representation

$$T_n(z) = \sum_{k=0}^{[n/2]} \binom{n}{2k} z^{n-2k} \left(z^2-1\right)^k \quad \text{is unique.} \tag{2.1}$$

Since

$$\begin{aligned}
T_0(z) &= 1 = \sum_{k=0}^{0} \binom{0}{2k} z^{0-2(0)} \left(z^2-1\right)^0 = 1 \\
T_1(z) &= \sum_{k=0}^{0} \binom{1}{2k} z^{1-2(0)} \left(z^2-1\right)^0 = z \\
T_2(z) &= \sum_{k=0}^{1} \binom{2}{2k} z^{2-2(k)} (z^2-1)^k = 2z^2-1 \\
&\vdots \\
T_n(z) &= \sum_{k=0}^{[n/2]} \binom{n}{2k} z^{n-2k} (z^2-1)^k.
\end{aligned}$$

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In general, given any polynomial

$$T_n(z) = \sum_{i=0}^{[n/2]} t_i T_i(z) \tag{2.2}$$

and using $T_n(z)$ representation, we can write

$$\begin{aligned} T_n(z) &= \sum_{i=0}^{[n/2]} t_i \left( \sum_{j=0}^{[i/2]} \begin{pmatrix} i \\ 2j \end{pmatrix} z^{i-2j}(z^2-1)^j \right) \\ &= \sum_{i=0}^{[n/2]} t_i \left\{ \begin{pmatrix} i \\ 0 \end{pmatrix} z^i + \begin{pmatrix} i \\ 2 \end{pmatrix} z^{i-2(1)}(z^2-1) + \begin{pmatrix} i \\ 4 \end{pmatrix} z^{i-2(2)}(z^2-1)^2 + \cdots \right\} \\ &= \sum_{i=0}^{[n/2]} t_i \begin{pmatrix} i \\ 0 \end{pmatrix} z^i + \sum_{i=0}^{[n/2]} t_i \begin{pmatrix} i \\ 2 \end{pmatrix} z^{i-2}(z^2-1) + \sum_{i=0}^{[n/2]} t_i \begin{pmatrix} i \\ 4 \end{pmatrix} z^{i-2(2)}(z^2-1)^2 + \cdots \\ &= \sum_{k=0}^{[n/2]} t_k \begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1). \end{aligned} \tag{2.3}$$

Hence, the set$\{T_n(z)\}_{n\geq 0}$ is well represented. To complete the proof, we state the following

*Lemma 2.2 The sequence $\{T_n(z)\}_{n\geq 0}$ of Chebychev polynomials in which $T_n(z)$ is of degree $n$ is basic.*

We have $T_0(z) = 1T_0$, since $T_0(z)$ is a polynomial of degree zero. Dividing through by $T_0$, we have $1 = \frac{1}{T_0}T_0(z) = \pi_{0,0}T_0(z)$ with $\pi_{0,0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1$. Let also $T_1(z) = T_{1,0} + T_{1,1}z$ where $T_{1,1} \neq 0$. Dividing through by $T_{1,1}$, we have $\frac{1}{T_{1,1}}T_1(z) = \frac{T_{1,0}}{T_{1,1}} \cdot 1 + z$ so that $z = -\frac{T_{1,0}}{T_{1,1}} \cdot 1 + \frac{1}{T_{1,1}}T_1(z) = \frac{-T_{1,0}}{T_{1,1}} \cdot \pi_{0,0}T_0(z) + \frac{1}{T_{1,1}}T_1(z) = -\pi_{1,0}T_0(z) + \pi_{1,1}T_1(z)$.
Hence the representation is true for $n = 0, 1$. Suppose it is true for $2, 3, \cdots n-1$ and let

$$\begin{aligned} T_n^*(z) &= T_{n,0} + T_{n,1}z + T_{n,2}(2z^2-1) + T_{n,3}(4z^2-3z) + \cdots \\ &\quad + T_{n,n-1}\left[ \sum_{k=n-1}^{[n/2]} \begin{pmatrix} n-1 \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k \right] \\ &\quad + T_{n,n} \sum_{k=n}^{[n/2]} \begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k \end{aligned}$$

where $T_{n,n} \neq 0$. Then

$$\begin{aligned} T_n^*(z) &= T_{n,0} \begin{pmatrix} 0 \\ 0 \end{pmatrix} z^0 + T_{n,1} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{1-0}(z^2-1)^0 \right\} \\ &\quad + T_{n,2} \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} z^{2-0}(z^2-1)^0 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} (z^2-1) \right\} + \cdots \end{aligned}$$

$$+T_{n,n-1}\left\{\sum_{k=n-1}^{[n/2]}\begin{pmatrix} n-1 \\ 2k \end{pmatrix} z^2(z^2-1)^k\right\}$$

$$+T_{n,n}\left\{\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k\right\}, \tag{2.4}$$

what we can write in a compact form as

$$\begin{aligned} T_n^*(z) &= T_{n,0}+T_{n,1}z+T_{n,2}(2z^2-1)+T_{n,3}(4z^3-3z)+\cdots \\ &\quad +T_{n,n}\left\{\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k\right\}. \end{aligned} \tag{2.5}$$

Let $T_{n,j}=\alpha_j$. Then we get

$$T_{n,0}=\alpha_0, T_{n,1}=\alpha_1, \cdots, T_{n,n}=\alpha_n,$$

so that

$$T_n^*(z)=\alpha_0T_0(z)+\alpha_1T_1(z)+\alpha_2T_2(z)+\cdots+\alpha_nT_n(z)$$

where

$$T_0(z)=1, T_1(z)=z, T_2(z)=(2z^2-1), \cdots.$$

Consequently,

$$\begin{aligned} T_{n,n}\left\{\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k\right\} &= T_n^*(z)-\{\alpha_0T_0(z) \\ &\quad +\alpha_1T_1(z)+\cdots+\alpha_{n-1}T_{n-1}(z)\} \end{aligned}$$

from which we get on dividing by $T_{n,n}$:

$$\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k=\frac{T_n^*(z)}{T_{n,n}}-\frac{1}{T_{n,n}}\left\{\alpha_0T_0(z)+\alpha_1T_1(z)+\cdots+\alpha_nT_n(z)\right\}. \tag{2.6}$$

The right hand side can be denoted by $\tilde{T}_n^\star(z)$. Hence, we obtain

$$\tilde{T}_n^\star(z)=\sum_{k=n}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k. \tag{2.7}$$

For $0<k<n$, we have

$$\tilde{T}_n^\star(z)=\sum_{k=0}^{[n/2]}\begin{pmatrix} n \\ 2k \end{pmatrix} z^{n-2k}(z^2-1)^k. \tag{2.8}$$

By unique representation $T_n^\star(z)=T_n(z)$. Furthermore, the polynomials are linearly independent. Indeed, one can immediately prove that

Notes

$$\begin{aligned} a_0T_0(z)+a_1T_1(z)+\cdots+a_nT_n(z) &= a_01+a_1z+a_2(2z^2-1)+a_3(4z^3-3z)\\ &\quad +a_4(8z^4-8z^2+1)\\ &\quad +\cdots+a_n\sum_{k=0}^{[n/2]}\binom{n}{2k}z^{n-2k}(z^2-1)^k\\ &= 0 \end{aligned} \tag{2.9}$$

Notes

leads to $a_i=0$ for $i=0,1,2,\cdots,n$. Thus, the set $\{T_n(z)\}_{n\geq 0}$ of Chebychev polynomials is basic.

## III. Associated Basic Series of $\{T_n(z)\}_{n\geq 0}$

In this section, we are interested in the investigation of existence of basic series associated with the basic set $\{T_n(z)\}_{n\geq 0}$.

*Theorem 3.1* *The set $\{T_n(z)\}_{n\geq 0}$ has an associated basic series.*

*Proof:* Consider the function

$$f(z)=\sum_{k=0}^{\infty}a_nT_n(z). \tag{3.1}$$

Expanding it, we get

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}a_n\left(\sum_{k=0}^{[n/2]}\binom{n}{2k}z^{n-2k}\left(z^2-1\right)^k\right)\\ &= \sum_{n=0}^{\infty}a_n\left\{\binom{n}{0}z^n+\binom{n}{2}z^{n-2}(z^2-1)+\cdots+\binom{n}{2j}z^{n-2j}(z^2-1)^j+\cdots\right\}\\ &= \sum_{n=0}^{\infty}a_n\left\{\binom{n}{0}Q_0(z)+\binom{n}{2}Q_1(z)+\cdots+\binom{n}{2j}Q_j(z)+\cdots\right\}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} Q_0(z) &= z^n, \quad Q_1(z)=z^{n-2}(z^2-1), \quad Q_2(z)=z^{n-4}(z^2-1)^2\\ &\vdots\\ Q_j(z) &= z^{n-2j}(z^2-1)^j. \end{aligned} \tag{3.3}$$

Then, we can write

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\left\{a_n\binom{n}{0}Q_0(z)+a_n\binom{n}{2}Q_1(z)+\cdots+a_n\binom{n}{2j}Q_j(z)+\cdots\right\}\\ &= a_0\binom{0}{0}Q_0(z)+a_1\binom{1}{0}Q_0(z)+\cdots+a_n\binom{n}{0}Q_0(z)\\ &+ a_0\binom{0}{2}Q_1(z)+a_1\binom{1}{2}Q_1(z)+\cdots+a_n\binom{n}{2}Q_1(z)\\ &\quad +a_0\binom{0}{4}Q_2(z)+a_1\binom{1}{4}Q_2(z)+\cdots+a_n\binom{n}{4}Q_2(z)+\cdots \end{aligned}$$

$$= \sum_{k=0}^{\infty} a_k \begin{pmatrix} k \\ 2n \end{pmatrix} Q_n(z) \equiv \sum_{n=0}^{\infty} \pi_n f(0) Q_n(z),$$

where $\pi_n f(0) = a_k \begin{pmatrix} k \\ 2n \end{pmatrix}$. Hence, the basic series associated with the basic set $\{T_n(z)\}_{n\geq 0}$ is $f(z) = \sum_{k=0}^{\infty} a_k \begin{pmatrix} k \\ 2n \end{pmatrix} Q_n(z)$, where $\{Q_n(z)\}_{n\geq 0}$ forms a basis for $\{T_n(z)\}_{n\geq 0}$

Notes

## IV. Effectiveness of Chebychev Polynomials $D(R)$ or $D_+(r)$

In this section, we investigate the effectiveness of $\{T_n(z)\}_{n\geq 0}$ in the domain $D(R)$ or $D_+(r)$.

*Theorem 4.1* *Let $\{T_n(z)\}_{n\geq 0}$ be a basic set of polynomials and suppose that, for any value $R > 0, \lambda(R) = \sigma \geq R$, then the basic series is effective in $\tilde{D}(R)$ for the class $\bar{H}(\sigma)$.*

*Proof:*

Let $f(z) = \sum_{n=0}^{\infty} a_n T_n(z)$ be any function regular in $|z| < \sigma$. Then $\sum_{n=0}^{\infty} a_n T_n(z)$ converges and $\lim_{n\to\infty} a_n z^{n-2k}(z^2-1)^k = 0$. So, we can choose $n$ large enough so that $|a_n z^{n-2k}(z^2 - 1)^k| \to 0$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} |a_n z^{n-2k}(z^2-1)^k| \leq \lim_{n\to\infty} |a_n|\sigma^n < 1$$

This implies that

$$\lim_{n\to\infty} |a_n|^{1/n} < \frac{1}{\sigma}. \tag{4.1}$$

Now, consider the series

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z).$$

We have, for $|z| \leq R$,

$$\left| a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z) \right| \leq |a_n| W_n(R).$$

By (4.1)

$$\overline{\lim}_{n\to\infty} \{|a_n| W_n(R)\}^{\frac{1}{n}} \leq \overline{\lim}_{n\to\infty} |a_n|^{\frac{1}{n}} \cdot \lambda(R) < \frac{\sigma}{\sigma} = 1.$$

Thus, the series

$$\sum_{n=0}^{\infty} |a_n| W_n(R)$$

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is convergent and since

$$|\sum_{n=0}^{\infty} a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z)| \leq \sum_{n=0}^{\infty} |a_n| W_n(R),$$

then by Weierstrass-M-test, the series

$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{[n/2]} \pi_{n,2k} Q_k(z)$$

is uniformly and absolutely convergent in $|z| \leq R$. Re-arranging the terms of the series, we conclude that the resulting basic series

$$\sum_{n=0}^{\infty} \pi_n f(0) Q_n(z)$$

converges uniformly to $f(z)$ in $|z| \geq R$. Thus, the basic series represents $f(z)$ in $|z| \leq R$.

*Corollary 4.2* *If for any value of $R > 0, \lambda(R) = R$, then the basic set $\{T_n(z)\}_{n \geq 0}$ will be effective in $|z| \leq R$.*

*Proof:* If $R > 0$ and $\lambda(R) = R$, from Theorem 4.1, the basic series represents in $|z| \leq R$ every function regular in $|z| \leq R$. That is to say, the basic series (or the basic set) will be effective in $|z| \leq R$. Thus the condition that $\lambda(R) = R$ is a sufficient condition for effectiveness in $|z| \leq R$.

*Theorem 4.3* *The necessary and sufficient condition for the set $\{T_n(z)\}_{n \geq 0}$ to be effective in $|z| \geq R$ is that $\lambda(R) = R$.*

*Proof:*

*Necessity:* If the set $\{T_n(z)\}_{n \geq 0}$ is effective in $|z| \leq R$, then $\lambda(R) = R$. If on the contrary, $\lambda(R) > R$, then for any number $\rho$ for which $R < \rho < \lambda(R)$, there exists a function $f(z)$ of radius of regularity $\rho$, that is $f(z)$ is regular in $|z| \leq R$ and that the basic series cannot represent in $|z| \leq R$. Thus, the set will not be effective in $|z| \leq R$.
*Sufficienty:* This follows directly from Corollary 4.2 since for any value of $R > 0, \lambda(R) = R$. Then the basic set $\{T_n(z)\}_{n \geq 0}$ will be effective in $|z| \leq R$. Since, $\{T_n(z)\}_{n \geq 0}$ is effective in $D(R)$ (or $D_+(r)$), then it is represented in $D(R)$ (or $D_+(r)$) by a basic series of the form

$$f(z) = \sum_{n=0}^{\infty} \pi_n f(0) Q_n(z),$$

where $f(z)$ belongs to $H(R)$ (or $H(r)$), the class of all holomorphic functions. Hence, there exists $f(z) \in H(R)$ (or $H(r)$) denoted by $f(z) = \sum_{n=0}^{\infty} \pi_n f(0) Q_n(z)$, representing the $\{T_n(z)\}_{n \geq 0}$ in $D(R)$ (or $D_+(r)$). Hence, $\{T_n(z)\}_{n \geq 0}$ is effective in $D(R)$ (or $D_+(r)$).

Suppose by definition

$$M_n(R) = \max_{|z|=R} |T_k(z)| = \max_{|z|=R} \left| z^{n-2k}(z^2-1)^k \right| \le R^{n-2k}(R^2-1)^k \le R^n(R^2-1)^k.$$

Then

$$W_n(R) = \sum_{k=0}^{[n/2]} |\pi_{n,2k}|\, M_n(R) < \sum_{k=0}^{[n/2]} |\pi_{n,2k}|\, R^n(R^2-1)^k$$

and the Cannon function $\lambda(R)$ is

$$\lambda(R) = \overline{\lim}_{n\to\infty} \{W_n(R)\}^{1/n}.$$

This implies that

$$\lambda(R) \le \overline{\lim}_{n\to\infty} \left\{ \sum_{k=0}^{[n/2]} |\pi_{n,2k}| \right\}^{1/n} \{R^n\}^{1/n} (R^2-1)^{k/n} = R.$$

Since $\lambda(R)$ is non-negative, i.e. $\lambda(R) > 0$, we have that $\lambda(R) = R$. Then the domain of effectiveness $D(R) = (0, 1)$ is a disc.

## V. Cannon Condition

In this section, we investigate the condition for the set $\{T_n(z)\}_{n\ge 0}$ to be a Cannon set. This can be stated in the following result.

*Theorem 5.1 The set $\{T_n(z)\}_{n\ge 0}$ forms a Cannon set.*

*Proof:* It suffices to show that if $N_n = C^n_{2k}$ is the number of non-zero terms in the unique representation of $\{T_n(z)\}_{n\ge 0}$, $N^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$. Using Stirling formula, i.e. $n! = \sqrt{2\pi n^n}\, e^{-n}$, we have

$$N_n = \frac{n!}{(2k)!(n-2k)!} = \frac{\sqrt{2\pi n^n} e^{-n}}{\sqrt{2\pi (2k)^{2k}}\, e^{-2k}} \times \left( \sqrt{2\pi (n-2k)^{(n-2k)}}\, e^{-(n-2k)} \right)^{-1}.$$

As $n$ approaches infinity, we have that , $N_n^{1/n} \longrightarrow 1$. Hence $\{T_n(z)\}_{n\ge 0}$ is a Cannon set.

## V. Improved Whittaker's Constant using Chebychev Polynomials of the First Kind

### a) Generalities

Over the years there has been intensive investigation [3]-[12] [18], on the best approximation of the so called Whittaker's constant noted $W$. This constant was introduced by Whittaker in his work [12] on interpolation of function. The problem is the following: given a function $f(z)$ in a complex plane $\mathbb{C}$, what is the upper bound for which this function is entire and is of exponential type $c$ such that in a domain $D(R)$ which could be a disc, one has $f(z) = f'(z) = f''(z) = \cdots = f^{(n)}(z) = 0$. In other words, what range does the constant $W$ lie? Ever since there remains the question of the range which best approximates this constant. To this question tackle many others since the

Ref

18. Nassif, M, and Adepoju, J.A., (1978), "Zeros of polynomials belonging to simple set" National Academy of Science letters ; Vol. 1, pp. 223-224.

historical work by Whittaker. Up till now, to our best knowledge of the literature, the best range is obtained using Goncharov polynomials and this has been computed to be $0.7259 \leq W \leq 0.7380$. In this section, we state this range can be improved using Chebychev polynomials of the first kind instead of the other classical polynomials. The Chebychev polynomials are of considerable interest in interpolation theory [12]. The degree of accuracy of the interpolation result obtained depends on the bound which can be found for these polynomials. With $|z_k| \leq 1 \quad k = 0, 1, \cdots$ the maximum value of these polynomials will be denoted by $M_n$ as follows:

$$M_n = |T_n(z, z_0, z_1, \cdots, z_n)|.$$

We claim that for $n \geq 1$, there exists $r$ such that

$$M_n < r^{n+1}, \tag{6.1}$$

where $r$ is some positive number, $r > 0$. This leads to an improved value of the Whittaker's constant $W$ defined as the least upper bound of a number $c$ such that the function $f(z)$ is an entire function of exponential type $c$ and if $f(z)$ and each of its derivatives have at least one zero in the unit circle, then $f(z) = f'(z) = \cdots = f^{(n)}(z) = 0$ or equivalently $f(z) \equiv 0$. Let now the set $\{T_n(z)\}_{n\geq 0}$ be a basic set of Chebychev polynomial, effective in the circle $|z| \leq 1$. Then $\{T_n(z)\}_{n\geq 0}$ is said to be of order 1 if its zeros lie in $|z| \leq 1$. In this case, the polynomial $\{T_n(z)\}_{n\geq 0}$ can be expanded in the power series

$$T_n(z) = \sum_{k=0}^{\infty} e^{\pi i k/2} \frac{z^k}{k!}.$$

Let $W_j$ be the Whittaker's constant, $0.7259 \leq W \leq 0.7380$ with $j = 0, 1, 2, \cdots, n$ and $W_1 \neq W_2 \neq \cdots \neq W_n$. Then the polynomial corresponding to $W_j$ is of the type $\sigma = \frac{1}{W_j}$, where $W_j$ is the modulos of the zeros of $\{T_n(z)\}_{n\geq 0}$. Assume $f(z)$ be an entire function exponential type c. Then if

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!},$$

it follows that $a_n = O(c+\epsilon)^n, \epsilon > 0$. That is, for any $b > c$, it follows that for sufficiently large $n$

$$|a_n| < b^n. \tag{6.2}$$

Denote by $\{z_k, k = 0, 1, \cdots, n\}$ the points, inside the unit circle where $f(z)$ and its derivative vanish. Then $\{T_n(z)\}_{n\geq 0}$ will be represented by the power series

$$f(z) = \sum_{k=0}^{\infty} \frac{a_{n+k}}{k!} T_k(z).$$

For large $n$ and $|z| \leq 1$, we get from (6.1) and (6.2)

$$|f(z)| \quad \leq \quad b^{n+k} r^{n+k+1} = \frac{(br)^n r}{(1 - br)}.$$

Ref

12. J. M. Whittaker, "Interpolatory Function Theory", The University Press, Cambridge, 1935.

Thus as $n \longrightarrow \infty$ we obtain

$$|f(z)| \leq \frac{r}{1-br}.$$

In this case the Whittaker's constant $W > \frac{1}{r}$.

### b) Main result

Here, we aim at proving that the Levinson method [9] applied to Chebychev polynomials instead of Goncharov one's, provides a better approximation of the range of the Whittaker's constant. Furthermore, the Chebychev polynomials reveal to be the best set for the computation of the closest boundary value of this range than the other classical polynomials (Lagauerre, Legendry, Jacobi, Bessel etc.) commonly used in Mathematical Physics. Following Levinson method, we get the following statement.

*Theorem 6.1 For an entire function $T_n(z)$ of Chebychev polynomials, there exists a positive number $r$ such that $|T_n(z)| = M_n < r^{n+1}$. Then, for Chebychev polynomials of the first kind, the Whittaker's constant has an upper bound not exceeding 0.7380 and a lower bound not exceeding 0.73778.*

*Proof:* Using Chebychev polynomial representation, we obtain

$$\frac{\partial T_n}{\partial z_k} = -T_k(z, z_1, z_2, \cdots, z_n)T_{n-k-1}(z_k, z_{k+1}, \cdots, z_{k-1})$$

Using Euler's formula for homogeneous function of degree $n$ allows us to write

$$\begin{aligned} nT_n(z) &= \frac{z\partial T_n}{\partial z} + \frac{z_0\partial T_n}{\partial z_0} + \cdots + z_{n-1}\frac{\partial T_n}{\partial z_{n-1}} \\ nT_n(z) &= zT_{n-1}(z, z_1, \cdots, z_{n-1}) \\ &\quad -\sum_{k=0}^{n-1} z_k T_k(z, \cdots, z_k)T_{n-k-1}(z_k, \cdots, z_{n-1}) \end{aligned} \quad . \tag{6.3}$$

Taking absolute value of both sides of the relation (6.3), we obtain

$$M_n \leq M_{n-1} + \sum_{k=0}^{n-1} M_k M_{n-k-1}. \tag{6.4}$$

To get $M_n$ we will make the use of the Taylor series expansion of the function $T_n(z, z_0, z_1, \cdots, z_n)$. First, let us define the new function $H_n$ such that

$$\begin{aligned} H_n(z_0, z_1, \cdots, z_{n-1}) &= T_n(0, z_0, \cdots, z_{n-1}) \\ \frac{\partial H_n}{\partial z_0} &= -T_{n-1}(z_0, z_1. \cdots, z_{n-1}) \\ \frac{\partial^2 H_n}{\partial z_0^2} &= -T_{n-2}(z_0, z_2. \cdots, z_{n-1}) \end{aligned} \tag{6.5}$$

Ref

9. N. Levinson, A Theorem of Boars, Duke Math. J. 8(1941) 181-182.

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Thus, by Taylor's theorem, we get

$$\begin{aligned} H_n(z_0, z_1, \cdots, z_{n-1}) &= -z_0 H_{n-1}(z_1, \cdots, z_{n-1}) - \frac{z_0^2}{2!} H_{n-2}(z_2, \cdots, z_{n-1}) \\ &\quad - \cdots - \frac{z_0^{n-1}}{(n-1)!} H_1(z_{n-1}). \end{aligned} \tag{6.6}$$

so that

$$\begin{aligned} T_n(z, z_0, z_1, \cdots, z_{n-1}) &= (z - z_0) H_{n-1}(z_1, z_2, \cdots, z_{n-1}) + \frac{(z-z_0)^2}{2!} H_{n-2}(z_2, \cdots z_{n-1}) \\ &\quad + \tfrac{(z-z_0)^3}{3!} H_{n-3}(z_3, z_4, \cdots, z_{n-1}) + \cdots + \frac{(z-z_0)^n}{n!} H_0, \end{aligned} \tag{6.7}$$

with

$$H_1(z) = -z_0, \quad H_2(z_0, z_1) = z_0 z_1 - \frac{z_0^2}{2!}, \quad H_3(z_0, z_1, z_2) = -z_0 z_1 + \frac{z_0 z_1^2}{2!} + \frac{z_0^2 z_1}{2!} - \frac{z_0^3}{3!}.$$

Now, as $|z| \leq 1$, assuming $z = -1, z_0 = 1$ and using (6.4), we obtain

$$M_1 \leq |1 - (-1)| = 2.$$

Similarly

$$M_2 \leq \left(\frac{2|z - z_0|}{2}\right), H_1(z_0) = 2 \sin\theta.1 \equiv f(\theta)$$

which attains its maximum at $\theta = \pi/2$. Hence $M_2 \leq 2$. By the same way, we find $M_3 \leq 3, M_4 \leq 5.06759$, etc. so that by (6.4), we get

$$M_n \leq 3M_{n-1} + 4M_{n-2} + 4M_{n-3} + 6M_{n-4} + 10.13518 M_{n-5}.$$

Then, assuming the existence of a number $r_0$ such that $M_k < r_0^{k+1}$, we obtain

$$M_n \leq 3r^n + 4r^{n-1} + 4r^{n-2} + 6r^{n-3} + 10.13518 r^{n-4} \cdots \tag{6.8}$$

As the right side of (6.8) is not greater than $nr^{n+1}$, for $r_0 < r$, we have

$$nr^{n+1} > M_n$$

so that for $n = 10$

$$10r^{11} > 3r^{10} + 4r^9 + 4r^8 + 6r^7 + 10.13518 r^6 \cdots \tag{6.9}$$

Dividing by $r^6$, (6.9) gives four complex roots with one real root stated below

$$\begin{aligned} &(i)\ -0.75884 - 0.49888895i; \quad (ii)\ -0.75884 + 0.49888895i; \\ &(iii)\ 0.231357 - 0.923815i; \quad (iv)\ 0.231357 + 0.923815i; \\ &(v)\ 1.35497 \cong 1.3550. \end{aligned} \tag{6.10}$$

The inverse of the real root gives the Whittaker's constant $W$ such that

$$W > \frac{1}{r} = \frac{1}{1.3550} = 0.7380. \tag{6.11}$$

By Newton Ralpson iteration there exists $r^*$ such that

$$1.354967 \leq r^* \leq 1.35542 \quad \text{and} \quad \frac{1}{1.354967} \geq \frac{1}{r^*} \geq \frac{1}{1.35542}$$

which gives

$$0.7380 \geq W \geq 0.73778.$$

So taking into account (6.11), we obtain the Whittaker constant $W = 0.7380$ which improves the earlier known result by Levinson [9] and Macintyre [6]. So, the new range of the Whittaker's constant becomes

$$0.73778 \leq W \leq 0.7380. \tag{6.12}$$

*Theorem 6.2 Let $\{T_n(z)\}$ be the set of Chebychev polynomials associated with the points $\alpha\beta^n$ where $\alpha$ and $\beta$ are complex numbers. If $\beta > 1$, the set will be of infinite order and if $\beta = 1$, the set will be of order 1 and type $\sigma = \frac{|\alpha|}{\tau}$ where $\tau$ is the modulo of a zero of the function*

$$f(\theta) = \sum_{t=0}^{\infty} \beta^{t\pi i/2} \theta^t / t!$$

*of the least modulo.*

*Proof:* The Chebychev polynomial of the first kind can be written as

$$f(z) = \cos(n \cos^{-1}(z)).$$

Let $\cos^{-1}(z) = \theta$ so that

$$f(\theta) = \cos(n\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}$$

or equivalently, putting $t = 2n$ and replacing the exponential function $e$ by $\beta$,

$$f(\theta) = \sum_{t=0}^{\infty} \beta^{\pi i t/2} \frac{\theta^t}{t!}.$$

Hence, the Chebychev polynomial is of order 1. Suppose $f(\theta)$ has no zero on the finite plane $-1 \leq \theta \leq 1$. Then $f(\theta)$ is constant. It follows that $\tau$ is the modulo of a zero of $f(\theta)$ nearest to the origin. Thus there should exist a function $h(\theta) = \frac{1}{f(\theta)}$ regular on the $|\theta| < \tau$, where

$$h(\theta) = \sum_{t=0}^{\infty} \beta^{-\pi i t/2} t! \theta^- t,$$

which can be written as

$$h(\theta) = \sum_{t=0}^{\infty} c_t \theta^{-r}, \tag{6.13}$$

Ref

6. S.A. Macintyre, An upper bound for the Whittaker's constant, London Maths. Soc. J. 22 (1947), 305-311.

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with $c_t = t!\beta^{-\pi i t/2}$. We assume that

$$\lim_{t\to\infty} \{c_t\}^{\frac{1}{t}} = \frac{1}{\tau} > 0.$$

Suppose now there exist a number $q < \dfrac{1}{\tau}$ and a positive integer $m$ such that

$$c_t > q^m; 0 \leq m \leq t. \tag{6.14}$$

The Cannon function $\lambda(h(\theta) : [r])$ is defined as

$$\lambda(h(\theta) : [r]) = \overline{\lim_{t\to\infty}} \{W_t(h(\theta) : [r])\}^{\frac{1}{t}},$$

where $W_t(h(\theta) : [r])$ is the Cannon sum expressed as

$$W_t(h(\theta) : [r]) \;=\; \sum_{t=0}^{\infty} \left| t!\beta^{-t\pi i/2} \right| M_t(\theta) = \sum_{t=0}^{\infty} |c_t| \, M_t(\theta),$$

where

$$M_t(\theta) = \max_{|\theta|=r} |\theta^t|.$$

Hence

$$W_t(h(\theta) : [r]) \geq |q^m| \, r^{-t},$$

that ensures that the order of the set $\{T_n(z)\}_{n\geq 0}$ is infinite in the case $\beta > 1$.
Now, let us examine the case $\beta = 1$. Then, by similar arguments, an integral function defined by

$$f(\theta) = \sum_{t=0}^{\infty} \frac{\theta^t}{t!}$$

is of order 1.
By analogy, following step by step the previous development, the case $|\alpha| = 1$ and $\beta \to 1$ as $n \to \infty$ leads to the same conclusion as above. Therefore, the order and type of the Chebychev set of polynomials are well defined.
To complete the full analysis regarding the better approximation of the lower and upper bounds of the Whittaker's constant, we achieve the computation of range on which lies this constant for all classical commonly used polynomials in Mathematical Physics i.e. Laguerre, Jacobi and Bessel polynomials. These sets of polynomials can be categorised into two groups: A-basic set and B-basic set. A-basic set contains those polynomials of the form $\sum_{k=0}^{n} a_k z^k$, (e.g. monomials ($z^n$), Laguerre polynomials $\left(\sum_{k=0}^{n} \dfrac{(-n)_k z^k}{(k!)^2}\right)$, etc), while B-basic set is the set of the polynomials of the form $\sum_{k=0}^{[n/2]} a_k z^k$ (such as Chebychev polynomials of the first kind, Hermite, Legendre Jacobi, Gaugenbauer polynomials, etc). Hence, the polynomials belonging to B-basic set have a better Whittaker's constant than those polynomials of A-basic set. See data provided in the table.

| Polynomials | Range of Whittaker's constant | Order | Type |
|---|---|---|---|
| A-basic set | $0.7220 \leq W \leq 0.7378$ | 1 if of the form $\sum_{t=0}^{\infty} \frac{\beta^{\pi it/2}\theta^t}{t!}$ or otherwise $\infty$. | $1.35538 \leq \sigma \leq 1.3589$ |
| B-basic set | $0.7230 \leq W \leq 0.7380$ | 1 if ultraspherical or the form $\sum_{t=0}^{\infty} \frac{\beta^{\pi it/2}\theta^t}{t!}$ or otherwise $\infty$ | $1.35542 \leq \sigma \leq 1.38408$ |

## VI. Concluding Remarks

In this paper, we have shown that the set of Chebychev polynomials of the first kind $\{T_n(z)\}_{n \geq 0}$ forms a basic sets and provides the best improved Whittaker's constant than other classical basic sets. The classical basic sets can be splited into two classes denoted, respectively, A-basic sets and B-basic sets of polynomials. In general, the Whittaker's constant is best improved upon by the B-basic sets.

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# Note on Intuitionistic Fuzzy (Normal) Subgroups or Vague (Normal) Subgroups

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*Keywords: intutionistic fuzzy or vague subset, intutionistic fuzzy/vague sub (normal) group.*

*GJSFR-F Classification : FOR Code : MSC 2010: 03F55, 06C05, 16D25.*

NOTEONINTUITIONISTICFUZZYNORMALSUBGROUPSORVAGUENORMALSUBGROUPS

*Strictly as per the compliance and regulations of :*

# Note on Intuitionistic Fuzzy (Normal) Subgroups or Vague (Normal) Subgroups

K. Lakshmi [α] & Dr. G. Vasanti [σ]

*Abstract-* The aim of this paper is basically to study some of the standard properties of the intuitionistic fuzzy subgroups under a crisp map. Also, we study some properties of intuitionistic fuzzy normal subgroups.

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## I. Introduction

Zadeh, in his pioneering paper, introduced the notion of Fuzzy Subset of a set $X$ as a function $\mu$ from $X$ to the closed interval [0,1] of real numbers. The function $\mu$, he called, the membership function which assigns to each memebr $x$ of $X$ its membership value, $\mu x$ in $[0, 1]$.

In 1983, Atanassov[1] generalized the notion of Zadeh fuzzy subset of a set further by introducing an additional function $\nu$ which he called a nonmembership function with some natural conditions on $\mu$ and $\nu$, calling these new generalized fuzzy subsets of a set, intutionistic fuzzy subsets. Thus according to him an intutionistic fuzzy subset of a set $X$, is a pair $A = (\mu_A, \nu_A)$, where $\mu_A, \nu_A$ are functions from the set $X$ to the closed interval $[0, 1]$ of real numbers such that for each $x \in X$, $\mu x + \nu x \leq 1$, where $\mu_A$ is called the membership function of $A$ and $\nu_A$ is called the nonmembership function of $A$.

Later on in 1984, Atanassov and Stoeva[3], further generalized the notion intuitionistic fuzzy subset to L-intuitionistic fuzzy subset, where L is any complete lattice with a complete order reversing involution N. Thus an L-intutionistic fuzzy subset $A$ of a set $X$, is a pair $(\mu_A, \nu_A)$ where $\mu_A, \nu_A$: $X \to L$ are such that $\mu_A \leq N\nu_A$. Let us recall that a complete order reversing involution is a map $N$: $L \to L$ such that (1) $N0_L = 1_L$ and $N1_L = 0_L$ (2) $\alpha \leq \beta$ implies $N\beta \leq N\alpha$ (3) $NN\alpha = \alpha$ (4) $N(\vee_{i\in I}\alpha_i) = \wedge_{i\in I}N\alpha_i$ and $N(\wedge_{i\in I}\alpha_i) = \vee_{i\in I}N\alpha_i$.

