Online ISSN: 2249-4626
Print ISSN: 0975-5896

Global Journal

of Science Frontier Research: F

Mathematics and Decision Sciences

Highlights

A Summation Formula Combinatorial Partition Identities

Bayesian Regression Method Involving Contiguous Relation

Discovering Thoughts, Inventing Future

VOLUME 15 ISSUE 4 VERSION 1.0

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences
Volume 15 Issue 4 (Ver. 1.0)

Open Association of Research Society

Incorporation No.: 0423089
License No.: 42125/022010/1186
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Employer Identification Number (EIN):
USA Tax ID: 98-0673427

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USA Toll Free: +001-888-839-7392
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Offset Typesetting

Global Journals Incorporated
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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 4 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Direct Estimation of Effects and Tests of their Statistical Significance for the Case of One Autosomal Locus with Two Alleles

By Charles J. Mode
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Abstract- In previous papers, see the references, the author introduced methods for estimating effects directly in samples of individuals whose genomes had be sequenced for the cases of one and two or more quantitative traits. In these papers, no attention was given to developing procedures of testing the statistical significance of the estimated effects. This paper is devoted to the development of statistical tests of significance of estimated effects for the simple case of one autosomal locus with two alleles, using Monte Carlo simulation methods. Because no real data was available to the author, artificial data for the three genotypes was simulated by using a Monte Carlo simulation procedures with fixed sample size for each genotypes as well as expectations and variances. In all cases considered, the null hypothesis was described in detail so as to inform a reader on the basic concepts underlying the proposed tests of statistical significance.

Keywords: estimating effects, estimated heritability, tests of statistical significance, definitions of null hypotheses, Monte Carlo simulations methods, simulated data, absolute normal distribution.

GJSFR-F Classification : FOR Code : MSC 2010: 37A60

Strictly as per the compliance and regulations of :

Ref

3. Mode, C. J. (2014-a) Estimating Effects and Variance Components in Models of Quantitative Genetics in an Era of Sequenced Genomes. Global Journal of Science Frontier Research-Mathematics and Decisions Sciences. Issue 5 Version 1.0 Online ISSN 2249-4626.

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Charles J. Mode

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Keywords: estimating effects, estimated heritability, tests of statistical significance, definitions of null hypotheses, monte carlo simulations methods, simulated data, absolute normal distribution.

I. Introduction

As was suggested in previous papers, when an investigator is dealing with a sample of individuals whose genomes have been sequenced and a set of regions of the genome have been identified that affect the expression of a quantitative trait or traits, then it becomes possible to provide a working definition of set of loci in each individual at the genomic level, see Mode [3] and [4]. Moreover, if it is also possible to use makers in the DNA of each individual to provide working definitions of at least two alleles at each locus, then an investigator can develop a concrete working definition of the set of loci with two alleles at each that have shown to have an effect on the expression of a quantitative trait of traits as expressed in a numerical measurement or measurements on each individual in the sample.

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In principle, a quantitative trait or traits may be analyzed statistically for any combination of the set of loci under consideration, but because the number of genotypes that can be identified increase at a fast rate as the number of loci under consideration increase, the sample of individuals may not be sufficiently large to assure that the number of individuals of each genotype is large enough to obtain statistically significant results as discussed in chapters 3 and 4. Such a situation will usually arise whenever the number of loci under consideration is greater than 5 or 6. Therefore, if an investigator has 6 loci or more loci under consideration in a sample of data, it would be prudent to perform a preliminary analysis of the data for each locus under consideration in order to develop an understanding as to which combination of loci would be most informative and fruitful to explore.

To execute such an experiment, an investigator would need software to estimate each effect whose square is a component of the genetic variance for each locus under consideration. In the papers presented in chapters 3 and 4, it was assumed that any allele in a genotype could be identified as to whether it was contributed by the father or mother of any individual in the sample. In many data sets, however, this assumption is not valid so that an investigator could identify only three genotypes per locus for the case of two alleles per locus. These three genotypes consist of two homozygotes and a heterozygote for which it was not possible to identify whether each allele was of maternal or paternal in origin. The purpose of this chapter is to provide an overview of the software necessary to carry out a preliminary exploration of a data set, such that the genome of each individual in the set has been sequenced, for each locus under consideration. The main focus of this chapter is to implement software to do the necessary computations for the simplest case of one autosomal locus with two alleles with a view towards extending the software to cases of two or more autosomal loci. A mathematical description of this software is provided for those investigators who write code using a programing language based on the manipulation of arrays such as APL or MATLAB.

A Monte Carlo simulation procedure was used to provide data for illustrating how the software may be used to analyze real data when it is available. Contained in this software are procedures of estimating the squares of effects and a description of a procedures to formulate null hypotheses to test the statistical significance of the estimated squares of effects. After a null hypothesis is defined, a Monte Carlo simulation procedure was used to estimate p-values to judge whether each estimated square of an effect was statistically significant, given some null hypothesis. Detailed technical descriptions of null hypotheses that were used in tests of statistical significance are also provided for each reported experiment.

II. Non-Identifiabilty of Heterozygotes in the Case of one Autosomal Locus

In previous work on defining effects in connection with components of variance models in quantitative genetics, it was assumed that two kinds of heterozygotes could be identified when working with a sample of individuals whose genomes have been sequenced. For example, let the symbol 1 denote the presence of a marker on a haplotype with respect to some locus, and let the symbol 0 denote the absence of this marker. Then, if it is assumed that it is possible to detect in each individual whether each of the two alleles in a genotype that was contributed by the maternal or paternal parent, then a genotype could to represented by the symbol (x, y), where x and y denote, respectively, the maternal and paternal allele. Thus, if both x and y are assigned the symbols 1 or 0, then four genotypes in the set

$$\mathbb{G}_1 = \{(1,1),(1,0),(0,1),(0,0)\} \tag{2.1}$$

can be identified in a sample of individuals. But, if it is not possible to identify the maternal and paternal alleles in an individual, then the two possible heterozygotes, $(1,0)$ and $(0,1)$ would be lumped into a single category called heterozygotes.

But, in such samples, the two homozygotes, $(1,1)$ and $(0,0)$, could be identified unambiguously. In what follows, the genotypes in the three categories will be denoted by the symbols $(1,1),(x \neq y)$ and $(0,0)$, where the symbol $(x \neq y)$ stands for heterozygotes in the non-identifiable case. In this case,

$$\mathbb{G}_2 = \{(1,1),(x \neq y),(0,0)\} = \{g_1, g_2, g_3\} \tag{2.2}$$

is the set of genotypes of recognizable genotypes. To simplify the notation in this case, the symbols g_1, g_2, g_3 will stand for three genotypes under consideration as indicated in (2.2)

For the case of non-identifible heterozygotes, let $n > 1$ denote the number of individuals in a sample, and let $n(1,1)$, $n(x \neq y)$ and $n(0,0)$ denote, respectively, the number of individuals of each of the three genotypes. To simply the notation, the numbers of each of these three genotypes is indicated in the set

$$\{n(g_1), n(g_2), n(g_3)\} \tag{2.3}$$

denote the set of genotypes and let g, with or without subscripts, denote an element in the set $\mathbb{G}_2$ as indicated on the right in (2.2). Then,

$$n = \sum\nolimits_{g \in \mathbb{G}_2} n(g) \ , \tag{2.4}$$

is the number of individuals in the sample, and

$$p_\nu = \frac{n(g_\nu)}{n} \tag{2.5}$$

is the estimated frequency of genotype g_ν in the population for $\nu = 1,2,3$. This collection of estimates will be referred to as the estimated genotypic distribution.

To take into account the problem setting up a structure that incorporates phenotypic variation among individuals of the same genotype in sample with respect to some quantitative trait, let W denote and random variable taking values in the set of $\mathbb{R}_W$ of possible values of the phenotype. Usually, $\mathbb{R}_W$ is a set of rational real numbers. Given genotype $g \in \mathbb{G}_2$, let $f(w \mid g)$ denote the conditional probability density function of the random phenotypic variable W, and suppose there are $n(g)$ observed realizations of the random variable W denoted by the symbols W_i for $i = 1,2,\cdots,n(g)$. The conditional expectation of the random variable W, given the genotype $g \in \mathbb{G}$, is

$$E[W \mid g] = \int_{\mathbb{R}_W} f(w \mid g)\, dw \tag{2.6}$$

for every genotype $g \in \mathbb{G}_2$.

Therefore, an estimator of the conditional expectation $E[W \mid g]$ is

$$\mu(g) = \frac{1}{n(g)} \sum\nolimits_{i=1}^{n(g)} W_i \tag{2.7}$$

for all $g \in \mathbb{G}$. In what follows, these estimates will be referred to as the genetic values. Let μ denote the unconditional expectation of the genetic values. Then,

$$\mu = \sum\nolimits_{g \in \mathbb{G}_2} p(g)\, \mu(g) \tag{2.8}$$

Observe that this formula of this type would also be valid if the set $\mathbb{G}_1$ of genotypes were under consideration, in this case there would be 4 genotypes to take into account.

One objectives of this chapter is to suggest ways of write software to implement the formulas above, using a array manipulating programing language. Included in the class of array manipulating programing languages are APL and MATLAB. For such languages, a good starting point is to represent the genotypic distribution in (2.5) and the set of estimated expectations in (2.7) in matrix forms. To cast the genotypic distribution in matrix form, suppose the set of genotypes under consideration is $\mathbb{G}_2$. Then, the matrix of estimated of genetic values will have the form

$$\boldsymbol{\mu}_{\mathbb{G}_2} = \begin{bmatrix} \mu(g_1) & \mu(g_2) \\ 0 & \mu(g_3) \end{bmatrix} . \tag{2.9}$$

If both of the genotypes $(1,0)$ and $(0,1)$, i.e., both maternal and paternal alleles in a sample of individuals can be identified, then the set of genotypes under consideration would be the set $\mathbb{G}_1$ as defined above. In this case, the matrix of genetic values has the form

$$\boldsymbol{\mu}_{\mathbb{G}_1} = \begin{bmatrix} \mu(1,1) & \mu(1,0) \\ \mu(0,1) & \mu(0,0) \end{bmatrix} = [\mu(i,j)] \ , \tag{2.10}$$

where in this case $\mu(0,1\ )$ may be positive.