Interestingly the same notion of intutionistic fuzzy subset of set was also introduced by Gau and Buehrer[6] in 1993 under a different name called Vague subset. Thus whether we called intutionistic fuzzy subset of a set or if-subset of a set for short, or vague subset of a set, they are one and the same.

*In order to make the document more readable, hereonwards we use the phrase if-subset for intuitionistic fuzzy or vague subset of a set. Obviously, if/v-subset only means intuitionistic fuzzy/vague subset, if/v-(normal)subgroup only means intuitionistic fuzzy/vague (normal) subgroup.*

Coming to generalizations of algebraic structures on to the intuitionitic fuzzy/vague sets:

as early as 1989, Biswas[7] introduced the notion of if/v-subgroup of a group and studied some properties of the same.

Ref

1. Atanassov, K., Intuitionistic fuzzy sets, in: V. Sgurev, Ed., VII ITKR's Session, Sofia, June 1983 (Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

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In 2004, Hur-Jang-Kang[15] introduced and studied if/v-normal subgroup of a group and Hur etal.[10,11,16] continued their studies of the same. In Hur etal.[16], they established a one-one correspondence between, if/v-normal subgroups and if/v-congrunces.

In 2003, Banergee-Basnet[6] introduced and studied the notions of if/v-subrings and if/v-ideals of a ring. The same year Hur-Jang-Kang[10] introduced and studied the notion if/v-subring of a ring. In Hur etal.[17,18] continued their studies of if/v-ideals. In Hur etal.[18], they introduced and studied the notions of if/v-prime ideals, if/v-completely prime ideals and if/v-weakly completely prime ideals.

Coming back to the studies of intuitionistic fuzzy/vague subgroups of a group, Feng[8] and Palaniappan etal.[22] initiated the study intuitionistic L-fuzzy/L-vague subgroups of a group.

In this paper we studied some propertities of intuitionistic fuzzy subgrups and intuitionistic fuzzy normal subgrups of an intuitionistic fuzzy subset.

For any set $X$, the set of all if/v-subsets of $X$ be denoted by $A(X)$. By defining, for any pair of if/v-subsets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ of $X$, $A \leq B$ iff $\mu_A \leq \mu_B$ and $\nu_B \leq \nu_A$, $A(X)$ becomes a complete infinitely distributive lattice. In this case for any family $(A_i)_{i\in I}$ of if/v-subsets of $X$, $(\vee_{i\in I}A_i)x = \vee_{i\in I}A_i x$ and $(\wedge_{i\in I}A_i)x = \wedge_{i\in I}A_i x$.

For any set $X$, one can naturally associate, with $X$, the if/v-subset $(\mu_X, \nu_X) = (1_X, 0_X)$, where $1_X$ is the constant map assuming the value 1 for each $x \in X$ and $0_X$ is the constant map assuming the value 0 for each $x \in X$, which turns out to be the largest element in $A(X)$. Observe that then, the if/v-empty subset $\phi$ of $X$ gets naturally associated with the if/v-subset $(\mu_\phi, \nu_\phi) = (0_X, 1_X)$, which turns out to be the least element in $A(X)$.

Let $A = (\mu_A, \nu_A)$ be an if/v-subset of $X$. Then the if/v-complement of $A$, denoted by $A^c$ is defined by $(\nu_A, \mu_A)$. Observe that $A^c = X - A = X \wedge A^c$.

Throughout this paper the capital letters $X$, $Y$ $Z$ stand for arbitrary but fixed (crisp) sets, the small letters $f$, $g$ stand for arbitrary but fixed (crisp) maps $f : X \to Y$ and $g : Y \to Z$, the capital letters $A$, $B$, $C$, $D$, $E$, $F$ together with their suffixes stand for if/v-subsets and the capital letters $I$ and $J$ stand for the index sets. Ingeneral whenever $P$ is an if-subset of a set $X$, *always* $\mu_P$ and $\nu_P$ denote the membership and nonmembership function of the if-subset $P$ respectively. Also we frequently use the standard convention that $\vee\phi = 0$ and $\wedge\phi = 1$.

## II. Intuitionistic Fuzzy/Vague-subgroups

In this section, first we give some definitions and statements. In the Lemma that follows this, we give equivalent statements which are quite frequently used in several prepositions later on without an explicit mention. Then analogues of some crisp theoretic results are established. In the end, Lagranges theorem is generalized to fuzzy setup.

**Definitions and Statements 2.1** *(a) Let $A$, $B$ be a pair of if/v-subsets of $G$. Let $C$ be defined by, $\mu_C x = \vee_{x=yz}\{\mu_A y \wedge \mu_B z\}$ and $\nu_C x = \wedge_{x=yz}\{\nu_A y \vee \nu_B z\}$, for each $x \in G$. Then the if/v-subset $C$ of $G$ is called the if/v-product of $A$ by $B$ and is denoted by $A \circ B$.*

*(b) For any if/v-subset $A$ of $G$, the if/v-inverse of $A$, denoted by $A^{-1}$, defined by $(\mu_{A^{-1}}, \nu_{A^{-1}})$ is in fact an if/v-subset of $G$, where for each $x \in G$ $\mu_{A^{-1}}(x) = \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1})$.*

*(c) For any $y \in G$ and for any pair $\alpha, \beta$ of [0, 1], the if/v-point of $G$, denoted by $y_{\alpha,\beta}$, is defined by the if/v-subset $y_{\alpha,\beta} = (\chi_y^\alpha, \chi_y^\beta)$ where $\chi_y^\alpha(x) = \alpha$, $\chi_y^\beta(x) = \beta$ when $x = y$ and $\chi_y^\alpha(x) = \chi_y^\beta(x) = 0$ when $x \neq y$.*

*(d) An if/v-subset $A$ of $G$ is called an if/v-subgroup of $G$ iff:*

*(1) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$, for each $x, y \in G$.*

*(2) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$, for each $x \in G$.*

Ref

6. Banerjee B. and Basnet D.Kr., Intuitionistic fuzzy subrings and ideals, Journal of Fuzzy Mathematics Vol.11(1)(2003), 139-155.

*(e) For any if/v-subgroup A of a group G,* $A_* = \{x \in G/\mu_A(x) = \mu_A(e)$ *and* $\nu_A(x) = \nu_A(e)\}$ *and* $A^* = \{x \in G/\mu_A(x) > 0$ *and* $\nu_A(x) < 1\}$.
*(f) For any if/v-subset A of G and for any* $\alpha, \beta \in [0,1]$, *the* $(\alpha, \beta)$*-level subset of A, denoted by* $A_{\alpha,\beta}$, *is defined by* $A_{\alpha,\beta} = \{g \in G/\mu_A g \geq \alpha, \nu_A g \leq \beta\}$.

The following Lemma, which provides alternative equivalent statements for some of the above definitions and statements, is quite useful and is frequently used without an explicit mention of it in several proofs in later chapters.

Notes

*Lemma 2.2* *Let* $A, B, (A_i)_{i\in I}$ *be if/v-subsets of a group G. Let* $\alpha = \vee\mu_A G$, $\beta = \wedge\nu_A G$, $y_{\alpha,\beta} = (\chi_y^\alpha, \chi_y^\beta)$. *Then the following are true:*

1. $(\mu_{A\circ B})(x) = \vee_{y\in G}(\mu_A(y) \wedge \mu_B(y^{-1}x)) = \vee_{y\in G}(\mu_A(y^{-1}) \wedge \mu_B(yx))$ $= \vee_{y\in G}(\mu_A(xy^{-1}) \wedge\mu_B(y)) = \vee_{y\in G}(\mu_A(xy) \wedge \mu_B(y^{-1}))$ *and* $(\nu_{A\circ B})(x) = \wedge_{y\in G}(\nu_A(y) \vee \nu_B(y^{-1}x)) = \wedge_{y\in G}(\nu_A(y^{-1}) \vee \nu_B(yx))$ $= \wedge_{y\in G}(\nu_A(x y^{-1}) \vee \nu_B(y)) = \wedge_{y\in G}(\nu_A(xy) \vee \nu_B(y^{-1}))$, *for each* $x \in G$. *In particular,* $(\mu_{A\circ B})(xy) = \vee_{z\in G}(\mu_A(xz) \wedge \mu_B(z^{-1}y)) = \vee_{z\in G}(\mu_A(xz^{-1}) \wedge\mu_B(zy))$ *and* $(\nu_{A\circ B})(xy) = \wedge_{z\in G}(\nu_A(xz) \vee \nu_B(z^{-1}y)) = \wedge_{z\in G}(\nu_A(xz^{-1}) \vee\nu_B(zy))$.
2. $A \circ (B \circ C) = (A \circ B) \circ C$.
3. $y_{\alpha,\beta} \circ A = (\chi_y^\alpha \circ \mu_A, \chi_y^\beta \circ \nu_A)$, $(\chi_y^\alpha \circ \mu_A)x = \mu_A(y^{-1}x)$ *and* $(\chi_y^\beta \circ \nu_A)x = \nu_A(y^{-1}x)$, *for each* $x, y \in G$.
*In particular* $e_{\alpha,\beta} \circ A = A$.
4. $A \circ y_{\alpha,\beta} = (\mu_A \circ \chi_y^\alpha, \nu_A \circ \chi_y^\beta)$, $(\mu_A \circ \chi_y^\alpha)x = \mu_A(xy^{-1})$ *and* $(\nu_A \circ \chi_y^\beta)x = \nu_A(xy^{-1})$, *for each* $x, y \in G$. *In particular* $A \circ e_{\alpha,\beta} = A$.
5. $(A^{-1})^{-1} = A$;
6. $A \leq A^{-1}$ *iff* $A^{-1} \leq A$ *iff* $A = A^{-1}$;
7. $A \leq B$ *iff* $A^{-1} \leq B^{-1}$;
8. $(\vee_{i\in I}A_i)^{-1} = \vee_{i\in I}A_i^{-1}$;
9. $(\wedge_{i\in I}A_i)^{-1} = \wedge_{i\in I}A_i^{-1}$;
10. $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$;
11. $g_{\alpha,\beta} \circ h_{\gamma,\delta} = (gh)_{\alpha\wedge\gamma,\beta\vee\delta}$.

*Proof*: (1): Since $G$ is a group and hence for each $x \in G$, $\{(a, b) \in G\times G/x = ab\} = \{(a, a^{-1}x) \in G \times G/a \in G\} = \{(a^{-1}, ax) \in G \times G/a \in G\} = \{(xb^{-1}, b) \in G \times G/b \in G\} = \{(xb, b^{-1}) \in G \times G/b \in G\}$, this assertion follows.
(2): $\mu_{A\circ(B\circ C)}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_{(B\circ C)}(y)) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge (\vee_{z\in G} (\mu_B(yz^{-1}) \wedge \mu_C z))) = \vee_{y\in G} \vee_{z\in G} (\mu_A(xy^{-1}) \wedge \mu_B(yz^{-1}) \wedge \mu_C(z))$ and $\mu_{(A\circ B)\circ C}(x) = \vee_{z\in G}(\mu_{(A\circ B)}(xz^{-1}) \wedge \mu_C z) = \vee_{z\in G}(\vee_{y\in G}(\mu_A(xy^{-1})\wedge\mu_B(yz^{-1})) \wedge\mu_C(z)) = \vee_{z\in G} \vee_{y\in G} (\mu_A(xy^{-1}) \wedge \mu_B(yz^{-1}) \wedge \mu_C(z))$, since $\alpha_i \wedge (\vee_{j\in J}\beta_j) = \vee_{j\in J}(\alpha_i \wedge \beta_j)$, when [0,1] is a complete infinite meet distributive lattice. Hence $\mu_{A\circ(B\circ C)}(x) = \mu_{(A\circ B)\circ C}(x)$.
Similarly, $\nu_{A\circ(B\circ C)}(x) = \wedge_{y\in G}(\nu_A(xy^{-1})\vee\nu_{(B\circ C)}(y)) = \wedge_{y\in G}(\nu_A(xy^{-1})\vee(\wedge_{z\in G} (\nu_B (yz^{-1}) \vee\nu_C z))) = \wedge_{y\in G}\wedge_{z\in G}(\nu_A(xy^{-1})\vee\nu_B(yz^{-1})\vee\nu_C(z)) = \nu_{(A\circ B)\circ C}(x)$, since $\alpha_i \vee (\wedge_{j\in J}\beta_j) = \wedge_{j\in J}(\alpha_i \vee \beta_j)$, when [0, 1] is a complete infinite join distributive lattice. Therefore $A \circ (B \circ C) = (A \circ B) \circ C$.
(3): $(\chi_y^\alpha \circ \mu_A)x = \vee_{x=ba}(\chi_y^\alpha(b) \wedge \mu_A(a)) = \vee_{b\in G}(\chi_y^\alpha(b) \wedge \mu_A(b^{-1}x)) = \alpha \wedge \mu_A(y^{-1}x) = (\vee\mu_A G) \wedge (\mu_A(y^{-1}x)) = \mu_A(y^{-1}x)$.
Similarly $(\chi_y^\beta \circ \nu_A)x = \wedge_{x=ba}(\chi_y^\beta(b) \vee \nu_A(a)) = \wedge_{b\in G}(\chi_y^\beta(b) \vee \nu_A(b^{-1}x)) = \beta \vee \nu_A(y^{-1}x) = (\wedge\nu_A G) \vee (\nu_A(y^{-1}x)) = \nu_A(y^{-1}x)$.
Letting $y = e$, $e_{\alpha,\beta} \circ A = A$.

(4): $(\mu_A \circ \chi_y^\alpha)x = \vee_{x=ab}(\mu_A(a) \wedge \chi_y^\alpha(b)) = \vee_{b\in G}(\mu_A(xb^{-1}) \wedge \chi_y^\alpha(b)) = \mu_A(xy^{-1}) \wedge \alpha = \mu_A(xy^{-1}) \wedge (\vee \mu_A G) = \mu_A(xy^{-1})$.
Similarly $(\nu_A \circ \chi_y^\beta)x = \wedge_{x=ab}(\nu_A(a) \vee \chi_y^\beta(b)) = \wedge_{b\in G}(\nu_A(xb^{-1}) \vee \chi_y^\beta(b)) = (\nu_A(xy^{-1})) \vee \beta = (\nu_A(xy^{-1})) \vee (\wedge \nu_A G) = \nu_A(xy^{-1})$.
Letting $y = e$, $A \circ e_{\alpha,\beta} = A$.
(5): For each $x \in G$, $\mu_{A^{-1}}(x) = \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1})$. $\mu_{(A^{-1})^{-1}}(x) = \mu_{A^{-1}}(x^{-1}) = \mu_A(x^{-1})^{-1} = \mu_A x$ and $\nu_{(A^{-1})^{-1}}(x) = \nu_{A^{-1}}(x^{-1}) = \nu_A(x^{-1})^{-1} = \nu_A x$. Hence $(A^{-1})^{-1} = A$.
(6): Let $A \le A^{-1}$. Then for each $x \in G$, $\mu_A(x) \le \mu_{A^{-1}}(x) = \mu_A(x^{-1})$ and $\nu_A(x) \ge \nu_{A^{-1}}(x) = \nu_A(x^{-1})$. Hence $\mu_{A^{-1}}(x^{-1}) = \mu_A x \le \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x^{-1}) = \nu_A x \ge \nu_A(x^{-1})$ implies $\mu_{A^{-1}} \le \mu_A$ and $\nu_{A^{-1}} \ge \nu_A$ or $A^{-1} \le A$. Thus $A \le A^{-1}$ implies $A^{-1} \le A$.
Similarly $A^{-1} \le A$ implies for each $x \in G$, $\mu_{A^{-1}}(x^{-1}) \le \mu_A(x^{-1})$ and $\nu_{A^{-1}}(x^{-1}) \ge \nu_A(x^{-1})$ which implies $\mu_A(x) = \mu_{A^{-1}}(x^{-1}) \le \mu_A(x^{-1}) = \mu_{A^{-1}}(x)$ and $\nu_A(x) = \nu_{A^{-1}}(x^{-1}) \ge \nu_A(x^{-1}) = \nu_{A^{-1}}(x)$. or $A \le A^{-1}$. Thus $A^{-1} \le A$ implies $A \le A^{-1}$. Now $A \le A^{-1}$ iff $A^{-1} \le A$ iff $A = A^{-1}$ is clear.
(7): $(\Rightarrow)$: Let $A \le B$. Then for each $x \in G$, $\mu_A(x^{-1}) \le \mu_B(x^{-1})$ and $\nu_A(x^{-1}) \ge \nu_B(x^{-1})$. Hence $\mu_{A^{-1}}(x) = \mu_A(x^{-1}) \le \mu_B(x^{-1}) = \mu_{B^{-1}}(x)$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1}) \ge \nu_B(x^{-1}) = \nu_{B^{-1}}(x)$ or $A^{-1} \le B^{-1}$.
$(\Leftarrow)$: Let $A^{-1} \le B^{-1}$. Then for each $x \in G$, $\mu_{A^{-1}}(x) \le \mu_{B^{-1}}(x)$ and $\nu_{A^{-1}}(x) \ge \nu_{B^{-1}}(x)$. Hence $\mu_A(x^{-1}) = \mu_{A^{-1}}(x) \le \mu_{B^{-1}}(x) = \mu_B(x^{-1})$ and $\nu_A(x^{-1}) = \nu_{A^{-1}}(x) \ge \nu_{B^{-1}}(x) = \nu_B(x^{-1})$ or $A \le B$.
(8): Let $A_i = (\mu_{A_i}, \nu_{A_i})$, $A_i^{-1} = (\mu_{A_i^{-1}}, \nu_{A_i^{-1}})$. Then for each $x \in G$, $(\vee_{i\in I}\mu_{A_i})^{-1}(x) = (\vee_{i\in I}\mu_{A_i})(x^{-1}) = \vee_{i\in I}\mu_{A_i}(x^{-1}) = \vee_{i\in I}\mu_{A_i^{-1}}(x) = (\vee_{i\in I}\mu_{A_i^{-1}})(x)$ and $(\wedge_{i\in I}\nu_{A_i})^{-1}(x) = (\wedge_{i\in I}\nu_{A_i})(x^{-1}) = \wedge_{i\in I}\nu_{A_i}\ (x^{-1}) = \wedge_{i\in I}\nu_{A_i^{-1}}(x) = (\wedge_{i\in I}\ \nu_{A_i^{-1}})(x)$.
Hence $(\vee_{i\in I}A_i)^{-1} = \vee_{i\in I}A_i^{-1}$.
(9): Let $A_i = (\mu_{A_i}, \nu_{A_i})$, $A_i^{-1} = (\mu_{A_i^{-1}}, \nu_{A_i^{-1}})$. Then for each $x \in G$, $(\wedge_{i\in I}\mu_{A_i})^{-1}(x) = (\wedge_{i\in I}\mu_{A_i})(x^{-1}) = \wedge_{i\in I}\mu_{A_i}(x^{-1}) = \wedge_{i\in I}\mu_{A_i^{-1}}(x) = (\wedge_{i\in I}\ \mu_{A_i^{-1}})(x)$ and $(\vee_{i\in I}\nu_{A_i})^{-1}(x) = (\vee_{i\in I}\nu_{A_i})(x^{-1}) = \vee_{i\in I}\nu_{A_i}\ (x^{-1}) = \vee_{i\in I}\nu_{A_i^{-1}}(x) = (\vee_{i\in I}\ \nu_{A_i^{-1}})(x)$.
Hence $(\wedge_{i\in I}A_i)^{-1} = \wedge_{i\in I}A_i^{-1}$.
(10): Let $(A\circ B)^{-1} = (\mu_{(A\circ B)^{-1}}, \nu_{(A\circ B)^{-1}})$, $B^{-1}\circ A^{-1} = (\mu_{B^{-1}\circ A^{-1}}, \nu_{B^{-1}\circ A^{-1}})$.
Then for each $x \in G$, $\mu_{(A\circ B)^{-1}}(x) = \mu_{A\circ B}(x^{-1}) = \vee_{y\in G}(\mu_A(x^{-1}y) \wedge \mu_B(y^{-1}))$ and $\nu_{(A\circ B)^{-1}}(x) = \nu_{A\circ B}(x^{-1}) = \wedge_{y\in G}\ (\nu_A(x^{-1}y) \vee \nu_B(y^{-1}))$.
On the other hand, $\mu_{(B^{-1}\circ A^{-1})}(x) = \vee_{y\in G}(\mu_{B^{-1}}(y) \wedge \mu_{A^{-1}}(y^{-1}x))$
$=\vee_{y\in G}(\mu_B\ (y^{-1}) \wedge \mu_A(y^{-1}x)^{-1}) = \vee_{y\in G}(\mu_B(y^{-1}) \wedge \mu_A(x^{-1}y))$
$= \vee_{y\in G}(\mu_A(x^{-1}y) \wedge \mu_B(y^{-1})) = \mu_{(A\circ B)}(x^{-1}) = \mu_{(A\circ B)^{-1}}(x)$ and
$(\nu_{B^{-1}\circ A^{-1}})(x) = \wedge_{y\in G}(\nu_{B^{-1}}(y) \vee \nu_{A^{-1}}(y^{-1}x)) = \wedge_{y\in G}(\nu_B(y^{-1}) \vee \nu_A(y^{-1}x)^{-1})$
$= \wedge_{y\in G}(\nu_B(y^{-1}) \vee \nu_A(x^{-1}y)) = \wedge_{y\in G}(\nu_A(x^{-1}y) \vee \nu_B(y^{-1})) = \nu_{(A\circ B)}(x^{-1}) = \nu_{(A\circ B)^{-1}}(x)$. Therefore $(A\circ B)^{-1} = B^{-1}\circ A^{-1}$.
(11): $(\chi_g^\alpha \circ \chi_h^\gamma)(x) = \vee_{z\in G}(\chi_g^\alpha(xz^{-1}) \wedge \chi_h^\gamma(z)) = \chi_g^\alpha(xh^{-1}) \wedge \gamma = \chi_{gh}^\alpha(x) \wedge \gamma = \alpha \wedge \gamma = \chi_{gh}^{\alpha\wedge\gamma}(x)$ where the third equality follows because $g = xh^{-1}$ or $gh = x$ and $\chi_g^\beta \circ \chi_h^\delta(x) = \wedge_{z\in G}(\chi_g^\beta(xz^{-1}) \vee \chi_h^\delta(z)) = \chi_g^\beta(xh^{-1}) \vee \delta = \chi_{gh}^\beta(x) \vee \delta = \beta \vee \delta = \chi_{gh}^{\beta\vee\delta}(x)$.
Hence $g_{\alpha,\beta} \circ h_{\gamma,\delta} = (gh)_{\alpha\wedge\gamma,\beta\vee\delta}$.

*Lemma 2.3 For any if/v-subset $A$ of a group $G$ such that $\mu_A(xy) \ge \mu_A(x) \wedge \mu_A(y)$, $\nu_A(xy) \le \nu_A(x) \vee \nu_A(y)$:*
*(1) $\mu_A(x^n) \ge \mu_A(x)$ (2) $\nu_A(x^n) \le \nu_A(x)$*
*for each $x \in G$ and $n \in N$.*

*Proof:* (1): $\mu_A(x^n) = \mu_A(x^{n-1}x) \ge \mu_A(x^{n-1}) \wedge \mu_A(x) \ge \mu_A(x) \wedge \mu_A(x)...\wedge \mu_A(x) = \mu_A(x)$ for each $x \in G$.
(2): $\nu_A(x^n) = \nu_A(x^{n-1}x) \le \nu_A(x^{n-1}) \vee \nu_A(x) \le \nu_A(x) \vee \nu_A(x)...\vee \nu_A(x) = \nu_A(x)$ for each $x \in G$.

Notes

Notes

*Lemma 2.4* *Whenever $A$ is an if/v-subgroup of a group $G$, for each $x \in G$, $\mu_A(x^{-1}) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(x)$.*

*Proof:* Let $A$ be an if/v-subgroup of $G$. Then for each $x \in G$, $\mu_A(x^{-1}) \geq \mu_A(x)$, $\nu_A(x^{-1}) \leq \nu_A(x)$. $\mu_A(x) = \mu_A((x^{-1})^{-1}) \geq \mu_A(x^{-1})$ and $\nu_A(x) = \nu_A((x^{-1})^{-1}) \leq \nu_A(x^{-1})$.
Hence $\mu_A(x^{-1}) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(x)$.

*Corollary 2.5* *For any if/v-subgroup $A$ of a group $G$, the following are true for each $x \in G$:*

1. *$\mu_A(e) \geq \mu_A(x)$ and $\nu_A(e) \leq \nu_A(x)$;*
2. *$\mu_{A\circ A} \geq \mu_A$ and $\nu_{A\circ A} \leq \nu_A$.*

*Proof:* (1): $\mu_A(e) = \mu_A(xx^{-1}) \geq \mu_A(x) \wedge \mu_A(x^{-1}) = \mu_A(x) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(e) = \nu_A(xx^{-1}) \leq \nu_A(x) \vee \nu_A(x^{-1}) = \nu_A(x) \vee \nu_A(x) = \nu_A(x)$.
(2): $\mu_{A\circ A}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_A(y)) \geq \mu_A(xe) \wedge \mu_A(e) \geq \mu_A(x)$ and $\nu_{A\circ A}(x) = \wedge_{y\in G}(\nu_A(xy^{-1}) \vee \nu_A(y)) \leq \nu_A(xe) \vee \nu_A(e) \leq \nu_A x$ for each $x \in G$.

*Lemma 2.6* *For any if/v-subset $A$ of a group $G$, $A$ is an if/v-subgroup iff $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$.*

*Proof:* ($\Rightarrow$): Suppose $A$ is an if/v-subgroup. Then by 2.4, $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y^{-1}) = \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y^{-1}) = \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$.
($\Leftarrow$): First, by hypothesis and 2.5(1), $\mu_A(x^{-1}) = \mu_A(ex^{-1}) \geq \mu_A(e) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(ex^{-1}) \leq \nu_A(e) \vee \nu_A(x) = \nu_A x$ for each $x \in G$.

Letting $x^{-1}$ inplace of $x$, $\mu_A(x) \geq \mu_A(x^{-1})$ and $\nu_A(x) \leq \nu_A(x^{-1})$ for each $x \in G$ or $\mu_A(x) = \mu_A(x^{-1})$ and $\nu_A(x) = \nu_A(x^{-1})$ for each $x \in G$.
Next, $\mu_A(xy) = \mu_A(x(y^{-1})^{-1}) \geq \mu_A(x) \wedge \mu_A(y^{-1}) = \mu_A(x) \wedge \mu_A(y)$.
Similarly $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. Therefore $A$ is an if/v-subgroup of $G$.

*Lemma 2.7* *For any if/v-subgroup $A$ of a group $G$,*

1. *$A_* = \{x \in G/\mu_A(x) = \mu_A(e), \nu_A(x) = \nu_A(e)\}$ is a subgroup of $G$;*
2. *$A^* = \{x \in G/\mu_A(x) > 0, \nu_A(x) < 1\}$ is a subgroup of $G$ whenever $L$ is strongly regular.*

*Proof:* (1): Let $x, y \in A_*$. Then $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(e)$, $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y) = \nu_A(e)$. By 2.5(1), $\mu_A(xy^{-1}) \leq \mu_A e$, $\nu_A(xy^{-1}) \geq \nu_A e$ for each $x, y \in G$. So, $\mu_A(xy^{-1}) = \mu_A e$ and $\nu_A(xy^{-1}) = \nu_A e$ or $xy^{-1} \in A_*$ implying $A_*$ is a subgroup of $G$.
(2): Since $L$ is strongly regular, by 2.1(f), for each $x, y \in A^*$, $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y) > 0$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y) < 1$ or $xy^{-1} \in A^*$ implying, $A^*$ is a subgroup of $G$.

*Lemma 2.8* *For any if/v-subset $A$ of a group $G$, $A$ is an if/v-subgroup of $G$ iff $A$ satisfies the following conditions:*
*(1) $\mu_{A\circ A} = \mu_A$ and $\nu_{A\circ A} = \nu_A$ or equivalently $A \circ A = A$.*
*(2) $\mu_{A^{-1}} = \mu_A$ and $\nu_{A^{-1}} = \nu_A$ or equivalently $A^{-1} = A$.*

*Proof:* ($\Rightarrow$): Let $A$ be an if/v-subgroup of $G$. Then for each $x, y \in G$, $\mu_A x = \mu_A(xy^{-1}y) \geq \mu_A(xy^{-1}) \wedge \mu_A(y)$, $\nu_A x = \nu_A(xy^{-1}y) \leq \nu_A(xy^{-1}) \vee \nu_A(y)$, $\mu_A(x^{-1}) = \mu_A(x)$ and $\nu_A(x^{-1}) = \nu_A(x)$ .
(1): $\mu_{A\circ A}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_A(y)) \leq \vee_{y\in G}\mu_A(x) = \mu_A(x)$ or $\mu_{A\circ A} \leq \mu_A$ and $(\nu_{A\circ A})(x) = \wedge_{y\in G}(\nu_A(xy^{-1}) \vee \nu_A(y)) \geq \wedge_{y\in G}\nu_A(x) = \nu_A(x)$ or $\nu_{A\circ A} \geq \nu_A$.
Now by 2.5(2), we get that $\mu_{A\circ A} = \mu_A$ and $\nu_{A\circ A} = \nu_A$.
(2): 2.4 implies for each $x \in G$, $\mu_{A^{-1}}(x) = \mu_A(x^{-1}) = \mu_A(x)$ or $\mu_A = \mu_{A^{-1}}$ and $\nu_{A^{-1}}(x) = \nu_A(x^{-1}) = \nu_A(x)$ or $\nu_{A^{-1}} = \nu_A$.

($\Leftarrow$): 2.2(1) and the facts that $\mu_{A^{-1}} = \mu_A$, $\nu_{A^{-1}} = \nu_A$, $\mu_{A\circ A} \leq \mu_A$, $\nu_{A\circ A} \geq \nu_A$ imply, for each $x \in G$ $\mu_A(xy^{-1}) \geq \mu_{A\circ A}(xy^{-1}) \geq \mu_A(xy^{-1}y) \wedge \mu_A(y^{-1}) = \mu_A x \wedge \mu_A y$ and $\nu_A(xy^{-1}) \leq \nu_{A\circ A}(xy^{-1}) \leq \nu_A(xy^{-1}y) \vee \nu_A(y^{-1}) = \nu_A x \vee \nu_A y$.

*Lemma 2.9 For any pair of if/v-subgroups $A$ and $B$ of a group $G$, $A \circ B$ is an if/v-subgroup of $G$ iff $A \circ B = B \circ A$.*

*Proof:* ($\Rightarrow$): Since $A$, $B$ and $A \circ B$ are if/v-subgroups of $G$, $A^{-1} = A$, $B^{-1} = B$, $A \circ B = (A \circ B)^{-1} = B^{-1} \circ A^{-1} = B \circ A$.
($\Leftarrow$): Let $A \circ B = B \circ A$. Then (a) $(A \circ B) \circ (A \circ B) = A \circ (B \circ A) \circ B = A \circ (A \circ B) \circ B = (A \circ A) \circ (B \circ B) = A \circ B$ and (b) $(A \circ B)^{-1} = (B \circ A)^{-1} = A^{-1} \circ B^{-1} = A \circ B$. By 2.8, $A \circ B$ is an if/v-subgroup of $G$.

*Lemma 2.10 For any pair of groups $G$ and $H$ and for any crisp homomorphism $f:G \rightarrow H$ the following are true:*

1. *$A$ is an if/v-subgroup of $G$ implies $f(A)$ is an if/v-subgroup of $H$, whenever [0, 1] is a complete infinite distributive lattice;*
2. *$B$ is an if/v-subgroup of $H$ implies $f^{-1}(B)$ is an if/v-subgroup of $G$.*

*Proof:* (1): Let $fA = B$. Then $\mu_B y = \vee \mu_A f^{-1} y$, $\nu_B y = \wedge \nu_A f^{-1} y$. Now we show that $\mu_B(xy^{-1}) \geq \mu_B(x) \wedge \mu_B(y)$ and $\nu_B(xy^{-1}) \leq \nu_B(x) \vee \nu_B(y)$. Let us recall that $\mu_B(x) = \vee \mu_A f^{-1} x = \vee_{a \in f^{-1}x} \mu_A a$, $\mu_B y = \vee \mu_A f^{-1} y = \vee_{b \in f^{-1}y} \mu_A b$ and $\nu_B(x) = \wedge \nu_A f^{-1} x = \wedge_{a \in f^{-1}x} \nu_A a$, $\nu_B y = \wedge \nu_A f^{-1} y = \wedge_{b \in f^{-1}y} \nu_A b$. If one of $f^{-1}x$ or $f^{-1}y$ is empty, we are done because $\vee \phi = 0_L$ and $\wedge \phi = 1_L$. So, let both of them be non-empty. $a \in f^{-1}x$, $b \in f^{-1}y$ imply $fa = x$, $fb = y$ which implies $fab^{-1} = fafb^{-1} = xy^{-1}$ which in turn implies $c = ab^{-1} \in f^{-1}(xy^{-1})$. Since $A$ is an if/v-subgroup of $G$, $\mu_B(xy^{-1}) = \vee_{c \in f^{-1}(xy^{-1})} \mu_A c \geq \mu_A(ab^{-1}) \geq \mu_A(a) \wedge \mu_A(b)$ and similarly $\nu_B(xy^{-1}) \leq \nu_A(a) \vee \nu_A(b)$ for each $a \in f^{-1}x$, $b \in f^{-1}y$.
Observe that in any complete infinite distributive lattice,
(1) $\gamma \geq \alpha \wedge \beta$ for each $\alpha \in M \subseteq [0, 1]$, for each $\beta \in N \subseteq [0, 1]$ implies $\gamma \geq (\vee_{\alpha \in M} \alpha) \wedge (\vee_{\beta \in N} \beta) = (\vee M) \wedge (\vee N)$,
(2) $\gamma \leq \alpha \vee \beta$ for each $\alpha \in M \subseteq [0, 1]$, for each $\beta \in N \subseteq [0, 1]$ implies $\gamma \leq (\wedge_{\alpha \in M} \alpha) \vee (\wedge_{\beta \in N} \beta) = (\wedge M) \vee (\wedge N)$.
So, we will get that $\mu_B(xy^{-1}) \geq \mu_B x \wedge \mu_B y$ and $\nu_B(xy^{-1}) \leq \nu_B x \vee \nu_B y$.
Hence $fA = B$ is an if/v-subgroup of $G$.

(2): Let $f^{-1}B = A$. Then $\mu_A x = \mu_B fx$, $\nu_A x = \nu_B fx$. Now we show that $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$ .
Since $f$ is a homomorphism and $B$ is an if/v-subgroup of $H$, $\mu_A(xy^{-1}) = \mu_B f(xy^{-1}) = \mu_B (fx)(fy)^{-1} \geq \mu_B fx \wedge \mu_B fy = \mu_A x \wedge \mu_A y$ and
$\nu_A(xy^{-1}) = \nu_B f(xy^{-1}) = \nu_B (fx)(fy)^{-1} \leq \nu_B fx \vee \nu_B fy = \nu_A x \vee \nu_A y$.
Hence $f^{-1}B = A$ is an if/v-subgroup of $G$.

*Lemma 2.11 For any family of if/v-subgroups $(A_i)_{i \in I}$ of a group $G$, $\wedge_{i \in I} A_i$ is an if/v-subgroup of $G$.*

*Proof:* Let $C = \wedge_{i \in I} A_i$. Then $\mu_C = \wedge_{i \in I} \mu_{A_i}$, $\nu_C = \vee_{i \in I} \nu_{A_i}$. Now we show that, $\mu_C(xy^{-1}) \geq \mu_C(x) \wedge \mu_C(y)$ and $\nu_C(xy^{-1}) \leq \nu_C(x) \vee \nu_C(y)$.
Let us recall that in any complete lattice,
(1) $\wedge_{i \in I}(\alpha_i \wedge \beta_i) = (\wedge_{i \in I} \alpha_i) \wedge (\wedge_{i \in I} \beta_i)$
(2) $\vee_{i \in I}(\alpha_i \vee \beta_i) = (\vee_{i \in I} \alpha_i) \vee (\vee_{i \in I} \beta_i)$
(3) $\alpha_i \leq \beta_i$ for each $i \in I$ implies $\wedge_{i \in I}\ \alpha_i \leq \wedge_{i \in I}\ \beta_i$
(4) $\alpha_i \leq \beta_i$ for each $i \in I$ implies $\vee_{i \in I}\ \alpha_i \leq \vee_{i \in I}\ \beta_i$
Now the above and $A_i$ is an if/v-subgroup of $G$ imply, $\mu_C(xy^{-1}) = (\wedge_{i \in I} \mu_{A_i})(xy^{-1}) = \wedge_{i \in I} \mu_{A_i}(xy^{-1}) \geq \wedge_{i \in I}(\mu_{A_i}(x) \wedge \mu_{A_i}(y)) = (\wedge_{i \in I} \mu_{A_i} x) \wedge (\wedge_{i \in I} \mu_{A_i} y) = \mu_C x \wedge \mu_C y$ and $\nu_C(xy^{-1}) = (\vee_{i \in I} \nu_{A_i})(xy^{-1}) = \vee_{i \in I} \nu_{A_i}(xy^{-1}) \leq \vee_{i \in I}(\nu_{A_i}(x) \vee \nu_{A_i}(y)) = (\vee_{i \in I} \nu_{A_i} x) \vee (\vee_{i \in I} \nu_{A_i} y) = \nu_C x \vee \nu_C y$. Hence $C = \wedge_{i \in I} A_i$ is an if/v-subgroup of $G$.

Notes

It may so happen that the $\wedge_{i\in I}A_i$ may be the empty if/v-subset which is trivially an if/v-subgroup of $G$ as shown in the following example:

*Example 2.12* $A_n = (\frac{1}{n}, 1-\frac{1}{n})$, $\wedge_{n=1}^{\infty}A_n = (0,1) = \phi$, *the empty subgroup of* $G$.

The $A \vee B$ of if/v-subgroups $A$, $B$ of a group $G$ need *not* be an if/v-subgroup as shown in the following example:

Notes

*Example 2.13 Let* $A = (\chi_{2z}, 1-\chi_{2z})$, $B = (\chi_{3z}, 1-\chi_{3z})$ *be the* $I$*-if/v-subgroups of* $Z$*, the additive group of integers, where* $I$ = *[0,1], the closed interval of real numbers. Then* $A \vee B = (\chi_{2z} \vee \chi_{3z}, (1-\chi_{2z}) \wedge (1-\chi_{3z}))$ *and* $\mu_{A\vee B}(5) = (\chi_{2z} \vee \chi_{3z})5 = 0 \vee 0 = 0$.
*If* $A \vee B$ *is an if/v-subgroup of* $G$*, then* $0 = \mu_{A\vee B}(3+2) \geq \mu_{A\vee B}(3) \wedge \mu_{A\vee B}(2) = (\chi_{2z} \vee \chi_{3z})3 \wedge (\chi_{2z} \vee \chi_{3z})2 = (0 \vee 1) \wedge (1 \vee 0) = 1 \wedge 1 = 1$*, a contradiction. So* $A \vee B$ *is* not *an if/v-subgroup of* $G$.

*Lemma 2.14 For any family of if/v-subgroups* $(A_i)_{i\in I}$ *of* $G$*,* $\vee_{i\in I}A_i$ *is an if/v-subgroup of* $G$ *whenever* $(A_i)_{i\in I}$ *is a sup/inf assuming chain of if/v-subgroups.*

*Proof:* Let $A = \vee_{i\in I}A_i$. Then $\mu_A = \vee_{i\in I}\mu_{A_i}$, $\nu_A = \wedge_{i\in I}\nu_{A_i}$. Now we show that $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$ for each $x, y \in G$. If one of $\mu_A x$ or $\mu_A(y) = 0$ and one of $\nu_A x$ or $\nu_A(y) = 1$ then anyway the inequalities hold good.

Let $\mu_A x$, $\mu_A y > 0$ and $\nu_A x$, $\nu_A y < 1$. Then $\vee_{i\in I}(\mu_{A_i}x)$, $\vee_{i\in I}(\mu_{A_i}y) > 0$ and $\wedge_{i\in I}(\nu_{A_i}x)$, $\wedge_{i\in I}(\nu_{A_i}y) < 1$. Then there exists $i_0 \in I$ such that $\mu_{A_{i_0}}x = \vee_{i\in I}\mu_{A_i}x$, $\nu_{A_{i_0}}x = \wedge_{i\in I}\nu_{A_i}x$ and there exists $j_0 \in I$ such that $\mu_{A_{j_0}}y = \vee_{i\in I}\mu_{A_i}y$, $\nu_{A_{j_0}}y = \wedge_{i\in I}\nu_{A_i}y$ because $(A_i)_{i\in I}$ is a sup/inf assuming chain.
Now (1) $A_{i_0} \leq A_{j_0}$ or (2) $A_{j_0} \leq A_{i_0}$ because $(A_i)_{i\in I}$ is a chain.
(1) Suppose $A_{i_0} \leq A_{j_0}$ or $\mu_{A_{i_0}} \leq \mu_{A_{j_0}}$ and $\nu_{A_{j_0}} \leq \nu_{A_{i_0}}$. Then $\mu_A(xy^{-1}) \geq \mu_{A_{j_0}}(xy^{-1}) \geq \mu_{A_{j_0}}x \wedge \mu_{A_{j_0}}y \geq \mu_{A_{i_0}}x \wedge \mu_{A_{j_0}}y = (\vee_{i\in I}\mu_{A_i}x) \wedge (\vee_{i\in I}\mu_{A_i}y) = \mu_A x \wedge \mu_A y$ and $\nu_A(xy^{-1}) \leq \nu_{A_{j_0}}(xy^{-1}) \leq \nu_{A_{j_0}}x \vee \nu_{A_{j_0}}y \leq \nu_{A_{i_0}}x \vee \nu_{A_{j_0}}y = (\wedge_{i\in I}\nu_{A_i}x) \vee (\wedge_{i\in I}\nu_{A_i}y) = \nu_A x \vee \nu_A y$.
2) Suppose $A_{j_0} \leq A_{i_0}$ or $\mu_{A_{j_0}} \leq \mu_{A_{i_0}}$ and $\nu_{A_{i_0}} \leq \nu_{A_{j_0}}$. Then $\mu_A(xy^{-1}) \geq \mu_{A_{i_0}}(xy^{-1}) \geq \mu_{A_{i_0}}x \wedge \mu_{A_{i_0}}y \geq \mu_{A_{i_0}}x \wedge \mu_{A_{j_0}}y = (\vee_{i\in I}\mu_{A_i x}) \wedge (\vee_{i\in I}\mu_{A_i}y) = \mu_A x \wedge \mu_A y$ and $\nu_A(xy^{-1}) \leq \nu_{A_{i_0}}(xy^{-1}) \leq \nu_{A_{i_0}}x \vee \nu_{A_{i_0}}y \leq \nu_{A_{i_0}}x \vee \nu_{A_{j_0}}y = (\wedge_{i\in I}\nu_{A_i}x) \vee (\wedge_{i\in I}\nu_{A_i}y) = \nu_A x \vee \nu_A y$.

## *If/V-Cosets And If/V-Index Of An If/V-Subgroup*

*Definitions 2.15 (1) For any if/v-subgroup* $A$ *of a group* $G$ *and for any* $g \in G$*, the if/v-subset* $gA = (\mu_{gA}, \nu_{gA})$ *of* $G$*, where* $\mu_{gA}$*,* $\nu_{gA} : G \to$ *[0, 1], are defined by* $\mu_{gA}x = \mu_A(g^{-1}x)$ *and* $\nu_{gA}x = \nu_A(g^{-1}x)$*, is called the if/v-left coset of* $A$ *by* $g$ *in* $G$*. The if/v-subset* $Ag = (\mu_{Ag}, \nu_{Ag})$ *of* $G$*, where* $\mu_{Ag}x = \mu_A(xg^{-1})$ *and* $\nu_{Ag}x = \nu_A(xg^{-1})$ *is called the if/v-right coset of* $A$ *by* $g$ *in* $G$.
*(2) The set of all if/v-left cosets of* $A$ *in* $G$ *is denoted by* $(G/A)_L$*. The set of all if/v-right cosets of* $A$ *in* $G$ *is denoted by* $(G/A)_R$.
*(3) (Later on we show, as in the crisp set up, that) The number of if/v-left cosets of* $A$ *in* $G$ *is the same as the number of if/v-right cosets of* $A$ *in* $G$ *and this common number, denoted by* $(G : A)$*, is called the if/v-index of* $A$ *in* $G$.

*Theorem 2.16 For any if/v-subgroup* $A$ *of a group* $G$ *and for any pair of elements* $g, h$ *of* $G$*, the following are true:*

1. $gA = g_{\mu_A e, \nu_A e} \circ A$ *and* $Ag = A \circ g_{\mu_A e, \nu_A e}$.
2. $gA = hA$ *iff* $gA_* = hA_*$.
3. $Ag = Ah$ *iff* $A_*g = A_*h$.

*Proof:* (1): From 2.2(3) and 2.15(1), $(\chi_g^{\mu_A e} \circ \mu_A)(x) = \mu_A(g^{-1}x) = \mu_{gA}x$ and $(\chi_g^{\nu_A e} \circ \nu_A)(x) = \nu_A(g^{-1}x) = \nu_{gA}x$ or $\mu_{gA} = \chi_g^{\mu_A e} \circ \mu_A$ and $\nu_{gA} = \chi_g^{\nu_A e} \circ \nu_A$.
Hence $gA = g_{\mu_A e, \nu_A e} \circ A$.
From 2.2(4) and 2.15(1), $(\mu_A \circ \chi_g^{\mu_A e})(x) = \mu_A(xg^{-1}) = \mu_{Ag}x$ and $(\nu_A \circ \chi_g^{\nu_A e})(x) = \nu_A(xg^{-1}) = \nu_{Ag}x$ or $\mu_{Ag} = \mu_A \circ \chi_g^{\mu_A e}$ and $\nu_{Ag} = \nu_A \circ \chi_g^{\nu_A e}$.
Hence $Ag = A \circ g_{\mu_A e, \nu_A e}$.
(2): ($\Rightarrow$): Suppose $gA = hA$. Then $\mu_{gA} = \mu_{hA}$ and $\nu_{gA} = \nu_{hA}$ or for each $x \in G$, $\mu_{gA}(x) = \mu_{hA}(x)$ and $\nu_{gA}x = \nu_{hA}x$ which implies $\mu_A(g^{-1}x) = \mu_A(h^{-1}x)$ and $\nu_A(g^{-1}x) = \nu_A(h^{-1}x)$.
Choosing $x = h$, $\mu_A(g^{-1}h) = \mu_A(h^{-1}h) = \mu_A(e)$ and $\nu_A(g^{-1}h) = \nu_A(h^{-1}h) = \nu_A(e)$ implying $g^{-1}h \in A_*$, where $A_* = \{x \in G/\mu_A(x) = \mu_A(e), \nu_A(x) = \nu_A(e)\}$.
Hence $gA_* = hA_*$.
($\Leftarrow$): From 2.7, $A$ is an if/v-subgroup of $G$ implies $A_*$ is a subgroup of $G$. Suppose $gA_* = hA_*$. Then $g^{-1}h \in A_*$ or $\mu_A(g^{-1}h) = \mu_A(e)$, $\nu_A(g^{-1}h) = \nu_A(e)$. Hence for each $z \in G$, $\mu_A(g^{-1}z) = \mu_A(g^{-1}hh^{-1}z) \geq \mu_A(g^{-1}h) \wedge \mu_A(h^{-1}z) = \mu_A(e) \wedge \mu_A(h^{-1}z) = \mu_A(h^{-1}z)$ and $\nu_A(g^{-1}z) = \nu_A(g^{-1}hh^{-1}z) \leq \nu_A(g^{-1}h) \vee \nu_A(h^{-1}z) = \nu_A(e) \wedge \nu_A(h^{-1}z) = \nu_A(h^{-1}z)$ because, $\mu_A e$ is the largest of $\mu_A G$ and $\nu_A e$ is the least of $\nu_A G$.
Similarly, for each $z \in G$, $\mu_A(h^{-1}z) \geq \mu_A(g^{-1}z)$ and $\nu_A(h^{-1}z) \leq \nu_A(g^{-1}z)$.
Hence for each $z \in G$, $\mu_A(g^{-1}z) = \mu_A(h^{-1}z)$, $\nu_A(g^{-1}z) = \nu_A(h^{-1}z)$ or $\mu_{gA}(z) = \mu_{hA}(z)$, $\nu_{gA}(z) = \nu_{hA}(z)$ for each $z$ or $\mu_{gA} = \mu_{hA}$, $\nu_{gA} = \nu_{hA}$ or $gA = hA$.
(3) ($\Rightarrow$): Suppose $Ag = Ah$. Then $\mu_{Ag} = \mu_{Ah}$, $\nu_{Ag} = \nu_{Ah}$ or for each $x \in G$, $\mu_{Ag}(x) = \mu_{Ah}(x)$ and $\nu_{Ag}x = \nu_{Ah}x$ which implies $\mu_A(xg^{-1}) = \mu_A(xh^{-1})$ and $\nu_A(xg^{-1}) = \nu_A(xh^{-1})$.
Choosing $x = h$, $\mu_A(hg^{-1}) = \mu_A(hh^{-1}) = \mu_A(e)$ and $\nu_A(hg^{-1}) = \nu_A(hh^{-1}) = \nu_A(e)$ implying $hg^{-1} \in A_*$ or $A_*g = A_*h$.
($\Leftarrow$): Suppose $A_*g = A_*h$. Then $hg^{-1} \in A_*$ or $\mu_A(hg^{-1}) = \mu_A(e)$ and $\nu_A(hg^{-1}) = \nu_A(e)$.
Hence for each $z \in G$, $\mu_A(zg^{-1}) = \mu_A(zh^{-1}hg^{-1}) \geq \mu_A(zh^{-1}) \wedge \mu_A(hg^{-1}) = \mu_A(zh^{-1}) \wedge \mu_A(e) = \mu_A(zh^{-1})$ and $\nu_A(zg^{-1}) = \nu_A(zh^{-1}hg^{-1}) \leq \nu_A(zh^{-1}) \vee \nu_A(hg^{-1}) = \nu_A(zh^{-1})$.
Similarly, for each $z \in G$, $\mu_A(zh^{-1}) \geq \mu_A(zg^{-1})$ and $\nu_A(zh^{-1}) \leq \nu_A(zg^{-1})$.
Hence for each $z \in G$, $\mu_A(zg^{-1}) = \mu_A(zh^{-1})$, $\nu_A(zh^{-1}) = \nu_A(zg^{-1})$ or $\mu_{Ag}(z) = \mu_{Ah}(z)$, $\nu_{Ag}(z) = \nu_{Ah}(z)$ for each $z$ or $Ag = Ah$.

Corollary 2.17 *For any if/v-subgroup $A$ of a group $G$, the following are true:*
*(1) The number of if/v-left(right) cosets of $A$ in $G$ is the same as the number of left(right) cosets of $A_*$ in $G$.*
*(2) $(G : A) = (G : A_*)$.*

*Proof*: (1): Let $\Im$ be the set of all if/v-left cosets of $A$ in $G$ and $\aleph$ be the set of all if/v-left cosets of $B$ in $G$. Define $\phi : \Im \to \aleph$ by $\phi(gA) = gA_*$. Then by 2.16(2), $\phi$ is both well defined and one-one. But clearly, $\phi$ is onto. Thus $\phi$ is a bijection implying our assertion.
(2): For any subgroup $H$ of a group $G$, the number of left coset of $H$ in $G$ is the same as the number of right coset of $H$ in $G$. Now the assertion follows from (1).

In the crisp set up, when $G$ is a finite group, for any subgroup $H$ of $G$, $|H| = \frac{|G|}{(G:H)}$. If one were to define the order for an if/v-subgroup of a finite group, the preceeding equation suggests that $|A| = \frac{|G|}{(G:A)}$. But $(G : A) = (G : A_*)$ and consequently $|A| = |A_*|$. Thus the definition of if/v-order of an if/v-subgroup is as follows:

Definition 2.18 *For any if/v-subgroup $A$ of a group $G$, the* order of A, *denoted by $|A|$, is defined to be the order of $A_*$ or $|A_*|$. In other words $|A| = |A_*|$.*

An if/v-subgroup $A$ of a group $G$ is *finite* or *infinite* according as its order $|A|$ is finite or infinite.

Notes

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*Lagranges Theorem*

*Theorem 2.19* *For any finite group $G$ and for any if/v-subgroup $A$, order of $A$, $|A|$ divides the order of $G$, $|G|$.*

## III. Intuitionistic Fuzzy/Vague-Normal Subgroups

In this section, we begin with equivalent conditions for if/v-normality for a subgroup and several of these conditions will be used in some subsequent results, sometimes, without an explicit mention. Later on we proceed to generalize various crisp theoretic results mentioned in the beginning of this chapter.

The following is a theorem which gives equivalent statements for an if/v-normal subgroup, some what similarly as in crisp set up.

*Theorem 3.1* *Let $A$ be an if/v-subgroup of $G$. Then the following are equivalent:*

1. $\mu_A(xy) = \mu_A(yx)$ *and* $\nu_A(xy) = \nu_A(yx)$ *for each* $x, y \in G$,
2. $\mu_A(xyx^{-1}) = \mu_A(y)$ *and* $\nu_A(xyx^{-1}) = \nu_A(y)$ *for each* $x, y \in G$,
3. $\mu_A[xy] \geq \mu_A x$ *and* $\nu_A[xy] \leq \nu_A x$ *for each* $x, y \in G$, *where* $[x, y] = x^{-1}y^{-1}xy$ *is the commutator of* $x$, $y$,
4. $\mu_A(xyx^{-1}) \geq \mu_A(y)$ *and* $\nu_A(xyx^{-1}) \leq \nu_A(y)$ *for each* $x, y \in G$,
5. $\mu_A(xyx^{-1}) \leq \mu_A(y)$ *and* $\nu_A(xyx^{-1}) \geq \nu_A(y)$ *for each* $x, y \in G$,
6. $A \circ B = B \circ A$ *for each if/v-subset* $B$ *of* $G$,
7. $Ag \circ Ah = Agh$, $gA \circ hA = ghA$, $Agh = ghA$ *and* $Ag \circ Ah = Ah \circ Ag$ *for each* $g, h \in G$,
8. $gA = Ag$ *for each* $g \in G$,
9. $A = g_{\mu_A e, \nu_A e} \circ A \circ g^{-1}_{\mu_A e, \nu_A e}$ *for each* $g \in G$.

*Proof*: Let $x, y \in G$.

(1)⇒(2): $\mu_A(xyx^{-1}) = \mu_A(x^{-1} \cdot xy) = \mu_A(y)$ and $\nu_A(xyx^{-1}) = \nu_A(x^{-1} \cdot xy) = \nu_A(y)$.

(2)⇒(3): $\mu_A(x^{-1}y^{-1}xy) = \mu_A(x^{-1}(y^{-1}xy)) \geq \mu_A(x^{-1}) \wedge \mu_A(y^{-1}xy) = \mu_A(x^{-1}) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(x^{-1}y^{-1}xy) = \nu_A(x^{-1}(y^{-1}xy)) \leq \nu_A(x^{-1}) \vee \nu_A(y^{-1}xy) = \nu_A(x^{-1}) \vee \nu_A(x) = \nu_A(x)$, by 2.4 and 2.6.

(3)⇒(4): $\mu_A(y^{-1}xy) = \mu_A(xx^{-1}y^{-1}xy) \geq \mu_A(x) \wedge \mu_A(x^{-1}y^{-1}xy) \geq \mu_A(x)$ and $\nu_A(y^{-1}xy) = \nu_A(xx^{-1}y^{-1}xy) \leq \nu_A(x) \vee \nu_A(x^{-1}y^{-1}xy) \leq \nu_A(x)$.

(4)⇒(5): $\mu_A(xyx^{-1}) \leq \mu_A(x^{-1} \cdot xyx^{-1} \cdot (x^{-1})^{-1}) = \mu_A(y)$ and $\nu_A(xyx^{-1}) \geq \nu_A(x^{-1} \cdot xyx^{-1} \cdot (x^{-1})^{-1}) = \nu_A(y)$.

(5)⇒(1): $\mu_A(xy) = \mu_A(xyxx^{-1}) = \mu_A(x \cdot yx \cdot x^{-1}) \leq \mu_A(yx)$ and $\mu_A(yx) = \mu_A(y \cdot xy \cdot y^{-1}) \leq \mu_A(xy)$, implying $\mu_A(xy) = \mu_A(yx)$.
$\nu_A(xy) = \nu_A(xyxx^{-1}) = \nu_A(x \cdot yx \cdot x^{-1}) \geq \nu_A(yx)$ and $\nu_A(yx) = \nu_A(y \cdot xy \cdot y^{-1}) \geq \nu_A(xy)$, implying $\nu_A(xy) = \nu_A(yx)$.

(1)⇒(6): $\mu_{A\circ B}(x) = \vee_{y\in G}(\mu_A(xy^{-1}) \wedge \mu_B(y)) = \vee_{y\in G}(\mu_A(y^{-1}x) \wedge \mu_B(y)) = \vee_{y\in G}(\mu_B(y) \wedge \mu_A(y^{-1}x)) = \mu_{B\circ A}(x)$ and $\nu_{A\circ B}(x) = \wedge_{y\in G}(\nu_A(xy^{-1}) \vee \nu_B(y)) = \wedge_{y\in G}(\nu_A(y^{-1}x) \vee \nu_B(y)) = \wedge_{y\in G}(\nu_B(y) \vee \nu_A(y^{-1}x)) = \nu_{B\circ A}(x)$, implying $A \circ B = B \circ A$.

(6)⇒(7): 2.16, 2.8, 2.2(11) imply $Ag \circ Ah = A \circ g_{\mu_A e, \nu_A e} \circ A \circ h_{\mu_A e, \nu_A e} = A \circ A \circ g_{\mu_A e, \nu_A e} \circ h_{\mu_A e, \nu_A e} = A \circ g_{\mu_A e, \nu_A e} \circ h_{\mu_A e, \nu_A e} = A \circ (gh)_{\mu_A e, \nu_A e} = Agh$. Similarly $gA \circ hA = ghA$.

Now letting $B = (gh)_{\mu_A e, \nu_A e}$, by the hypothesis, the above implies $Agh = ghA$. Again by hypothesis, $Ag \circ Ah = A \circ g_{\mu_A e, \nu_A e} \circ h_{\mu_A e, \nu_A e} = A \circ h_{\mu_A e, \nu_A e} \circ g_{\mu_A e, \nu_A e} = Ahg = Ah \circ Ag$.

(7)⇒(8): $h = e$ implies $Ag = gA$.

(8)⇒(9): By 2.16, 2.2(4) and 2.2(11), $g_{\mu_A e, \nu_A e} \circ A \circ g^{-1}_{\mu_A e, \nu_A e} = gA \circ g^{-1}_{\mu_A e, \nu_A e} = Ag \circ g^{-1}_{\mu_A e, \nu_A e} = A \circ g_{\mu_A e, \nu_A e} \circ g^{-1}_{\mu_A e, \nu_A e} = A \circ (gg^{-1})_{\mu_A e, \nu_A e} = A \circ (e)_{\mu_A e, \nu_A e} = Ae = A$.

(9)⇒(1): By 2.15, $\mu_A(xy) = \mu_A(y^{-1}yxy) = \mu_{yAy^{-1}}(yx) = \mu_A(yx)$ and $\nu_A(xy) = \nu_A(y^{-1}yxy) = \nu_{yAy^{-1}}(yx) = \nu_A(yx)$.

*Definition and Statements 3.2* *(1) For any if/v-subgroup $A$ of a group $G$, $A$ is an $L$-if/v-normal subgroup of $G$ iff it satisfies any one of the previous nine equivalent conditions. In particular, $A$ is an if/v-normal subgroup of $G$ iff for each $g \in G$, $Ag = gA$.*
*(2) The set of all if/v-cosets of $G$, denoted by $G/A$ or $\frac{G}{A}$, whenever $A$ is an if/v-normal subgroup of $G$, is called the if/v-quotient set of $G$ by $A$.*
*(3) Whenever $G$ is a finite group and $A$ is an if/v-normal subgroup of $G$, from the generalized Lagranges Theorem 2.19, $|(G/A)| = \frac{|G|}{|A|}$.*

*Proposition 3.3* *The following are true for any group $G$:*
*(a) If $G$ is abelian then every if/v-subgroup of $G$ is if/v-normal subgroup of $G$, but not conversely.*
*(b) For an if/v-subgroup $A$ of $G$ and for any $z \in G$, the if/v-subset $zAz^{-1} = (\mu_{zAz^{-1}}, \nu_{zAz^{-1}})$ where $\mu_{zAz^{-1}}x = \mu_A(z^{-1}xz)$ and $\nu_{zAz^{-1}}x = \nu_A(z^{-1}xz)$ for each $x \in G$, is an if/v-subgroup of $G$.*
*(c) For any if/v-subgroup $A$ of $G$, for each $z \in G$, $zAz^{-1} = z_{\mu_A e, \nu_A e} \circ A \circ z^{-1}_{\mu_A e, \nu_A e}$.*

*Proof:* (a): It follows from 3.1(1) and 3.2(1).
(b): Since $\mu_{zAz^{-1}}x = \mu_A(z^{-1}xz) \leq N\nu_A(z^{-1}xz) = N\nu_{zAz^{-1}}x$, it follows that $zAz^{-1}$ is an if/v-subset of $G$.
$\mu_{zAz^{-1}}(xy) = \mu_A(z^{-1}xyz) = \mu_A(z^{-1}xzz^{-1}yz) \geq \mu_A(z^{-1}xz) \wedge \mu_A(z^{-1}yz) = \mu_{zAz^{-1}}(x) \wedge \mu_{zAz^{-1}}(y)$ and $\nu_{zAz^{-1}}(xy) = \nu_A(z^{-1}xyz) = \nu_A(z^{-1}xzz^{-1}yz) \leq \nu_A(z^{-1}xz) \vee \nu_A(z^{-1}yz) = \nu_{zAz^{-1}}(x) \vee \nu_{zAz^{-1}}(y)$.
$\mu_{zAz^{-1}}(x) = \mu_A(z^{-1}xz) = \mu_A(z^{-1}x^{-1}z) = \mu_{zAz^{-1}}(x^{-1})$ and $\nu_{zAz^{-1}}(x) = \nu_A(z^{-1}xz) = \nu_A(z^{-1}x^{-1}z) = \nu_{zAz^{-1}}(x^{-1})$ for each $z \in G$.Hence $zAz^{-1}$ is an if/v-subgroup of $G$.
(c): It follows from 2.16(1).

*Definition 3.4* *For any pair of if/v subgroups $A$ and $B$ of a group $G$, $A$ is said to be an -if/v-conjugate of $B$ iff there exists $y \in G$ such that $A = yBy^{-1}$ or simply $A = B_y$.*
*It is easy to see that being conjugate to an arbitrary but fixed if/v-subgroup $A$, is an equivqlence relation on the set of all if/v-subgroups of $G$.*

*Theorem 3.5* *For any if/v-normal subgroup $A$ of $G$, the following are true:*
*(1) $A_* = \{x/\mu_A(x) = \mu_A(e),\ \nu_A(x) = \nu_A(e)\}$ is a normal subgroup of $G$.*
*(2) $A^* = \{x \in G/\mu_A(x) > 0,\ \nu_A(x) < 1\}$ is a normal subgroup of $G$, whenever $L$ is a strongly regular complete lattice.*

*Proof:* By 2.7, $A_*$ is subgroup of $G$ and $A^*$ is a subgroup of $G$ when $L$ is a strongly regular complete lattice. Since $A$ is an if/v-normal subgroup of $G$, by 3.1(2),

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(1): For each $y \in A_*$, $\mu_A(xyx^{-1}) = \mu_A y = \mu_A(e)$ and $\nu_A(xyx^{-1}) = \nu_A y = \nu_A(e)$ or $xyx^{-1} \in A_*$ or $A_*$ is a normal subgroup of $G$.
(2): For each $y \in A^*$, $\mu_A(xyx^{-1}) = \mu_A y > 0$ and $\nu_A(xyx^{-1}) = \nu_A(y) < 1$ or $xyx^{-1} \in A^*$ or $A^*$ is a normal subgroup of $G$.

## *If/V-Normalizer*

*Theorem 3.6* *For any if/v-subgroup $A$ of a group $G$, $N_G(A) = \{x \in G/\mu_A(xy) = \mu_A(yx),\ \nu_A(xy) = \nu_A(yx),$ for each $y \in G\}$ is a subgroup of $G$ and the restriction of $A$ to $N_G(A)$, denoted by $A|N_G(A)$, defined by $(\mu_A|N_G(A), \nu_A|N_G(A))$, is an if/v-normal subgroup of $N_G(A)$.*

*Proof:* Since $\mu_A(ey) = \mu_A(y) = \mu_A(ye)$ and $\nu_A(ey) = \nu_A(ye)$ for each $y \in G$, $e \in N_G(A)$.
Let $x, y \in N_G(A)$ and $z \in G$. Then $x \in N_G(A)$ implies $\mu_A(x \cdot y^{-1}z) = \mu_A(y^{-1}z \cdot x)$, $\nu_A(x \cdot y^{-1}z) = \nu_A(y^{-1}z \cdot x)$ and $y \in N_G(A)$ implies $\mu_A(x^{-1}z^{-1} \cdot y) = \mu_A(y \cdot x^{-1}z^{-1})$, $\nu_A(x^{-1}z^{-1} \cdot y) = \nu_A(y \cdot x^{-1}z^{-1})$.
From the above, $\mu_A(xy^{-1} \cdot z) = \mu_A(x \cdot y^{-1}z) = \mu_A(y^{-1}z \cdot x) = \mu_A((y^{-1}zx)^{-1}) = \mu_A(x^{-1}z^{-1} \cdot y) = \mu_A(y \cdot x^{-1}z^{-1}) = \mu_A((z \cdot xy^{-1})^{-1}) = \mu_A(z \cdot xy^{-1})$ and $\nu_A(xy^{-1} \cdot z) = \nu_A(x \cdot y^{-1}z) = \nu_A(y^{-1}z \cdot x) = \nu_A((y^{-1}zx)^{-1}) = \nu_A(x^{-1}z^{-1} \cdot y) = \nu_A(y \cdot x^{-1}z^{-1}) = \nu_A((z \cdot xy^{-1})^{-1}) = \nu_A(z \cdot xy^{-1})$.
Thus $xy^{-1} \in N_G(A)$ and $N_G(A)$ is a subgroup of $G$.
Now we show that $A|N_G(A)$ is an if/v-normal subgroup of $N_G(A)$.
But first $A|N_G(A)$ is an if/v-subgroup of $N_G(A)$ because for each $x, y \in N_G(A)$, $(\mu_A|N_G(A))(xy^{-1}) = \mu_A(xy^{-1}) \geq \mu_A x \wedge \mu_A y = (\mu_A|N_G(A))x \wedge (\mu_A|N_G(A))y$ and
$(\nu_A|N_G(A))(xy^{-1}) = \nu_A(xy^{-1}) \leq \nu_A x \vee \nu_A y = (\nu_A|N_G(A))x \vee (\nu_A|N_G(A))y$.
Next for each $x, y \in N_G(A)$,
$(\mu_A|N_G(A))(xy) = \mu_A(xy) = \mu_A(yx) = (\mu_A|N_G(A))(yx)$ and $(\nu_A|N(A))(xy) = \nu_A(xy) = \nu_A(yx) = (\nu_A|N_G(A))(yx)$ implying $A|N_G(A)$ is an if/v-normal subgroup of $N_G(A)$.

*Definition 3.7* *For any if/v-subgroup $A$ of a group $G$, the subgroup $N_G(A)$ of $G$ defined as above is called the normalizer of $A$ in $G$ and $A|N_G(A)$ is called the if/v-normalizer of $A$.*

*lemma 3.8* *For any if/v-subgroup $A$ of a group $G$, $A$ is an if/v-normal subgroup of $G$ iff $N_G(A) = G$.*

*Proof:* ($\Rightarrow$): Always $N_G(A) \subseteq G$. On the other hand, $x \in G$ implies for each $y \in G$, by 3.1(1), $\mu_A(xy) = \mu_A(yx)$ and $\nu_A(xy) = \nu_A(yx)$. So, $x \in N_G(A)$.
($\Leftarrow$): Again by 3.1(1), we get that $A$ is an if/v-normal subgroup of $G$.

*Theorem 3.9* *For any if/v-subgroup $B$ of a group $G$, the number of if/v-conjugates of $B$ in $G$ is equal to the index $(G : N_G(B))$ of the normalizer $N_G(B)$ in $G$.*

*Proof:* Let $u, v \in G$. Then $v^{-1}Gu = G$. Now $uBu^{-1} = vBv^{-1}$ iff for each $x \in G$, $\mu_B(u^{-1}xu) = \mu_B(v^{-1}xv)$ and $\nu_B(u^{-1}xu) = \nu_B(v^{-1}xv)$ iff (put $x = vxu^{-1}$) $\mu_B(u^{-1}v \cdot x) = \mu_B(x \cdot u^{-1}v)$ and $\nu_B(u^{-1}v \cdot x) = \nu_B(x \cdot u^{-1}v)$ iff $u^{-1}v \in N_G(B)$ iff $u^{-1}N_G(B) = v^{-1}N_G(B)$. Hence $B_u \to u^{-1}N_G(B)$ is a bijection from $\{uBu^{-1}/u \in G\}$ onto $\{uN_G(B)/u \in G\}$.

*Theorem 3.10* *For any if/v-subgroup $B$ of a group $G$, $\wedge_{u \in G} uBu^{-1}$ is an if/v-normal subgroup of $G$ and is the largest if/v-normal subgroup of $G$ that is contained in $B$.*

*Proof:* First observe that $uBu^{-1}$ is an if/v-subgroup of $G$ for each $u \in G$ by 6.1.3(b). So $\wedge_{u \in G} uBu^{-1}$ is an if/v-subgroup of $G$, by 2.11.
Since $\{uBu^{-1}/u \in G\} = \{(xu)B(xu)^{-1}/u \in G\}$ for each $x \in G$,
$\wedge_{u \in G} \mu_{uBu^{-1}}(x^{-1}yx) = \wedge_{u \in G} \mu_B(u^{-1}\ x^{-1}yx\ u) = \wedge_{u \in G} \mu_B((xu)^{-1}y(xu)) = \wedge_{u \in G}\ \mu_{(xu)B(xu)^{-1}}(y) = \wedge_{u \in G} \mu_{uBu^{-1}}(y)$ and

$\vee_{u\in G}\nu_{uBu^{-1}}(x^{-1}yx) = \vee_{u\in G}\nu_B(u^{-1}\ x^{-1}yx\ u) = \vee_{u\in G}\nu_B((xu)^{-1}y(xu))$
$= \vee_{u\in G}\nu_{(xu)B(xu)^{-1}}(y) = \vee_{u\in G}\nu_{uBu^{-1}}(y)$ for each $x, y \in G$.
Hence $\wedge_{u\in G}uBu^{-1}$ is an if/v-normal subgroup of $G$.
Next, let $A$ be an if/v-normal subgroup of $G$, with $A \leq B$. Since $A$ is an if/v-normal subgroup of $G$, $A = uAu^{-1}$ for each $u \in G$. Since $A \leq B$, $A = uAu^{-1} \leq uBu^{-1}$ for each $u \in G$ or $A \leq \wedge_{u\in G}uBu^{-1}$ or $\wedge_{u\in G}uBu^{-1}$ is the largest if/v-normal subgroup of $G$ that is contained in $B$.