Let the array

$$\boldsymbol{p}_{\mathbb{G}_1} = \begin{bmatrix} p(1,1) & p(1,0) \\ p(0,1) & p(0,0) \end{bmatrix} = [p(i,j)] \tag{2.11}$$

represent the matrix form of the genotypic distribution for case 1. For the case the set of genotypes $\mathbb{G}_1$ is under consideration, let a matrix operation of element by element multiplication be denoted by

$$\boldsymbol{p} \times \mu = [p(i,j)\,\mu(i,j)] . \tag{2.12}$$

The symbol on the left in (2.12) would be the format for doing the operation of matrix of element by element implication in APL. Then, for case 1, the unconditional expectation of the matrix of genetic values may be written in the form

$$\mu_{\mathbb{G}_1} = \sum_{(i,j)\in\mathbb{G}_1} [p(i,j)\,\mu(i,j)] \ . \tag{2.13}$$

It is important to observe that if the set $\mathbb{G}_2$ of genotypes were under consideration, then formula of type (2.13) could also be used to compute unconditional expectation $\mu_{\mathbb{G}_2}$. From the point of view of writing software with an array processing programming language, it is important to observe that the same program could be used for these cases, except case 2, were the matrix of expected genetic values would have the form in (2.9) and the matrix form of the genotypic distribution would be represented in the form

$$\boldsymbol{p}_{\mathbb{G}_2} = \begin{bmatrix} p(g_1) & p(g_2) \\ 0 & p(g_3) \end{bmatrix} . \tag{2.14}$$

In what follows in this section, a general notation will be used to partition the phenotypic variance of a trait into the genetic and environmental variances, using a general notation that includes cases 1 and 2 described above. Let $\mathbb{A}$ denote the set of alleles at some autosomal locus, let (x,y) any genotype, where $x \in \mathbb{A}$ and $y \in \mathbb{A}$ and let

$$\mathbb{G} = \{(x,y) \mid x \in \mathbb{A} \text{ and } y \in \mathbb{A}\} \tag{2.15}$$

Notes

be the set of genotypes under consideration. Given genotype $(x,y) \in \mathbb{G}$, a set of genotypes under consideration, let $W(x,y)$ denote a realization of the phenotypic random variable W, given (x,y). Then observe that the equation

$$W(x,y) - \mu = (\mu(x,y) - \mu) + (W(x,y) - \mu(x,y)) \tag{2.16}$$

is valid for all genotypes $(x,y) \in \mathbb{G}$.

The phenotypic variance, genetic and environmental variances are defined as

$$var_P[W] = \sum_{(x,y)} p(x,y) (W(x,y) - \mu)^2 \tag{2.17}$$

$$var_G[W] = \sum_{(x,y)} p(x,y) (\mu(x,y) - \mu)^2 \tag{2.18}$$

and

$$var_E[W] = \sum_{(x,y)} p(x,y) (W(x,y) - \mu(x,y))^2 \tag{2.19}$$

respectively.

By using a conditioning argument given (x,y) and (2.16), it can be shown that the equation

$$var_P[W] = var_G[W] + var_E[W] \tag{2.20}$$

is valid. If a reader is interested in a detailed derivation of the formula in (2.20), consult chapter 3, which contains a detailed account for the case of one quantitative trait. Observe that if the variances on right of equation (2.20) are estimates based on data, then the sum on the right in (2.20) is an estimator of $var_P[W]$, the phenotypic variance. By definition, H, the heritability of a trait, is

$$H = \frac{var_G[W]}{var_G[W] + var_E[W]} = \frac{var_G[W]}{var_P[W]} . \tag{2.21}$$

From this formula, it can be seen that to estimate H, it suffices to compute $var_G[W]$ and $var_E[W]$, and then use equation (2.20) to check the validity of formula (2.20) numerically. Note that any estimate of H will satisfy the condition $0 \leq H \leq 1$.It should also be observed that from formula (2.17) that the phenotypic variance $var_P[W]$ may also be computed directly.

The formulas just derived are theoretical and when a sample of data is available to an investigator, the genotypic distribution may be estimated as well as means and variances. For example, for each genotype $(i,j) \in \mathbb{G}_2$, let $W_\nu(i,j)$ for $\nu = 1, 2 \cdot \cdot, n(i,j)$ be a set of quantitative observations on some trait. Then, for every genotype $(i,j) \in \mathbb{G}_2$,

$$\widehat{\mu}(i,j) = \frac{1}{n(i,j)} \sum_{\nu=1}^{n(i,i)} W(i,j) \tag{2.22}$$

is an estimate of the expectation $\mu(i,j)$, and an estimate of $\sigma^2(i,j)$ is

$$\widehat{\sigma^2}(i,j) = \frac{1}{n(i,j) - 1} \sum_{\nu=1}^{n(i,j)} (W(i,j) - \widehat{\mu}(i,j))^2 \tag{2.23}$$

is an estimate of $\sigma^2(i,j)$ for $n(i,j) > 1$, and

$$\widehat{p}(i,j) = \frac{n(i,j)}{n} \tag{2.24}$$

is an estimate of the genotypic distribution. In the sections that follow the symbol $\widehat{\circ}$ will be dropped to lighten the notation, but it will be tacitly understood that when a set of data is under consideration, the symbols $\mu(i,j), \sigma^2(i,j)$ and $p(i,j)$ are estimates of parameters.

III. Estimating Effects Directly for Case of One Autosomal Locus

In classical quantitative genetics, the variances in variance component models were usually estimated indirectly by using analysis of variance-covariance procedures, but within this framework it was not possible to estimate the effects, because they were squared terms in the weighted sums that were, by definition, variance components. See chapter 1 for a specific example of an analysis of variance-covarince procedure applied to data. But, as will be shown in this section, when the genotype of each individual in a sample is known at the DNA level, it is possible to estimate effects directly from the data for cases of at least two alleles at the locus as was suggested in chapters 3 and 4.

Suppose that the maternal and paternal alleles are not identified in a sample, and let $n(1,1)$, $n(0,0)$ and $n(het)$ denote, respectively, the total number of individuals that were homozygous for the alleles 1 or 0 and heterozygous for these alleles. Then, the total number of individuals in the sample is

$$n = n(1,1) + n(0,0) + n(het) \ . \tag{3.1}$$

Let $p(1,1,)\,, p(0,0)$ and $p(het)$ denote, respectively, the frequencies of the three genotypes under consideration. Then, $p(1,1) = n(1,1)/n$, $p(0.0) = n(0,0)/n$ and $p(het) = n(het)/n.$ are estimators of these frequencies. It is also essential to have estimates of the frequencies of alleles in the sample. Let $p(1)$ and $p(0)$ denote the estimated frequencies of alleles 1 and 0 in the sample. The number of copies of allele 1 in individuals of genotype $(1,1)$ is 2, and the number of copies of this allele in each heterozygote is one. Therefore, the estimated frequency of allele 1 in the sample is

$$p(1) = \frac{2n(1,1) + n(het)}{2n} = p(1,1) + \frac{1}{2}p(het) \ . \tag{3.2}$$

Similarly, the estimate of allele 0 in the sample is

$$p(0) = p(0,0) + \frac{1}{2}p(het) \tag{3.3}$$

Observe that

$$p(1) + p(0) = 1 \tag{3.4}$$

as they should.

In order to define all effects for the case of one autosomal locus, it will be necessary to define conditional expectations of the estimated means defined in equation (2.7) in section 2 with respect to the genotypic distribution defined below. When programming in an array processing language, it is convenient to represent the data and estimates in a matrix form. Let $p(1,0) = p(0,1) = 0.5p(het)$. Then, the matrix form of the genotypic distribution is

$$\boldsymbol{p}_{\mathbb{G}_2} = \begin{bmatrix} p(1,1) & p(1,0) \\ p(0,1) & p(0,0) \end{bmatrix} \ . \tag{3.5}$$

The subscript $\mathbb{G}_2$ denotes the set of genotypes defined in (2.2) in section 2. Given this matrix, the frequency of allele 1 has the form

$$p(1) = p(1,1) + p(1,0) \ . \tag{3.6}$$

Similarly, the frequency of allele 0 has the form

$$p(0) = p(0,1) + p(0,0) \tag{3.7}$$

From the point of view of writing computer code in an array processing language, one could use a single command to compute the sum of the rows of the matrix $\boldsymbol{p}_{\mathbb{G}_2}$ that would result in an array with the elements $p(1)$ and $p(0)$.

To expedite the writing of code in an array processing programing language, let $\mu(1,1), \mu(0,0)$ and $\mu(het)$ denote the genetic means for three genotypes under consideration, and, to cast these means in matrix form as in (3.5), let $\mu(0,1) = \mu(1,0) = \mu(het)$. Then, the matrix of these means may be represented in the form

$$\boldsymbol{\mu}_{\mathbb{G}_2} = \begin{bmatrix} \mu(1,1) & \mu(1,0) \\ \mu(0,1) & \mu(0,0) \end{bmatrix} . \tag{3.8}$$

An essential step in defining the effects in what follows is to estimate the mean μ defined in (2.8) of section 2. To this end consider the matrix product

$$\boldsymbol{p}_{\mathbb{G}_2} \times \boldsymbol{\mu}_{\mathbb{G}_2} , \tag{3.9}$$

where the symbol $\times$ stands for element by element multiplication. Then, in this notation, the mean μ has the form

$$\mu = \sum \boldsymbol{p}_{\mathbb{G}_2} \times \boldsymbol{\mu}_{\mathbb{G}_2} = \sum_{(i,,j)\in\mathbb{G}_2} p(i,j)\,\mu(i,j) , \tag{3.10}$$

see (2.8) of section 2. When writing code in an array processing programing language, only a few symbols would be required to write the code to do the operations defined in (3.10).

Another essential step in defining effects is to define the conditional distributions based in terms of the elements of the matrix $\boldsymbol{p}_{G_2}$in (3.5). By definition, the conditional distribution of the genotypic distribution, given allele 1, is

$$\frac{1}{p(1)} (p(1,1), p(1,0)) . \tag{3.11}$$

Let $\mu(1)$ denote the conditional expectation of the means $(\mu(1,1)), \mu(1,0)$, given allele 1. Then, by definition

$$\mu(1) = \frac{1}{p(1)} (p(1,1)\,\mu(1,1)) + p(1,0)\,\mu(1,0) . \tag{3.12}$$

The conditional expectation $\mu(0)$ is defined similarly. Observe that if the sample is in a Hardy-Weinberg equilibrium so that $p(i,j) = p(i)\,p(j)$ for all genotypes $(i,j) \in \mathbb{G}_2$,then $\mu(1)$ has the form

$$\mu(1) = \frac{1}{p(1)} p^2(1)\,\mu(1,1) + p(1)\,p(0)\,\mu(1,0) = p(1)\,\mu(1,1) + p(0)\,\mu(1,0) \tag{3.13}$$

The conditional expectation $\mu(0)$ has a similar form when the sample is in a Hardy-Weinberg equilibrium.

Given the above definitions, the effect of allele 1 is defined by

$$\alpha(1) = \mu(1) - \mu . \tag{3.14}$$

Similarly, the effect of allele 0 is defined by

$$\alpha(0) = \mu(0) - \mu . \tag{3.15}$$

In what follows, the effects defined in (3.14) and (3.15) will be referred to as first order effects. Observe that the expectation of these effects with respect to the genotypic distribution is

$$E_{\mathbb{G}_2}[\alpha] = p(1)\,\alpha(1) + p(0)\,\alpha(0) = 0 . \tag{3.16}$$

It is clear that the effects just defined can be estimated directly from the data, given that the genotype of every individual in the sample can be identified.

For reasons that will be made clear subsequently, the additive variance is defined by

$$var_A[W] = p(1)\alpha^2(1) + p(0)\alpha^2(0) \tag{3.17}$$

The symbol W has been included in left the side of this equation as a reminder that effects on the right side of the equation have been estimated from data. From a perspective of using an analysis of variance procedure to estimate the additive variance in (3.17), it would be impossible to estimate the effects defined in (3.14) and 3. from an estimate of $var_A[W]$. This observation clearly differentiates classical methods of estimating variance components, based on some analysis of variance procedure, from the direct method of estimating effects from the data as outlined above.