*lemma 3.11 For any if/v-normal subgroup $A$ of a group $G$ and for any $x, y \in G$ such that $xA = yA$, $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.*

*Proof:* By 2.16(2), $xA = yA$ implies $xA_* = yA_*$ which implies $x^{-1}y \in A_*$ and $y^{-1}x \in A_*$ or $\mu_A(x^{-1}y) = \mu_A e = \mu_A(y^{-1}x)$ and $\nu_A(x^{-1}y) = \nu_A e = \nu_A(y^{-1}x)$. Since $A$ is an if/v-normal subgroup of $G$, $\mu_A(x) = \mu_A(y^{-1}xy) \geq \mu_A(y^{-1}x) \wedge \mu_A(y) = \mu_A(e) \wedge \mu_A(y) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y^{-1}xy) \leq \nu_A(y^{-1}x) \vee \nu_A(y) = \nu_A(e) \vee \nu_A(y) = \nu_A(y)$. Similarly, $\mu_A(y) = \mu_A(x^{-1}yx) \geq \mu_A(x^{-1}y) \wedge \mu_A(x) = \mu_A(e) \wedge \mu_A(x) = \mu_A(x)$ and $\nu_A(y) = \nu_A(x^{-1}yx) \leq \nu_A(x^{-1}y) \vee \nu_A(x) = \nu_A(e) \vee \nu_A(x) = \nu_A(x)$. Hence $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$.

*Theorem 3.12 For any if/v-normal subgroup $A$ of a group $G$. The following are true in $G/A$:*

1. *$(xA) \circ (yA) = (xy)A$ for each $x, y \in G$;*
2. *$(G/A, \circ)$ is a group;*
3. *$G/A \cong G/A_*$;*
4. *Let $A^{(*)}$ be an if/v-subset of $G/A$ be defined by $\mu_{A^{(*)}}(xA) = \mu_A(x)$ and $\nu_{A^{(*)}}(xA) = \nu_A(x)$ for each $x \in G$. Then $A^{(*)}$ is an if/v-normal subgroup of $G/A$.*

*Proof:* (1): Since $A$ is an if/v-normal subgroup, by 3.1(7), this follows.

(2): By (1), $G/A$ is closed under the operation $\circ$.
For each $x, y, z \in G$, $xA \circ (yA \circ zA) = xA \circ (yz)A = (xyz)A = (xy)A \circ zA = (xA \circ yA) \circ zA$. So $G/A$ is associative under the operation $\circ$.
By 2.2(3), $eA = A$. Further by (1), for each $x \in G$, $A \circ xA = eA \circ xA = exA = xA$ and $xA \circ A = xA \circ eA = xeA = xA$ or $A$ is the identity element for $G/A$.
$(x^{-1}A) \circ (xA) = (x^{-1}x)A = eA = A = (xA) \circ (x^{-1}A)$ or $x^{-1}A$ is the inverse of $xA$ in $G/A$. Hence $(G/A, \circ)$ is a group.
(3): Let $\eta : G/A \to G/A_*$, defined by $\eta(xA) = xA_*$. Then $\eta$ is well defined and 1-1 because $xA = yA$ iff $xA_* = yA_*$.
Now we show that $\eta$ is a homomorphism or $xyA_* = xA_*\ yA_*$. But by 3.5(1), $A_*$ is a normal subgroup of $G$ and so it follows that $\eta$ is a homomorphism.

Now we show that $\eta$ is onto. $\beta \in G/A_*$ implies $\beta = gA_*$, $g \in G$. Then $gA \in \frac{G}{A}$ such that $\eta(gA) = gA_* = \beta$ or $\eta$ is onto.
(4): First we show that $A^{(*)}$ is an if/v-subgroup of $G/A$.
Since $A$ be an if/v-subgroup of $G$,
(a): $\mu_{A^{(*)}}(gA \circ hA) = \mu_{A^{(*)}}(ghA) = \mu_A(gh) \geq \mu_A(g)\ \wedge\mu_A(h) = \mu_{A^{(*)}}(gA) \wedge\mu_{A^{(*)}}(hA)$ and $\nu_{A^{(*)}}(gA \circ hA) = \nu_{A^{(*)}}(ghA) = \nu_A(gh) \leq \nu_A(g)\vee\nu_A(h) = \nu_{A^{(*)}}(gA)\vee\nu_{A^{(*)}}(hA)$.
(b): $\mu_{A^{(*)}}((gA)^{-1}) = \mu_{A^{(*)}}(g^{-1}A) = \mu_A(g^{-1}) = \mu_A(g) = \mu_{A^{(*)}}(gA)$ and
$\nu_{A^{(*)}}((gA)^{-1}) = \nu_{A^{(*)}}(g^{-1}A) = \nu_A(g^{-1}) = \nu_A(g) = \nu_{A^{(*)}}(gA)$.
Therefore $A^{(*)}$ is an if/v-subgroup of $G/A$.
Now we show that $A^{(*)}$ is an if/v-normal subgroup of $G/A$.
Since $A$ is an if/v-normal subgroup of $G$, for each $g, h \in G$, $\mu_{A^{(*)}}((gA)^{-1} \circ (hA) \circ (gA)) = \mu_{A^{(*)}}(g^{-1}A \circ hA \circ gA) = \mu_{A^{(*)}}(g^{-1}hgA) = \mu_A(g^{-1}hg) \geq \mu_A(h) = \mu_{A^{(*)}}(hA)$ and $\nu_{A^{(*)}}((gA)^{-1} \circ (hA) \circ (gA)) = \nu_{A^{(*)}}(g^{-1}A \circ hA \circ gA) = \nu_{A^{(*)}}(g^{-1}hgA) = \nu_A(g^{-1}hg) \leq \nu_A(h) = \nu_{A^{(*)}}(hA)$. Hence $A^{(*)}$ is an if/v-normal subgroup of $G/A$.

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*Theorem 3.13* *For any if/v-subgroup $B$ of a group $G$ and for any normal subgroup $N$ of $G$, the if/v-subset $C$: $G/N \to L$ where for each $x \in G$, $\mu_C(xN) = \vee\mu_B(xN)$ and $\nu_C(xN) = \wedge\nu_B(xN)$, is an if/v-subgroup of $G/N$ when $L$ is a complete infinite distributive lattice.*

*Proof:* Since $B$ is an if/v-subgroup of $G$ and $N$ is a normal subgroup of $G$ and hence for each $x \in G$, $(xN)^{-1} = x^{-1}N$,

$\mu_C((xN)^{-1}) = \mu_C(x^{-1}N) = \vee\ \mu_B(x^{-1}N) = \vee_{z\in x^{-1}N}\mu_B z$
$= \vee_{w^{-1}\in x^{-1}N(=(xN)^{-1})}\mu_B w^{-1} = \vee_{w\in xN}\mu_B w = \vee\ \mu_B(xN) = \mu_C(xN)$
and $\nu_C((xN)^{-1}) = \nu_C(x^{-1}N) = \wedge\ \nu_B(x^{-1}N) = \wedge_{z\in x^{-1}N}\nu_B z$
$= \wedge_{w^{-1}\in x^{-1}N(=(xN)^{-1})}\ \nu_B w^{-1} = \wedge_{w\in xN}\nu_B w = \wedge\ \nu_B(xN) = \nu_C(xN)$

where the $5^{th}$ equality in both cases is due to the fact that $w \in xN$ iff $w^{-1} \in (xN)^{-1}$. Hence $C(xN)^{-1} = C(xN)$.

Since [0,1] is a complete infinite distributive lattice and $N$ is a normal subgroup of $G$, for each $x, y \in G$

$\mu_C((xN)(yN)) = \vee\mu_B(xyN) = \vee_{z\in xyN}\mu_B z = \vee_{u\in xN,v\in yN}\mu_B(uv)$
$\geq \vee_{u\in xN,v\in yN}(\mu_B(u) \wedge \mu_B(v)) = (\vee_{u\in xN}\mu_B(u)) \wedge (\vee_{v\in yN}\ \mu_B(v))$
$= (\vee\mu_B(xN)) \wedge (\vee\mu_B(yN)) = (\mu_C(xN)) \wedge (\mu_C(yN))$ and
$\nu_C((xN)(yN)) = \wedge\nu_B(xyN) = \wedge_{z\in xyN}\nu_B z = \wedge_{u\in xN,v\in yN}\nu_B(uv)$
$\leq \wedge_{u\in xN,v\in yN}(\nu_B(u) \vee \nu_B(v)) = (\wedge_{u\in xN}\nu_B(u)) \vee (\wedge_{v\in yN}\nu_B(v))$
$= (\wedge\nu_B(xN)) \vee (\wedge\nu_B(yN)) = (\nu_C(xN)) \vee (\nu_C(yN))$.

Hence $C$ is an if/v-subgroup of $G/N$.

*Definition 3.14* *For any if/v-subgroup $B$ of a group $G$ and for any normal subgroup $N$ of $G$, the if/v-subgroup $C$:$G/N \to L$, where $L$ is a complete infinite distributive lattice, defined by $\mu_C(xN) = \vee\mu_B(xN)$ and $\nu_C(xN) = \wedge\nu_B(xN)$ for each $x \in G$, is called the if/v-quotient subgroup of $G/N$ relative to $B$ and is denoted by $B/N$ or $\frac{B}{N}$.*
*In other words when $N$ is a normal subgroup of $G$ and $B$ is any if/v-subgroup of $G$, and [0,1] is a complete infinite distributive lattice, $\frac{B}{N}$: $\frac{G}{N} \to$ [0, 1] is defined by $\mu_{\frac{B}{N}}(gN) = \vee\mu_B(gN)$ and $\nu_{\frac{B}{N}}(gN) = \wedge\nu_B(gN)$ for each $g \in G$.*

*Lemma 3.15* *For any pair of groups $G$ and $H$ and for any crisp homomorphism $f : G \to H$, the following are true:*

1. *$A$ is an if/v-normal subgroup of $G$ implies $f(A)$ is an if/v-normal subgroup of $H$ when $f$ is onto.*

2. *$B$ is an if/v-normal subgroup of $H$ implies $f^{-1}(B)$ is an if/v-normal subgroup of $G$.*

*Proof:* (1): $A$ is an if/v-normal subgroup of $G$ implies $\mu_A(g^{-1}hg) \geq \mu_A(h)$ and $\nu_A(g^{-1}hg) \leq \nu_A(h)$ for each $h, g \in G$.

Let $fA = B$. Then $\mu_B y = \vee\mu_A f^{-1}y$ and $\nu_B y = \wedge\nu_A f^{-1}y$. Since the if/v-image of an if/v-subgroup is an if/v-subgroup, we only show that $\mu_B(g^{-1}hg) \geq \mu_B(h)$ and $\nu_B(g^{-1}hg) \leq \nu_B(h)$ for each $g, h \in G$.
Since $f$ is onto, for each $y \in H$, $f^{-1}y \neq \phi$. Let $a \in f^{-1}g$, $b \in f^{-1}h$. Then $fa = g$, $fb = h$ and $fa^{-1} = g^{-1}$. Since $f$ is a homomorphism, $g^{-1}hg = f(a^{-1}ba)$ and $a^{-1}ba \in f^{-1}(g^{-1}hg)$. So, for each $b \in f^{-1}h$, $\mu_B(g^{-1}hg) = \vee\mu_A f^{-1}(g^{-1}hg)$ $= \vee_{c\in f^{-1}(g^{-1}hg)}\mu_A c \geq \mu_A(a^{-1}ba) \geq \mu_A(b)$ and $\nu_B(g^{-1}hg) = \wedge\nu_A f^{-1}(g^{-1}hg) =$ $\wedge_{c\in f^{-1}(g^{-1}hg)}\nu_A c \leq \nu_A(a^{-1}ba) \leq \nu_A(b)$ implying $\mu_B(g^{-1}hg) \geq \vee_{b\in f^{-1}h}\ \mu_A(b)$ $= \mu_B(h)$ and $\nu_B(g^{-1}hg) \leq \wedge_{b\in f^{-1}h}\ \nu_A(b) = \nu_B(h)$
or $B = f(A)$ is an if/v-normal subgroup of $H$ when $f$ is onto.

(2): Let $f^{-1}B = A$. Then for each $g \in G$, $\mu_A g = \mu_B fg$ and $\nu_A g = \nu_B fg$. Since the if/v-inverse image of an if/v-subgroup is an if/v-subgroup we only show that $\mu_A(g^{-1}hg) \geq \mu_A(h)$ and $\nu_A(g^{-1}hg) \leq \nu_A(h)$.

Since $f$ is a homomorphism and $B$ is an if/v-normal subgroup of $H$, for each $g, h \in G$, $\mu_A(g^{-1}hg) = \mu_B f(g^{-1}hg) = \mu_B((fg)^{-1}(fh)(fg)) \geq \mu_B fh = \mu_A h$ and $\nu_A(g^{-1}hg) = \nu_B f(g^{-1}hg) = \nu_B((fg)^{-1}(fh)(fg)) \leq \nu_B fh = \nu_A h$ or $A = f^{-1}B$ is an if/v-normal subgroup of $G$.

*Definition 3.16* *For any pair of if/v-subgroups $A$ and $B$ of a group $G$ such that $A \leq B$, $A$ is called an if/v-normal subgroup of $B$ iff for each $x, y \in G$, $\mu_A(xyx^{-1}) \geq \mu_A(y) \wedge \mu_B(x)$ and $\nu_A(xyx^{-1}) \leq \nu_A(y) \vee \nu_B(x)$.*

*Theorem 3.17* *For any pair of if/v-subgroups $A$ and $B$ of a group $G$ such that $A \leq B$, the following are equivalent:*

1. *$A$ is an if/v-normal subgroup of $B$.*
2. *$\mu_A(yx) \geq \mu_A(xy) \wedge \mu_B(x)$ and $\nu_A(yx) \leq \nu_A(xy) \vee \nu_B(x)$ for each $x, y \in G$.*
3. *$(\chi_x^{\mu_A e} \circ \mu_A) \geq (\mu_A \circ \chi_x^{\mu_A e}) \wedge \mu_B$ and $(\chi_x^{\nu_A e} \circ \nu_A) \leq (\nu_A \circ \chi_x^{\nu_A e}) \vee \nu_B$ for each $x \in G$.*

*Proof:* (1)⇒(2): Since $A$ is an if/v-normal subgroup of $B$, for each $x, y \in G$, $\mu_A(yx) = \mu_A(x^{-1}xyx) = \mu_A(x^{-1}(xy)x) \geq \mu_A(xy) \wedge \mu_B(x)$ and $\nu_A(yx) = \nu_A(x^{-1}xyx) = \nu_A(x^{-1}(xy)x) \leq \nu_A(xy) \vee \nu_B(x)$.
(2)⇒(3): By 2.2(3) and 2.2(4), we have $(\chi_x^{\mu_A e} \circ \mu_A)y = \mu_A(x^{-1}y) \geq \mu_A(yx^{-1}) \wedge \mu_B(y) = (\mu_A \circ \chi_x^{\mu_A e})y \wedge \mu_B y = ((\mu_A \circ \chi_x^{\mu_A e}) \wedge \mu_B)y$ and $(\chi_x^{\nu_A e} \circ \nu_A)y = \nu_A(x^{-1}y) \leq \nu_A(yx^{-1}) \vee \nu_B(y) = (\nu_A \circ \chi_x^{\nu_A e})y \vee \nu_B y = ((\nu_A \circ \chi_x^{\nu_A e}) \vee \nu_B)y$ or for each $x \in G$, $(\chi_x^{\mu_A e} \circ \mu_A) \geq (\mu_A \circ \chi_x^{\mu_A e}) \wedge \mu_B$ and $(\chi_x^{\nu_A e} \circ \nu_A) \leq (\nu_A \circ \chi_x^{\nu_A e}) \vee \nu_B$.
(3)⇒(1): Letting $z^{-1} = x^{-1}y$ and by 2.2(3) and 2.2(4), we have
$\mu_A(x^{-1}yx) = \mu_A(z^{-1}x) = (\chi_z^{\mu_A e} \circ \mu_A)x \geq (\mu_A \circ \chi_z^{\mu_A e})x \wedge \mu_B x = \mu_A(xz^{-1}) \wedge \mu_B(x) = \mu_A(xx^{-1}y) \wedge \mu_B(x) = \mu_A(y) \wedge \mu_B(x)$ and
$\nu_A(x^{-1}yx) = \nu_A(z^{-1}x) = (\chi_z^{\nu_A e} \circ \nu_A)x \leq (\nu_A \circ \chi_z^{\nu_A e})x \vee \nu_B x = \nu_A(xz^{-1}) \vee \nu_B(x) = \nu_A(xx^{-1}y) \vee \nu_B(x) = \nu_A(y) \vee \nu_B(x)$ or for each $x, y \in G$, $\mu_A(x^{-1}yx) \geq \mu_A(y) \wedge \mu_B(x)$ and $\nu_A(x^{-1}yx) \leq \nu_A(y) \vee \nu_B(x)$ or $A$ is an if/v-normal subgroup of $B$.

*Theorem 3.18* *For any pair of if/v-subgroups $A$ and $B$ of a group $G$ such that $A$ is an if/v-normal subgroup of $B$:*

1. *$A_*$ is a normal subgroup of $B_*$.*
2. *$A^*$ is a normal subgroup of $B^*$ whenever [0, 1] is strongly regular.*

*Proof:* (1): Since $\mu_A e$ is the largest of $\mu_A G$, $\nu_A e$ is the smallest of $\nu_A G$ and $A$ is an if/v-subgroup of $G$, we get for each $x, y \in A_*$, $\mu_A(xy^{-1}) \geq \mu_A x \wedge \mu_A y = \mu_A e$ and $\nu_A(xy^{-1}) \leq \nu_A x \vee \nu_A y = \nu_A e$, so we have $\mu_A xy^{-1} = \mu_A e$ and $\nu_A xy^{-1} = \nu_A e$ or $xy^{-1} \in A_*$. Hence $A_*$ is a subgroup of $B_*$.
Again since $\mu_A e$ is the largest of $\mu_A G$, $\nu_A e$ is the smallest of $\nu_A G$, $A$ is an if/v-normal subgroup of $B$; we get for each $b \in B_*$ and $a \in A_*$, $\mu_A(bab^{-1}) \geq \mu_A a \wedge \mu_B b = \mu_A e \wedge \mu_B e = \mu_A e$ and $\nu_A(bab^{-1}) \leq \nu_A a \vee \nu_B b = \nu_A e \vee \nu_B e = \nu_A e$, so we have $\mu_A(bab^{-1}) = \mu_A e$ and $\nu_A(bab^{-1}) = \nu_A e$ or $bab^{-1} \in A_*$. Therefore $A_*$ is a normal subgroup of $B_*$.
(2): Since [0, 1] is strongly regular, for each $x, y \in A^*$, $\mu_A(xy^{-1}) \geq \mu_A x \wedge \mu_A y > 0$ and $\nu_A(xy^{-1}) \leq \nu_A x \vee \nu_A y < 1$ or $xy^{-1} \in A^*$. Hence $A^*$ is a subgroup of $B^*$.
Again, since [0,1] is strongly regular, for each $b \in B^*$ and $a \in A^*$, we get $\mu_A(bab^{-1}) \geq \mu_A a \wedge \mu_B b > 0$ and $\nu_A(bab^{-1}) \leq \nu_A a \vee \nu_B b < 1$ or $bab^{-1} \in A^*$. Hence $A^*$ is a normal subgroup of $B^*$ when [0,1] is strongly regular.

*Lemma 3.19* *For any pair of if/v-subgroups $A$ and $B$ of a group $G$ such that $A$ is an if/v normal subgroup of $B$, the if/v-subset $C$: $\frac{B^*}{A^*} \to$[0, 1] defined by, for each $b \in B^*$ $\mu_C bA^* = \vee \mu_B bA^*$ and $\nu_C bA^* = \wedge \nu_B bA^*$, is an if/v-subgroup of $\frac{B^*}{A^*}$, whenever [0,1] is a strongly regular complete infinite distributive lattice.*

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*Proof:* Since [0,1] is strongly regular, by 3.18, $A^*$ is a normal subgroup of $B^*$. Now in 3.13 set $G = B^*$, $N = A^*$, $B = B$. Then since [0,1] is a complete infinite distributive lattice, $C$ is an if/v-subgroup of $\frac{B^*}{A^*}$.

*Definition 3.20 For any pair of if/v-subgroups $A$ and $B$ of a group $G$ such that $A$ is an if/v normal subgroup of $B$ and $L$ is a strongly regular complete infinite distributive lattice, the if/v-quotient subgroup of $B|B^*$ relative to $A^*$, denoted by $B/A$ or $\frac{B}{A}$, is defined by $B/A$: $B^*/A^* \to L$ with $\mu_{B/A}(bA^*) = \vee\mu_B(bA^*)$ and $\nu_{B/A}(bA^*) = \wedge\nu_B(bA^*)$ for each $b \in B^*$ and is called L-if/v-quotient subgroup of $B$ relative to $A$.*

In what follows we prove a natural relation between $(\frac{B}{A})^*$ and $\frac{B^*}{A^*}$ which is used in the Third Isomorphism Theorem.

*Lemma 3.21 For any pair of if/v-subgroups $A$ and $B$ of a group $G$ such that $A$ is an if/v-normal subgroup of $B$, $(B/A)^* = B^*/A^*$.*

*Proof:* Let us recall that $(B/A)^* = \{bA^* \in (B^*/A^*)\ /b \in B^*, \mu_{B/A}(bA^*) > 0$ and $\nu_{B/A}(bA^*) < 1\}$. So always, $(B/A)^* \subseteq B^*/A^*$.
$\alpha \in B^*/A^*$ implies $\alpha = bA^*$ for some $b \in B^*$. Now as $e \in A^*$ and $b \in B^*$, $\mu_{B/A}(bA^*) = \vee\mu_B(bA^*) \geq \mu_B b > 0$ and $\nu_{B/A}(bA^*) = \wedge\nu_B(bA^*) \leq \nu_B b < 1$ implying $\alpha \in (B/A)^*$. Hence $(B/A)^* = B^*/A^*$.

*Theorem 3.22 For any if/v-normal subgroup $A$ of $G$ and an if/v-subgroup $B$ of $G$, $A \wedge B$ is an if/v-normal subgroup of $B$.*

*Proof:* By 3.11, if $A, B$ are if/v-subgroups of $G$ then $A \wedge B$ is an if/v-subgroup of $G$ and $A \wedge B \leq B$. Now we show that $C = A \wedge B$ is an if/v-normal subgroup of $B$ or for each $x, y \in G$, $\mu_C(xyx^{-1}) \geq \mu_C(y) \wedge \mu_B(x)$ and $\nu_C(xyx^{-1}) \leq \nu_C(y) \vee \nu_B(x)$. Since $A$ is an if/v-normal subgroup of $G$, for each $x, y \in G$,

$$\begin{aligned}
&\mu_C(xyx^{-1}) = (\mu_A \wedge \mu_B)(xyx^{-1}) = \mu_A(xyx^{-1}) \wedge \mu_B(xyx^{-1})\\
&\geq \mu_A(y) \wedge \mu_B(xyx^{-1}) \geq \mu_A(y) \wedge \mu_B(x) \wedge \mu_B(y) \wedge \mu_B(x^{-1})\\
&= (\mu_A(y) \wedge \mu_B(y)) \wedge \mu_B(x) = \mu_{A\wedge B}(y) \wedge\mu_B(x) = \mu_C(y) \wedge \mu_B(x) \text{ and}\\
&\nu_C(xyx^{-1}) = (\nu_A \vee \nu_B)(xyx^{-1}) = \nu_A(xyx^{-1}) \vee \nu_B(xyx^{-1})\\
&\leq \nu_A(y) \vee \nu_B(xyx^{-1}) \leq \nu_A(y) \vee \nu_B(x) \vee \nu_B(y) \vee \nu_B(x^{-1})\\
&= (\nu_A(y) \vee \nu_B(y)) \vee \nu_B(x) = \nu_{A\wedge B}(y) \vee\nu_B(x) = \nu_C(y) \vee \nu_B(x).
\end{aligned}$$

Therefore $\mu_C(xyx^{-1}) \geq \mu_C(y) \wedge \mu_B(x)$ and $\nu_C(xyx^{-1}) \leq \nu_C(y) \vee \nu_B(x)$ or $C = A \wedge B$ is an if/v-normal subgroup of $B$.

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# 4-Point Correlations of Dusty Fluid MHD Turbulent Flow in a 1st order Chemical-Reaction

By M. A. Bkar Pk, Abdul Malek & M. A. K. Azad
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*Keywords:* *MHD turbulence; dusty fluid; correlations; chemical reaction; matlab; decay law.*

*GJSFR-F Classification :* *FOR Code : MSC 2010: 00A69*

4POINTCORRELATIONSOFDUSTYFLUIDMHDTURBULENTFLOWINA1STORDERCHEMICALREACTION

*Strictly as per the compliance and regulations of :*

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*Keywords*: *MHD turbulence; dusty fluid; correlations; chemical reaction; matlab; decay law.*

## I. Introduction

Chemical reaction as used in chemistry, chemical engineering, physics, fluid mechanics, heat and mass transport. The mathematical models that describe chemical reaction kinetics provide chemists and chemical engineers with tools to better understand and describe chemicals processes such as food decomposition, stratospheric ozone decomposition, the complex chemistry of biological systems and MHD turbulence. In recent year, the motion of dusty viscous fluids has developed rapidly. The motion of dusty fluid occurs in the movement of dust − laden air, in problems of fluidization, in the use of dust in a gas cooling system and in the sedimentation problem of tidal rivers. The behavior of dust particles in a turbulent flow depends on the concentrations of the particles and the size of the particles with respect to the scale of turbulent fluid. Kishore and Golsefid [1, 1988] obtained and expression for the effect of Coriolis force on acceleration covariance in MHD turbulent flow of a dusty incompressible fluid. Kumar and Patel [2, 1974] derived expressions the first order reactant in homogeneous turbulence before the final period of decay. Kumar and Patel [3, 1975] also studied the first order reactant in homogeneous turbulence before the final period for the case of multi-point and multi-time. Chandrasekhar [4, 1951] obtained the invariant theory of isotropic turbulence in magneto-hydrodynamics. Corrsin [5, 1951] established on the spectrum of isotropic temperature fluctuations in isotropic turbulence. Bkar PK *et al.*,[6,2012]calculated for the first-order reactant in homogeneous dusty fluid turbulence prior to the ultimate phase of decay for four-point correlation in a rotating system. Sarker *et al.,* [7, 2012] discussed the homogeneous dusty fluid turbulence in a first-order reactant for the case of multi point and multi time prior to the final period of decay. Bkar Pk *et al.*, [8, 2013] also established the homogeneous turbulence in a first-order reactant for the case of multi point and multi time prior to the final period of decay in a rotating system. Bkar Pk *et al.*,[9, 2014]further enlarge the previous problem for the first-order reactant of homogeneous dusty fluid turbulence prior to the final period of decay in a rotating system for the case of multi-point and multi-time at four-point

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Ref

1. N. Kishore and Y.T. Golsefid, Effect of coriolis force on acceleration covariance in MHD turbulent flow of a dusty incompressible fluid. Astrophysics and Space Sciences (Astr. Space Sci.) 150, 89-101 (1988). http://dx.doi.org/10.1007/BF00714156

correlation. Sarker and Kishore [10, 1991] studied the decay of MHD turbulence before the final period. Bkar Pk *et al.*, [11, 2012] discussed the decay of energy of MHD turbulence for four-point correlation. Bkar Pk *et al.*, [12, 2013] also pointed out that the decay of MHD turbulence prior to the ultimate phase in presence of dust particle for four-point correlation. Bkar Pk *et al.*, [13, 2013] further calculated the decay of dusty fluid MHD turbulence for four-point correlation in a rotating system. Sarker and Islam [14, 2001] obtained the decay of dusty fluid MHD turbulence before the final period in a rotating system. Sarker and Ahmed [15, 2011] pointed out that the fiber motion in dusty fluid turbulent flow with two point correlation. Dixit and Upadhyay [16, 1989] obtained the effect of Coriolis force on acceleration covariance in MHD turbulent dusty flow with rotational symmetry. Azad *et al.* [17, 2011] studied the statistical theory of certain distribution functions in MHD turbulent flow for velocity and concentration undergoing a first order reaction in a rotating system. Islam and Sarker [18, 2001] studied the first order reactant in MHD turbulence before the final period of decay for the case of multi-point and multi-time. Bkar Pk *et al.*, [19,2015] discussed the effects of first-order reactant on MHD turbulence at four-point correlation. Deissler [20, 21 1958, 1960] developed a theory 'On the decay of homogeneous turbulence before the final period.' Sengupta and Ahmed [22, 2014] studied the MHD free convective chemically reactive flow of a dissipative fluid with thermal diffusion, fluctuating wall temperature and concentrations in velocity slip regime. Mukhopadhyay [23, 2013] obtained the chemically reactive solute transfer in MHD boundary layer flow along a stretching cylinder with partial slip. Azad et al., [24, 2010] discussed first order reactant in magneto-hydrodynamic turbulence before the final Period of decay in presence of dust particles. Poornima and Bhaskar Reddy [25, 2013] investigated the effects of thermal radiation and chemical reaction on MHD free convection flow past a semi-infinite vertical porous moving plate.

For first order chemical reaction, most of the author has been discussed their problems in two and three point correlation and some author has been done in a porous moving plate. Bkar PK *et al.*, has been investigated their problems for MHD turbulence with the present of dust particles, rotating systems, dust particles in rotating systems and first order chemical reaction for point correlation.

To the best of author's knowledge, the interaction between dusty fluid MHD turbulence and first order chemical reaction at four point correlations has received little attention. Hence in our present paper we have studied the decay of dusty fluid MHD turbulence in a first order chemical reaction for four-point correlation. The expressions for the fluctuation of velocity components and concentration have been obtained and effects of chemical reactions have been computed numerically and discussed in detail. Finally we have obtained the decay of dusty fluid of magnetic energy fluctuation of concentration undergoing a first order chemical reaction for four-point correlation in the form

$$\frac{\langle h^2 \rangle}{2} = \left( AT_0^{\frac{-3}{2}} + BT_0^{-5} \right) \exp(-RT_0) + \left( CT^{\frac{-15}{2}} + DT^{\frac{-17}{2}} \right) \exp\{(-R+M)T\}.$$

where $R$ is the chemical reaction, $M$ is the dust particle parameter, $\langle h^2 \rangle$ denotes the total energy that is, mean square of the magnetic field fluctuation, $t$ is the time, and $A$ , $B$, $C$, $D$, $t_0$ and $t_1$ are arbitrary constants determined by the initial conditions.

## II. Two-Point Correlation and Spectral Equations-

First we discussed two and three point correlations with spectral equations in briefly next calculated our main problem elaborately. Induction equation at the point $P$ and the corresponding equation for the point $P'$ in the magnetic are given by

Ref

11. M. A. Bkar. PK,M. A. K. Azad and M.S.Alam Sarker. Decay of energy of MHD turbulence for four-point correlation. International Journal of Engineering Research and Technology.1(9): pp 1-13. 2012.

$$\frac{\partial h_i}{\partial t}+u_k\frac{\partial h_i}{\partial x_k}-h_k\frac{\partial u_i}{\partial x_k}=\left(\frac{\nu}{p_M}\right)\frac{\partial^2 h_i}{\partial x_k\partial x_k}, \tag{1}$$

$$\frac{\partial h'_j}{\partial t}+u'_k\frac{\partial h'_j}{\partial x'_k}-h'_k\frac{\partial u'_j}{\partial x'_k}=\left(\frac{\nu}{p_M}\right)\frac{\partial^2 h'_j}{\partial x'_k\partial x'_k}. \tag{2}$$

Notes

Multiplying equation (1) by $h'_j$ (2) by $h_i$ ,adding and taking ensemble average and using

$$\frac{\partial}{\partial r_k}=-\frac{\partial}{\partial x_k}=\frac{\partial}{\partial x'_k},$$

with the relations

$$\langle u_k h_i h'_j\rangle=\langle -u'_k h_i h'_j\rangle,\langle u'_j h_i h'_k\rangle=\langle -u_i h_k h'_j\rangle$$

we get

$$\frac{\partial}{\partial t}\langle h_i h'_j\rangle+2[\frac{\partial}{\partial r_k}\langle u'_k h_i h'_j\rangle-\frac{\partial}{\partial r'_k}\langle u_i h_k h'_j\rangle]=2\left(\frac{\nu}{p_M}\right)\frac{\partial^2}{\partial r_k\partial r_k}\langle h_i h'_j\rangle, \tag{3}$$

Interchanging the points $p$ and $p'$ with indices $i$ and $j$, then taking contraction of the indices $i$ and $j$, we get the spectral equation corresponding to two point correlation is

The spectral equation corresponding to the two point correlation equation taking contraction of the indices is

$$\frac{\partial}{\partial t}\langle\varphi_i\varphi'_i(\hat{k})\rangle+\frac{2\nu}{p_M}k^2\langle\varphi_i\varphi'_i(\hat{k})\rangle=2ik_k[\langle\alpha_i\varphi_k\varphi'_i(\hat{k})\rangle-\langle\alpha_k\varphi_i\varphi'_i(-\hat{k})\rangle] \tag{4}$$

where,

$\varphi_i\varphi'_i$ and $\alpha_i\phi_k\phi'_i$ are defined by

$$\langle h_i h'_i(\hat{r})\rangle=\int_{-\infty}^{\infty}\langle\varphi_i\varphi'_i(\hat{k})\rangle\exp[i(\hat{k}.\hat{r})]d\hat{k} \tag{5}$$

$$\langle u_i h_i h'_i(\hat{r})\rangle=\int_{-\infty}^{\infty}\langle\alpha_i\varphi_k\varphi'_i(\hat{k})\rangle\exp[i(\hat{k}.\hat{r})]d\hat{k} \tag{6}$$

and $\langle u'_k h_i h'_i\rangle=\langle u_k h_i h'_i(-\hat{r})\rangle=\int_{-\infty}^{\infty}<\alpha_k\varphi_i\varphi'_i(-\hat{k})>\exp(i\hat{k}.\hat{r})d\hat{k}.$

## III. Three-Point Correlation and Spectral Equations-

We take momentum equation of MHD turbulence at the point p, and the induction equations in the magnetic field at the and $p''$ as

$$\frac{\partial u_l}{\partial t}+u_k\frac{\partial u_l}{\partial x_k}-h_k\frac{\partial h_l}{\partial x_k}=-\frac{\partial W}{\partial t}+\nu\frac{\partial^2 u_l}{\partial x_k\partial x_k}, \tag{7}$$

$$\frac{\partial h'_i}{\partial t}+u'_k\frac{\partial h'_i}{\partial x'_k}-h'_k\frac{\partial u'_i}{\partial x'_k}=\left(\frac{\nu}{p_M}\right)\frac{\partial^2 h'_i}{\partial x'_k\partial x'_k}, \tag{8}$$

$$\frac{\partial h_j''}{\partial t} + u_k''\frac{\partial h_j''}{\partial x_k''} - h_k''\frac{\partial u_j''}{\partial x_k''} = \left(\frac{\nu}{p_M}\right)\frac{\partial^2 h_j''}{\partial x_k''\partial x_k''}. \tag{9}$$

We multiplying equation (7) by $h_i'h_j''$, (8) by $u_l h_j''$,and (9) by $u_l h_i'$ , then adding and taking ensemble average and using

$$\frac{\partial}{\partial x_k'} = \frac{\partial}{\partial x_k'}, \frac{\partial}{\partial r_k'} = \frac{\partial}{\partial x_k''}, \frac{\partial}{\partial x_k} = -(\frac{\partial}{\partial r_k} + \frac{\partial}{\partial r_k'})$$

and interchanging of points $p'$ and $p''$, in the subscript $i$ and $j$, with the relations

$$\langle u_l u_k'' h_j'' h_i' \rangle = \langle u_l u_k' h_i' h_j' \rangle \text{ and } \langle u_l u_j'' h_j h_i' \rangle = \langle u_l u_k h_i h_k'' \rangle .$$

After simplifying the obtained results and then using Fourier transforms as

$$\langle u_l h_i'(\hat{r}) h_j''(\hat{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \phi_l \beta_i'(\hat{k})\beta_j''(\hat{k}')\rangle \exp[i(\hat{k}.\hat{r} + \hat{k}'.\hat{r}')d\hat{k}d\hat{k}', \tag{10}$$

$$\langle u_l u_k'(\hat{r}) h_i'(\hat{r}) h_j''(\hat{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \phi_l\phi_k'(\hat{k}) \beta_i'(\hat{k})\beta_j''(\hat{k}')\rangle \exp[i(\hat{k}.\hat{r} + \hat{k}'.\hat{r}')d\hat{k}d\hat{k}', \tag{11}$$

$$\langle u_l u_i'(\hat{r}) h_i'(\hat{r}) h_j''(\hat{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \phi_l\phi_i'(\hat{k}) \beta_i'(\hat{k})\beta_j''(\hat{k}')\rangle \exp[i(\hat{k}.\hat{r} + \hat{k}'.\hat{r}')d\hat{k}d\hat{k}', \tag{12}$$

$$\langle u_l h_i'(\hat{r}) h_j''(\hat{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \phi_l \beta_i'(\hat{k})\beta_j''(\hat{k}')\rangle \exp[i(\hat{k}.\hat{r} + \hat{k}'.\hat{r}')d\hat{k}d\hat{k}', \tag{13}$$

$$\langle u_l h_k(\hat{r}) h_i'(\hat{r}) h_j''(\hat{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \phi_l\beta_k(\hat{k}) \beta_i'(\hat{k})\beta_j''(\hat{k}')\rangle \exp[i(\hat{k}.\hat{r} + \hat{k}'.\hat{r}')d\hat{k}d\hat{k}', \tag{14}$$

$$\langle w h_i'(\hat{r}) h_j''(\hat{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \gamma \beta_i'(\hat{k})\beta_j''(\hat{k}')\rangle \exp[i(\hat{k}.\hat{r} + \hat{k}'.\hat{r}')d\hat{k}d\hat{k}'. \tag{15}$$

we get

$$\frac{\partial}{\partial t}\left(\overline{\phi_l\beta_i'\beta_j''}\right) + \frac{\nu}{p_M}[(1+p_M)(k^2+k'^2) + 2p_M kk' + R]\left(\overline{\phi_l\beta_i'\beta_j''}\right) =$$

$$i(k_k + k_k')(\overline{\phi_l\phi_k\beta_i'\beta_j'}) - i(k_k + k_k')(\overline{\beta_l\beta_k\beta_i'\beta_j'}) - i(k_k + k_k')(\overline{\phi_l\phi_k'\beta_i'\beta_j'})$$

$$+ i(k_k + k_k')(\overline{\phi_l\phi_i'\beta_k'\beta_j'}) + i(k_l + k_l')(\overline{\gamma\beta_i'\beta_j'})$$

Taking contraction of the indices $i$ and $j$ ,we get spectral equations corresponding to the three-point correlation equations

$$\frac{\partial}{\partial t}\left(\overline{\phi_l\beta_i'\beta_j''}\right) + \frac{\nu}{p_M}[(1+p_M)(k^2+k'^2) + 2p_M kk' + R]\left(\overline{\phi_l\beta_i'\beta_j''}\right) =$$

$$i(k_k + k_k')(\overline{\phi_l\phi_k\beta_i'\beta_j''}) - i(k_k + k_k')(\overline{\beta_l\beta_k\beta_i'\beta_j''}) - i(k_k + k_k')(\overline{\phi_l\phi_k'\beta_i'\beta_j''})$$

$$+ i(k_k + k_k')(\overline{\phi_l\phi_i'\beta_k'\beta_j''}) + i(k_l + k_l')(\overline{\gamma\beta_i'\beta_j''}) \tag{16}$$

and

$$-(\overline{\gamma\beta_i'\beta_j''}) = \frac{(K_l K_k + K_l' K_k + K_l k_k' + K_l' K_k')}{(K_l^2 + K_l'^2 + 2K_l K_l')}(\overline{\phi_l\phi_k\beta_i'\beta_i''} - \overline{\beta_l\beta_k\beta_i'\beta_j''}) \tag{17}$$

Notes

## IV. Mathematical Formulation

To find the four point correlation equation, following Deissler's [17] we take the momentum equation of dusty fluid MHD turbulence in a first order chemical reaction at the point p and the induction equation of magnetic field fluctuation at $p', p''$ and $p'''$ as

$$\frac{\partial u_l}{\partial t}+u_k\frac{\partial u_l}{\partial x_k}-h_k\frac{\partial h_l}{\partial x_k}=-\frac{\partial w}{\partial x_l}+\nu\frac{\partial^2 u_l}{\partial x_k\partial x_k}-Ru_l+f(u_l-v_l) \tag{18}$$

$$\frac{\partial h_i'}{\partial t}+u_k'\frac{\partial h_i'}{\partial x_k'}-h_k'\frac{\partial u_i'}{\partial x_k'}=\frac{\nu}{p_M}\frac{\partial^2 h_i'}{\partial x_k'\partial x_k'} \tag{19}$$

$$\frac{\partial h_j''}{\partial t}+u''\frac{\partial h_j''}{\partial x_k''}-h''\frac{\partial u_j''}{\partial x_k''{}_k}=\frac{\nu}{p_M}\frac{\partial^2 h_j''}{\partial x_k''\partial x_k''} \tag{20}$$

$$\frac{\partial h_m'''}{\partial t}+u_k'''\frac{\partial h_m'''}{\partial x_k'''}-h_k'''\frac{\partial u_m'''}{\partial x_k'''}=\frac{\nu}{p_M}\frac{\partial^2 h_m'''}{\partial x_k'''} \tag{21}$$

where $w=\frac{p}{\rho}+\frac{1}{2}\langle h^2\rangle$ is the total MHD pressure, $p(\hat{x},t)$ is the hydrodynamic pressure, $\rho$ is the fluid density, $P_M=\frac{\nu}{\lambda}$ is the Magnetic Prandtl number, $\Omega_s$ is the angular velocity components, $m_i=\frac{4}{3}\pi R^3{}_i\rho_i$ is the mass of a single spherical dust particle of radius $R_i$ and $\rho_i$ constant density of the material in dust particles, $R$ is the first order chemical reaction $f=\frac{KN}{\rho}$, is the dimensions of frequency, $K$ is the Stock's drug resistance, $N$ is the constant number density of dust particle. $\nu$, is the kinematics viscosity, $\lambda$ is the magnetic diffusivity, $h_i(x,t)$ is the magnetic field fluctuation, $u_k(x,t)$ is the turbulent velocity, $v_l$ dust velocity component, $t$ is the time, $x_k$ is the space co-ordinate and repeated subscripts are summed from 1 to 3.

Multiplying equation (18) by $h_i'h_j''h_m'''$ (19) by $u_lh_j''h_m'''$ (20) by $u_lh_i'h_m'''$ (21) by $u_lh_i'h_j''$ and adding the four equations, we than taking the angular bracket $(\overline{\ldots\ldots})$ or $\langle\ldots\ldots\ldots\rangle$, we get

$$\frac{\partial}{\partial t}(\overline{u_lh_i'h_j''h_m'''})+\frac{\partial}{\partial x_k}(\overline{u_lu_kh_i'h_j''h_m'''})-\frac{\partial}{\partial x_k}(\overline{h_kh_lh_i'h_j''h_m'''})+$$

$$\frac{\partial}{\partial x_k'}(\overline{u_lu_kh_i'h_j''h_m'''})-\frac{\partial}{\partial x_k'}(\overline{u_lu_i'h_k'h_j''h_m'''})+\frac{\partial}{\partial x_k''}(\overline{u_lu_k''h_i'h_j''h_m'''})-$$

$$\frac{\partial}{\partial x_k''}(\overline{u_lu_j''h_i'h_k''h_m'''})+\frac{\partial}{\partial x_k'''}(\overline{u_lu_k''h_i'h_j''h_m'''})-\frac{\partial}{\partial x_k'''}(\overline{u_lu_j''h_i'h_j''h_m'''})=$$

$$-\frac{\partial}{\partial x_l}(\overline{wh_i'h_j''h_m'''})+\frac{\partial^2}{\partial x_k\partial x_k}(\overline{u_lh_i'h_j''h_m'''})+\frac{\nu}{p_M}[\frac{\partial^2}{\partial x_k'\partial x_k'}(\overline{u_lh_i'h_j''h_m'''})+$$

$$\frac{\partial^2}{\partial x_k''\partial x_k''}(\overline{u_lh_i'h_j''h_m'''})+\frac{\partial^2}{\partial x_k'''\partial x_k'''}(\overline{u_lh_i'h_j''h_m'''})]-R\left[\overline{u_lh_i'h_j''h_m'''}\right]$$

$$+\mathrm{f}\left[\left(\overline{u_lh_i'h_j''h_m'''}\right)-\left(\overline{v_lh_i'h_j''h_m'''}\right)\right] \tag{22}$$





Ref

17. M. A. K. Azad, M. A. Aziz and M. S. Alam Sarker, Statistical theory of certain distribution functions in MHD turbulent flow for velocity and concentration undergoing a first order reaction in a rotating system, Bangladesh Journal of Scientific & Industrial Research 46(1)59-68-68, 2011.

Notes

Using the transformations,

$$\frac{\partial}{\partial x_k''}=\frac{\partial}{\partial r_k'},\ \frac{\partial}{\partial x_k'}=\frac{\partial}{\partial r_k'},\ \frac{\partial}{\partial x_k}=-(\frac{\partial}{\partial r_k'}+\frac{\partial}{\partial r_k'}+\frac{\partial}{\partial r_k''})$$

and Fourier transforms

$$\langle u_l h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\phi_l\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{23}$$

$$\langle u_l u_k' h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\phi_l\phi_k'(\hat k)\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{24}$$

$$\langle u_l u_i' h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\phi_l\phi_i'(\hat k)\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{25}$$

$$\langle u_l u_k'' h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\phi_l\phi_k'(\hat k')\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{26}$$

$$\langle u_l u_j'' h_i'(\hat r)h_k''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\phi_l\phi_j''(\hat k')\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{27}$$

$$\langle u_l u_k h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\phi_l\phi_k\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{28}$$

$$\langle u_l u_i'' h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\phi_l\phi_i'(\hat k')\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{29}$$

$$\langle w h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\delta\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{30}$$

$$\langle v_l h_i'(\hat r)h_j''(\hat r')h_m'''(\hat r'')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\eta_l\gamma_i'(\hat k)\gamma_j''(\hat k')\gamma_m'''(\hat k'')\rangle\exp[i(\hat k.\hat r+\hat k'.\hat r'+\hat k''.\hat r'')d\hat k d\hat k' d\hat k'', \tag{31}$$

with the fact

$$\langle u_l u_k''' h_i' h_j'' h_m'''\rangle=\langle u_l u_k' h_i' h_j'' h_m'''\rangle,$$
$$\langle u_l u_k''' h_i' h_j'' h_m'''\rangle=\langle u_l u_k' h_i'' h_j'' h_m'''\rangle,$$
$$\langle u_l u_m''' h_i' h_j'' h_m'''\rangle=\langle u_l u_i' h_i' h_k' h_j'' h_m'''\rangle,$$
$$\langle u_l u_j'' h_i' h_k'' h_m'''\rangle=\langle u_l u_i' h_i' h_k' h_j'' h_m'''\rangle,$$

and by taking contraction of the indices $i$ and $j$, $i$ and $m$, we obtained four-point correlation equation as

$$\frac{\partial}{\partial t}\left(\overline{\phi_l\gamma_i'\gamma_i''\gamma_m'''}\right)+\frac{\nu}{p_M}[(1+p_M)(k^2+k'^2+k''^2)+2p_M kk'$$
$$+2p_M kk''+2p_M kk'''](\overline{\phi_l\gamma_i'\gamma_i''\gamma_m'''})+(R-f)(\overline{\phi_l\gamma_i'\gamma_i''\gamma_m'''})$$
$$+f(\overline{\eta_l\gamma_i'\gamma_i''\gamma_m'''})=i(k_k+k_k'+k_k'')(\overline{\phi_l\phi_k\gamma_i'\gamma_i''\gamma_m'''})$$
$$-i(k_k+k_k'+k_k'')(\overline{\gamma_l\gamma_k\gamma_i'\gamma_i''\gamma_m'''})-i(k_k+k_k'+k_k'')(\overline{\phi_l\phi_k'\gamma_i'\gamma_i''\gamma_m'''})$$

$$+i(k_k+k'_k+k''_k)(\overline{\phi_l\phi'_i\gamma'_k\gamma'_i\gamma'''_m})+i(k_k+k'_k+k''_k)(\overline{\delta\gamma'_k\gamma''_i\gamma'''_m}). \tag{32}$$

If we take the derivative with respect to $x_l$ of equation (18) and multiplying by $h'_ih''_jh'''_m$, using time averages and writing the equation in terms of the independent variables $\hat{r},\hat{r}',\hat{r}''$, we have

Notes

$$-(\overline{\delta\gamma'_i\gamma''_j\gamma'''_m})=$$

$$\frac{(k_lk_k+k_lk'_k+k_lk''_k+k'_lk_k+k'_lk'_k+k'_lk''_k+k''_lk_k+k''_lk_k+k''_lk'_k+k''_lk''_k)}{k_lk_l+k'_lk'_l+k''_lk''_l+2k_lk'_l+2k'_lk''_l+2k_lk''_l}$$

$$\times(\overline{\phi_l\phi_k\gamma'_i\gamma''_j\gamma'''_m}-\overline{\gamma_l\gamma_k\gamma'_i\gamma''_j\gamma'''_m}). \tag{33}$$

Equation (32) and (33) are the spectral equation corresponding to the four-point correlation equation.

A relation between $\phi_l\phi'_k\beta'_i\beta''_j and\ \phi_l\gamma'_i\gamma''_j\gamma'''_m$ can be obtained by letting $\hat{r}''=0$ in equation (23) and comparing the result with equation (11), we get

$$\left\langle\phi_l\phi'_k(\hat{k})\beta'_i(\hat{k})\beta''_j(\hat{k}')\right\rangle=\int_{-\infty}^{\infty}\left\langle\phi_l\gamma'_i(\hat{k})\gamma''_j(\hat{k}')\gamma'''_m(\hat{k}'')\right\rangle d\hat{k}'' \tag{34}$$

The relation between $\alpha_i\varphi_k\varphi'_j(\hat{k})$ and $\varphi_l\beta'_i\beta''_j$ is obtained by letting $\hat{r}'=0$ in equation (17) and comparing the result with equation (5), then

$$\left\langle\alpha_i\varphi_k\varphi'_i(\hat{k})\right\rangle=\int_{-\infty}^{\infty}\left\langle\phi_i\beta'_i(\hat{k})\beta''_i(\hat{k}')\right\rangle d\hat{k}' \tag{35}$$

## V. Solution Neglecting Quintuple Correlations

Using $f\left(\overline{\eta_l\gamma'_i\gamma''_i\gamma'''_m}\right)=L\left(\overline{\phi_l\gamma'_i\gamma''_i\gamma'''_m}\right)$, 1-$L$=s, and neglecting all the terms on the right side of equation (32), then integrating between $t_1$ and t, we get

$$\left\langle\phi_l\gamma'_i\gamma''_j\gamma'''_m\right\rangle=$$

$$\left\langle\phi_l\gamma'_i\gamma''_j\gamma'''_m\right\rangle_1\exp\{\frac{-\nu}{p_M}(1+p_M)\left(k^2+k'^2+k''^2+2kk'+2k'k''+2kk'\right)-R+fs\}(t-t_1) \tag{36}$$

For small values of $k$, $k'$ and $k''$ $\left\langle\phi_l\gamma'_i\gamma''_j\gamma'''_m\right\rangle_1$ is the value of $\left\langle\phi_l\gamma'_i\gamma''_j\gamma'''_m\right\rangle$ at $t=t_1$. Substituting of equations (17), (33),(34) (35),(36) in equation (16) and integrating with respect to $k''_1,k''_2,k''_3$ and farther integrating with respect to time, and in order to simplify calculations, we will assume that $[a]_1=0$ and the integration is performed, then substituting the obtained equation in equation (4) and setting $H=2\pi k^2\varphi_i\varphi'_i$ ,we obtain

$$\frac{\partial H}{\partial t}+(\frac{2\nu k^2}{p_M})H=G \tag{37}$$

where,

$$G=k^2\int_{-\infty}^{\infty}2\pi.i[\left\langle k_k\phi_l\beta'_i\beta''_i(\hat{k},\hat{k}')\right\rangle-\left\langle k_k\phi_l\beta'_i\beta''_i(-\hat{k},-\hat{k}')\right\rangle]_0\cdot\exp\{-(R-fs)(t-t_o)\}$$

$$\exp[-\frac{\nu}{p_M}(t-t_0)\{(1+p_M)(k^2+k'^2)+2p_M kk'\}]dk'$$

$$+k^2\int_{-\infty}^{\infty}\frac{2p_M.\pi^{\frac{5}{2}}}{\nu}i\left[b(\hat{k}.\hat{k}')-b(-\hat{k}.-\hat{k}')\right]_1.\exp\{-R(t-t_1\}$$

$$.-\omega^{-1}\exp[-\omega^2\left(\frac{(1+2p_M)k^2}{(1+p_M)^2}+\frac{2P_m kk'}{1+p_M}+k'^2\right)]$$

$$+k.\exp\ [-\omega^2(\quad(1+p_M)(k^2+k'^2)+2p_M kk']\int_0^{\frac{\omega k}{2}}\exp(x^2)dx\}dk'$$

$$+k^2\int_{-\infty}^{\infty}\frac{2p_M.\pi^{\frac{5}{2}}}{\nu}i\left[c(\hat{k}.\hat{k}')-c(-\hat{k}.-\hat{k}')\right]_1\exp\{-(R-fs)(t-t_1\}$$

$$-\omega^{-1}\exp[-\omega^2\left(k^2+\frac{2P_m kk'}{1+p_M}+\frac{(1+2p_M)k'^2}{(1+p_M)^2}\right)]$$

$$+k'\exp[-\omega^2(\ (1+p_M)(k^2+k'^2)+2p_M kk')].\int_0^{\frac{\omega k'}{2}}\exp(x^2)dx\}dk' \tag{38}$$

where $G$ is the energy transfer function and $H$ is the magnetic energy spectrum function. In order to make further calculations, an assumption must be made for the forms of the bracketed quantities with the subscripts 0 and 1 in equation (38) which depends on the initial conditions.

$$(2\pi)^2[\langle k_k\phi_l\beta_i'\beta_i''(\hat{k},\hat{k}')\rangle-\langle k_k\phi_l\beta_i'\beta_i''(-\hat{k},-\hat{k}')\rangle]_0=-\xi_0(k^2k'^4-k^4k'^2) \tag{39}$$

where $\xi_0$ is a constant depending on the initial conditions for the other bracketed quantities in equation (38), we get

$$\frac{4p_M.\pi^{\frac{7}{2}}}{\nu}i\left[b(\hat{k}.\hat{k}')-b(-\hat{k}.-\hat{k}')\right]_1=\frac{4p_M.\pi^{\frac{7}{2}}}{\nu}i\left[c(\hat{k}.\hat{k}')-c(-\hat{k}.-\hat{k}')\right]_1=-2\xi_1(k^4k'^6-k^6k'^4) \tag{40}$$

Remembering, $d\hat{k}'=2\pi\hat{k}'^2d(\cos\theta)dk'$ and $kk'=kk'\cos\theta,\theta$ is the angle between $\hat{k}$ and $\hat{k}'$ and carrying out the integration with respect to $\theta$, we get

$$G=\int_0^{\infty}[\frac{\xi_0(k^2k'^4-k^4k'^2)kk'}{\nu(t-t_0)}\{\exp[-\frac{\nu}{p_M}(t-t_0)\{(1+p_M)(k^2+k'^2)-2p_M kk'\}]$$

$$-\exp[-\frac{\nu}{p_M}(t-t_0)\{(1+p_M)(k^2+k'^2)+2p_M kk'\}]$$

$$+\frac{\xi_1(k^4k'^6-k^6k'^4)kk'}{\nu(t-t_0)}\exp[fs(t-t_1)](\omega^{-1}\exp[-\omega^2\left(\frac{(1+2p_M)k^2}{(1+p_M)^2}-\frac{2p_M kk'}{(1+p_M)}+k'^2\right)]$$

$$-\omega^{-1}\exp[-\omega^2\left(\frac{(1+2p_M)k^2}{(1+p_M)^2}+\frac{2p_M kk'}{(1+p_M)}+k'^2\right)]$$

Notes

Notes

$$+\omega^{-1}\exp[-\omega^2\left(k^2-\frac{2p_M kk'}{(1+p_M)}+\frac{(1+2p_M)k'^2}{(1+p_M)^2}\right)]$$

$$-\omega^{-1}\exp[-\omega^2\left(k^2+\frac{2p_M kk'}{(1+p_M)}+\frac{(1+2p_M)k'^2}{(1+p_M)^2}\right)]$$

$$-\omega^{-1}\exp[-\omega^2\left(k^2+\frac{2p_M kk'}{(1+p_M)}+\frac{(1+2p_M)k'^2}{(1+p_M)^2}\right)]$$

$$+\{k\exp[-\omega^2((1+p_M)(k^2+k'^2)-2p_M kk')]$$

$$-k\exp[-\omega^2((1+p_M)(k^2+k'^2)+2p_M kk')]\}\int_0^{\frac{\omega k}{2}}\exp(x^2)dx$$

$$+\{k'\exp[-\omega^2((1+p_M)(k^2+k'^2)-2p_M kk')]$$

$$-k'\exp[-\omega^2((1+p_M)(k^2+k'^2)+2p_M kk')]\}\int_0^{\frac{\omega k'}{2}}\exp(x^2)dx)]dk' \qquad (41)$$

here, $\omega=\left(\frac{\nu(t-t_1)(1+p_M)}{p_M}\right)^{\frac{1}{2}}$.

Integrating equation (41) with respect to $k'$, we have

$$G=G_\beta+G_\gamma\exp\{-(R-fs)(t-t_1\} \qquad (42)$$

where,

$$G_\beta=-\frac{\pi^{\frac{1}{2}}\xi_0 p_M^{\frac{5}{2}}}{\nu^{\frac{3}{2}}(t-t_0)^{\frac{3}{2}}(1+p_M)^{\frac{5}{2}}}\exp\left\{-\frac{\nu(t-t_0)(1+2p_M)k^2}{p_M(1+p_M)}\right\}$$

$$\left[\frac{15p_M k^4}{4\nu^2(t-t_0)^2(1+p_M)}+\left\{\frac{5p_M{}^2}{(1+p_M)^2\nu(t-t_0)}-\frac{3}{2\nu(t-t_0)}\right\}k^6+\frac{p_M}{1+p_M}\left\{\frac{p_M{}^2}{(1+p_M)^2}-1\right\}k^8\right]$$

and

$$G_\gamma=G_{\gamma_1}+G_{\gamma_2}+G_{\gamma_3}+G_{\gamma_4}$$

$$G_{\gamma_1}=\frac{\xi_1\sqrt{\pi}p^5{}_M}{8\nu^2(t-t_1)^2(1+p_M)^5}\exp\left(\frac{-\nu(t-t_1)(1+2p_M-p^2{}_M)}{p_M(1+p_M)}\right)k^2$$

$$[\frac{90p_M k^6}{\nu^4(t-t_1)^4(1+p_M)}+3\left\{\frac{4p_M}{\nu^2(t-t_1)^2(1+p_M)}+\frac{2p^2{}_M}{\nu^3(t-t_1)^3(1+p_M)^2}-\frac{1}{\nu^3(t-t_1)^3}\right\}k^8$$

$$+\left\{\frac{64p^2{}_M}{\nu(t-t_1)(1+p_M)^2}+\frac{10p^3{}_M}{\nu^2(t-t_1)^2(1+p_M)^3}-\frac{40}{\nu(t-t_1)}\right\}k^{10}$$

$$+8\left\{\left(\frac{p_M}{1+p_M}\right)^2-\left(\frac{p_M}{1+p_M}\right)\right\}k^{12}$$

$$G_{\gamma_2}=\frac{\xi_1\sqrt{\pi}\,p^5{}_M(1+p_M)^4}{8\nu^2(t-t_1)^2(1+2p_M)^{9/2}}\exp\left(\frac{-\nu(t-t_1)(1+p_M)(1+2p_M-p^2{}_M)}{p_M(1+p_M)}\right)k^2$$

$$[\frac{90p_M(1+p_M)k^6}{\nu^4(t-t_1)^4(1+2p_M)}+\left\{\frac{120p_M(1+p_M)}{\nu^2(t-t_1)^2(1+2p_M)}+\frac{2p^2{}_M(1+p_M)^2}{\nu^3(t-t_1)^3(1+2p_M)^2}-\frac{1}{\nu^3(t-t_1)^3}\right\}k^8$$

$$+\left\{\frac{64p^2{}_M(1+p_M)^2}{\nu(t-t_1)(1+2p_M)^2}-\frac{40}{\nu(t-t_1)}+\frac{10p^3{}_M(1+p_M)^3}{\nu^2(t-t_1)^2(1+2p_M)^3}\right\}k^{10}$$

$$+\left\{8p^3{}_M\left(\frac{1+p_M}{1+2p_M}\right)^3-\left(\frac{p_M(1+p_M)}{1+2p_M}\right)\right\}k^{12}]$$

$$G_{\gamma_3}=\frac{\xi_1\pi^{\frac{1}{2}}p^{\frac{9}{2}}{}_M}{8\nu^{\frac{3}{2}}(t-t_1)^{\frac{3}{2}}(1+p_M)^8}\exp\left(\frac{-\nu(t-t_1)(1+p_M)(1+2p_M-2p^2{}_M)}{p_M(1+p_M)}\right)k$$

$$[\frac{90p_Mk^7}{\nu^2(t-t_1)^2(1+p_M)^2}+\{\frac{120p_M}{\nu^2(t-t_1)^2}+\frac{60p^2{}_M}{\nu^3(t-t_1)^3(1+p_M)^2}-\frac{30}{\nu^3(t-t_1)^3}\}k^9$$

$$+\{\frac{64p^2{}_M}{\nu(t-t_1)}+\frac{10p^3{}_M}{\nu^2(t-t_1)^2(1+p_M)^2}-\frac{40(1+p_M)^2}{\nu(t-t_1)}\}k^{11}+\{p^2{}_M-p_M(1+p_M)^2\}k^{13}]\int_0^{\frac{\omega_1}{2}}\exp(y^2)dy$$

here, $\omega_1=\left(\frac{\nu(t-t_1)(1+p_M)}{p_M}\right)^{\frac{1}{2}}k$

$$G_{\gamma_4}=\frac{\xi_1\pi^{1/2}p^{15/2}{}_M}{2^8\nu(t-t_1)(1+p_M)^{29/2}}\exp\left(\frac{-\nu(t-t_1)(1+p_M)(1+2p_M)}{p_M}\right)k^2$$

$$[\frac{7560(1+p_M)^3}{\nu^4(t-t_1)^2p^2{}_M}k^6+\{\frac{20160(1+p_M)^5}{\nu^3(t-t_1)^3p_M}-\frac{4233600(1+p_M)^7}{\nu^3(t-t_1)^3p^3{}_M}\}k^8+$$

$$\{\frac{12096(1+p_M)^5}{\nu^2(t-t_1)^2}-\frac{3360(1+p_M)^7}{\nu^2(t-t_1)^2p^2{}_M}\}k^{10}+\{\frac{2304(1+p_M)^5p_M}{\nu(t-t_1)}-\frac{1344(1+p_M)^9}{p^2{}_M}\}k^{12}$$

$$\{128(1+p_M)^5p^2{}_M-128(1+p_M)^7\}k^{14}+\ldots]$$

In equation (42), the quantity $G_\beta$ represents the transfer function arising due to consideration of magnetic field at 3 and $G_\gamma$ for four-point correlation equation in a chemical reaction. Integration of equation (42) over all wave numbers shows that

$$\int_0^\infty G\,d\mathbf{k}=0 \tag{43}$$

Since $G$ is a measure of transfer of energy and the numbers must be zero it satisfies the conditions of continuity and homogeneity, from (37),

$$H=\exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right]\int G\exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right]dt+J(k)\exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right],$$

Notes

where, $J(k)=\dfrac{N_0k^2}{\pi}$ is a constant of integration and can be obtained as by Corrsin [5]. Therefore we obtain,

$$H=\frac{N_0k^2}{\pi}\exp[\frac{-2\nu k^2(t-t_0)}{p_M}]+\exp[\frac{-2\nu k^2(t-t_0)}{p_M}]\int[G_\beta+(G_{\gamma_1}+G_{\gamma_2}+G_{\gamma_3}+G_{\gamma_4})$$

$$\exp\{-(R_s-fs)(t-t_1)\}]\exp[\frac{-2\nu k^2(t-t_0)}{p_M}]dt \qquad (44)$$

From equation (42), we get

$$H=H_1+H_2\exp\{(-R+fs)(t-t_1)\} \qquad (45)$$

In equation (45) $H_1$ and $H_2$ magnetic energy spectrum arising from consideration of the three and four–point correlation equations in a first order chemical reaction respectively. Equation (45) can be integrated over all wave numbers to give the total magnetic turbulent energy.

$$\frac{\langle h_ih_i'\rangle}{2}=\left[\frac{N_0p_M^{\frac{3}{2}}\nu^{-\frac{3}{2}}(t-t_0)^{-\frac{3}{2}}}{8\sqrt{2\pi}}+\xi_0Q\nu^{-6}(t-t_0)^{-5}\right]\exp[-R(t-t_0)]$$

$$+[\xi_1L_1\nu^{-\frac{17}{2}}(t-t_1)^{-\frac{15}{2}}+\xi_1L_2\nu^{-\frac{19}{2}}(t-t_1)^{-\frac{17}{2}}]\exp\{-(R-fs)(t-t_1)\} \qquad (46)$$

This represents the equation of the decay of dusty fluid MHD turbulence in a first order chemical reaction for four point correlation.

$$Q=\frac{\pi.p^6M}{(1+P_M)(1+2P_M)^{5/2}}$$

$$\left\{\frac{9}{16}+\frac{5P_M(7P_M-6)}{(1+2P_M)}-\frac{35P_M(3p^2{}_M-2p_M+3)}{8(1+2p_M)^2}+\frac{8p_M(3p^2{}_M-2p_M+3)}{3.2^6.(1+2p_M)^3}+\ldots\right\}$$

here,

$L_1=Q_2+Q_4+Q_6+Q_7, L_2=Q_1+Q_3+Q_5$ and $Q^{,s}$ values

$$Q_1=-\frac{\pi.p^6{}_M}{(1+p_M)^{5/2}(1+2p_M-p^2{}_M)^{7/2}}$$

$$[\frac{15.9}{2^6}+\frac{15.7(15-6p_M+21p^2{}_M)}{2^{10}(1+2p_M-p^2{}_M)}+\frac{15.7.3(15-6p_M+36p^2{}_M-6p^3{}_M+61p^4{}_M)}{2^{11}(1+2p_M-p^2{}_M)^2}+$$

$$\left(\frac{11.9.7(1+p^2{}_M)(75-30p_M+180p^2{}_M-30p^3{}_M+305p^4{}_M)}{2^{13}(1+2p_M-p^2{}_M)^3}\right)+$$

$$\left(\frac{13.11.9.7(1+p^2{}_M)^2(75-3p_M+90p^2{}_M-30p^3{}_M+15p^4{}_M)}{2^{14}(1+2p_M-p^2{}_M)^4}\right)-\ldots\ldots\ldots\ldots]$$

$$Q_2=-\frac{\pi.p^{21/2}{}_M}{(1+p_M)^{3/2}(1+2p_M-p^2{}_M)^{9/2}}$$

Ref

5. S. Corrsin, On the spectrum of isotropic temperature fluctuations in isotropic turbulence. J. Appl. Phys. **22**, 469-473 (1951). http://dx.doi.org/10.1063/1.1699986

$$[\frac{15.7}{2^6}+\frac{15.9.7(14p^2{}_M-18-40p_M)}{2^9(1+2p_M-p^2{}_M)}+\frac{15.11.9.7(14p^4{}_M-56p^3{}_M-12p^2{}_M-40p_M-18)}{2^{10}(1+2p_M-p^2{}_M)^2}-\quad\ldots]$$

$$Q_3=-\frac{\pi.p^{19/2}{}_M(1+p_M)^{1/2}}{(1+2p_M)^2(1+2p_M-p^2{}_M)^{7/2}}\quad.$$

$$[\frac{9.15}{2^6}+\frac{15.7(17+32p_M-2p^2{}_M+4p^3{}_M+20p^4{}_M)}{2^{10}(1+p_M)^2(1+2p_M-p^2{}_M)}$$

$$+\frac{9.7.5(17+49p_M+13p^2{}_M-13p^3{}_M+98p^4{}_M+134p^5{}_M+104p^6{}_M+60p^7{}_M)}{2^{11}(1+p_M)^3(1+2p_M-p^2{}_M)^2}$$

$$+(\frac{(11.9.7.5(1+p_M-p^2{}_M+p^3{}_M)(17+49p_M+13p^2{}_M-13p^3{}_M+98p^4{}_M+134p^5{}_M+104p^6{}_M+60p^7{}_M)}{2^{13}(1+p_M)^4(1+2p_M-p^2{}_M)^3}$$

$$+(\frac{(13.11.9.7.5(1+p_M-p^2{}_M+p^3{}_M)^2(17+49p_M+13p^2{}_M-13p^3{}_M+98p^4{}_M+134p^5{}_M+104p^6{}_M+60p^7{}_M)}{2^{14}(1+p_M)^5(1+2p_M-p^2{}_M)^4})-\ldots]$$

$$Q_4=-\frac{\pi.p^{21/2}{}_M}{(1+p_M)^{1/2}(1+2p_M)(1+2p_M-p^2{}_M)^{9/2}}\quad[\frac{25.7.3}{2^5}+$$

$$\frac{15.9.7(-40p_M-48p^2{}_M+64p^3{}_M+52p^4{}_M)}{2^9(1+p_M)^2(1+2p_M-p^2{}_M)}+$$

$$\frac{15.11.9.7(-40p_M-89p^2{}_M+51p^3{}_M+124p^4{}_M-40p^5{}_M+36p^6{}_M+60p^7{}_M)}{2^{10}(1+p_M)^3(1+2p_M-p^2{}_M)^2}-\ldots..]$$

$$Q_5=-\frac{\pi.p^{19/2}{}_M}{(1+p_M)^{19/2}(1+2p_M)^{9/2}}$$

$$\left\{\begin{array}{l}\frac{45.7.5.3}{2^{10}}+\frac{9.7.5.3(20p^2{}_M-70p_M-5)}{2^{11}(1+2p_M)}+\frac{11.9.7.5.3(20p^4{}_M-40p^3{}_M+160p^2{}_M-60p_m-5)}{2^{13}(1+2p_M)^2}+\\ \frac{13.11.9.7.5.3(1-2p_M)(20p^4{}_M-40p^3{}_M+160p^2{}_M-60p_M-5)}{2^{14}.(1+2p_M)^3}-\ldots\ldots\ldots\ldots\end{array}\right\}$$

$$Q_6=-\frac{\pi.p^{21/2}{}_M}{(1+p_M)^{15/2}(1+2p_M)^{11/2}}\left\{\frac{15.9.7.5.3}{2^8}+\frac{11.9.7.5.3(24p^2{}_M-200p_M+20)}{2^{11}(1+2p_M)}-\ldots\ldots\ldots\ldots\ldots\ldots\right\}$$

$$Q_7=-\frac{\pi.p^9{}_M}{(1+p_M)^{23/2}(1+2p_M)^{7/2}}$$

$$[\{\frac{9.7.5.3}{2^{11}}-\frac{7.5.3(4231710+16938180p_M+25381440p^2{}_M+1689480p^3{}_M+4213440p^4{}_M}{2^{13}(1+2p_M)}\}$$

$$(9.7.5.3(2115855+4237380p_M-4245780p^2{}_M-16927680p^3{}_M-14783328p^4{}_M$$

$$-\{\frac{-4218816p^5{}_M-4368p^6{}_M)}{2^{14}(1+2p_M)^2}\}\ldots]$$

Equation (46) can be written as

$$\frac{\langle h^2\rangle}{2}=\left(AT_0^{\frac{-3}{2}}+BT_0^{-5}\right)\exp(-RT_0)+\left(CT^{\frac{-15}{2}}+DT^{\frac{-17}{2}}\right)\exp\{(-R+M)T\}. \qquad (47)$$

Notes

Where, $T_0 = (t - t_0)$ and $T = (t - t_1)$

This is the equation of 4-point correlations of dusty fluid MHD turbulence in a first order chemical reaction.