Additional effects can also be defined as measures of the presence of interactions among the two alleles in the genotype of any individual in the sample. For any genotype (i,j), a second order effect $\alpha(i,j)$ is defined as

$$\alpha(i,j) = \mu(i,j) - \mu - \alpha(i) - \alpha(j) \tag{3.18}$$

for all genotypes $(i,j) \in \mathbb{G}_2$. From this equation, it follows that

$$\mu(i,j) = \mu + \alpha(i) + \alpha(j) + \alpha(i,j) \tag{3.19}$$

for all genotypes $(i,j) \in \mathbb{G}_2$. If there are no interactions among the alleles i and j, then $\alpha(i,j) = 0$ and equation (3.19) reduces to

$$\mu(i,j) = \mu + \alpha(i) + \alpha(j) \tag{3.20}$$

If this equation holds, then it is said that the alleles i and j act additively, but if $\alpha(i,j) \neq 0$, then there is some interaction among the alleles i and j.

If a sample of individuals is in a Hardy-Weinberg equilibrium,see definition below, then in the additive case, the genetic variance has the form

$$var_G[W] = \sum_{(i.j)\in\mathbb{G}_2} p(i,j)(\mu(i,j)-\mu)^2 = p(1)\alpha^2(1) + p(0)\alpha^2(0) = var_A[W] \tag{3.21}$$

which justifies equation (3.17). By definition

$$var_{IAI}[W] = \sum_{(i,j)\in G_2} p(i,j)\alpha^2(i,j) \tag{3.22}$$

is the intra-allelic interaction variance. If the sample is in a Hardy-Weinberg equilibrium, then, by definition $p(i,j) = p(i)p(j)$ for all alleles i and j, and it can be shown that

$$var_G[W] = var_A[W] + var_{IAI}[W] \ . \tag{3.23}$$

A proof of these results may be found in chapter 3.

If the population is not in a Hardy-Weinberg equilibrium, then from equation it can be shown that $var_G[W]$ would also contain covariance terms, but the details will be omitted, but if a reader is interested in more details, chapter 3 may again be consulted. In classical quantitative genetics, the objective of an experiment would be the estimation of the variance components on the left side of equation (3.23), using some analysis of variance procedure. But, as was shown above, when the genotype of every individual in the sample is known, then all second order effects defined above can be estimated directly from the data.

For example, from equation (3.19), it follows that the formula for estimating the second order effect for genotype $(1,1)$ is

$$\alpha(1,1) = \mu(1,1) - \mu - 2\alpha(1) \ . \tag{3.24}$$

A similar formula for an estimator of the effect $\alpha(0,0)$ would also have the form of equation (3.24). The formula for estimating the second order effect for genotype (i,j) such that $(i \neq j)$ directly has the form

$$\alpha(1,2) = \mu(1,2) - \mu - \alpha(1) - \alpha(2) \ . \tag{3.25}$$

As can be seen from the symmetric matrices in (3.5) and (3.8), it follows that $\alpha(2,1) = \alpha(1,2)$.

Notes

For the case in which maternal and paternal alleles can be identified in any genotype, the set of genotypes $\mathbb{G}_1 = \{(1,1),(1,0),(0,1),(0,0)\}$, see section 2 for more detailed comments, would be under consideration. In this case the 2×2 matrix $\boldsymbol{P}_{\mathbb{G}_1}$ would not be symmetric, because the relation $p(1,2) \neq p(2,1)$ would hold except for rare coincidences. Similarly, the 2×2 matrix $\boldsymbol{\mu}_{G_1}$ genetic values, conditional means, would also be non-symmetric. In this case it would also be necessary to distinguish the frequencies of maternal and paternal alleles. For example, if the rows of the matrix $\boldsymbol{P}_{\mathbb{G}_1}$ were summed, the result would be the distribution of maternal alleles $(p_{mat}(1), p_{mat}(0))$. Similarly, if the columns of this matrix were summed, the result would be the distribution of $(p_{pat}(1), p_{pat}(0))$ of paternal alleles. Given these allelic distributions, to write the computer code to compute the effects in this case would require only a few changes in the code for the case in which the maternal and paternal cannot be distinguished as outlined above.

A question that naturally arises at this point formulating the model under consideration is how can we construct procedures to test the statistical significance of the estimated effects? Because the squares of the effects are summed when defining variance components, it seem fitting to use squared effect in designing tests of statistical significance. Another advantage of using squared effects is that their signs are always non-negative. For the case in which the maternal and paternal alleles cannot be distinguished, let the 5×1 column vector

$$\boldsymbol{E} = \begin{bmatrix} \alpha^2(1) \\ \alpha^2(0) \\ \alpha^2(1,1) \\ \alpha^2(0,0) \\ \alpha^2(1,0) \end{bmatrix} \tag{3.26}$$

denote the squared effects. For the case the maternal and paternal alleles can be distinguished, this vector would contain 8 elements. In the next section, procedures for testing the statistical significance of the elements in the vector in (3.26) will be presented.

IV. Permutation Tests for Statistical Significance of Estimated Effects

In this section, a class of tests of statistical significance based on computer intensive methods will be discussed. In the statistical literature, this class of tests described in this section are often referred to as permutation tests. For example, consider the case in which the maternal and paternal alleles cannot be distinguished. Then, the set of genotypes in the model would be $\mathbb{G}_2 = \{(1,1),(0,0),(1,0)\}$, where the symbol $(1,0)$ represents heterozygotes. Let $n(1,1), n(0,0)$ and $n(1,0)$ denote the number of individuals of the three genotypes in a sample of

$$n = n(1,1) + n(0,0) + n(1,0) \tag{4.1}$$

individuals whose genomes have been sequenced. For each genotype $(i,j) \in \mathbb{G}_2$, let $\boldsymbol{W}(i,j) = \{W_\nu(i,j) \mid \nu = 1,2,\cdots,n(i,j)\}$ denote the the sample of $n(i,j)$ realizations of the phenotypic random variable W describing the variability in

the expression of some quantitative trait for individuals of genotype (i,j).The combined data set under consideration may be represented as

$$DATA = \boldsymbol{W}(1,1), \boldsymbol{W}(0,0), \boldsymbol{W}(1,0) \tag{4.2}$$

and consists of n observations.

The first step in setting up a permutation test of the data is to compute a random permutation denoted by $PERM$ of the data set in (4.2). Given this random permutation of the data, the next step consists of choosing the first $n(1,1)$ elements of $PERM$ as a sample of the quasi observations of the $n(1,1)$ individuals of genotype $(1,1)$. Similarly, the next $n(0,0)$ elements of $PERM$ would represent quasi observations on the $n(0,0)$ of genotype $(0,0)$. Finally, the last $n(1,0)$ elements of $PERM$ would represent the $n(1,0)$ quasi observations on individuals of genotype $n(1,0)$. Then suppose these operations are repeated N times to generate a set of random mutations of the data in (4.2).

As is well known, the set $\mathbb{S}$ of all permutations of the data chosen in this manner contains

$$M = \frac{n!}{n(1,1)!n(0,0)!n(1,0)!} \tag{4.3}$$

elements. For example, for the case $n(1,1) = 33, n(0,0) = 33$ and $n(1,0) = 34$, this number is

$$M = \frac{100!}{33!33!34!} = 4.1924 \times 10^{45}, \tag{4.4}$$

which is a very large number. Indeed it is so large that most current computers would be unable compute this many permutations of the data in an acceptably short time span. Consequently, when doing a permutation test of the data, an investigator would need to compute some number N of permutations that is much less than M.Most programming languages contain programs to compute random numbers, which can be used to write code for computing a sample of random permutations of a data set.

When doing a permutation test, the null hypothesis H_0 is that the observed data set is a random sample from the set $\mathbb{S}$ of all possible permutations of the data. To carry out such a test, a computer would need to be programmed in such a way that some number N of random permutations of the data would be computed and for each permutation of the data estimates of the effects of the vector $\boldsymbol{E}$ in equation (3.26) of section 3 would be computed. Let $\boldsymbol{SIM}$ denote a $5 \times N$ matrix of simulated estimates of the five effects in the vector $\boldsymbol{E}$ such that each column of this matrix is an estimated realization of the observed vector $\boldsymbol{E}$ based on a random permutation of the data. Similarly, let $\boldsymbol{ALPHA}$ denote a $5 \times N$ matrix such that each column is a copy of the vector $\boldsymbol{E}$. Then consider the relationship $\geq$ and a $5 \times N$ matrix $\boldsymbol{R}$ defined by

$$\boldsymbol{R} = \boldsymbol{SIM} \geq \boldsymbol{ALPHA}\,. \tag{4.5}$$

Let $\boldsymbol{SIM}(i,j)$ denote the element from the i-th row and j-th column of the matrix $\boldsymbol{SIM}$, and define the element $\boldsymbol{ALPHA}(i,j)$ analogously. Each elements of the matrix $\boldsymbol{R}$ is 0 or 1. If the relation

$$\boldsymbol{SIM}(i,j) \geq \boldsymbol{ALPHA}(i,j) \tag{4.6}$$

is true, then the element $\boldsymbol{R}(i,j) = 1$, and if this relation is false, $\boldsymbol{R}(i,j) = 0$. As will be demonstrated in what follows, by using the matrix $\boldsymbol{R}$ various types of p-values may used to judge statistical significance of each of the estimated effects.

For example, let the $SUMROWS$ denote the array with 5 elements that results from summing the rows of the matrix $\boldsymbol{R}$, and let $r(\nu)$ denote the ν-th element in this array, where $\nu = 1, 2, \cdots, 5$. For example, according the ordering

used in defining the 5×1 vector $\boldsymbol{E}$ in section 3, the number $r(\nu)$ denotes the number of times among the N sample values that the inequality

$$\boldsymbol{SIM}(1,j) \geq \boldsymbol{ALPHA}(1,j) \tag{4.7}$$

was satisfied for effect 1.

Observe that for every $\nu = 1.2.\cdots, 5$ the value of $r(\nu)$ will be a member of the set $\{x \mid x = 0, 1, 2 \cdot \cdot, N\}$. Consequently, $p(\nu) = r(\nu)/N$ is the estimated p-value for the estimated effect $\alpha^2(\nu)$ for $\nu = 1, 2, \cdots, 5$, see the vector $\boldsymbol{E}$ in (3.26) in section 3 for details. Note that according to the elements in this vector, $p(1)$ is the p-value for the estimated effect $\alpha^2(1)$ of the marker allele 1. If for any $\nu = 1, 2, \cdots, 5$, $p(\nu) < 0.05$, then the null hypothesis H_0 that the data are a sample from the set $\mathbb{S}$ of all permutations of the data will be rejected and the estimate of the effect $\alpha(\nu)$ will be said to be statistically significant. In this example, the probability 0.05 was chosen arbitrarily, but an investigator would be free to choose any other small probability as a bench mark for declaring statistical significance. Alternatively, an investigator may want to observe each p-value, and make a judgement a to whether an estimated effect was statistically significant. At this point in the discussion, note that each of the probabilities estimated from the data, pertains to only the statistical significance of one of the five estimated effects under consideration. But, as will be demonstrated below, it will also be possible, to obtain joint probabilities that some sets of effects are jointly significant that will be illustrated in the next paragraph.