## VI. Results and Discussion

The first term of right hand side of equation (47) corresponds to the energy of magnetic field fluctuation of two-point correlation; the second term represents magnetic energy for the three-point correlation; the third and fourth term represents magnetic energy for four-point correlation. For large times, the second term in the equation becomes negligible, leaving the -3/2 power decay law for the ending phase.

If Chemical reaction and dust particles are absent then equation (47) is of the form

$$\frac{\langle h^2 \rangle}{2} = AT_0^{\frac{-3}{2}} + BT_0^{-5} + CT^{\frac{-15}{2}} + DT^{\frac{-17}{2}}. \tag{48}$$

this is the energy decay of MHD turbulence for four-point correlation. If $\xi_1 = 0$ then the equation (47) becomes

$$\frac{\langle h^2 \rangle}{2} = (AT_0^{\frac{-3}{2}} + BT_0^{-5})\exp(-RT_0) \tag{49}$$

This was obtained earlier by Islam and Sarker [18] for 3-point correlation.

This study shows that the terms associated with the higher-order correlation's die out faster than those associated with the lower order ones. Here three and four-point correlations between fluctuating quantities have been considered and the quintuple correlations are neglected in comparison to the third and fourth order correlations. If the quadruple and quintuple correlations were not neglected, equation (46) contains more terms in negative higher power of $(t - t_1)$ and $(t - t_0)$ would be added to equation (47). In the Figures *h1, h2, h3, h4* and *h5* represents the energy decay curves in a first order chemical reaction of equation (47) at $t_0 = t_1 =$0.5, 1, 1.5, 2, and 2.5 respectively.

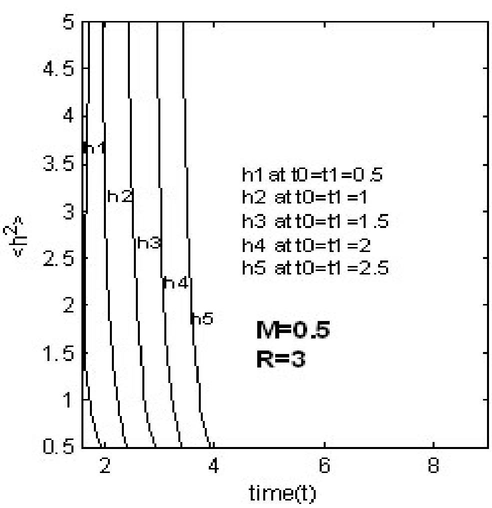
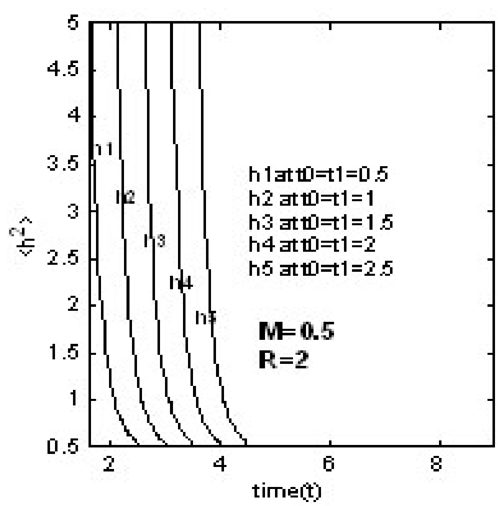



*Figure 1 :* Decomposing curves for *M*=0.5, *R*=0.50.




*Figuyre 2 :* Energy decay curves of equation (47) if *M*=0.5, *R*=2.

Ref

18. Islam, M.A. and M.S.A. Sarker. First order reactant in MHD turbulence before the final period of decay for the case of multi-point and multi-time. Indian J.Pure Appl.Math., 32:1173-1184.2001.

Notes

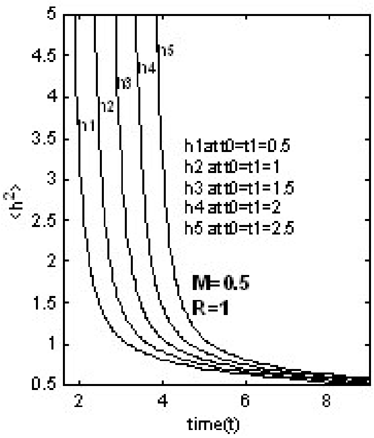



*Figure 3 :* Decomposing curves of equation $M$=0.5, $R$=1

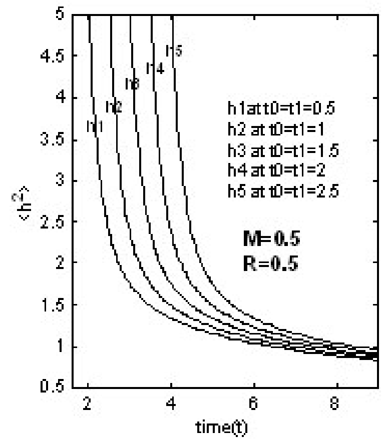



*Figure 4:* Energy decay curves of equation. (47) if $M$=0.5, $R$=0.5

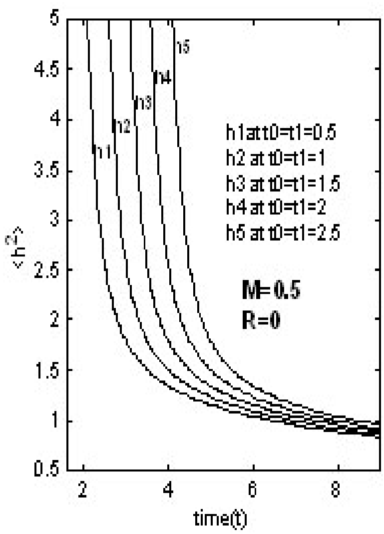



*Figure 5 :* Decomposing curves of equation (48) if $M$=0.5, $R$=0

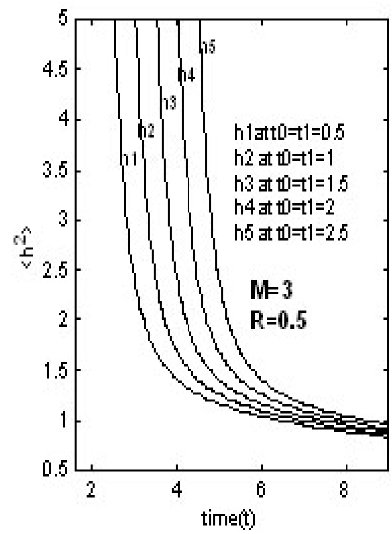



*Figure 6 :* Energy decay curves of equation (47) if $M$=3, $R$=0.5

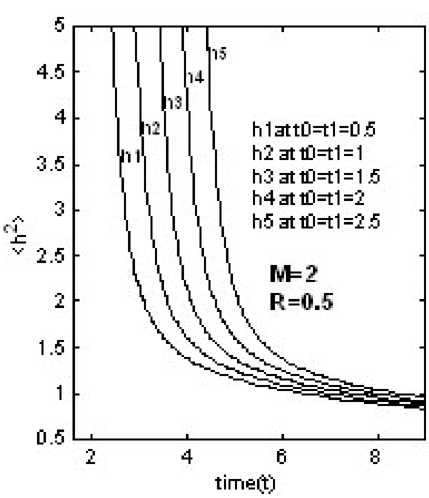



*Figure 7 :* Energy decay curves of equation (47) if $M$=2, $R$=0.5

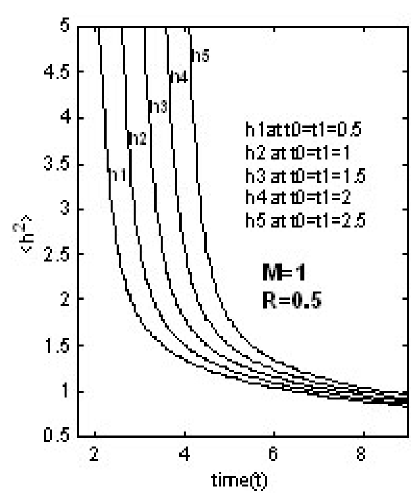



*Figure 8 :* Decay curves of equation (47) if $M$=1, $R$=0.5

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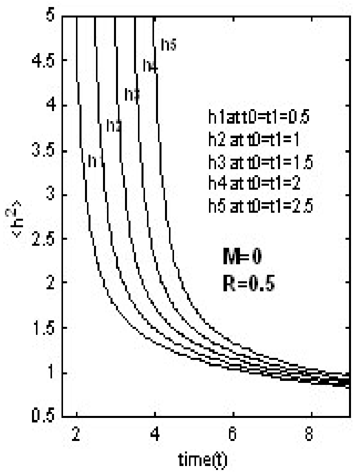


*Figure 9 :* Decomposing curves of equation (47) if $M$=0, $R$=0.5

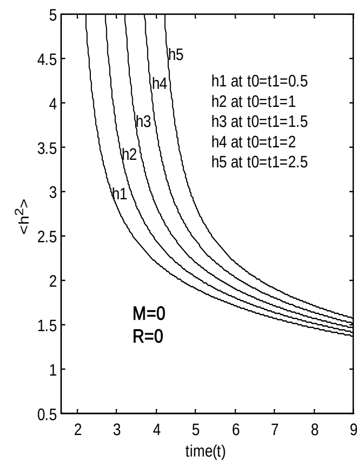


*Figure 10 :* Decay curves of equation (47) if $M$=0, $R$=0

In the Figures *h1, h2, h3, h4* and *h5* represents the energy decay curves in a first order chemical reaction of equation (47) at $t_0 = t_1$ =0.5, 1, 1.5, 2, and 2.5 respectively. From the Figures (1-5) we observed that if $M$=0.5 energy decay increases for the decreases of the values $R$ and maximum if the chemical reaction is absent. If $M$=3, 2, 1, 0 then the decay of energy decreases slowly at the point where $R$=0.5 that are indicated in the Figures (6-9). From Figure: 10 we see that energy decay very rapidly in the clean fluid.

## VII. Conclusion

We conclude that if the concentration selected in the chemical reactant of dusty fluid MHD turbulent flow of the first order at four point correlations, then the result is that the decaying of the concentration fluctuation is much more slow and the slower rate of decay is governed by $\exp[-(R-M)T]$. In the case of clean combination, the decay of concentration fluctuation is much more rapid and the faster rate of decay is due to absent of chemical reaction and dust particles.

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# 9×9 Composite Loubéré Magic Squares Infinite Abelian Group as a Miscellany Case of the 3×3 Loubéré Magic Squares Infinite Abelian Group

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*Abstract-* In this paper, mystic miscellaneous algebraic properties of the set of $9 \times 9$ Composite (Nested) Loubéré Magic Squares are vividly visualized. And, verbatim virtuoso of algebraic properties of the $3 \times 3$ Loubéré Magic Squares viz: Eigen group, Magic Sum group and Centre Pieces group viewed the algebraic properties of its $9 \times 9$ Composite. It is also showcased that both the 2 sets equipped with the matrix binary operation of addition form infinite additive abelian groups.

*Keywords: composite–, loubéré –, abelian group, eigen group, magic sums, centre pieces.*

*GJSFR-F Classification : FOR Code : MSC 2010: 06F20*

*Strictly as per the compliance and regulations of :*

# $9 \times 9$ Composite Loubéré Magic Squares Infinite Abelian Group as a Miscellany Case of the $3 \times 3$ Loubéré Magic Squares Infinite Abelian Group

Babayo A. M [α], G. U. Garba [σ] & Abdul Aziz Garba Ahmad [ρ]

*Abstract-* In this paper, mystic miscellaneous algebraic properties of the set of $\mathbf{9 \times 9}$ Composite (Nested) Loubéŕe Magic Squares are vividly visualized. And, verbatim virtuoso of algebraic properties of the $\mathbf{3 \times 3}$ Loubéré Magic Squares viz: Eigen group, Magic Sum group and Centre Pieces group viewed the algebraic properties of its $\mathbf{9 \times 9}$ Composite. It is also showcased that both the 2 sets equipped with the matrix binary operation of addition form infinite additive abelian groups.

*Keywords: composite–, loubéŕe –, abelian group, eigen group, magic sums, centre pieces.*

## I. Introduction

It is remarkable that almost trivially the sets of eigen values, centre pieces and magic sums of the $3 \times 3$ Loubéré Magic Squares Infinite Abelian Group form Infinite Additive Abelian Groups. For Loubéŕe Magic Squares eigen values computations, see [2].

We highlighted consortium of miscellany effects of rotations and/or reflections [5] and/or enumerations of the $3 \times 3$ Loubéré Magic Squares to figure out the consortium of the $9 \times 9$ composites.

Establishing such a fact relationships set us conjecture that the $9 \times 9$ Composite Loubéré Magic Squares [1] Infinite Additive Abelian Group is a miscellany case of the $3 \times 3$ Loubéré Magic Squares Infinite Abelian Group.

## II. Preliminaries

### a) *Definition 2.1*

A basic magic square of order n can be defined as an arrangement of arithmetic sequence of common difference of 1 from 1 to $n^2$ in an $n \times n$ square grid of cells such that every row, column and diagonal add up to the same number, called the magic sum M(S) expressed as $M(S) = \frac{n^3+n}{2}$ and a centre piece C as $C = \frac{M(S)}{n}$.

### b) *Definition 2.2*

A Composite Loubéŕe Magic Square is a magic square such that each of its cell (grid) is a Loubéré Magic Square. See also [1].

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1. Maya Ahmed (2004). Algebraic Combinatorics of Magic Squares, A Doctor of Philosophy Dissertation, University of California, Davis, pp. 85.

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*c) Definition 2.3*

Main Row or Column is the column or row of the Loubéré Magic Squares containing the first term and the last term of the arithmetic sequence in the square.

*d) Definition 2.4*

A Loubéŕe Magic Square of type I is a magic square of arithmetic sequence entries such that the entries along the main column or row have a common difference and the main column or row is the central column or central row respectively.

*e) Loubéré Procedure (NE-W-S or NW-E-S, the cardinal points)*

Consider an empty $n \times n$ square of grids (or cells). Start, from the central column or row at a position $\left\lceil \frac{n}{2} \right\rceil$ where $\lceil \square \rceil$ is the greater integer number less than or equal to, with the number 1. The fundamental movement for filling the square is diagonally up, right (clock wise or NE or SE) or up left (anti clock wise or NW or SW) and one step at a time. If a filled cell (grid) is encountered, then the next consecutive number moves vertically down ward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S ( depending on the position of the first term of the sequence) to the last row or first row or first column or last column.

The square grid of cells $[a_{ij}]_{n \times n}$ is said to be Loubéré Magic Square if the following conditions are satisfied.

i. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = k$;

ii. $\mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n \times n}} = \mathrm{trace}[\mathrm{a_{ij}}]_{\mathrm{n \times n}}^{\mathrm{T}} = \mathrm{k}$; and

iii. $a_{1,\lceil \frac{n}{2} \rceil}$, $a_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$, $a_{n,\lceil \frac{n}{2} \rceil}$ are on the same main column or row and $a_{\lceil \frac{n}{2} \rceil, n}$, $a_{\lceil \frac{n}{2} \rceil \lceil \frac{n}{2} \rceil}$, $a_{\lceil \frac{n}{2} \rceil, 1}$ are on the same main column or row

,where $\lceil \square \rceil$ is the greater integer less or equal to, T is the transpose (of the square), k is the magic sum (magic product is defined analogously) usually expressed as $k = \frac{n}{2}[2a + (n-1)j]$ − from the sum of arithmetic sequence, where j is the common difference along the main column or row and a is the first term of the sequence− and $a_{\lceil \frac{n}{2} \rceil \lceil \frac{n}{2} \rceil} = \frac{k}{n}$.

*f) Definition 2.6*

Loubéré Magic Squares of type II are magic squares constructed with Loubéré Procedure with repeating – pattern - sequence.

*g) Definition 2.7*

A least subelement magic square of Loubéré Magic Square is a $3 \times 3$ Magic Square formed by removing boarder cells of the Loubéré Magic Squares.

*h) Remark 2.8*

The least subelements magic square of Loubéré Magic Squares are subsets of the semi pancolumn magic squares and the least subelement magic square of the composite Loubéré Magic Square is a $3 \times 3$ Loubéré Magic Square. If we use a repeating pattern sequence a, a, a, ... n times, b, b, b,...n times, c, c, c, ... n times, ... n number; we get Type II(a) Loubéré Magic Square and if instead we use a,b,c,... n number, a, b, c,... n number, ... n times; we get the Type II(b) Loubéré Magic Square.

*i) Group*

A non empty set G together with an operation * is known as a group if the following properties are satisfied.

i. G is closed with respect to *.i.e., $a * b \in G, \forall a, b \in G$.

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ii. $*$ is associative in G. i.e., $a*(b*c)=(a*b)*c, \forall a,b,c \in G$.

iii. $\exists\, e \in G$, such that $e*a=a*e=a, \forall a \in G$. Here e is called the identity element in G with respect to $*$.

iv. $\forall a \in G, \exists b \in G$ such that $a*b=b*a=e$, where e is the identity element. Here b is called the inverse of a and similarly vise versa. The inverse of the element a is denoted as $a^{-1}$.

The above definition of a group is given in [3]. If in addition to the above axioms, the following axiom is satisfied, we call $(G,*)$ an abelian group where $(G,*)$ is a denotation of a group.

v. $a*b=b*a, \forall a,b \in G$. That is all (not some of) the elements of G commutes.

*j) The Proof of the General* $\left\lceil\frac{m^2}{2}\right\rceil = a+\left(\frac{m-1}{2}\right)j$ *and of the General* $M(S)=\frac{m}{2}[2a+(m-1)j]$, *Where* $j=\frac{l-a}{m-1}$

i. *Theorem 2.11*

Let the arithmetic sequence $a, a+d, \dots, l=a+(n-1)d$ be arranged in an $m \times m$ Loubéré Magic Square. Then the magic sum of the square is expressed as $M(S)=\frac{m}{2}\,[\,2a+(m-1)j\,]$ and the middle term of the sequence (centre piece of the square) is expressed as $C=a+\left(\frac{m-1}{2}\right)j$ where $j$ denotes the common difference of entries along the main column or row and is given as $j=\frac{l-a}{m-1}$.

*Proof.* Consider any arbitrary General Loubéré Magic Square [4] (here we consider $3 \times 3$) as follows:

| | | |
|---|---|---|
| $c+b$ | $c-b-d$ | $c+d$ |
| $c-b+d$ | $c$ | $c+b-d$ |
| $c-d$ | $c+b+d$ | $c-b$ |

Let $a=c-b-d$ and $l=c+b+d$. Then we have (from the square) an arithmetic sequence: $c-b-d, c-b, \dots, c+b+d$ having the sums $S$ as

$$S=(c-b-d)+(c-b)+\cdots+(c+b)+(c+b+d) \to (1)$$
$$+$$
$$S=(c+b+d)+(c+b)+\cdots+(c-b)+(c-b-d) \to (2)$$

..............................................

Adding (1)and (2), $\quad 2s=2c+2c+\cdots \quad$ n times

i.e. $2s=2nc \Longrightarrow c=\frac{s}{n} \dots (3)$ and $s=\frac{n}{2}(a+l)\dots(4)$ from the Gaussian High School (Elementary) Method.

Since our square is $m \times m$, m number of cells (terms) are on the main column whence $a=c-b-d$. Thus, (3) and (4) become $C=\frac{M(S)}{m} \dots (5)$ and $M(S)=\frac{m}{2}[a+l]\dots(6)$ respectively. And, $l=a+(m-1)j\dots(7)$ where $j$ is along the main column. Substituting (7) in (6), we have: $M(S)=\frac{m}{2}[2a+(m-1)j] \dots (8)$. Substituting (8) in (5), we get: $C=a+\left(\frac{m-1}{2}\right)j\dots(9)$ From (3) and (4), $C=\frac{1}{2}(a+l)=\left(a-\frac{a}{2}\right)+\frac{l}{2}=a+\frac{(l-a)}{2}=a+\frac{l-a}{m-1}\frac{m-1}{2}$, i.e. $C=a+\left(\frac{m-1}{2}\right)\frac{l-a}{m-1} \dots (10)$. Comparing (9) and (10), we have: $j=\frac{l-a}{m-1} \dots (11)$.

We consider $m \times m$ for the square is more general than the $n \times n$ considered initially.

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*k) Centre Pieces and Magic Sums Abelian Groups*

i. *Centre Pieces Abelian Group*

The set of the centre pieces $c_1, c_2, c_3, \ldots$ of $m \times m$ Loubéré Magic Squares equipped with integer addition forms an infinite additive abelian group. Given the centre pieces $c_1, c_2, c_3, \ldots$ of $m \times m$ Loubéré Magic Squares with corresponding formula $c_1 = a_1 + \left(\frac{m-1}{2}\right) j_1, c_2 = a_2 + \left(\frac{m-1}{2}\right) j_2,$

$$c_3 = a_3 + \left(\frac{m-1}{2}\right) j_3, \ldots; \text{then}$$

- $c_1 + c_2 = (a_1 + a_2) + \left(\frac{m-1}{2}\right)(j_1 + j_2)$ is the centre piece of the $m \times m$ Loubéré Magic Square with first term $a_1 + a_2$ and common difference along the main column as $j_1 + j_2$. Hence, the set is *closed.*
- *Associativity.* This is an inherited property of the set of integer numbers:

$$c_1 + (c_2 + c_3) = (a_1 + a_2 + a_3) + \left(\frac{m-1}{2}\right)(j_1 + j_2 + j_3) = (c_1 + c_2) + c_3$$

- The *identity element* is the zero centre piece e.g.

$$\begin{bmatrix} \mathrm{c} & -\mathrm{d} & \mathrm{a} \\ -\mathrm{b} & 0 & \mathrm{b} \\ -\mathrm{a} & \mathrm{d} & -\mathrm{c} \end{bmatrix}$$

- Given an arbitrary centre piece $c_n = a_n + \left(\frac{m-1}{2}\right) j_n$ of the $m \times m$ Loubéré Magic Square, there exists another centre piece $c_{-n}$ of another $m \times m$ Loubéré Magic Square having first term as $-a_n$ and common difference along the main column or row as $-j_n$ , thus its formulae is $c_{-n} = -a_n + \left(\frac{m-1}{2}\right)(-j_n)$ and is such that $c_n + c_{-n} = c_{-n} + c_n = (a_n - a_n) + \left(\frac{m-1}{2}\right)[j_n - j_n] = 0 = c_i$ , the identity centre piece. $\mathrm{c_n}$ and $\mathrm{c_{-n}}$ are *inverses* of each other.
- Clearly $c_1 + c_2 = a_1 + a_2 + \left(\frac{m-1}{2}\right)(j_1 + j_2) = a_2 + a_1 + \left(\frac{m-1}{2}\right)(j_2 + j_1) = c_2 + c_1$

The set equipped with the operation is an abelian group.

ii. *Magic Sum Abelian Groups*

The set of the magic sums $M(s_1), M(s_2), M(s_3), \ldots$of $m \times m$ Loubéré Magic Squares equipped with the integer number binary operation of addition forms an infinite additive abelian group. Given the magic sums $M(s_1), M(s_2), M(s_3), \ldots$of $m \times m$ Loubéré Magic Squares with corresponding formula

$$M(s_1) = \frac{m}{2}\left[2a_1 + (m-1)j_1, M(s_2) = \frac{m}{2}[2a_2 + (m-1)j_2], M(s_3\right) = \frac{m}{2}[2a_3 + (m-1)j_3, \ldots;$$

then (as in the above).

- $M(s_1) + M(s_2) = M(s_?)$ where $\mathrm{M}(s_?)$ is a magic sum of another $\mathrm{m} \times \mathrm{m}$ Loubéré Magic Square with first term $a_1 + a_2$ and common difference along the main column as $j_1 + j_2$.

The axioms: ii, iii, iv and v follow, by analogy to the centre piece infinite additive abelian group properties, immediately.

iii. *Eigen Values Abelian Group*

The Eigen values computation in the magic squares is what is zealotly prophesized that magic squares are special type of matrices, hence the definition of the magic squares, we do not love to like such a sudden conclusion if loving to liking forces choosing the definitions in terms of just the square grids (or cells).

Notes

We want to show through concrete examples that the set of Eigen Values of the Loubéré Magic Squares with usual integer numbers binary operation of addition forms a group. Consider the following arbitrary two $3 \times 3$ Loubéré Magic Squares −which we let

$$\mathrm{a} = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 1 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \mathrm{b} = \begin{bmatrix} 2 & -5 & 0 \\ -3 & -1 & 1 \\ -2 & 3 & -4 \end{bmatrix} \text{ and their sum } \mathrm{c} = \begin{bmatrix} 6 & -8 & 2 \\ -4 & 0 & 4 \\ -2 & 8 & -6 \end{bmatrix}$$

We compute the eigen values for $a$ as follows: The corresponding matrix of $a$ is $(a) = \begin{pmatrix} 4 & -3 & 2 \\ -1 & 1 & 3 \\ 0 & 5 & -2 \end{pmatrix}$, its eigen vector is $|a - \lambda I| = \begin{vmatrix} 4-\lambda & -3 & 2 \\ -1 & 1-\lambda & 3 \\ 0 & 5 & -2-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 3\lambda^2 - 24\lambda - 72 = (\lambda - 3)(\lambda^2 - 24) = 0$ having eigen values $\lambda_{a_1} = 3$, $\lambda_{a_2} = 4.9 \text{ and } \lambda_{a_3} = -4.9$.

We compute the eigen values for b as follows: The corresponding matrix of b is $(b) = \begin{pmatrix} 2 & -5 & 0 \\ -3 & -1 & 1 \\ -2 & 3 & -4 \end{pmatrix}$, its eigen vector is $|b - \lambda I| = \begin{vmatrix} 2-\lambda & -5 & 0 \\ -3 & -1-\lambda & 1 \\ -2 & 3 & -4-\lambda \end{vmatrix} = 0$ i.e. $\lambda^3 + 3\lambda^2 - 24\lambda - 72 = (\lambda + 3)(\lambda^2 - 24) = 0$ with eigen values $\lambda_{b_1} = -3, \lambda_{b_2} = 4.9 \text{ and } \lambda_{b_3} = -4.9$ .

We compute the eigen values for c as follows: The corresponding matrix of c is $(c) = \begin{pmatrix} 6 & -8 & 2 \\ -4 & 0 & 4 \\ -2 & 8 & -6 \end{pmatrix}$, its eigen vector is $|c - \lambda I| = \begin{vmatrix} 6-\lambda & -8 & 2 \\ -4 & -\lambda & 4 \\ -2 & 8 & -6-\lambda \end{vmatrix} = 0$, i.e. $\lambda^3 - 96\lambda = 0$ with corresponding eigen values $\lambda_{c_1} = 0, \lambda_{c_2} = 9.8$ and $\lambda_{c_3} = -9.8$ .

We now conclude this session by showing that the set of eigen values satisfies *The Properties of an Additive Abelian Group* as follows:

*Closure Property.* Consider any 3 arbitrary Loubéré Magic Squares a, b, c; such that $a + b = c$; then from the example above, the corresponding eigen values of a; $\lambda_{a_1}, \lambda_{a_2} \lambda_{a_3}$; the corresponding eigen values of b; $\lambda_{b_1}, \lambda_{b_2}, \lambda_{b_3}$; are such that $\lambda_{a_1} + \lambda_{b_1} = \lambda_{c_1}$ , $\lambda_{a_2} + \lambda_{b_2} = \lambda_{c_2}$, and $\lambda_{a_3} + \lambda_{b_3} = \lambda_{c_3}$ where $\lambda_{c_1}, \lambda_{c_2}, \lambda_{c_3}$ are the corresponding eigen values of c.

*Associativity Property.* Since Loubéré Magic Squares are a semi group (which is easy to observe), the eigen values are associative.

*Identity Element Property.* The eigen value 0 is the identity element that corresponds to the sum of the Loubéré Magic Squares of opposite eigen values as in the above.

*Inverse Elements Property.* For any arbitrary eigen value $\lambda_{m}$ corresponding to a Loubéré Magic Square m, there exist a $-\lambda_{m}$ eigen value corresponding to another Loubéré Magic Square such that $\lambda_{m} + (-\lambda_{m})$ gives the identity element which is formed as a result of matrix addition of the aforementioned Loubéré Magic Squares.

*Commutativity.* Integer numbers binary operation of addition is commutative.

This completes the proof. The idea of eigen values computation of a magic square is conceived from the work of [2].

## III. 9 × 9 Composite Loubéré Magic Squares Infinite Abelian Group as a Miscellany Case of the 3 × 3 Loubéré Magic Squares Infinite Abelian Group

Let n stands for number of columns, d stands for common difference of entries and f stands for first term of the aforementioned square. Then $\mathcal{S}_{(n,\ d,\ f)}$ denotes the

Ref

2. Daryl Lynn Stephens (1993). Matrix Properties of Magic Squares, Master of Science Professional Paper, College of Arts and Sciences, Denton, Texas, pp. 32.

sequences of $n \times n$ Loubéré Magic Squares of type I of d common difference of its entries and of first terms f, ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ denotes the sequences of $n \times n$ Loubéré Magic Squares of type II(a) of d common difference of its entries and of first terms f, ${}^{2}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ denotes the sequence of type II(b), $\mathcal{CS}_{(n,\ d,\ f)}$ denotes the sequence of the composites of $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{CT}_{(n,\ d,\ f)}$ denotes the sequence of composites of ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$, $\mathcal{CST}_{(n,\ d,\ f)}$ denotes the sequence of the composites of $\mathcal{S}_{(n,\ d,\ f)}$ having entries ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ or simply T(f), and $\mathcal{CTS}_{(n,\ d,\ f)}$ denotes the sequence of the composites of ${}^{1}_{\square}\mathcal{T}_{(n,\ d,\ f)}$ having entries $\mathcal{S}_{(n,\ d,\ f)}$. Then, the sequences are as follows:

$$\mathcal{S}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} 5 & -2 & 3 \\ 0 & 2 & 4 \\ 1 & 6 & -1 \end{bmatrix}, \begin{bmatrix} 6 & -1 & 4 \\ 1 & 3 & 5 \\ 2 & 7 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 5 \\ 2 & 4 & 6 \\ 3 & 8 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 9 & 2 & 7 \\ 4 & 6 & 8 \\ 5 & 10 & 3 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} 12 & -2 & 8 \\ 2 & 6 & 10 \\ 4 & 14 & 0 \end{bmatrix}, \begin{bmatrix} 13 & -1 & 9 \\ 3 & 7 & 11 \\ 5 & 15 & 1 \end{bmatrix}, \begin{bmatrix} 14 & 0 & 10 \\ 4 & 8 & 12 \\ 6 & 16 & 2 \end{bmatrix}, \begin{bmatrix} 15 & 1 & 11 \\ 5 & 9 & 13 \\ 7 & 17 & 3 \end{bmatrix}, \begin{bmatrix} 16 & 2 & 12 \\ 6 & 10 & 14 \\ 8 & 18 & 4 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} 19 & -2 & 13 \\ 4 & 10 & 16 \\ 7 & 22 & 1 \end{bmatrix}, \begin{bmatrix} 20 & -1 & 14 \\ 5 & 11 & 17 \\ 8 & 23 & 2 \end{bmatrix}, \begin{bmatrix} 21 & 0 & 15 \\ 6 & 12 & 18 \\ 9 & 24 & 3 \end{bmatrix}, \begin{bmatrix} 22 & 1 & 16 \\ 7 & 13 & 19 \\ 10 & 25 & 4 \end{bmatrix}, \begin{bmatrix} 23 & 2 & 17 \\ 8 & 14 & 20 \\ 11 & 26 & 5 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{S}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} -9 & -2 & -7 \\ -4 & -6 & -8 \\ -5 & -10 & -3 \end{bmatrix}, \begin{bmatrix} -8 & -1 & -6 \\ -3 & -5 & -7 \\ -4 & -9 & -2 \end{bmatrix}, \begin{bmatrix} -7 & 0 & -5 \\ -2 & -4 & -6 \\ -3 & -8 & -1 \end{bmatrix}, \begin{bmatrix} -6 & 1 & -4 \\ -1 & -3 & -5 \\ -2 & -7 & 0 \end{bmatrix}, \begin{bmatrix} -5 & 2 & -3 \\ 0 & -2 & -4 \\ -1 & -6 & 1 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} -16 & -2 & -12 \\ -6 & -10 & -14 \\ -8 & -18 & -4 \end{bmatrix}, \begin{bmatrix} -15 & -1 & -11 \\ -5 & -9 & -13 \\ -7 & -17 & -3 \end{bmatrix}, \begin{bmatrix} -14 & 0 & -10 \\ -4 & -8 & -12 \\ -6 & -16 & -2 \end{bmatrix}, \begin{bmatrix} -13 & 1 & -9 \\ -3 & -7 & -11 \\ -5 & -15 & -1 \end{bmatrix}, \begin{bmatrix} -12 & 2 & -8 \\ -2 & -6 & -10 \\ -4 & -14 & 0 \end{bmatrix}, \ldots$$

$$\mathcal{S}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} -23 & -2 & -17 \\ -8 & -14 & -20 \\ -11 & -26 & -5 \end{bmatrix}, \begin{bmatrix} -22 & -1 & -16 \\ -7 & -13 & -19 \\ -10 & -25 & -4 \end{bmatrix}, \begin{bmatrix} -21 & 0 & -15 \\ -6 & -12 & -18 \\ -9 & -24 & -3 \end{bmatrix}, \begin{bmatrix} -20 & 1 & -14 \\ -5 & -11 & -17 \\ -8 & -23 & -2 \end{bmatrix}, \begin{bmatrix} -19 & 2 & -13 \\ -4 & -10 & -16 \\ -7 & -22 & -1 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{T}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & -2 \\ -2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & -1 \\ -1 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 4 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 & 5 \\ 5 & 3 & 1 \\ 1 & 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 2 & 6 \\ 6 & 4 & 2 \\ 2 & 6 & 4 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 5 \\ 5 & 2 & -1 \\ -1 & 5 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 0 & 6 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 7 \\ 7 & 4 & 1 \\ 1 & 7 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 2 & 8 \\ 8 & 5 & 2 \\ 2 & 8 & 5 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{T}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} -3 & -2 & -4 \\ -4 & -3 & -2 \\ -2 & -4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & -1 & -3 \\ -3 & -2 & -1 \\ -1 & -3 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -2 \\ -2 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 2 & 8 \\ 8 & 5 & 2 \\ 2 & 8 & 5 \end{bmatrix}, \ldots$$

$$\mathcal{T}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} -4 & -2 & -6 \\ -6 & -4 & -2 \\ -2 & -6 & -4 \end{bmatrix}, \begin{bmatrix} -3 & -1 & -5 \\ -5 & -3 & -1 \\ -1 & -5 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 0 & -4 \\ -4 & -2 & 0 \\ 0 & -4 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -3 \\ -3 & -1 & 1 \\ 1 & -3 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}, \ldots$$

Notes

$$\mathcal{T}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} -5 & -2 & -8 \\ -8 & -5 & -2 \\ -2 & -8 & -5 \end{bmatrix}, \begin{bmatrix} -4 & -1 & -7 \\ -7 & -4 & -1 \\ -1 & -7 & -4 \end{bmatrix}, \begin{bmatrix} -3 & 0 & -6 \\ -6 & -3 & 0 \\ 0 & -6 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 1 & -5 \\ -5 & -2 & 1 \\ 1 & -5 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -4 \\ -4 & -1 & 2 \\ 2 & -4 & -1 \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CS}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} \\ \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} \\ \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} & \mathcal{S}_{(3,\ 0,-1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} \\ \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} \\ \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} & \mathcal{S}_{(3,\ 0,\ 0)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} \\ \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} \\ \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} & \mathcal{S}_{(3,\ 0,\ 1)} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 1,\ 5)} & \mathcal{S}_{(3,1,-2)} & \mathcal{S}_{(3,\ 1,3)} \\ \mathcal{S}_{(3,\ 1,0)} & \mathcal{S}_{(3,\ 1,2)} & \mathcal{S}_{(3,\ 1,4)} \\ \mathcal{S}_{(3,1,1)} & \mathcal{S}_{(3,\ 1,6)} & \mathcal{S}_{(3,1,-1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 1,6)} & \mathcal{S}_{(3,\ 1,-1)} & \mathcal{S}_{(3,1,4)} \\ \mathcal{S}_{(3,\ 1,1)} & \mathcal{S}_{(3,\ 1,3)} & \mathcal{S}_{(3,\ 1,5)} \\ \mathcal{S}_{(3,\ 1,2)} & \mathcal{S}_{(3,\ 1,7)} & \mathcal{S}_{(3,\ 1,0)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 1,7)} & \mathcal{S}_{(3,\ 1,0)} & \mathcal{S}_{(3,1,5)} \\ \mathcal{S}_{(3,\ 1,2)} & \mathcal{S}_{(3,\ 1,4)} & \mathcal{S}_{(3,\ 1,6)} \\ \mathcal{S}_{(3,\ 1,3)} & \mathcal{S}_{(3,\ 1,8)} & \mathcal{S}_{(3,\ 1,1)} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 2,12)} & \mathcal{S}_{(3,\ 2,-2)} & \mathcal{S}_{(3,\ 2,8)} \\ \mathcal{S}_{(3,\ 2,2)} & \mathcal{S}_{(3,2,6)} & \mathcal{S}_{(3,\ 2,10)} \\ \mathcal{S}_{(3,\ 2,4)} & \mathcal{S}_{(3,\ 2,14)} & \mathcal{S}_{(3,\ 2,0)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 2,13)} & \mathcal{S}_{(3,\ 2,-1)} & \mathcal{S}_{(3,\ 2,9)} \\ \mathcal{S}_{(3,\ 2,3)} & \mathcal{S}_{(3,\ 2,7)} & \mathcal{S}_{(3,\ 2,11)} \\ \mathcal{S}_{(3,\ 2,5)} & \mathcal{S}_{(3,\ 2,15)} & \mathcal{S}_{(3,\ 2,1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 2,14)} & \mathcal{S}_{(3,\ 2,0)} & \mathcal{S}_{(3,\ 2,10)} \\ \mathcal{S}_{(3,\ 2,4)} & \mathcal{S}_{(3,\ 2,8)} & \mathcal{S}_{(3,\ 2,12)} \\ \mathcal{S}_{(3,\ 2,6)} & \mathcal{S}_{(3,\ 2,16)} & \mathcal{S}_{(3,2,2} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} \mathcal{S}_{(3,\ 3,19)} & \mathcal{S}_{(3,\ 3,-2)} & \mathcal{S}_{(3,\ 3,13)} \\ \mathcal{S}_{(3,\ 3,4)} & \mathcal{S}_{(3,\ 3,10)} & \mathcal{S}_{(3,\ 3,16)} \\ \mathcal{S}_{(3,\ 3,7)} & \mathcal{S}_{(3,\ 3,22)} & \mathcal{S}_{(3,3,1)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 3,20)} & \mathcal{S}_{(3,\ 3,-1)} & \mathcal{S}_{(3,\ 3,14)} \\ \mathcal{S}_{(3,3,5)} & \mathcal{S}_{(3,\ 3,11)} & \mathcal{S}_{(3,\ 3,17)} \\ \mathcal{S}_{(3,\ 3,8)} & \mathcal{S}_{(3,3,23)} & \mathcal{S}_{(3,\ 3,2)} \end{bmatrix}, \begin{bmatrix} \mathcal{S}_{(3,\ 3,21)} & \mathcal{S}_{(3,3,0)} & \mathcal{S}_{(3,3,15)} \\ \mathcal{S}_{(3,\ 3,6)} & \mathcal{S}_{(3,3,122)} & \mathcal{S}_{(3,3,18)} \\ \mathcal{S}_{(3,\ 3,9)} & \mathcal{S}_{(3,\ 3,24)} & \mathcal{S}_{(3,3,3)} \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CS}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} -\mathcal{S}_{(3,\ 1,\ 5)} & -\mathcal{S}_{(3,1,-2)} & -\mathcal{S}_{(3,\ 1,3)} \\ -\mathcal{S}_{(3,\ 1,0)} & -\mathcal{S}_{(3,\ 1,2)} & -\mathcal{S}_{(3,\ 1,4)} \\ -\mathcal{S}_{(3,1,1)} & -\mathcal{S}_{(3,\ 1,6)} & -\mathcal{S}_{(3,1,-1)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 1,6)} & -\mathcal{S}_{(3,\ 1,-1)} & -\mathcal{S}_{(3,1,4)} \\ -\ \mathcal{S}_{(3,\ 1,1)} & -\mathcal{S}_{(3,\ 1,3)} & -\mathcal{S}_{(3,\ 1,5)} \\ \mathcal{S}_{(3,\ 1,2)} & -\mathcal{S}_{(3,\ 1,7)} & -\mathcal{S}_{(3,\ 1,0)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 1,7)} & -\mathcal{S}_{(3,\ 1,0)} & -\mathcal{S}_{(3,1,5)} \\ -\ \mathcal{S}_{(3,\ 1,2)} & -\mathcal{S}_{(3,\ 1,4)} & -\mathcal{S}_{(3,\ 1,6)} \\ -\mathcal{S}_{(3,\ 1,3)} & -\mathcal{S}_{(3,\ 1,8)} & -\mathcal{S}_{(3,\ 1,1)} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} -\mathcal{S}_{(3,\ 2,12)} & -\mathcal{S}_{(3,\ 2,-2)} & -\mathcal{S}_{(3,\ 2,8)} \\ -\mathcal{S}_{(3,\ 2,2)} & -\mathcal{S}_{(3,2,6)} & -\mathcal{S}_{(3,\ 2,10)} \\ -\mathcal{S}_{(3,\ 2,4)} & -\mathcal{S}_{(3,\ 2,14)} & -\mathcal{S}_{(3,\ 2,0)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 2,13)} & -\mathcal{S}_{(3,\ 2,-1)} & -\mathcal{S}_{(3,\ 2,9)} \\ -\mathcal{S}_{(3,\ 2,3)} & -\mathcal{S}_{(3,\ 2,7)} & -\mathcal{S}_{(3,\ 2,11)} \\ -\mathcal{S}_{(3,\ 2,5)} & -\mathcal{S}_{(3,\ 2,15)} & -\mathcal{S}_{(3,\ 2,1)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 2,14)} & -\mathcal{S}_{(3,\ 2,0)} & -\mathcal{S}_{(3,\ 2,10)} \\ -\mathcal{S}_{(3,\ 2,4)} & -\mathcal{S}_{(3,\ 2,8)} & -\mathcal{S}_{(3,\ 2,12)} \\ -\mathcal{S}_{(3,\ 2,6)} & -\mathcal{S}_{(3,\ 2,16)} & -\mathcal{S}_{(3,2,2} \end{bmatrix}, \ldots$$

$$\mathcal{CS}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} -\mathcal{S}_{(3,\ 3,19)} & -\mathcal{S}_{(3,\ 3,-2)} & -\mathcal{S}_{(3,\ 3,13)} \\ -\mathcal{S}_{(3,\ 3,4)} & -\mathcal{S}_{(3,\ 3,10)} & -\mathcal{S}_{(3,\ 3,16)} \\ -\mathcal{S}_{(3,\ 3,7)} & -\mathcal{S}_{(3,\ 3,22)} & -\mathcal{S}_{(3,3,1)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 3,20)} & -\mathcal{S}_{(3,\ 3,-1)} & -\mathcal{S}_{(3,\ 3,14)} \\ -\mathcal{S}_{(3,3,5)} & -\mathcal{S}_{(3,\ 3,11)} & -\mathcal{S}_{(3,\ 3,17)} \\ -\mathcal{S}_{(3,\ 3,8)} & -\mathcal{S}_{(3,3,23)} & -\mathcal{S}_{(3,\ 3,2)} \end{bmatrix}, \begin{bmatrix} -\mathcal{S}_{(3,\ 3,21)} & -\mathcal{S}_{(3,3,0)} & -\mathcal{S}_{(3,3,15)} \\ \mathcal{S}_{(3,\ 3,6)} & -\mathcal{S}_{(3,3,122)} & -\mathcal{S}_{(3,3,18)} \\ -\mathcal{S}_{(3,\ 3,9)} & -\mathcal{S}_{(3,\ 3,24)} & -\mathcal{S}_{(3,3,3)} \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CT}_{(3,\ 0,\ f)} = \cdots, \begin{bmatrix} T(-2) & T(-2) & T(-2) \\ T(-2) & T(-2) & T(-2) \\ T(-2) & T(-2) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(-1) & T(-1) \\ T(-1) & T(-1) & T(-1) \\ T(-1) & T(-1) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(0) & T(0) \\ T(0) & T(0) & T(0) \\ T(0) & T(0) & T(0) \end{bmatrix}, \begin{bmatrix} T(1) & T(1) & T(1) \\ T(1) & T(1) & T(1) \\ T(1) & T(1) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(2) & T(2) \\ T(2) & T(2) & T(2) \\ T(2) & T(2) & T92) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,\ 1,\ f)} = \cdots, \begin{bmatrix} T(-1) & T(-2) & T(0) \\ T(0) & T(-1) & T(-2) \\ T(-2) & T(0) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(-1) & T(1) \\ T(1) & T(0) & T(-1) \\ T(-1) & T(1) & T(0) \end{bmatrix}, \begin{bmatrix} T(1) & T(0) & T(2) \\ T(2) & T(1) & T(0) \\ T(0) & T(2) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(1) & T(3) \\ T(3) & T(2) & T(1) \\ T(1) & T(3) & T(2) \end{bmatrix}, \begin{bmatrix} T(3) & T(2) & T(4) \\ T(4) & T(3) & T(2) \\ T(2) & T(4) & T(3) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,\ 2,\ f)} = \cdots, \begin{bmatrix} T(0) & T(-2) & T(2) \\ T(2) & T(0) & T(-2) \\ T(-2) & T(2) & T(0) \end{bmatrix}, \begin{bmatrix} T(1) & T(-1) & T(3) \\ T(3) & T(1) & T(-1) \\ T(-1) & T(3) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(0) & T(4) \\ T(4) & T(2) & T(0) \\ T(0) & T(4) & T(2) \end{bmatrix}, \begin{bmatrix} T(3) & T(1) & T(5) \\ T(5) & T(3) & T(1) \\ T(1) & T(5) & T(3) \end{bmatrix}, \begin{bmatrix} T(4) & T(2) & T(6) \\ T(6) & T(4) & T(2) \\ T(2) & T(6) & T(4) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,\ 3,\ f)} = \cdots, \begin{bmatrix} T(1) & T(-2) & T(4) \\ T(4) & T(1) & T(-2) \\ T(-2) & T(4) & T(1) \end{bmatrix}, \begin{bmatrix} T(2) & T(-1) & T(5) \\ T(5) & T(2) & T(-1) \\ T(-1) & T(5) & T(2) \end{bmatrix}, \begin{bmatrix} T(3) & T(0) & T(6) \\ T(6) & T(3) & T(0) \\ T(0) & T(6) & T(3) \end{bmatrix}, \begin{bmatrix} T(4) & T(1) & T(7) \\ T(7) & T(4) & T(1) \\ T(1) & T(7) & T(4) \end{bmatrix}, \begin{bmatrix} T(5) & T(2) & T(8) \\ T(8) & T(5) & T92) \\ T(2) & T(8) & T(5) \end{bmatrix}, \ldots$$

$\vdots$

$$\mathcal{CT}_{(3,-1,\ f)} = \cdots, \begin{bmatrix} T(-3) & T(-2) & T(-4) \\ T(-4) & T(-3) & T(-2) \\ T(-2) & T(-4) & T(-3) \end{bmatrix}, \begin{bmatrix} T(-2) & T(-1) & T(-3) \\ T(-3) & T(-2) & T(-1) \\ T(-1) & T(-3) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(0) & T(-2) \\ T(-2) & T(-1) & T(0) \\ T(0) & T(-2) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(1) & T(-1) \\ T(-1) & T(0) & T(1) \\ T(1) & T(-1) & T(0) \end{bmatrix}, \begin{bmatrix} T(5) & T(2) & T(8) \\ T(8) & T(5) & T(2) \\ T(2) & T(8) & T(5) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,-2,\ f)} = \cdots, \begin{bmatrix} T(-4) & T(-2) & T(-6) \\ T(-6) & T(-4) & T(-2) \\ T(-2) & T(-6) & T(-4) \end{bmatrix}, \begin{bmatrix} T(-3) & T(-1) & T(-5) \\ T(-5) & T(-3) & T(-1) \\ T(-1) & T(-5) & T(-3) \end{bmatrix}, \begin{bmatrix} T(-2) & T(0) & T(-4) \\ T(-4) & T(-2) & T(0) \\ T(0) & T(-4) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(1) & T(-3) \\ T(-3) & T(-1) & T(1) \\ T(1) & T(-3) & T(-1) \end{bmatrix}, \begin{bmatrix} T(0) & T(2) & T(-2) \\ T(-2) & T(0) & T(2) \\ T(2) & T(-2) & T(0) \end{bmatrix}, \ldots$$

$$\mathcal{CT}_{(3,-3,\ f)} = \cdots, \begin{bmatrix} T(-5) & T(-2) & T(-8) \\ T(-8) & T(-5) & T(-2) \\ T(-2) & T(-8) & T(-5) \end{bmatrix}, \begin{bmatrix} T(-4) & T(-1) & T(-7) \\ T(-7) & T(-4) & T(-1) \\ T(-1) & T(-7) & T(-4) \end{bmatrix}, \begin{bmatrix} T(-3) & T(0) & T(-6) \\ T(-6) & T(-3) & T(0) \\ T(0) & T(-6) & T(-3) \end{bmatrix}, \begin{bmatrix} T(-2) & T(1) & T(-5) \\ T(-5) & T(-2) & T(1) \\ T(1) & T(-5) & T(-2) \end{bmatrix}, \begin{bmatrix} T(-1) & T(2) & T(-4) \\ T(-4) & T(-1) & T(2) \\ T(2) & T(-4) & T(-1) \end{bmatrix}, \ldots$$

$\vdots$

### a) Remarks 3.1

$\mathcal{CTS}_{(\mathrm{n},\ \mathrm{d},\ \mathrm{f})}$ and $\mathcal{CST}_{(\mathrm{n},\ \mathrm{d},\ \mathrm{f})}$ are enumerated analogously.

Notes

*b) The Generalized* $3 \times 3$ *Loubéŕe Magic Square*

Let $\mathbb{Z}$ denotes the set of integer numbers, $\overline{\vee}$ denotes exclusive 'or' and $\vee$ denotes inclusive 'or'. Then the general $3 \times 3$ Loubéré Magic Square is given by

$$G_{3\times 3\,L} := \left\{ \left( \begin{bmatrix} f+7d & f & f+5d \\ f+2d & f+4d & f+6d \\ f+3d & f+8d & f+d \end{bmatrix} \vee \begin{bmatrix} c+b & c-(b+d) & c+d \\ c-b+d & c & c+b-d \\ c-d & c+(b+d) & c-b \end{bmatrix} \right)^{1 \text{ or } C} \overline{\vee} \right.$$

$$\left. \vee \left( \begin{bmatrix} d & f & c \\ c & d & f \\ f & c & d \end{bmatrix} \vee \begin{bmatrix} c & f & d \\ f & d & c \\ d & c & f \end{bmatrix} \right)^{1 \text{ or } M \text{ or } C} : c, d, f \in \mathbb{Z} \right\}$$

Where $S^M$ denotes the miscellany effects of rotations and"or reflections of $S$ and $S^C$ denotes composition of S.

The advantage of this generalization is that it has covered both miscellany effects and composites. It also consider $9 \times 9$ Composite Loubéré a special case of the $3 \times 3$ Loubéré .

*c) Theorem 3.3.*

The set of $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$ , $\mathcal{CS}_{(n,\ d,\ f)}$, $\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ form Infinite Additive Abelian Groups.

*Proof.* The sum of two sequences of the types $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$ , $\mathcal{CS}_{(n,\ d,\ f)}$, $\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ is a sequence of their type. Thus, *Closure Property* is exhibited.

*Associativity Property.* This is an inherited property of closure above whence we have integer number entries in the sequences.

*Identity Property.* The identities are the sequence $\mathcal{S}_{(n,\ 0,\ f)}$, $\mathcal{T}_{(n,\ 0,\ f)}$, $\mathcal{CS}_{(n,\ 0,\ f)}$, $\mathcal{CT}_{(n,\ 0,\ f)}$, $\mathcal{CTS}_{(n,\ 0,\ f)}$ and $\mathcal{CST}_{(n,\ 0,\ f)}$ respectively.

*Inverse Property.* Each element in $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$ , $\mathcal{CS}_{(n,\ d,\ f)}$, $\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ has an inverse. Examples. $\mathcal{S}_{(n,\ z,\ y)}$, $\mathcal{T}_{(n,\ ,z,\ y)}$, $\mathcal{CS}_{(n,\ z,\ y)}$, $\mathcal{CT}_{(n,\ z,\ y)}$, $\mathcal{CTS}_{(n,\ z,\ y)}$ and $\mathcal{CST}_{(n,\ z,\ y)}$ have inverses $\mathcal{S}_{(n,-z,\ y)}$, $\mathcal{T}_{(n,\ ,-z,\ y)}$, $\mathcal{CS}_{(n,-z,\ y)}$, $\mathcal{CT}_{(n,-z,\ y)}$, $\mathcal{CTS}_{(n,-z,\ y)}$ and $\mathcal{CST}_{(n,-z,\ y)}$ respectively.

*Commutativity.* Integer numbers binary operation of addition is commutative.
This completes the proof.

*d) Conjecture 3.4*

The 3 Eigen Values, Magic Sums and Centre Pieces of the $9 \times 9$ Composite Loubéré Magic Square are 3 times that of the $3 \times 3$ Loubéré Magic Square.

*Proof.* This is manifested clearly in the enumeration of $\mathcal{S}_{(n,\ d,\ f)}$, $\mathcal{T}_{(n,\ d,\ f)}$, $\mathcal{CS}_{(n,\ d,\ f)}$, $\mathcal{CT}_{(n,\ d,\ f)}$, $\mathcal{CTS}_{(n,\ d,\ f)}$ and $\mathcal{CST}_{(n,\ d,\ f)}$ above.

*e) Theorem 3.5*

The 3 Eigen Values, Magic Sums and Centre Pieces of the $9 \times 9$ Composite Loubéré Magic Square that are multiples of that of the $3 \times 3$ Loubéré Magic Square form Infinite Additive Abelian Groups.

*Proof.* If a set of integer numbers equipped with an operation is a group, then 3 times the set of corresponding elements of the set equipped with the same operation is also a group.

Notes

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Notes


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# A Class of Multivalent Harmonic Functions Involving Salagean Operator

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*Introduction-* A continuous complex valued function f=u+iv defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. Let F and G be analytic in D so that F(0)=G(0)=0, ReF = Ref=u, ReG = Imf=v by writing (F+iG)/2 = h, (F-iG)/2 = g, The function f admits the representation f = h + g , where h and g are analytic in D. h is called the analytic part of f and g, the co-analytic part of f.

*GJSFR-F Classification : FOR Code : MSC 2010: 30F15*

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*Strictly as per the compliance and regulations of :*

# A Class of Multivalent Harmonic Functions Involving Salagean Operator

Noohi Khan

## I. Introduction

A continuous complex valued function f=u+iv defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. Let F and G be analytic in D so that F(0)=G(0)=0, ReF = Ref=u, ReG = Imf=v by writing (F+iG)/2 = h, (F-iG)/2 = g, The function f admits the representation $f = h + \overline{g}$ where h and g are analytic in D. h is called the analytic part of f and g, the co-analytic part of f.

Ahuja and Jahangiri [1], [2] introduce and studied certain subclasses of the family SH(m), $m \geq 1$ of all multivalent harmonic and orientation preserving functions in $\Delta = \{z : |z| < 1\}$. A function f in SH(m) can be expressed as $f = h + \overline{g}$, where h and g are analytic functions of the form

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}$$

$$g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1. \tag{1}$$

For analytic function $h(z) \in S(m)$ Salagean [3] introduced an operator $D_m^v$ defined as follows:

$$D_m^0 h(z) = h(z), \; D_m^1 h(z) = D_m(h(z)) = \frac{z}{m} h'(z) \text{ and}$$

$$D_m^v h(z) = D_m(D_m^{v-1} h(z)) = \frac{z(D_m^{v-1} h(z))'}{m} \tag{2}$$

$$= z + \sum_{n=2}^{\infty} \left( \frac{n+m-1}{m} \right)^v a_{n+m-1} z^{n+m-1}, \quad v \in N.$$

Whereas, Jahangiri et al. [4] defined the Salagean operator $D_m^v f(z)$ for multivalent harmonic function as follows:

$$D_m^v f(z) = D_m^v h(z) + (-1)^v D_m^v g(z) \tag{3}$$

where,

$$D_m^v h(z) = z^m + \sum_{n=2}^{\infty} \left( \frac{n+m-1}{m} \right)^v a_{n+m-1} z^{n+m-1}$$

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Ref

1. Ahuja,O.P. and Jahangiri,J.M., Multivalent Harmonic starlike Functions, Ann. Univ Mariae Curie – Sklodowska, Section A, 55 (1) (2001), 1-13.

$$D_m^\nu g(z)=\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu} b_{n+m-1}z^{n+m-1}.$$

In this paper we define a sub class $H_m(\lambda,\nu,\alpha)$ of m-valent harmonic functions involving Salagean operator $D_m^\nu f(z)$ as follows:

*Definition 1*

Let $f(z)=h(z)+\overline{g(z)}$ be the harmonic multivalent function of the form (1), then $f\in H_m(\lambda,\nu,\alpha)$ if and only if

$$\operatorname{Re}\left\{(1-\lambda)\frac{D_m^\nu f(z)}{z^m}+\lambda\frac{\frac{\partial}{\partial\theta}D_m^\nu f(z)}{\frac{\partial}{\partial\theta}z^m}\right\}>\alpha \tag{4}$$

where $0\le\alpha<1,\lambda\ge 0$, $z=re^{i\theta}\in\Delta$ and $D_m^\nu f(z)$ is defined by (3) and

$$\frac{\partial}{\partial\theta}D_m^\nu f(z)=i\left[z(D_m^\nu h(z))'-(-1)^\nu\overline{z(D_m^\nu g(z))'}\right],\quad \frac{\partial}{\partial\theta}z^m=imz^m.$$

We denote the subclass $TH_m(\lambda,\nu,\alpha)$ consist of harmonic functions $f_\nu=h+\overline{g}_\nu$ in $H_m(\lambda,\nu,\alpha)$ so that h and $g_\nu$ are of the form

$$h(z)=z^m-\sum_{n=2}^{\infty}|a_{n+m-1}|z^{n+m-1},$$

$$g_\nu(z)=(-1)^\nu\sum_{n=1}^{\infty}|b_{n+m-1}|z^{n+m-1},\ |b_m|<1.$$

Also note that $TH_m(\lambda,\nu,o)\equiv TH_m(\lambda,\nu)$.

The class $H_m(\lambda,\nu,\alpha)$ provides a transition between two classes:

$$\operatorname{Re}\left\{\frac{D_m^\nu f(z)}{z^m}\right\}>\alpha \text{ and } \operatorname{Re}\left\{\frac{\frac{\partial}{\partial\theta}D_m^\nu f(z)}{\frac{\partial}{\partial\theta}z^m}\right\}>\alpha \text{ as } \lambda \text{ moves between } 0 \text{ and } 1.$$

Denote $H_m(0,\nu,\alpha)$ by $P_m(\nu,\alpha)$ and $H_m(1,\nu,\alpha)$ by $Q_m(\nu,\alpha)$.

In this paper first we obtained the sufficient coefficient condition for $f(z)\in H_m(\lambda,\nu,\alpha)$ and then it is shown that this coefficient condition is also necessary for $f(z)\in TH_m(\lambda,\nu,\alpha)$. Also distortion bounds, extreme points, convex combination, integral operator, convolution condition, radius of convexity, radius of starlikeness for the functions $f(z)\in TH_m(\lambda,\nu,\alpha)$ are obtained.

## II. Main Results

### a) *Theorem 1 (Sufficient coefficient condition for* $H_m(\lambda,\nu,\alpha)$*)*

Assume that $f=h+\overline{g}$, h and g be given by (1) and $\lambda\ge 0$, if

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]|a_{n+m-1}|+$$

$$\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left|\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right||b_{n+m-1}|\le 1-\alpha,0\le\alpha<1 \tag{6}$$

then, $f(z)\in H_m(\lambda,\nu,\alpha)$.

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*b) Remark 2*

The coefficient bound (6) in above theorem is sharp for the function

$$f(z)=z^{m}+\sum_{n=2}^{\infty}\frac{x_n}{\left(\frac{n+m-1}{m}\right)^{v}\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]}z^{n+m-1} \tag{7}$$

$$+\sum_{n=1}^{\infty}\overline{\frac{y_n}{\left(\frac{n+m-1}{m}\right)^{v}\left\|\left[\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right]\right\|}z^{n+m-1}}$$

where

$$\frac{1}{1-\alpha}\left(\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|\right)=1.$$

*c) Remark 3*

For $\lambda\geq 1$,

$$1\leq\left(\frac{n+m-1}{m}\right)\leq\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]\leq\left[\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right]. \tag{8}$$

*d) Corollary 4*

Let $f=h+\overline{g}$ be such that h and g are given by (1) and let

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{v}\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]|a_{n+m-1}|+ \tag{9}$$

$$\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{v}\left[\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right]|b_{n+m-1}|\leq 1-\alpha$$

for $\lambda\geq 1$ and $0\leq\alpha<1$, then $f\in H\,(\lambda,v,\alpha)$.

Putting $\lambda=0$ in Theorem 1 the following Corollary is obtained.

*e) Corollary 5*

Let $f=h+\overline{g}$ be such that h and g are given by (1) and let

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{v}|a_{n+m-1}|+\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{v}|b_{n+m-1}|\leq 1-\alpha$$

for $0\leq\alpha<1$, then $f\in P_m(v,\alpha)$.

Putting $\lambda=1$ in Theorem 1 the following Corollary is obtained.

*f) Corollary 6*

Let $f=h+\overline{g}$ be such that h and g are given by (1) and let

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{v+1}|a_{n+m-1}|+\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{v+1}|b_{n+m-1}|\leq 1-\alpha$$

for $0\leq\alpha<1$, then $f\in Q_m(v,\alpha)$.

*g) Remark 7*

$H_m(\lambda,v,\alpha_2)\subseteq H_m(\lambda,v,\alpha_1)$ for $\alpha_1\leq\alpha_2$. Also, $Q_m(v,\alpha)\subset P_m(v,\alpha)$.

*h) Theorem 8 (Coefficient inequality for* $TH_m(\lambda,\nu,\alpha)$)

Let $f_\nu = h + \overline{g}_\nu$ be so that h and $g_\nu$ are given by (5). Then, $f_\nu \in TH_m(\lambda,\nu,\alpha)$

$$\sum_{n=2}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]|a_{n+m-1}|+$$

$$\sum_{n=1}^{\infty}\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right]|b_{n+m-1}|\le 1-\alpha \tag{10}$$

where $0\le\alpha<1, \lambda\ge 1$ and $|a_m|=1$.

*i) Theorem 9 (Distortion Bounds)*

If $f_\nu \in TH_m(\lambda,\nu,\alpha)$ and $\lambda\ge 1, |z|=r<1$, then

$$|f_\nu(z)|\le(1+|b_m|)r^m+\frac{r^{m+1}}{\left(\frac{m+1}{m}\right)^{\nu}}\left[\frac{m(1-\alpha)}{(m+\lambda)}-\frac{m(2\lambda-1)}{(m+\lambda)}|b_m|\right] \tag{11}$$

and

$$|f_\nu(z)|\ge(1-|b_m|)r^m-\frac{r^{m+1}}{\left(\frac{m+1}{m}\right)^{\nu}}\left[\frac{m(1-\alpha)}{(m+\lambda)}-\frac{m(2\lambda-1)}{(m+\lambda)}|b_m|\right]. \tag{12}$$

*j) Corollary 10*

Let $f_\nu \in TH_m(\lambda,\nu,\alpha)$ then for $|z|=r<1$ and $\lambda\ge 1$

$$\left[w:|w|<\left\{\frac{(m+1)^{\nu}(m+\lambda)-m^{\nu+1}(1-\alpha)}{(m+1)^{\nu}(m+\lambda)}\right\}+\left\{\frac{(2\lambda-1)-(m+1)^{\nu}(m+\lambda)}{(m+1)^{\nu}(m+\lambda)}\right\}|b_m|\right]\subset f_\nu(\Delta). \tag{13}$$

*k) Theorem 11 (Extreme Points)*

Let $f_\nu$ be given by (5) then $f_\nu \in TH_m(\lambda,\nu,\alpha)$ ; $\lambda\ge 1$ if and only if.

$$f_\nu(z)=\sum_{n=1}^{\infty}\left[x_{n+m-1}h_{n+m-1}(z)+y_{n+m-1}g_{n+m-1,\nu}(z)\right], \tag{14}$$

where

$$h_m(z)=z^m,\ h_{n+m-1}(z)=z^m-\frac{1}{\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda+(1-\lambda)\right]}z^{n+m-1}, (n=2,3\ldots)$$

and

$$g_{n+m-1,\nu}(z)=z^m+(-1)^{\nu}\frac{1}{\left(\frac{n+m-1}{m}\right)^{\nu}\left[\left(\frac{n+m-1}{m}\right)\lambda-(1-\lambda)\right]}\overline{z}^{n+m-1}, (n=1,2,3\ldots.)$$

$$x_{n+m-1}\ge 0,\ y_{n+m-1}\ge 0,\ x_m=1-\sum_{n=2}^{\infty}x_{n+m-1}-\sum_{n=1}^{\infty}y_{n+m-1}.$$

In particular, the extreme points of $TH_m(\lambda,\nu,\alpha)$ are $\{h_{n+m-1}\}$ and $\{g_{n+m-1,\nu}\}$.

Notes

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*l) Theorem 12 (Convex Combination)*

If $f_{i,v}(i=1,2\ldots)$ belongs to $TH_m(\lambda,\nu,\alpha)$; $\lambda \geq 1$ then the function $\sum_{i=1}^{\infty} t_i f_{i,v}(z)$ is also in $TH_m(\lambda,\nu,\alpha)$ where $f_{i,v}$ is defined by

$$f_{i,v} = z^m - \sum_{n=2}^{\infty} |a_{n+m-1,i}| z^{n+m-1} + (-1)^{\nu} \sum_{n=1}^{\infty} |b_{n+m-1,i}| \overline{z}^{n+m-1} (i=1,2\ldots) \tag{15}$$

and $0 \leq t_i < 1, \sum_{i=1}^{\infty} t_i = 1$.

*m) Definition 2*

The harmonic generalized Bernardi-Libera-Livingston integral operator $L_c(f(z))$ for m-valent functions is defined by

$$L_c(f(z)) = \frac{c+m}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+m}{z^c}\int_0^z t^{c-1}g(t)dt}, \quad c > -1.$$

*n) Theorem 13 (Integral Operator)*

Let $f \in TH_m(\lambda,\nu,\alpha)$ ; $\lambda \geq 1$. Thus $L_c(D_m^{\nu} f_{\nu}(z))$ belongs to the class $TH_m(\lambda,\nu,\alpha)$.

*o) Theorem 14 (Convolution Condition)*

Let $f_{\nu} \in TH_m(\lambda,\nu,\alpha)$ and $F_{\nu} \in TH_m(\lambda,\nu,\alpha)$ ; $\lambda \geq 1$ then the convolution

$$(f_{\nu} * F_{\nu})(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}A_{n+m-1}| z^{n+m-1} +$$

$$+(-1)^{\nu}\sum_{n=1}^{\infty} |b_{n+m-1}B_{n+m-1}| \overline{z}^{n+m-1} \in TH_m(\lambda,\nu,\alpha).$$

*p) Theorem 15 (Radius of Convexity)*

The radius of convexity for the function $f_{\nu} \in TH_m(\lambda,\nu)$ is given by

$$r_0 = \frac{\left(\frac{m+1}{m}\right)^{\nu-2}}{1-(2\lambda-1)|b_m|}, \text{ for } \lambda \geq 1.$$

*q) Theorem 16 (Radius of Starlikeness)*

The radius of starlikeness for the function $f_{\nu} \in TH_m(\lambda,\nu)$ is given by

$$r_0 = \frac{\left(\frac{m+1}{m}\right)^{\nu-1}}{1-(2\lambda-1)|b_m|}, \text{ for } \lambda \geq 1.$$

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Notes

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# Numerical Method for Finding All Points of Extremum of Random as Smooth and Non-Smooth Functions of One Variable

By Roman Bihun & Gregory Tsehelyk
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*Abstract-* A device of non-classic Newton's minorant and their graphs of functions of two real table-like variables have been introduced and a new numerical method for finding extremum of random as smooth and non-smooth functions of one real variable has been constructed.