This other set of interesting joint p-values may be computed by summing the columns of the matrix $\boldsymbol{R}$. Let $SUMCOLUMNS$ denote an array with N elements that are sums of the columns of $\boldsymbol{R}$, and let $c(\nu)$ denote the sum of column ν. Then, the value of each $c(\nu)$ is an integer in the set $\{y \mid y = 0, 1, 2, 3, 4, 5\}$ for $\nu = 1, 2, \cdots, N$. For example, if for some column ν, $c(\nu) = 0$, then all the numbers in column ν of $\boldsymbol{R}$ would be 0, indicating that for every row i the inequality (4.6) was not satisfied for column $\nu = j$. But, if $c(\nu) = 5$ for some column $\nu = j$, then the inequality in (4.6) would be satisfied for all rows $i = 1, 2, 3, 4, 5$. Given the array $SUMCOLUMNS$, it would be possible to estimate a distribution that would provide insights into the joint statistical significance of the 5 estimated effects in the column $\boldsymbol{E}$ in (3.26) in section 3. For any fixed $\nu = 0, 1, 2, 3, 4, 5$, let $m(\nu)$ be the number of elements of the array $SUMCOLUMNS$ has the value ν. Let $p_{joint}(\nu) = m(\nu)/N$ denote the estimated probability for the values $\nu = 0, 1, 2, 3, 4, 5$. By viewing these joint probabilities, an investigator would be in a position to judge whether all the five estimated effects were jointly statistically significant. If, for example, this distribution were skewed to the left so that the probabilities $p_{joint}(\nu)$ for $\nu = 0, 1$ were larger than the probabilities $p_{joint}(\nu)$ for $\nu = 4, 5$, then an investigator could make a judgement as to whether all estimates of the five effects were jointly statistically significant.

Some investigators may wish to carry out a permutation test, but in this chapter the focus of attention will be focused on another class of tests of statistical significance based on Monte Carlo simulation methods that will be formulated in the next section. However, in practice, an investigator may want to carry out statistical test of significance belonging to different classes of tests to get some idea as to whether estimated effects are statistically significant for at least two classes of tests of statistical significance

V. Testing the Statistical Significance of Estimated Effects Based on Monte Carlo Simulation Methods

There is also another approach to judging whether estimates of the five effects are statistically significant by using Monte Carlo simulation methods. Suppose, for example, that there are $n(i,j)$ non-negative simulated realizations of the random variable $\boldsymbol{W}(i,j)$ for every genotype $(i,j) \in \mathbb{G}_2$. The rationale

under lying the choice of non-negative random variable is that most measures with respect to some quantitative trait are non-negative numbers. One of the simplest approaches to simulation realizations of non-negative random variables is to use the absolute or folded normal distribution. Suppose a random $X(i,j)$ has a normal distribution with an expectation $\mu(i,j)$ and variance $\sigma^2(i,j)$ for every genotype $(i,j) \in \mathbb{G}_2$. Let Z denote a standard normal random variable with expectation 0 and variance 1. Then, as is well known, if Z is a simulated realization from a standard normal distribution, then $X(i,j) = \mu(i,j) + \sigma(i,j) Z$ is a simulated realization of the random variable $X(i,j)$ for every genotype $(i,j) \in \mathbb{G}_2$. Given a simulated realization of a random variable $X(i,j)$, $W(i,j) =\mid X(i,j) \mid$ is a simulated realization of a random variable $W(i,j)$ with an absolute normal distribution for every genotype $(i,j) \in \mathbb{G}_2$. A more detailed description of the folded normal distribution will be given in a appendix.

The next step in setting up a Monte Carlo simulation experiment is to formulate a procedure for testing a null hypothesis. Suppose, for example, that the $n(i,j)$ observations of the random variable $\boldsymbol{W}(i,j)$ for every genotype $(i,j) \in \mathbb{G}_2$ are a random sample from a folded normal distribution. Moreover, suppose there are real data consisting of a sample of size $n(i,j)$ for each genotype $(i,j) \in \mathbb{G}_2$. To simplify the notation, from now on the symbols $\mu(i,j)$ and $\sigma^2(i,j)$ will denote estimates of the corresponding expectation and variance for each genotype $(i,j) \in \mathbb{G}_2$ based on the data. Given the data, an investigator could also estimate the genotypic distribution $\{p(i,j) \mid (i,j) \in \mathbb{G}_2\}$ as well as the 5×1 vector $\boldsymbol{E}$ of effects. In this situation, an investigator may wish to entertain the null hypothesis H_0 of homogeneity and suppose that there are positive numbers μ and σ^2 such that

$$\mu(i,j) = \mu_{uc} \text{ and } \sigma^2(i,j) = \sigma^2_{nc} \tag{5.1}$$

for all genotypes $(i,j) \in \mathbb{G}_2$, the subscript uc stands for unconditional.

At this point, to simulate samples from a normal distribution with expectation μ_{uc} and variance σ^2_{uc}, an investigator may decide to choose μ_{uc} and σ^2_{uc} as

$$\mu_{uc} = \sum_{(i,j)\in\mathbb{G}_2} p(i,j)\,\mu(i,j) \tag{5.2}$$

and

$$\sigma^2_{uc} = \sum_{(i,j)\in\mathbb{G}_2} p(i,j)\,\sigma^2(i,j) \tag{5.3}$$

as the parameters in the normal distribution to be used to simulate $n(i,j)$ of realizations of random variable the $X(i,j)$ for every genotype $(i,j) \in \mathbb{G}_2$. Given this simulated sample of realizations of random variables with a homogeneous distribution, the simulated $n(i,j)$ realizations of the phenotypic random variables $W(i,j)$ would be computed by using the formula $W(i,j) =\mid X(i,j) \mid$ for every genotype $(i,i) \in \mathbb{G}_2$.

This choice of σ^2_{uc}, for example, would result in a simulated samples with a variance that would be close to that in the original data so that unrealistic outliers would occur with small probabilities in the simulated data. The choice of μ_{uc}, however, is less sensitive than that for σ^2_{uc}. For it is interesting to note that if the hypothesis H_0 were true, then all effects in the vector $\boldsymbol{E}$ are 0 for any choice of μ. For example, consider the first order effect

$$\alpha(1) = \mu(1) - \mu \tag{5.4}$$

see (3.14) in section 3. By inspecting (3, 13) in section 3, it can be seen that if H_0 is true, then $\mu(1) = \mu$ so that $\alpha(1) = 0$.

Similar arguments may be used to see that if H_0 is true, then all the effects in the vector $\boldsymbol{E}$ would be 0. It is interesting to note that for any choice

of μ in a simulation procedure, all the effects in the vector $\boldsymbol{E}$ would be 0. From the point of view of simulating $n(i,j)$ realizations of the random variables $W(i,j) =\mid X(i,j) \mid$ for every genotype $(i,i) \in \mathbb{G}_2$, it can be seen that if the expectations of the random variables $X(i,j)$ are some constant μ, then the expectations of the random variables $W(i,j)$ would also be some constant. Hence, the estimated effects based on the means of folded normal distribution would also be constant. If a reader is interested in further details regarding the folded normal distribution, it is suggested that the appendix be consulted.

To carry out a test of statistical significance using the Monte Carlo simulation procedure just outlined, an investigator would start with the array $\boldsymbol{DATA}$ in (2.2) . Then the first step in a Monte Carlo simulation procedure would be that of simulating a quasi-data set, $\boldsymbol{QUASIDATA}$, consisting of n realizations of a random $\boldsymbol{W}$ with a normal distribution with expectation μ_{uc} and variance σ^2_{uc} as suggested as indicated above. Let $\boldsymbol{SIMW}(1,1)$ denote the first $n(1,1)$ elements of the array , $\boldsymbol{QUASIDATA}$ and define the arrays $\boldsymbol{SIMW}(0,0)$ and $\boldsymbol{SIMW}(1,0)$ similarly for the numbers of individuals $n(0,0)$ and $n(1,0)$ of genotypes $(0,0)$ and $(1,0)$, respectively. Then, the simulated quasi array of data for the three genotypes may be represented in the form

$$\boldsymbol{QUASIDATA} = \boldsymbol{SIMW}(1,1), \boldsymbol{SIMW}(0,0), \boldsymbol{SIMW}(1,0) \tag{5.5}$$

just as the real data in (4.2).

Given the simulated data set in (5.5), the next step in the Monte Carlo simulation procedure would be that of computing estimates of the five effects in the 5×1 vector $\boldsymbol{E}$. By repeating this step just outlined $N > 1$ times, a version of the $5 \times N$ matrix $\boldsymbol{R}$ in (4.5) could be computed. Then, by using the procedure outlined in section 4, a test statistical significance for any estimated effect the vector $\boldsymbol{E}$ could could be accomplished. Similarly, a test for the joint statistical significance of the five estimated effects could be carried out, by using the procedure outlined in section 4 by summing the rows of the matrix $\boldsymbol{R}$. In such an experiment, an investigator would be testing the null hypothesis that all the five effects in the vector $\boldsymbol{E}$ are zero.

To simplify the notation, from now on the symbols $\mu(i,j)$ and $\sigma^2(i,j)$ will denote estimates of the corresponding expectation and variance for each genotype $(i,j) \in \mathbb{G}_2$. The rational for considering simulated non-negative random variable is that the majority of measurements for some quantitative traits are usually non-negative numbers as stated above. Given this information, an investigator could estimate the genotypic distribution $\{p(i,j) \mid (i,j) \in \mathbb{G}_2\}$ as well as the 5×1 vector $\boldsymbol{E}$ of effects. In setting up this Monte Carlo experiments , an investigator may wish to entertain the null hypothesis H_0 that there are positive numbers μ and σ^2 such that

$$\mu(i,j) = \mu_{uc} \text{ and } \sigma^2(i,j) = \sigma^2_{uc} \tag{5.6}$$

for all genotypes $(i,j) \in \mathbb{G}_2$ see (5.2) and (5.3).

This choice of σ^2, for example, would result in a simulated samples with a variance that would be close to that in the original data so that unrealistic outliers would occur with small probabilities in the simulated data. The choice of μ, however, is less sensitive than that for σ^2. For it is interesting to note that if the hypothesis H_0 were true, then all effects in the vector $\boldsymbol{E}$ are 0 for any choice of μ. For example, consider the first order effect

$$\alpha(1) = \mu(1) - \mu \tag{5.7}$$

see (14) in section 3. By inspecting (3.16) in section 3, it can be seen that if H_0 is true, then $\mu(1) = \mu$ so that $\alpha(1) = 0$. Similar arguments may be used to see that if H_0 is true, then all the effects in the vector $\boldsymbol{E}$ would be 0. It is

interesting to note that for any choice of μ in a simulation procedure, all the effects in the vector $\boldsymbol{E}$ would be 0.