*Keywords: minorant and majorant of function, numerical analysis, optimization method.*

*GJSFR-F Classification : FOR Code : MSC 2010: 11F12*

NUMERICALMETHODFORFINDINGALLPOINTSOFEXTREMUMOFRANDOMASSMOOTHANDNONSMOOTHFUNCTIONSOFONEVARIABLE

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# Numerical Method for Finding All Points of Extremum of Random as Smooth and Non-Smooth Functions of One Variable

Roman Bihun [α] & Gregory Tsehelyk [σ]

*Abstract-* A device of non-classic Newton's minorant and their graphs of functions of two real table-like variables have been introduced and a new numerical method for finding extremum of random as smooth and non-smooth functions of one real variable has been constructed.

*Keywords: minorant and majorant of function, numerical analysis, optimization method.*

## I. Introduction

In [2,3] a device of non-classical Newton's majorants and diagrams of functions given in tabular form is constructed and its usage for: the approximation of functions; construction, calculation of the definite integrals and numerical methods for solving the Cauchy problem for ordinary differential equations and their systems, accurate to a certain class of functions is discussed (leaving aside the rounding transaction); optimization both smooth and non-smooth logarithmically concave functions of one and several real variables.

In [1,4] for the first time a device of non-classical Newton's minorants of functions given in tabular form is constructed, which is used for the approximation of functions and development of numerical optimization methods as smooth and non-smooth logarithmically convex functions of one and two real variables.

## II. Device of Non-Classical Newton's Majorants and Minorants of Functions, Given in Tabular Form, and Their Diagrams

Let consider the function of a real variable $y = f(x)$, which defined its values at some points $x_i$, $i = 0,1, \dots ,n$:

$$f(x_i) = y_i, \quad i = 0,\ 1,\ \dots ,n. \tag{1}$$

Let

$$|y_i| = a_i \le M, \quad i = 0,\ 1,\ \dots ,n, \quad a_1 \cdot a_n \ne 0, \tag{2}$$

where $M$ – certain constant.

*Definition 1.* Point $P_i$ $(x_i, -\ln a_i)$ coordinates $x = x_i$, $y = -\ln a_i$ in space $xy$ called *bitmaps value function* $y = f(x)$ in the point $x = x_i$.

Assume that the points of the image $P_i$ of the function $y = f(x)$ at points $x_i$, $i = 0,\ 1,\ \dots,\ n$, in plane $xy$ are built. From every point $P_i$ we draw a half-line in

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Ref

4. *Глебена М.І.* Апарат некласичних мінорант Ньютона та його використання / М.І. Глебена, Г.Г Цегелик // Наук. вісн. Ужгород. ун-ту. Сер. матем. та інформ.. – 2013. – Вип. 24.-N1. – С.16-21.

positive direction of the axis $Oy$, perpendicular to the axis $Ox$. The set of these half-lines is denoted by $S$, and its convex hull – by $C(S)$. For each point $x \in [x_0, x_n]$ we define the point $B_x(x, \chi_x)$, where

$$\chi_x = \inf_{(x,y) \in C(S)} y.$$

The set of points $B_x(x, \chi_x)$, $x \in [x_0, x_n]$, forms a line $\delta_f$, which limits $C(S)$ below. This line is continuous, convex, broken and its equation is

$$y = \chi(x), \quad x \in [x_0, x_n],$$

where $\chi(x) = \chi_x$.

*Definition 2.* Broken line $\delta_f$, defined on the interval $[x_0, x_n]$, called *non-classical Newton's diagram* of function $y = f(x)$ on this interval.

Newton's diagram $\delta_f$ of function $y = f(x)$ has the following properties:

– each vertex $\delta_f$ is placed in one of the *bitmaps* $P_i$ of value of the function $y = f(x)$ at the point $x_i$, $i = 0, 1, \ldots, n$;

– each bitmap $P_i$, $i = 0, 1, \ldots, n$, is located on $\delta_f$ or above it.

Let

$$M_f(x) = \exp\,(-\chi(x)), \quad x \in [x_0, x_n].$$

Then for each point $x_i$, $i = 0, 1, \ldots, n$, the inequality is performed

$$|f(x_i)| = a_i \le M_f(x_i).$$

In fact, with the construction of $\delta_f$ follows that

$$-\ln\,|f(x_i)| \ge \chi(x_i),$$

or

$$|f(x_i)| \le \exp\,(-\chi(x_i)) = M_f(x_i).$$

Besides,

$$M_f(x_0) = |f(x_0)|, \quad M_f(x_n) = |f(x_n)|.$$

*Definition 3.* Function $y = M_f(x)$, defined on the interval $[x_0, x_n]$, called *non-classical Newton's majorant* of function $y = f(x)$ on this interval.

Let

$$M_f(x_i) = T_i, \qquad i = 0, 1, \ldots, n.$$

*Definition 4.* Values

$$R_i = \left(\frac{T_{i-1}}{T_i}\right)^{\frac{1}{x_i - x_{i-1}}} \quad (i = 1, 2, \ldots, n;\ R_0 = 0)$$

and

$$D_i = \frac{R_{i+1}}{R_i} \quad (i = 1, 3, \ldots, n-1;\ D_0 = D_n = \infty)$$

called, respectively, *i-th numerical inclination* and *i-th deviation* of Newton's diagram $\delta_f$.

Notes

Notes

*Definition 5.* If the bitmap $P_i$, $i = 0,\ 1,\ \ldots,\ n$, is located at the top of $\delta_f$, then index $i$ is called *vertex index*, if it is placed on $\delta_f$, - then it is called *diagram index* of $\delta_f$. Indexes $i = 0$ та $i = n$ belong to vertex indexes.

The set of vertex indices we denote by $I$, and the set of diagram indexes – by $G$. Obviously, $I \subset G$ and $T_i = a_i$ for all $i \in G$.

Newton's diagram was constructed for function given in nine points in fig.1.

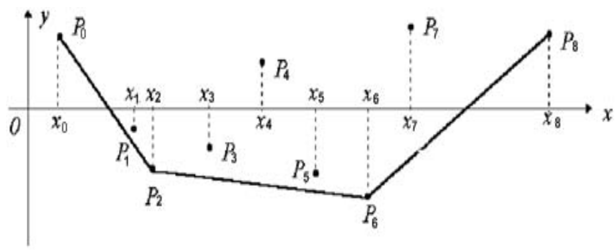


*Figure 1 :* Newton's diagram for function given in nine points

Now let the points of the image $P_i$ of the function $y = f(x)$ at points $x_i$, $i = 0,\ 1,\ \ldots,\ n$, in plane $xy$ are built. From every point $P_i$ we draw a half-line in negative direction of the axis $Oy$, perpendicular to the axis oci $Ox$. The set of these half-lines is denoted by $S$, and its convex hull – by $C(S)$. For each point $x \in [x_0,\ x_n]$ we define the point $D_x(x,\ \chi_x)$, where

$$\chi_x = \sup_{(x,y)\ \in\ C(S)} y.$$

The set of points $D_x(x, \chi_x)$, $x \in [x_0, x_n]$ forms a line $\delta_f$, which limits $C(S)$ top. This line is continuous, concave, broken and its equation is

$$y = \chi(x),\ x \in [x_0, x_n],$$

where $\chi(x) = \chi_x$.

Let $m_f(x) = \exp(-\chi(x)),\ x \in [x_0, x_n]$.

Then for each point $x_i$, $i = 0,\ 1,\ \ldots,\ n$, the inequality is performed

$$m_f(x_i) \leq |f(x_i)| = a_i.$$

In fact, with the construction of $\delta_f$ follows that

$$-\ln|f(x_i)| \leq \chi(x_i),$$

or

$$|f(x_i)| \geq \exp\big(-\chi(x_i)\big) = m_f(x_i).$$

Besides,

$$m_f(x_0) = |f(x_0)|,\ m_f(x_n) = |f(x_n)|.$$

*Definition 6.* The function $y = m_f(x)$, defined on the interval $[x_0,\ x_n]$, called *non-classical Newton's minorant* of function $y = f(x)$ on this interval, and broken line $\delta_f$ – its diagram.

Newton's minorant diagram was constructed for function given in nine points in fig.2.

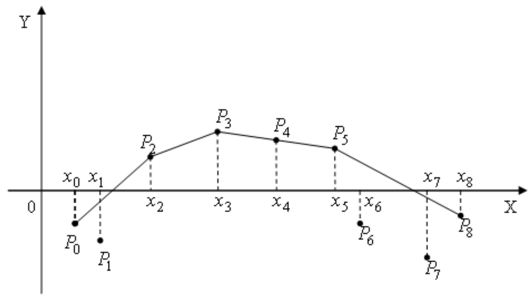


*Figure 2 :* Newton's minorant diagram for function given in nine points

Newton's minorant diagram $\delta_f$ of function $y = f(x)$ has the following properties:
– each vertex $\delta_f$ is placed in one of the *bitmaps* $P_i$ of value of the function $y = f(x)$ at the point $x_i$, $i = 0, 1, \ldots, n$;
– each bitmap $P_i$, $i = 0, 1, \ldots, n$, is located on $\delta_f$ or below it.
Let

$$m_f(x_i) = t_i, \quad i = 0,1, \ldots, n.$$

*Definition* 7. Values

$$r_i = \left(\frac{t_{i-1}}{t_i}\right)^{\frac{1}{x_i - x_{i-1}}} \quad (i = 1,2, \ldots, n; \quad r_0 = \infty)$$

and

$$d_i = \frac{r_{i+1}}{r_i} \quad (i = 1,2, \ldots, n-1; \quad d_0 = d_n = 0)$$

called, respectively, *i-th numerical inclination* and *i-th deviation* of Newton's minorant diagram $\delta_f$.

Let $f(x)$ is logarithmically concave function on the interval $[a, b]$. Let us choose on the interval $[a, b]$ points system $x_0, x_1, \ldots, x_n$, where $x_k = x_0 + kh$ $(k = 0, 1, \ldots, n)$, $x_0 = a$, $h = \frac{b-a}{n}$, and find the value of the function $y = f(x)$ at these points. Let

$$f(x_i) = c_i, \quad i = 0, 1, \ldots, n.$$

Since the $f(x)$ – logarithmically concave function on the interval $[a, b]$, then numerical inclinations of Newton's majorants, which were built on the values of the function at the points $x_1, x_2, \ldots, x_n$, are determined by the formula

$$R_k = \left(\frac{c_{k-1}}{c_k}\right)^{\frac{1}{h}} \quad (k = 1, 2, \ldots, n; R_0 = 0).$$

In this case

$$R_1 < R_2 < \ldots < R_n.$$

Deviations $D_k$ of Newton's majorants will satisfy the condition

$$D_k > 1 \quad (k = 1, 2, \ldots, n-1; D_0 = D_n = \infty).$$

If for some index $k$ $(0 < k < n)$ the conditions $R_k \leq 1$, $R_{k+1} > 1$ accomplish, then the point $x_k$ with accuracy $\varepsilon < h$ is a maximum point of function $f(x)$.

Notes

Now let $f(x)$ is logarithmically convex function on the interval $[a, b]$. Similarly choose on the interval $[a,\ b]$ points system $x_0,\ x_1, \ldots,\ x_n$, where $x_k = x_0 + kh$ $(k = 0,\ 1, \ldots,\ n)$, $x_0 = a$, $h = \frac{b-a}{n}$, and find the value of the function $y = f(x)$ at these points. Let

$$f(x_i) = c_i, \quad i = 0,\ 1,\ \ldots,\ n.$$

Notes

Since the $f(x)$ – logarithmically convex function on the interval $[a,\ b]$, then numerical inclinations of Newton's minorants, which were built on the values of the function at the points $x_1,\ x_2,\ \ldots,\ x_n$, are determined by the formula

$$r_k = \left(\frac{c_{k-1}}{c_k}\right)^{\frac{1}{h}} \quad (k = 1,\ 2,\ \ldots,\ n;\ r_0 = \infty).$$

In this case

$$r_1 > r_2 > \ .\ .\ .\ > r_n.$$

Deviations $d_k$ of Newton's minorants will satisfy the condition

$$0 < d_k < 1 \quad (k = 1,\ 2,\ \ldots,\ n-1;\ d_0 = d_n = 0).$$

If for some index $k$ $(0 < k < n)$ the conditions $r_k \geq 1,\ r_{k+1} < 1$ accomplish, then the point $x_k$ with accuracy $\varepsilon < h$ is a minimum point of function $f(x)$.

## III. Numerical Method for Finding All Points of Extremum of Random as Smooth and Non-Smooth Functions at Preset Interval

Let we have to find all points of extremum of function $y = f(x)$ at preset interval $[a, b]$. We assume that $f(x) > 0$ for all $x \in [a, b]$.

Choose on the interval $[a, b]$ points system $x_0, x_1, \ldots, x_n$, where $x_k = x_0 + kh$ $(k = 0,1, \ldots, n)$, $x_0 = a$, $h = \frac{b-a}{n}$, and find the value of function $y = f(x)$ at these points. Let

$$f(x_i) = c_i\,, \quad i = 0,1, \ldots, n.$$

Put

$$\tilde{r}_k = \left(\frac{c_{k-1}}{c_k}\right)^{\frac{1}{h}}, \qquad k = 1,2, \ldots, n.$$

Then on the intervals $[\alpha, \beta] \in [a, b]$, where the function $f(x)$ is convex,

$$\tilde{r}_i \geq \tilde{r}_{i+1},$$

and the intervals where the function $f(x)$ concave,

$$\tilde{r}_i \leq \tilde{r}_{i+1},$$

### *a) Algorithm of the method*

The algorithm of the method consists of series of steps. In the first step we choose the point $x_0$ and $x_1$ and find $\tilde{r}_1$. Then the following two possible cases:

$$1) \qquad \tilde{r}_1 \leq 1, \qquad\qquad 2)\ \tilde{r}_1 > 1$$

In the first case we calculate $\tilde{r}_2, \tilde{r}_3, \ldots$ until for some $i$ $(i \geq 1)$ condition $\tilde{r}_{i+1} > 1$ does not perform. Then point $x_i$ with accuracy $\varepsilon < h$ is taken as a point of local maximum of function $f(x)$.

In the second case we calculate $\tilde{r}_2, \tilde{r}_3, \ldots$ until for some $i$ $(i \geq 1)$ condition $\tilde{r}_{i+1} < 1$ does not perform. Then point $x_i$ with accuracy $\varepsilon < h$ s taken as a point of local minimum of function $f(x)$. In the second step the point as a starting point $x_i$, found in the first step. Then, if $\tilde{r}_{i+1} \leq 1$, we search $\tilde{r}_{i+2}, \tilde{r}_{i+3}, \ldots$ until for some $k$ $(k > 1)$ condition $\tilde{r}_{i+k} > 1$ does not perform. The point $x_{i+k-1}$ is taken as a point of local maximum with accuracy $\varepsilon < h$ of the function $f(x)$. If $\tilde{r}_{i+1} > 1$, we search $\tilde{r}_{i+2}, \tilde{r}_{i+3}, \ldots$ until for some $k$ $(k > 1)$ condition $\tilde{r}_{i+k} \leq 1$ does not performed. The point $x_{i+k-1}$ is taken as a point of local maximum with accuracy $\varepsilon < h$ of the function $f(x)$.

The process ends when we found the point $x_l$, which is a point of local extremum, and the sequence $\tilde{r}_{l+1}, \tilde{r}_{l+2}, \ldots, \tilde{r}_n$ is either decreasing or increasing.

*b) Example*

We will consider the problem of function optimization

$$f(x) = 8x^6 - 3x^5 - 4x^4 + x^3 - 5x^2 + 4x + 10; \tag{3}$$

on the interval $[-1; 1]$ with step $h = 0{,}1$ $(n = 20)$.

The graph of this function is shown in fig. 3.

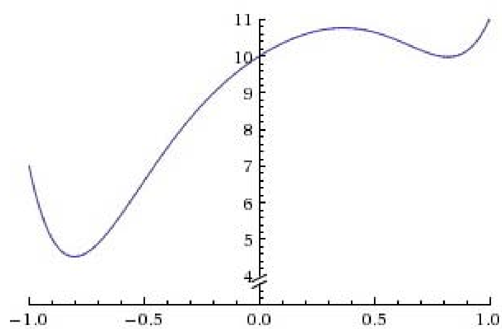


*Figure 3 :* The graph of function (3)

Values $x_i$ and $\tilde{r}_i$ $(i = 0, 1, \ldots, 20)$ are given in table 1.

*Table 1 :* Values for function (3)

| $i$ | $x_i$ | $\tilde{r}_i$ |
|---|---|---|
| 0 | -1 | |
| 1 | -0,9 | 27,81566 |
| 2 | -0,8 | 2,79197 |
| 3 | -0,7 | 0,46336 |
| 4 | -0,6 | 0,22772 |
| 5 | -0,5 | 0,22189 |
| 6 | -0,4 | 0,27695 |
| 7 | -0,3 | 0,35992 |
| 8 | -0,2 | 0,45347 |
| 9 | -0,1 | 0,54546 |
| 10 | 0 | 0,63011 |
| 11 | 0,1 | 0,70852 |
| 12 | 0,2 | 0,78726 |
| 13 | 0,3 | 0,87542 |
| 14 | 0,4 | 0,98111 |
| 15 | 0,5 | 1,10548 |

Notes

| | | |
|---|---|---|
| 16 | 0,6 | 1,23041 |
| 17 | 0,7 | 1,29707 |
| 18 | 0,8 | 1,19776 |
| 19 | 0,9 | 0,86411 |
| 20 | 1 | 0,44072 |

Notes

Let describe one iteration of the algorithm in detail. First, we choose a points $x_0$ and $x_1$ by initial. Then we find $\tilde{r}_1 = 27{,}81566 > 1$. Therefore compute $\tilde{r}_2, \tilde{r}_3, \dots$ until for some $i$ $(i \geq 1)$ condition $\tilde{r}_{i+1} < 1$ performs. We obtain $\tilde{r}_3 = 0{,}46336$ and take $x_2$ for local minimum point.

After completing the required number of iterations, we will find 3 extremum points: $x_2, x_{14}, x_{18}$. Function (3) reaches a local minimum at points $x_2, x_{18}$, and local maximum at point $x_{14}$.

## IV. Conclusion

In this paper, using device of non-classical Newton's majorants and minorants of functions of one real variable given in tabular form, numerical method for finding all points of extremum of random as smooth and nonsmooth functions of one real variable at the selected interval is constructed, also example of this method is shown.

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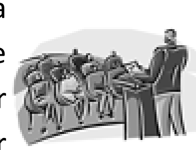

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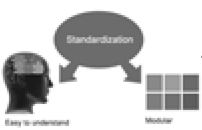


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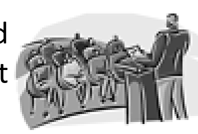

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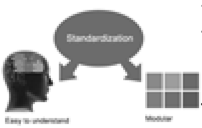


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The paper should be in proper format. The format can be downloaded from first page of 'Author Guideline' Menu. The Author is expected to follow the general rules as mentioned in this menu. The paper should be written in MS-Word Format (*.DOC,*.DOCX).

The Author can submit the paper either online or offline. The authors should prefer online submission.Online Submission: There are three ways to submit your paper:

**(A) (I) First, register yourself using top right corner of Home page then Login. If you are already registered, then login using your username and password.**

**(II) Choose corresponding Journal.**

**(III) Click 'Submit Manuscript'. Fill required information and Upload the paper.**

**(B) If you are using Internet Explorer, then Direct Submission through Homepage is also available.**

**(C) If these two are not conveninet , and then email the paper directly to dean@globaljournals.org.**

Offline Submission: Author can send the typed form of paper by Post. However, online submission should be preferred.

# Preferred Author Guidelines

**MANUSCRIPT STYLE INSTRUCTION <u>(Must be strictly followed)</u>**

Page Size: 8.27" X 11'"

- Left Margin: 0.65
- Right Margin: 0.65
- Top Margin: 0.75
- Bottom Margin: 0.75
- Font type of all text should be Swis 721 Lt BT.
- Paper Title should be of Font Size 24 with one Column section.
- Author Name in Font Size of 11 with one column as of Title.
- Abstract Font size of 9 Bold, “Abstract” word in Italic Bold.
- Main Text: Font size 10 with justified two columns section
- Two Column with Equal Column with of 3.38 and Gaping of .2
- First Character must be three lines Drop capped.
- Paragraph before Spacing of 1 pt and After of 0 pt.
- Line Spacing of 1 pt
- Large Images must be in One Column
- Numbering of First Main Headings (Heading 1) must be in Roman Letters, Capital Letter, and Font Size of 10.
- Numbering of Second Main Headings (Heading 2) must be in Alphabets, Italic, and Font Size of 10.

**You can use your own standard format also.**
**Author Guidelines:**

1. General,

2. Ethical Guidelines,

3. Submission of Manuscripts,

4. Manuscript’s Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

**1. GENERAL**

Before submitting your research paper, one is advised to go through the details as mentioned in following heads. It will be beneficial, while peer reviewer justify your paper for publication.

**Scope**

The Global Journals Inc. (US) welcome the submission of original paper, review paper, survey article relevant to the all the streams of Philosophy and knowledge. The Global Journals Inc. (US) is parental platform for Global Journal of Computer Science and Technology, Researches in Engineering, Medical Research, Science Frontier Research, Human Social Science, Management, and Business organization. The choice of specific field can be done otherwise as following in Abstracting and Indexing Page on this Website. As the all Global

Journals Inc. (US) are being abstracted and indexed (in process) by most of the reputed organizations. Topics of only narrow interest will not be accepted unless they have wider potential or consequences.

**2. ETHICAL GUIDELINES**

Authors should follow the ethical guidelines as mentioned below for publication of research paper and research activities.

Papers are accepted on strict understanding that the material in whole or in part has not been, nor is being, considered for publication elsewhere. If the paper once accepted by Global Journals Inc. (US) and Editorial Board, will become the copyright of the Global Journals Inc. (US).

**Authorship: The authors and coauthors should have active contribution to conception design, analysis and interpretation of findings. They should critically review the contents and drafting of the paper. All should approve the final version of the paper before submission**

The Global Journals Inc. (US) follows the definition of authorship set up by the Global Academy of Research and Development. According to the Global Academy of R&D authorship, criteria must be based on:

1) Substantial contributions to conception and acquisition of data, analysis and interpretation of the findings.

2) Drafting the paper and revising it critically regarding important academic content.

3) Final approval of the version of the paper to be published.

All authors should have been credited according to their appropriate contribution in research activity and preparing paper. Contributors who do not match the criteria as authors may be mentioned under Acknowledgement.

Acknowledgements: Contributors to the research other than authors credited should be mentioned under acknowledgement. The specifications of the source of funding for the research if appropriate can be included. Suppliers of resources may be mentioned along with address.

**Appeal of Decision: The Editorial Board's decision on publication of the paper is final and cannot be appealed elsewhere.**

**Permissions: It is the author's responsibility to have prior permission if all or parts of earlier published illustrations are used in this paper.**

Please mention proper reference and appropriate acknowledgements wherever expected.

If all or parts of previously published illustrations are used, permission must be taken from the copyright holder concerned. It is the author's responsibility to take these in writing.

Approval for reproduction/modification of any information (including figures and tables) published elsewhere must be obtained by the authors/copyright holders before submission of the manuscript. Contributors (Authors) are responsible for any copyright fee involved.

**3. SUBMISSION OF MANUSCRIPTS**

Manuscripts should be uploaded via this online submission page. The online submission is most efficient method for submission of papers, as it enables rapid distribution of manuscripts and consequently speeds up the review procedure. It also enables authors to know the status of their own manuscripts by emailing us. Complete instructions for submitting a paper is available below.

Manuscript submission is a systematic procedure and little preparation is required beyond having all parts of your manuscript in a given format and a computer with an Internet connection and a Web browser. Full help and instructions are provided on-screen. As an author, you will be prompted for login and manuscript details as Field of Paper and then to upload your manuscript file(s) according to the instructions.

To avoid postal delays, all transaction is preferred by e-mail. A finished manuscript submission is confirmed by e-mail immediately and your paper enters the editorial process with no postal delays. When a conclusion is made about the publication of your paper by our Editorial Board, revisions can be submitted online with the same procedure, with an occasion to view and respond to all comments.

Complete support for both authors and co-author is provided.

**4. MANUSCRIPT'S CATEGORY**

Based on potential and nature, the manuscript can be categorized under the following heads:

Original research paper: Such papers are reports of high-level significant original research work.

Review papers: These are concise, significant but helpful and decisive topics for young researchers.

Research articles: These are handled with small investigation and applications

Research letters: The letters are small and concise comments on previously published matters.

**5.STRUCTURE AND FORMAT OF MANUSCRIPT**

The recommended size of original research paper is less than seven thousand words, review papers fewer than seven thousands words also.Preparation of research paper or how to write research paper, are major hurdle, while writing manuscript. The research articles and research letters should be fewer than three thousand words, the structure original research paper; sometime review paper should be as follows:

**Papers**: These are reports of significant research (typically less than 7000 words equivalent, including tables, figures, references), and comprise:

(a)Title should be relevant and commensurate with the theme of the paper.

(b) A brief Summary, "Abstract" (less than 150 words) containing the major results and conclusions.

(c) Up to ten keywords, that precisely identifies the paper's subject, purpose, and focus.

(d) An Introduction, giving necessary background excluding subheadings; objectives must be clearly declared.

(e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition; sources of information must be given and numerical methods must be specified by reference, unless non-standard.

(f) Results should be presented concisely, by well-designed tables and/or figures; the same data may not be used in both; suitable statistical data should be given. All data must be obtained with attention to numerical detail in the planning stage. As reproduced design has been recognized to be important to experiments for a considerable time, the Editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned un-refereed;

(g) Discussion should cover the implications and consequences, not just recapitulating the results; conclusions should be summarizing.

(h) Brief Acknowledgements.

(i) References in the proper form.

Authors should very cautiously consider the preparation of papers to ensure that they communicate efficiently. Papers are much more likely to be accepted, if they are cautiously designed and laid out, contain few or no errors, are summarizing, and be conventional to the approach and instructions. They will in addition, be published with much less delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and to make suggestions to improve briefness.

It is vital, that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

**Format**

*Language: The language of publication is UK English. Authors, for whom English is a second language, must have their manuscript efficiently edited by an English-speaking person before submission to make sure that, the English is of high excellence. It is preferable, that manuscripts should be professionally edited.*

Standard Usage, Abbreviations, and Units: Spelling and hyphenation should be conventional to The Concise Oxford English Dictionary. Statistics and measurements should at all times be given in figures, e.g. 16 min, except for when the number begins a sentence. When the number does not refer to a unit of measurement it should be spelt in full unless, it is 160 or greater.

Abbreviations supposed to be used carefully. The abbreviated name or expression is supposed to be cited in full at first usage, followed by the conventional abbreviation in parentheses.

Metric SI units are supposed to generally be used excluding where they conflict with current practice or are confusing. For illustration, 1.4 l rather than 1.4 × 10-3 m3, or 4 mm somewhat than 4 × 10-3 m. Chemical formula and solutions must identify the form used, e.g. anhydrous or hydrated, and the concentration must be in clearly defined units. Common species names should be followed by underlines at the first mention. For following use the generic name should be constricted to a single letter, if it is clear.

**Structure**

All manuscripts submitted to Global Journals Inc. (US), ought to include:

Title: The title page must carry an instructive title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) wherever the work was carried out. The full postal address in addition with the e-mail address of related author must be given. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining and indexing.

*Abstract, used in Original Papers and Reviews:*

Optimizing Abstract for Search Engines

Many researchers searching for information online will use search engines such as Google, Yahoo or similar. By optimizing your paper for search engines, you will amplify the chance of someone finding it. This in turn will make it more likely to be viewed and/or cited in a further work. Global Journals Inc. (US) have compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Key Words

A major linchpin in research work for the writing research paper is the keyword search, which one will employ to find both library and Internet resources.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy and planning a list of possible keywords and phrases to try.

Search engines for most searches, use Boolean searching, which is somewhat different from Internet searches. The Boolean search uses "operators," words (and, or, not, and near) that enable you to expand or narrow your affords. Tips for research paper while preparing research paper are very helpful guideline of research paper.

Choice of key words is first tool of tips to write research paper. Research paper writing is an art.A few tips for deciding as strategically as possible about keyword search:

- One should start brainstorming lists of possible keywords before even begin searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in research paper?" Then consider synonyms for the important words.
- It may take the discovery of only one relevant paper to let steer in the right keyword direction because in most databases, the keywords under which a research paper is abstracted are listed with the paper.
- One should avoid outdated words.

Keywords are the key that opens a door to research work sources. Keyword searching is an art in which researcher's skills are bound to improve with experience and time.

Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

*Acknowledgements: Please make these as concise as possible.*

References

References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

References to information on the World Wide Web can be given, but only if the information is available without charge to readers on an official site. Wikipedia and Similar websites are not allowed where anyone can change the information. Authors will be asked to make available electronic copies of the cited information for inclusion on the Global Journals Inc. (US) homepage at the judgment of the Editorial Board.

The Editorial Board and Global Journals Inc. (US) recommend that, citation of online-published papers and other material should be done via a DOI (digital object identifier). If an author cites anything, which does not have a DOI, they run the risk of the cited material not being noticeable.

The Editorial Board and Global Journals Inc. (US) recommend the use of a tool such as Reference Manager for reference management and formatting.

Tables, Figures and Figure Legends

*Tables: Tables should be few in number, cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g. Table 4, a self-explanatory caption and be on a separate sheet. Vertical lines should not be used.*

*Figures: Figures are supposed to be submitted as separate files. Always take in a citation in the text for each figure using Arabic numbers, e.g. Fig. 4. Artwork must be submitted online in electronic form by e-mailing them.*

Preparation of Electronic Figures for Publication

Even though low quality images are sufficient for review purposes, print publication requires high quality images to prevent the final product being blurred or fuzzy. Submit (or e-mail) EPS (line art) or TIFF (halftone/photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Do not use pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings) in relation to the imitation size. Please give the data for figures in black and white or submit a Color Work Agreement Form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution (at final image size) ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs) : >350 dpi; figures containing both halftone and line images: >650 dpi.

Color Charges: It is the rule of the Global Journals Inc. (US) for authors to pay the full cost for the reproduction of their color artwork. Hence, please note that, if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a color work agreement form before your paper can be published.

*Figure Legends: Self-explanatory legends of all figures should be incorporated separately under the heading 'Legends to Figures'. In the full-text online edition of the journal, figure legends may possibly be truncated in abbreviated links to the full screen version. Therefore, the first 100 characters of any legend should notify the reader, about the key aspects of the figure.*

**6. AFTER ACCEPTANCE**

Upon approval of a paper for publication, the manuscript will be forwarded to the dean, who is responsible for the publication of the Global Journals Inc. (US).

**6.1 Proof Corrections**

The corresponding author will receive an e-mail alert containing a link to a website or will be attached. A working e-mail address must therefore be provided for the related author.

Acrobat Reader will be required in order to read this file. This software can be downloaded

(Free of charge) from the following website:

www.adobe.com/products/acrobat/readstep2.html. This will facilitate the file to be opened, read on screen, and printed out in order for any corrections to be added. Further instructions will be sent with the proof.

Proofs must be returned to the dean at dean@globaljournals.org within three days of receipt.

As changes to proofs are costly, we inquire that you only correct typesetting errors. All illustrations are retained by the publisher. Please note that the authors are responsible for all statements made in their work, including changes made by the copy editor.

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**6.4 Author Material Archive Policy**

Please note that if not specifically requested, publisher will dispose off hardcopy & electronic information submitted, after the two months of publication. If you require the return of any information submitted, please inform the Editorial Board or dean as soon as possible.

**6.5 Offprint and Extra Copies**

A PDF offprint of the online-published article will be provided free of charge to the related author, and may be distributed according to the Publisher's terms and conditions. Additional paper offprint may be ordered by emailing us at: editor@globaljournals.org .

Before start writing a good quality Computer Science Research Paper, let us first understand what is Computer Science Research Paper? So, Computer Science Research Paper is the paper which is written by professionals or scientists who are associated to Computer Science and Information Technology, or doing research study in these areas. If you are novel to this field then you can consult about this field from your supervisor or guide.

## TECHNIQUES FOR WRITING A GOOD QUALITY RESEARCH PAPER:

**1. Choosing the topic:** In most cases, the topic is searched by the interest of author but it can be also suggested by the guides. You can have several topics and then you can judge that in which topic or subject you are finding yourself most comfortable. This can be done by asking several questions to yourself, like Will I be able to carry our search in this area? Will I find all necessary recourses to accomplish the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

**2. Evaluators are human:** First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

**3. Think Like Evaluators:** If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

**4. Make blueprints of paper:** The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

**5. Ask your Guides:** If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

**6. Use of computer is recommended:** As you are doing research in the field of Computer Science, then this point is quite obvious.

**7. Use right software:** Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

**8. Use the Internet for help:** An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

**9. Use and get big pictures:** Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

**10. Bookmarks are useful:** When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

**11. Revise what you wrote:** When you write anything, always read it, summarize it and then finalize it.

**12. Make all efforts:** Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

**13. Have backups:** When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

**14. Produce good diagrams of your own:** Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

**15. Use of direct quotes:** When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

**16. Use proper verb tense:** Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

**17. Never use online paper:** If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

**18. Pick a good study spot:** To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

**19. Know what you know:** Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

**20. Use good quality grammar:** Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

**21. Arrangement of information:** Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

**22. Never start in last minute:** Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

**23. Multitasking in research is not good:** Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

**24. Never copy others' work:** Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

**25. Take proper rest and food:** No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

**26. Go for seminars:** Attend seminars if the topic is relevant to your research area. Utilize all your resources.

**27. Refresh your mind after intervals:** Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

**28. Make colleagues:** Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

**29. Think technically:** Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

**30. Think and then print:** When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

**31. Adding unnecessary information:** Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

**32. Never oversimplify everything:** To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

**33. Report concluded results:** Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

**34. After conclusion:** Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

## Informal Guidelines of Research Paper Writing

**Key points to remember:**

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

**Final Points:**

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

**General style:**

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

**Title Page:**

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

**Abstract:**

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

**Introduction:**

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

**Procedures (Methods and Materials):**

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

**Results:**

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

**Discussion:**

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

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| | **A-B** | **C-D** | **E-F** |
| ***Abstract*** | Clear and concise with appropriate content, Correct format. 200 words or below | Unclear summary and no specific data, Incorrect form<br><br>Above 200 words | No specific data with ambiguous information<br><br>Above 250 words |
| ***Introduction*** | Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited | Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter | Out of place depth and content, hazy format |
| ***Methods and Procedures*** | Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads | Difficult to comprehend with embarrassed text, too much explanation but completed | Incorrect and unorganized structure with hazy meaning |
| ***Result*** | Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake | Complete and embarrassed text, difficult to comprehend | Irregular format with wrong facts and figures |
| ***Discussion*** | Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited | Wordy, unclear conclusion, spurious | Conclusion is not cited, unorganized, difficult to comprehend |
| ***References*** | Complete and correct format, well organized | Beside the point, Incomplete | Wrong format and structuring |

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