To carry out a test of statistical significance using the Monte Carlo simulation procedure just outlined, an investigator would start with the array $\boldsymbol{DATA}$ in (4.2) . Then the first step in a Monte Carlo simulation procedure would be that of simulation a quai-data set, $\boldsymbol{QUASIDATA}$, consisting of $n(i.j)$ realizations of random variables $\boldsymbol{X}(i,j)$ for every genotype $(i,j) \in \mathbb{G}_2$ with a normal distribution with expectation μ and variance σ^2 that would be transformed to $n(i,j)$ folded normal random variables $W(i.j)$ for every $(i,j) \in \mathbb{G}_2$. Let $\boldsymbol{SIMW}(1,1)$ denote the first $n(1,1)$ elements of the array , $\boldsymbol{QUASIDATA}$, and define the arrays $\boldsymbol{SIMW}(0,0)$ and $\boldsymbol{SIMW}(1,0)$ similarly for the numbers of individuals $n(0,0)$ and $n(1,0)$ of genotypes $(0,0)$ and $(1,0)$, respectively. Then, the simulated quasi array of data for the three genotypes may be represented in the form

$$\boldsymbol{QUASIDATA} = \boldsymbol{SIMW}(1,1), \boldsymbol{SIMW}(0,0), \boldsymbol{SIMW}(1,0) \tag{5.8}$$

just as the real data in (4.2).

Given the simulated data set in (5.8), the next step in the Monte Carlo simulation procedure would be that of computing estimates of the five effects in the 5×1 vector $\boldsymbol{E}$. By continuing this step just outlined $N > 1$ times, a version of the $5 \times N$ matrix $\boldsymbol{R}$ in (4.5) could be computed. Then, by using the procedure outlined in section 4, a test of statistical significance for any estimated effect the vector $\boldsymbol{E}$ could could be accomplished. Similarly, a test for the joint statistical significance of the five estimated effects could be carried out, by summing the rows of the matrix $\boldsymbol{R}$ and counting the numbers of each of the values $0, 1, 2, 3, 4, 5$. In any computer experiment of the type under consideration, an investigator would be testing the null hypothesis that all the five effects in the vector $\boldsymbol{E}$ are 0.

One of the principal goals of the type of Monte Carlo simulation under consideration is that of computing p-values on which a judgment of statistical significance for each of the five effects that were estimated form the data can be made. The $5 \times N$ matrix $\boldsymbol{R}$ plays a fundamental role in estimating the p-values. Consider, for example, an element by element representation of this matrix of the form

$$\boldsymbol{R} = [r_{ij}] \tag{5.9}$$

and let $\alpha^2(i)$ denote squared effect estimated form the data in the vector $\boldsymbol{E}$ for $i = 1, 2, 3, 4, 5$. Similarly, let $\alpha^2(i,j)$ be an estimate of the squared effect i of replication j of the Monte Carlo simulation experiment for $j = 1, 2, \cdots, N$. Then, for each $i = 1, 2 \cdot \cdot, 5$ and $j = 1, 2, \cdots, N$, r_{ij} may be interpreted as a Bernoulli indicator such that $r_{ij} = 1$ if the event

$$\left[\alpha^2(i,j) \geq \alpha^2(i)\right] \tag{5.10}$$

occurs and $r_{ij} = 0$ if event defined in (5.10) does not occur.

Given a null hypothesis H_0 let

$$E[r_{ij} \mid H_0] = 1P[r_{ij} = 1 \mid H_0] + 0P[r_{ij} = 0 \mid H_0] = P[r_{ij} = 1 \mid H_0] = p(i \mid H_0) \tag{5.11}$$

denote the conditional probability that the event in (5.10) occurs, given H_0 for $j = 1, 2, \cdots, N$. The technical details of the random number generator used in the Monte Carlo simulation experiments reported in following sections will not be discussed here. But, because the randomness in the properties of the sequences of uniform random numbers taking values in the interval $[0, 1)$, the assumption that the events denoted in (5.10) are independent for all $j = 1, 2, \cdots, N$ so that for each $i = 1, 2, \cdots, 5$, it is highly plausible to assume that the sequence

of Bernoulli indicator functions are independently distributed with a common expectation $p\,(i \mid H_0)$ defined in (5.11) . Therefore, by invoking the law of large numbers, it follows that

$$\lim_{N\uparrow\infty} \frac{1}{N} \sum_{j-1}^{N} r_{ij} = p\,(i \mid H_0) \tag{5.12}$$

for every $i = 1, 2, \cdots, 5$. If the strong law of large numbers is invoked, then the limit in (5.12) holds with probability one. In general the larger the choice of the number N,the greater is the reliability of the estimate in (5.14) . In the experiments that will be reported in subsequent sections of this chapter, N was chosen as 10,000. With this choice of N the computing run time of each of the experiments reported in the sections to follow was in the range of 2 to 3 minutes, which would be acceptable if a quantitative trait under consideration involved repeating an experiments for 10 to 15 loci which were thought to be involved in the expression of the trait.

At this point in the development of ideas making up the procedure for tests of significance under consideration, it will be helpful to express the ideas in the last paragraph of section 4 more formally. Let

$$s_j = \sum_{i=1}^{5} r_{ij} \tag{5.13}$$

denote the sum of the indicators in column j or the matrix $\boldsymbol{R}$ for $j = 1, 2, \cdots, N$. Then the array $SUMCOLUMNS$ defined in the last paragraph of section 4 has the form

$$SUMCOLUMNS = (s_1, s_2, \cdots, s_N) \ . \tag{5.14}$$

Let

$$[j \mid s_j = x] \tag{5.15}$$

for $x = 0, 1, 2, 3, 4, 5$. Then

$$m\,(x) = \sum_{j \in [j \mid s_j = x]} s_j \tag{5.16}$$

for all x and

$$\sum_{x=0}^{5} m\,(x) = N \ . \tag{5.17}$$

Then in terms of the formal system developed in this section

$$p_{joint}\,[x \mid H_0] = \frac{m\,(x)}{N} \tag{5.19}$$

for all x is the conditional distribution for judging the joint statistically significance of the 5 effects under consideration. Observe that the conditional distribution defined in (5.19) has the property

$$\sum_{x=0}^{5} p_{joint}\,[x \mid H_0] = 1 \tag{5.21}$$

as it should. In any simulation experiment, it is useful to check that this equation holds as part of tests for the correctness of the software. Equation (5.19) is justified, because sets in the collection of sets in (5.15) are a disjoint partition of the set $SUMCOLUMNS$ in (5.14).

VI. Monte Carlo Simulation Experiments on Simulating Data and Testing the Null Hypotheses

There is a vast literature on Monte Carlo simulation procedures that have been used in many fields of science. For example, the paper by Mode and Gallop (2008) [1], as it turned out, provided the authors into a window on an extensive literature on Monte Carlo simulation procedures as used in many fields of science, see the internet link

$$http \quad : \quad //biomedupdater.com/urlu8c?srk = 2a20a8d6419d877$$
$$ad74198186ff9a1c0ae53b88acaa0ade5587ca0baf302b72$$

Furthermore, the book cited in [2] and edited by the author, also contains an extensive collection of papers on the application of Monte Carlo simulation methods in various fields of biology and related sciences. In this section, the random number generator with a very long period set forth in Mode and Gallop (2008) [1] in section on Monte Carlo simulation methods will be used as well as in all sections to follow, containing accounts of Monte Carlo simulation experiments used to test various version of the null hypothesis. There is a caveat that a reader should be aware of when reading the experimental results reported in this and the following sections is that the random number generator used in all experiments reported in this paper was designed for computers based on 32 bit words. The computers used to conduct Monte Carlo simulation experiments reported in this paper, however, were based on 64 bit words. Algorithms for random number generators for 64 bit words may be found in the papers cited in Mode and Gallop (2008) [1], but to implement these algorithms for use on computers based on 64 bit words would require an extensive period of development, using array manipulating programming languages such as APL. It seems plausible, however, that if the Monte Carlo simulation experiments reported in this paper were based on a random number generator designed for a 64 bit word computers rather than the 32 bit word generator, that the results and conclusions would not be significantly different.

The first step in setting up a Monte Carlo experiment to test some null hypothesis H_0 is to simulate the data that will used to estimate all parameters and effects as functions of parameters. In the best of all worlds, data of the type under consideration would be posted on the internet so it could be downloaded by investigators and used to present concrete examples of the application of new statistical procedures. But, unfortunately, getting permission to use such data is often impossible, unless you are a member of the group that has assembled the data. The parameter values used to simulate the data used in the experiments discussed in the section are shown in Table 6.1 below.

Table 6.1 Parameter Values Used to Simulate Data

$\mu(1,1)$	30
$\sigma(1,1)$	$0.25\mu(1,1)$
$\mu(0,0)$	40
$\sigma(0,0)$	$0.25\mu(0,0)$
$\mu(1,0)$	60
$\sigma(1,0)$	$0.25\mu(1,0)$

By way of interpreting the chosen values in Table 6.1, on some scale of hypothetical units used to measure the expression of some quantitative trait under consideration, the expected values for the three genotypes $(1,1)$, $(0,0)$ and $(1,0)$ were chosen as $\mu(1,1) = 30, \mu(0,0) = 40$ and $\mu(1,0) = 60$. The rational used in choosing these numbers was to assign different values to each of these expected values so that the estimated genetic variance would be positive. The rational for not choosing $\mu(1,0) = 50$ but as $\mu(1,0) = 60$ was to consider the

Ref

1. Mode, C. J. and Gallop, R. J. (2008) A Review on Monte Carlo Simulation Methods as They Apply to Mutation and Selection as Formulated in Wright- Fisher Models of Evolutionary Genetics. Mathematical Biosciences 211: 205-225.

case there was a heterotic effect for heterozygotes of genotype $(1,0)$, i.e., it was assumed that there was some interaction of alleles 1 and 0 in individuals, whose genotype was the heterozygote. The reason for choosing the standard deviations $\sigma(1,1)$, $\sigma(0,0)$ and $\sigma(1,0)$ as a common fraction 0.25 of the expectation for each genotype was that the estimates of the environmental valance for each genotype seemed plausible as observed in preliminary experiments. The sample sizes chosen for the genotypes were $n(1,1) = 100$, $n(0,0) = 200$ and $n(1,0) = 450$. These sample sizes resulted in the allele frequencies $p(1) = 0.433$ and $p(0) = 0.567$. The estimated heritability based on the simulated data for the three genotypes was $H = 0.4280$.

Two null hypotheses were considered in the illustrative examples on testing the statistical significance of the squared effects estimated from the simulated data. Presented in table 6.2 are the assigned parameter values used in testing the two null hypotheses under consideration.

Table 6.2 Parameter Valued Used in Testing Two Null Hypotheses Based on Monte Carlo Simulation Methods

$Parn$	$H_0(1)$	$H_0(2)$
$\mu(1,1)$	μ_{uc}	0
$\sigma(1,1)$	σ_{uc}	σ_{uc}
$\mu(0,0)$	μ_{uc}	0
$\sigma(0,0)$	σ_{uc}	σ_{uc}
$\mu(1,0)$	μ_{uc}	0
$\sigma(1,0)$	σ_{uc}	σ_{uc}

In table 6.2, the subscript uc stands for unconditional expectations and standard deviations as shown in (5.1). Moreover, the estimates of these parameters were computed from the simulated data, using formulas (5.2) and (5.3). The estimate σ_{uc} was computed using the formula

$$\sigma_{uc} = \left(\sigma_{un}^2\right)^{\frac{1}{2}}, \tag{6.1}$$

see (5.2). Table 6.3 contains the symbolic form of the squares of the estimated effects, the estimates of these parameters based on the simulated data and the p-values computed in tests of statistical significance of the null hypotheses $H_0(1)$ and $H_0(2)$ using Monte Carlo simulation methods under consideration.

Table 6.3. Statistical Test of Significance of the Estimates of the Squared Effects Based on Monte Carlo Methods

ESQ	EST	$H_0(1)$	$H_0(2)$
$\alpha^2(1)$	0.03185	0.6124	0.3948
$\alpha^2(0)$	0.00557	0.7796	0.64411
$\alpha^2(1,1)$	376.7118	0	0
$\alpha^2(0,0)$	92.48211	0	0
$\alpha^2(1,0)$	106.8706	0	0

In the simulation experiment designed to test the null hypothesis $H_0(1)$, the squares of the estimated effects were computed with 10,000 Monte Carlo replications, using the parameter assignments listed in the second column of table 6.2. Similarly, to test the null hypothesis $H_0(2)$ 10,000 Monte Carlo replications of the squared effects were again computed. The p-values listed in columns 3 and 4 of table 6.3 were computed using the 10,000 Monte Carlo replication as set forth in equation (5.12) with $N = 10,000$ for each null hypothesis being tested. From rows 1 and 2 of table 6.2, it can be seen that if the null hypothesis

$H_0(1)$ is true, then $\alpha^2(1) = 0$ and $\alpha^2(0) = 0$. The estimates of these two squared effects based on the simulated data are 0.03185 and 0.00557, with the corresponding p-values 0.6124 and 0.7796, respectively, under null hypothesis $H_0(1)$, and are not sufficiently small to reject the null hypothesis being tested. An investigator may therefore conclude that the additive effects of alleles 1 and 0 are not statistically different from 0. As can be seen from the second column of table 6.3, the estimates of the squared effects for interactions effects $\alpha^2(1,1)$, $\alpha^2(0,0)$ and $\alpha^2(1,0)$ are 376.7118, 92.48211 and 106.8706, respectively, with corresponding p-values $0, 0, 0$. The number 0 is the smallest possible p-value; consequently, the squared effects for allelic interaction are highly significantly different form zero under the null hypothesis $H_0(1)$. It is interesting to note if the null hypothesis $H_0(2)$ were tested using the same methods, the statistical conclusions just stated for the five squared effects under consideration would not change even though the p-values for the additive effects differ from those that were computed under the null hypothesis $H_0(1)$.

The last set of p-values that will be presented in this section are those described in (5.13) through (5.21), which were estimated by summing the columns $5 \times 10,000$ matrix $\boldsymbol{R}$ of realized Bernoulli indicator functions as described in section 5. Presented in table 6.4 are the are estimates of these p-values under null hypotheses $H_0(1)$ and $H_0(2)$.

Table 6.4 Estimates of the Joint p-Values Under Each Null Hypothesis

Values	0	1	2	3	4	5
$H_0(1)$	0.2204	0.1672	0.6124	0	0	0
$H_0(2)$	0.3559	0.2403	0.3948	0	0	0

Observe that for each row in this table the listed p-values are a distribution satisfying equation (5.21). For example, for null hypothesis $H_0(1)$

$$0.2204 + 0.1672 + 0.6124 = 1. \tag{6.2}$$

An equation of the same form would be also be valid for the row in table 6.4 corresponding to the null hypothesis $H_0(2)$. By way of interpreting this table, with an estimated probability 0.2204, the sum of a column of the indicator matrix $\boldsymbol{R}$ would be zero under the null hypothesis $H_0(1)$. Similarly, under this null hypotheses, 0.6124 was the estimated probability that the sum of a column of indicators was 2. Observe that, because this probability is largest in the distribution under the null hypothesis $H_0(1)$, the number 2 is the mode of this distribution of joint p-values. It is also interesting to note that the estimated probability that a column sum of indicators in the matrix $\boldsymbol{R}$ had the value $3, 4$ or 5 was 0 under both null hypotheses under consideration. It is interesting to also note that the number 2 was also the mode of the distribution of joint probabilities under null hypothesis $H_0(2)$. Moreover, under both null hypotheses, the events that two columns of the indicator matrix $\boldsymbol{R}$ were both occurred more often and has a greater influence on the p-values for judging the statistical significance of each squared effect that was estimated by summing the rows of the indicator matrix $\boldsymbol{R}$.

Software to test the statistical significance of the estimated heritability $H = 0.4280$ was also developed so that p-values of testing null hypotheses of the form H_0: $H = 0$ could be tested in Monte Carlo simulation experiments. For all the null hypotheses tests described in this section, the p-value was zero. Hence, the estimate of heritability was, in a statistical sense, highly significantly different from zero.

VII. Two Monte Carlo Simulation Experiments to Simulate Data and Test Null Hypotheses

In this section two sets of simulated data, for experiments A and B, will be used in testing null hypotheses. Presented in Table 7.1 are the chosen parameter values for simulation data used in experiments A and B reported in this section.

Table 7.1 Parameter Values Used for Simulation Data in Experiments A and B

Prams	*Exp A*	*Exp B*
$\mu(1,1)$	30	30
$\sigma(1,1)$	$0.25\mu(1,1)$	$\mu(1,1)$
$\mu(0,0)$	40	40
$\sigma(0,0)$	$0.25\mu(0,0)$	$\mu(0,0)$
$\mu(1,0)$	60	60
$\sigma(1,0)$	$0.25\mu(1,0)$	$\mu(1,0)$
$n(1,1)$	10	100
$n(0,0)$	1,000	200
$n(1,0)$	15	450

From this table it can see seen that, with the exception of the sample sizes $n(1,1), n(0,0)$ and $n(1,0)$, the values of the parameters chosen for experiment A in the second column of the table 7.1 are the same as those in Table 6.1. In experiment A, it was assumed allele 1 was a rare mutation that arose in some ancestral population from which the sample evolved. The number of individuals of genotype $(1,1)$ in the sample was chosen as $n(1,1) = 10$, and the sample sizes for genotypes $(0,0)$ and $(1,0)$ were chosen as $n(0,0) = 1,000$ and $n(1,0) = 15$. The objective of experiment A was to provide some insights concerning what impact the low frequency of allele 1 in the sample would have on the estimates and test of statistical significance reported in section 6. With the exceptions of the standard deviations, which were chosen as $\sigma(i,j) = \mu(i,j)$ for all genotypes $(i,j) \in \mathbb{G}_2$, the sample sizes and expectations of a quantitative trait under consideration were the same as those in table 6.1. The objective of experiment B was to provide some insights into the effects of higher environmental variances would have on the estimates of parameters when compared with the experiments and tests of statistical significance reported in section 6.

In experiment A, the estimated of the frequencies of alleles 1 and 0 were about $p(1) = 0.01$ and $p(0) = 0.9$. The estimate of heritability in experiment A was $H_A = 0.0902$. In experiment B, the estimates of the frequencies of alleles 1 and 0 were $p(1) = 0.4333$ and $p(0) = 0.5667$, and the estimate of heritability was $H_B = 0.0932$.

Contained in table 7.2 are the parameter values used to test the null hypotheses in experiments A and B.

Table 7.2 Parameter Values for Testing Null Hypotheses

Parms	*Exp A*	*Exp B*
$\mu(1,1)$	$\mu_{uc}(A)$	$\mu_{uc}(B)$
$\sigma(1,1)$	$\sigma_{uc}(A)$	$\sigma_{uc}(B)$
$\mu(0,0)$	$\mu_{uc}(A)$	$\mu_{uc}(B)$
$\sigma(0,0)$	$\sigma_{uc}(A)$	$\sigma_{uc}(B)$
$\mu(1,0)$	$\mu_{uc}(A)$	$\mu_{uc}(B)$
$\sigma(1,0)$	$\sigma_{uc}(A)$	$\sigma_{uc}(B)$

In table 7.2 the same subscripts are used of the designated parameter values used to test to define the null distributions for testing null hypotheses in experiments A and B. The values of these parameters were chosen by using the formulas in

equations (5.2) and (5.3), but it is clear from the assigned parameter values in table 7.1 that the estimated values of the parameters in table 2 would differ in experiments A and B.

Table 7.3 contains estimates of the squares of effects and p-values estimated by using 10, 000 Monte Carlo replications in both experiments A and B.

Table 7.3 Estimated Squared Effects and P-Values for Experiments A and B

Parms	*Est A*	$p-values\ A$	*Est B*	$p-values\ B$
$\alpha^2(1)$	101.5387	0	54.4873	0
$\alpha^2(0)$	99.4566	0.1525	56.0312	0
$\alpha^2(1,1)$	105.7899	0	1771.4965	0
$\alpha^2(0,0)$	403.1152	0	1043.0865	0
$\alpha^2(1,0)$	409.493	0	7.1615	0.0111

From table 7.3, the second column contains the estimates of the squared effect using the simulated data. With the exception of the estimate of additive effect $\alpha^2(0)$, which was 99.4566, the estimated squares of the remaining effects have zero p-values, are highly statistically different from zero. By way of contrast, in table 6.3 all the squared additive effects, are not statistically different under either null hypotheses $H_0(1)$ or $H_0(2)$. From this example, it can be seen that the small numbers of genotypes $(1,1)$ and $(1,0)$ had a significant effect on the reported p-values in column 3 of the table. In column 5 of the table where the p-values for experiment B are displayed, it can be seen from the estimated p-values that, with the exception that for estimate of the squared effect $\alpha^2(1,0)$, the estimates of the four other squared effect are highly statistically different from zero. It is also interesting to note from an estimated p-value of 0.0111, it may be concluded at about the one percent level, the estimate 7.1615 of $\alpha^2(1,0)$ statistically different from zero. The results of this experiments suggest that, even with a low estimate of heritability resulting from higher assigned values of the environmental variances, there may be interesting cases using real data that would lead to squared effects that were statistically different from zero.

The next set of p-values to be presented in this section are two joint distributions for each of the null hypotheses under consideration as shown in table 7.4

Table 7.4 Estimates of the Joint p-Values Under Each Null Hypothesis

Values	0	1	2	3	4	5
$H_0(A)$	0.8475	0.1525	0	0	0	0
$H_0(B)$	0.9889	0.0111	0	0	0	0

From table 7.4, it can be seen that for both hypotheses the mode of the distribution was 0. It is also interesting to note that for hypothesis $H_0(A)$ the p-value at the number 1 is the same as the p-value in table 7.3 corresponding to the estimate of the parameter $\alpha^2(0)$. Similarly, the p-value for hypothesis $H_0(B)$ at the number 1 is the same as the p-value corresponding to the estimate of the parameter $\alpha^2(1,0)$ in table 7.3. From these observations, it follows that for both null hypotheses there were no columns of the $5 \times 10,000$ matrix $\boldsymbol{R}$ with ones for the values 2,3,4,5. It should also be mentioned that the estimates of heritability for both hypotheses had zero p-values even though both estimates were small. Recall that $H_A = 0.0902$ and $H_B = 0.0932$.

VIII. Further Developments and Potential Applications

By using various statistical and other methods, researchers have identified a number regions in the human genome that are associated with diseases such

as Altzheimer's. A recent paper on such regions has been reported in Raj et al. (2012) [5], in which, among other things, eleven regions of the human genome, associated with susceptibility to Altzheimer's disease, have been identified. Evidence is also reported on the existence of a protein net work involving four of these of these regions that is sustained in the human genome by natural selection. The number of individuals in this sample are about 5,000 that the DNA of each individual in the sample has been sequenced. In a related paper Rossin et al. (2011) [6] report that proteins coded by identified regions of the human genome associated with immune-mediated diseases, physically interact and suggest some underlying basic biology. Altzheimer's disease may also be viewed as a quantitative trait whenever its expression is measured on some numerical scale. Moreover, if for each of the 11 genomic regions may be identified in two alternative forms, then from the point of view of quantitative genetics these 11 regions may be referred to as loci with two alleles at each locus.

When an investigator considers 11 loci and two alleles per locus, the number of effects that may be estimated directly will become very large even if only three genotypes per locus may be identified as discussed in the published in chapters 3 and 4. For example for the case of four loci for which only three genotypes may be identified per locus, the number of identifiable genotypes with respect to 11 loci would be

$$3^{11} = 1.771\,5 \times 10^5 \tag{8.1}$$

However, if only four loci were under consideration, then the number of identifiable genotypes would be

$$3^4 = 81\ . \tag{8.2}$$

As expected this is a much smaller number than that in (8.1), but, nevertheless, when an array with 81 cells is under consideration, a problem that may arise is whether the number of individuals in each cell are large enough to draw statistically reliable statistical inferences. Such problems suggest that an investigator should explore the data to estimate the genotypic frequency of each genotypes as well the frequencies of each allele at the four loci under consideration. Similar questions will arise whenever an investigator wishes to explore the data to determine whether there is a sufficient number of observations in each cell to draw reliable statistical inferences, when N, the number of loci under consideration, is such that $N > 4$. A step that should be included in any exploration of the data would be that of determining if all loci under consideration were autosomal, for if one or more sex linked loci are included in the sample, then such loci would need to treated separately.

When all the loci are autosomal, one approach to determine the number of loci that are such that each cell in a multidimensional would have a sufficiently large number of observations to draw reliable statistical inferences is to investigate each locus under consideration. In this investigation one of the goals would be to determine, among other things, whether the frequency of the two alleles at each locus are sufficiently large enough to be included in the construction of arrays of data with respect to two or more loci that will contain a sufficient number of observations in each cell to draw reliable statistical inferences. For the case of the data on Altzheimer's disease mentioned above, an investigator would need to do an exploratory experiment involving 11 loci with two alleles at each locus. But, even in a sample of 5,000 individuals, the frequency of some alleles at one locus or two or more loci may not be sufficiently large to construct multidimensional arrays that involve low frequency alleles.

These observations suggest that it would be expedient for the above case of a sample of 5,000 individuals to estimate the frequency at each of the 11 loci to obtain information as to whether each allele at each locus has a sufficiently high frequency to be included in multidimensional arrays with respect to two or more loci. But, there are other criteria that could also be used to judge as to what loci would be included in multidimensional arrays. For example, if an estimate of

Ref

5. Raj, T., Shulman, J. M., Keenan, B. T. ,Lori B. Chibnik, L. B., Evans, D. A. Bennett, D. A., Stranger, B. E. and De Jager, P. L. (2012) Alzheimer Disease Susceptibility Loci: Evidence for a Protein Network under Natural Selection DOI 10.1016/j.ajhg.2012.02.022. ˙2012 The American Society of Human Genetics.

heritability at some locus is low, then including this locus in a multidimensional array may not be fruitful. An investigator could also use estimates of the five effects that may be estimated when one autosomal locus is under consideration and carry out statistical test significance on the squares of the effects to judge which ones are statistically significant for each of the 11 loci. If there were loci for which none of squares of effects were not statistically significant, then an investigator may not want to include this locus in a multidimensional array involving two or more loci.

With regard to further developments of the software, it would be possible to create a front end to the APL programs to do the analyses reported in this paper so that the existing APL software could be used to carry out the type of exploratory experiment described above using any computer platform. But, before data consisting on multiple arrays involving two or more autosomal loci can be analyzed, an investigator would need to find either existing software or write software to accommodate multidimensional arrays of data on two or more autosomal loci. If APL were used to write this software, then the existing software for the case of two alleles at one autosomal locus could become part of the extended software when the additive effects and intra-locus effects are estimated at each of the loci under consideration. But, to estimate effects involving two or more loci, new programs would need to be written. It seems very plausible that an array manipulating programming language such as APL would be helpful writing succinct code designed to process multidimensional data on multiple loci. If an investigator wanted to consider one or more quantitative traits, then a modification of the ideas presented in chapter4 to accommodate the case in which only three genotypes per locus can be recognized, then the APL software used in this chapter for the case of one locus would need to be extended to handle two or more traits with respect to each locus. For those cases in which two or more loci and two or more traits are under consideration, an array manipulating programming such as APL would be very helpful in writing code to do the required matrix operation described in chapter 4. Just as the ordering of the three genotypes considered in this chapter which played a basic role in writing the software, some expeditious ordering of the genotypes with respect to two or more loci will also be a crucial step in developing computer code to accommodate cases of multiple loci with three recognizable genotypes at each locus.

Appendix

As an aid to developing a deeper understanding as to the properties of the absolute normal distribution that was used in Monte Carlo simulation experiments described in previous sections, in this appendix formulas for the expectation and variance of this distribution will be derived. In the literature on probability and statistics, the absolute normal distribution is called the folded normal and for the case $\mu = 0$ and $\sigma = 1$, this distribution is known as the half normal. The formulas the will be derived below may be found on the internet, and if a reader is interested in more details, it is suggested the web site: en.wikipedia.org/wiki/Folded_normal_distribution be consulted, where some references are also listed. Some proofs of the formulas derived below may also be found on the internet, but many of these proof lack transparency. In what follows, attempts will be made to included enough details with the hope that the derivation of the formulas will be transparent.

The first distribution to be described is the half normal. Let Z denote a normal random variable with expectation 0 and variance 1. In symbols $Z \sim N(0,1)$. The *pdf* of Z is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right] \tag{A.1}$$

for $z \in (-\infty, \infty) = \mathbb{R}$, the set of real numbers. The distribution function of Z is, therefore,

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left[-\frac{1}{2}y^2 dy\right] \tag{A.2}$$

for $z \in \mathbb{R}$. Let $\mu \in \mathbb{R}$ and $\sigma \in [0, \infty)$. Then, by definition, a random variable $X = \mu + \sigma Z$ has a normal distribution with expectation

$$E[X] = \mu + \sigma E[Z] = \mu \tag{A.3}$$

and variance

$$var[X] = E\left[(X-\mu)^2\right] = \sigma^2. \tag{A.4}$$

The random variable $Y = \mid Z \mid$, the absolute value of Z, maps $\mathbb{R}$ into $[0, \infty)$,and has the distribution function

$$\begin{aligned} F(y) &= P[Y \leq y] = P[-y \leq Z \leq y] = \frac{1}{\sqrt{2\pi}} \int_{-y}^{y} \exp[-\frac{1}{2}y^2] dy \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{y} \exp[-\frac{1}{2}y^2] dy = \sqrt{\frac{2}{\pi}} \int_{0}^{y} \exp[-\frac{1}{2}y^2] dy \end{aligned} \tag{A.5}$$

Therefore, the *pdf* of Y

$$f(y) = \frac{dF(y)}{dy} = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}y^2} \tag{A.6}$$

for $y \in [0, \infty)$. By definition, the expectation of Y is

$$E[Y] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} y e^{-\frac{1}{2}y^2} dy. \tag{A.7}$$

But,

$$\int_{0}^{\infty} y e^{-\frac{1}{2}y^2} dy = -\lim_{y \uparrow \infty} e^{-\frac{1}{2}y^2} - \left(-e^{-\frac{1}{2}0}\right) = 1 \tag{A.8}$$

so that

$$E[Y] = \sqrt{\frac{2}{\pi}}\ . \tag{A.9}$$

From the definition of Y, it follows that $Y^2 = Z^2$ so that

$$E\left[Y^2\right] = E\left[Z^2\right] = 1 \tag{A.10}$$

because $Z \sim N(0,1)$. As is well known, the variance of Y may be expressed in the form

$$var[Y] = E\left[Y^2\right] - (E[Z])^2 \tag{A.11}$$

so that

$$var[Y] = 1 - \frac{2}{\pi} < 1. \tag{A.12}$$

The numerical value of this expression is

$$1 - \frac{2}{\pi} = 0.363\,38\ , \tag{A.13}$$

which will be helpful in the numerical evaluation of some formulas to follow.

When formulating a distribution so that it yields non-negative realizations of random variables in Monte Carlo simulation experiments, one approach would be to consider the random variable defined by

$$W = \mu + \sigma Y = \mu + \sigma \mid Z \mid . \tag{A.14}$$

From the above results, it can be seen that

$$E[W] = \mu + \sigma E[Y] = \mu + \sigma\sqrt{\frac{2}{\pi}} . \tag{A.15}$$

Furthermore, the variance of W has the formula

$$\begin{aligned} var[W] &= E\left[(W - E[W])^2\right] \\ &= E\left[\left(\mu + \sigma Y - \mu - \sigma\sqrt{\frac{2}{\pi}}\right)^2\right] \\ &= \sigma^2 E\left[\left(Y - \sqrt{\frac{2}{\pi}}\right)^2\right] \\ &= \sigma^2 var[Y] = \sigma^2\left(1 - \frac{2}{\pi}\right) < \sigma^2 . \end{aligned} \tag{A.16}$$

An advantage of this formulation is that the theoretical expectation and variance are easy to evaluate numerically. Thus, if the random variable defined in $A.14$ were used in testing a null hypothesis, as described in previous sections, the formulas for the expectation and variance in $A.15$ and $A.16$ could be used to estimate the expectation and variance of the random variable W.

From now on attention will be devoted to the folded normal distribution. In the Monte Carlo simulation experiments described in the forgoing sections of this paper is a primary task was to compute realizations of a phenotypic random variable W defined by

$$W = \mid X \mid \ , \tag{A.17}$$

where $X = \mu + \sigma Z \sim N\left(\mu, \sigma^2\right)$ .Some authors call the distribution of the random variable W the folded normal distribution. As a first step in finding the expectation of the random variable W, is to recall the definition of the function $\mid x \mid$ and observe that $\mid x \mid = x$ if $x \geq 0$ and $\mid x \mid = -x$ if $x \leq 0$. Let A denote the set

$$A = [z \in \mathbb{R} \mid \mu + \sigma z \geq 0] . \tag{A.18}$$

Equivalently,

$$A = [z \in \mathbb{R} \mid \mu + \sigma z \geq 0] = \left[z \in \mathbb{R} \mid z \geq \frac{-\mu}{\sigma}\right] . \tag{A.19}$$

Similarly, let B denote the set

$$B = [z \in \mathbb{R} \mid \mu + \sigma z \leq 0] = \left[z \in \mathbb{R} \mid z \leq \frac{-\mu}{\sigma}\right] . \tag{A.20}$$

Then, then the expectation of the random variable W may be represented in the form

$$E[W] = E\left[W \mid Z \geq \frac{-\mu}{2}\right] - E\left[W \mid Z \leq \frac{-\mu}{2}\right] . \tag{A.21}$$

Notes

Observe that

$$E\left[W \mid Z \geq \frac{-\mu}{\sigma}\right] = z\int_{\frac{-\mu}{\sigma}}^{\infty}(\mu+\sigma z)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}d$$

$$= \mu\left(\int_{\frac{-\mu}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\right)+\sigma\int_{\frac{-\mu}{\sigma}}^{\infty}z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz \quad (A.22)$$

It can be seen that the coefficient of μ in $A.22$

$$\int_{\frac{-\mu}{\sigma}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz = 1-\Phi\left(\frac{-\mu}{\sigma}\right). \quad (A.23)$$

Next, observe that

$$\int_{\frac{-\mu}{\sigma}}^{\infty}z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz = -\left(-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}}. \quad (A.24)$$

Therefore

$$E\left[W \mid Z \geq \frac{-\mu}{\sigma}\right] = \mu\left(1-\Phi\left(-\frac{\mu}{\sigma}\right)\right)+\sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}}. \quad (A.25)$$

Next consider

$$E\left[W \mid Z \leq \frac{-\mu}{\sigma}\right] = \mu\left(\int_{-\infty}^{\frac{-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\right)+\sigma\int_{-\infty}^{\frac{-\mu}{\sigma}}z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz. \quad (A.26)$$

By definition

$$\int_{-\infty}^{\frac{-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz = \Phi\left(-\frac{\mu}{\sigma}\right), \quad (A.27)$$

and

$$\int_{-\infty}^{\frac{-\mu}{\sigma}}z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz = -\frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2}{2\sigma^2}}. \quad (A.28)$$

By applying $A.21$ and doing the algebra, it follows that

$$\mu_W = E[W] = \mu(\left(1-2\sigma^2\Phi\left(-\frac{\mu}{\sigma}\right)\right)+\sigma\sqrt{\frac{2}{\pi}}e^{-\frac{\mu^2}{2\sigma^2}}. \quad (A.29)$$

Because $W^2 = X^2$, it follows that

$$E\left[W^2\right] = E\left[X^2\right] = \sigma^2+\mu^2. \quad (A.30)$$

The validity of this equation follows form the equation

$$E[W^2] = var[W]+(E[W])^2. \quad (A.31)$$

Therefore,

$$var[W] = \sigma^2+\mu^2-(\mu_W)^2. \quad (A.32)$$

From this equation, it can be seen that if $\mu > 0$ and large, then for every fixed $\sigma > 0$

$$\Phi\left(-\frac{\mu}{\sigma}\right) \approx 0 \tag{A.33}$$

and

$$e^{-\frac{\mu^2}{2\sigma^2}} \approx 0\ . \tag{A.34}$$

Therefore,

$$\mu_W = E[W] \approx \mu\ , \tag{A.35}$$

and

$$var[W] \approx \sigma^2 \tag{A.36}$$

is valid. These results are of interest, but under the condition listed above, the mean and variance of the folded normal distribution would be near that of the original normal distribution.

In all Monte Carlo simulation experiments reported in the previous sections on testing null hypotheses, σ was chosen as $\sigma = p\mu$, where $0 < p < 1$. In such cases

$$\Phi\left(-\frac{\mu}{\sigma}\right) = \Phi\left(-\frac{1}{p}\right) \tag{A.37}$$

If p is small, then $\Phi\left(-\frac{1}{p}\right) \approx 0$. For example, suppose $p = 1/4$. Then

$$\Phi(-4) \tag{A.38}$$

would be small, and if an algorithm were available to evaluated $\Phi(z)$ for any $z \in \mathbb{R}$, then the number in $A.38$ could be computed. Next observe that if $\sigma = p\mu$ then

$$\exp\left[-\frac{\mu^2}{2p^2\mu^2}\right] = \exp\left[-\frac{1}{2p^2}\right]. \tag{A.39}$$

In particular, if $p = 1/4$, then

$$\exp[-8] = 3.3546 \times 10^{-4} \tag{A.40}$$

It seems plausible, therefore, that for values of p such that $p \leq 1/2$ the approximations in $A.35$ and $A.36$ would be near the actual values of μ_W and $var[W]$.

There is another approach to approximating μ_W and $var[W]$ by using Monte Carlo methods and the law of large numbers. For example, let $W_1, W_2, \cdots, W_N$ be independent realizations of the random variable W. Then

$$\widehat{\mu}_W = \lim_{N\to\infty} \frac{1}{N}\sum_{\nu=1}^{N} W_\nu = \mu_W \tag{A.41}$$

with probability one. Similarly,

$$\widehat{E}\left[W^2\right] = \lim_{N\to\infty} \frac{1}{N}\sum_{\nu=1}^{N} W_\nu^2 = E\left[W^2\right] \tag{A.42}$$

with probability one. Therefore, if N is large, $N \geq 10,000$, then

$$\widehat{\mu}_W \approx \mu_W \tag{A.43}$$

and

$$\widehat{E}\left[W^2\right] \approx E\left[W^2\right]\ . \tag{A.44}$$

Hence,

$$\widehat{var}[W] = \widehat{E}\left[W^2\right] - \left(\widehat{\mu}_W\right)^2 \approx var[W]\ . \tag{A.45}$$

Notes

When these approximations are compared with the numerical value of μ_W in $A.29$ and that of $var[W]$ in $A.31$, then an investigator may judge how well the approximations in $A.43$ and $A.45$ are acceptable.

Lastly observe that there is another check on the correctness of formulas $A.29$ and $A.52$.For if $\mu = 0$ and $\sigma = 1$, then from $A.29$ and $A.32$, it follows that

$$E[W] = \sqrt{\frac{2}{\pi}}\,. \tag{A.46}$$

and

$$var[W] = 1 - \frac{2}{\pi}\,. \tag{A.47}$$

By definition if $\mu = 0$ and $\sigma = 1$, the random variable W has a half normal distribution with an expectation given by $A.46$ and variance $A.47$ which match the formulas in $A.9$ and $A.12$. This demonstration shows that the half normal distribution is a special case of the folded normal distribution as was expected.

Notes

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 4 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A Comparative Analysis of the Contributions of the Agricltural Sector and Industrial Sector towards the Development/Growth of the Nigerian Economy

By Nwosu, Dozie F., Osuji, George A., Ajibade, Bright F. & Ogbumuo, Dennis T.
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Abstract- The comparison of the contribution of agricultural sector and industrial sector towards the development/growths in the Nigerian economy between 2005-2014 were determined using Hoteling T-distribution procedure. The data were collected from National Population Bureau of Statistics (NBS) and two populations were studied, namely agricultural sector and industrial sector. Under these two populations, four samples were selected from each population. For agricultural sector, the samples selected were crop production, livestock, forestry and fishing. For industrial sector, the samples selected were quarrying and other minerals, oil refining, cement and other manufacturing. The analysis shows that both sectors contribute to the development/growth of the Nigerian economy and the hypothesis were tested to know their level of contribution towards the growth/development of the Nigerian economy, which also shows that agricultural sector, contributes more to the economy than the industrial sector.

Keywords: Agricultural sector, Industrial sector, Economic growth, GDP, Mahalanobis D^2-statistics, Hotelling's T^2 Distribution.

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Strictly as per the compliance and regulations of :

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I. Introduction

In most developing countries, agriculture is the main traditional pursuit and the key to sustain the growth of the modern economy. Economic growth has gone hand in hand with agriculture and stagnation in agriculture is the principal explanation for poor economic performance in any developed country like Nigeria.

Economic growth is the increase per capital GDP or other measures of aggregate income, typically reported as the annual rate of change in real GDP. Economic growth is referred to the value of goods and services produced and does not account for working conditions such as education, politics, social, environmental degradation etc.

Agriculture is the backbone of the Nigerian economy and the growth of the economy is primarily driven by improvements in productivity which involves producing goods and services with fewer inputs of labour, energy and material per unit of growth. Population growth contributes to economic growth on a national level but population growth itself does not improve the standard of living.

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Nigerian manufacturing industry has suffered neglects, since the country's economy has depended on the petroleum sector since 1970's. As the government tries to diversify the economy, it is working to revive the manufacturing sector so as to increase its contribution to the Nigerian prosperity. Lagos and its surroundings are about 60% of the Nigerian industrial base. Other key industrial base or centers are Kano, Ibadan, and Kaduna. The Nigerian most manufacturing industries are Beverages, Cement, Cigarettes, Food Processing, Textiles and Detergent. The manufacturing industry contributed 3.6% GDP in 2008 and 4.2% in 2009. The contribution to GDP has changed little over decade. Industries like cement and Beverages attracts investment from home and abroad while other industries are closing up shop. Between 2000 and 2012 more than 850 manufacturing companies either shutdown or temporarily halted production. Capacity utilization in the manufacturing industry is about 53%. Imports of manufactured goods have constituted the biggest category of import since 1980's, but the governments are working to revitalize the ailing sector.

In 2010, the Nigerian government announced a USD 1.3 billion fund to help banks extend credit to the manufacturing sector, following the decline in availability of finance after the onset of the global economic crisis.

Nigeria as a country has an urgent need to increase the supply of capital; there is usually no shortage of labour, though there may be a shortage of particular skill. An increase in capital can increase the total output and average income. If people are transferred to a place where their marginal productivity is low to a developing industry where their marginal productivity is high, the labour force may be unsuited for industrial work. Therefore, it is necessary to improve the level of literacy of the population as a whole and to develop technical and managerial skills at all levels.

Another problem is that a movement of labour from agricultural sector to industrial sector may result in a downfall of the production of food with consequent food shortage in the expanding urban centers of production. A necessary industrialization may be agricultural reforms and modernization such as change in ownership of land. So that the economic growth may involve a radical change of social and political customs.

The problem of industrial sector over decade has been inadequate infrastructure and lack of power supply. The country set a target of generating 6000 MW of electricity by the end of 2009, but estimated national demand is 25000MW. Manufacturers mainly installed their own generator to compensate for spotty supply from the state and the industry as a whole generated around 72% of its own energy needs. But operating these generator increases the cost of manufacturing goods and the cost is passed on the consumer, making it difficult for Nigerian goods to compete with cheaper imports.

II. Methodology

In realizing the objective or purpose of this work, a method of data analysis was employed namely multivariate analysis. Multivariate analysis was used to study and compare the contributions of agricultural sector and industrial sector towards the growth of the Nigerian economy. Multivariate analysis is a major area in statistics where p-correlated variables must be analyzed jointly and large part of the analysis is concerned with inference on the basis of sample information. The multivariate method for this work follows the Hotelling' s T^2distribution with v-degree of freedom. Hotelling's T^2 distribution is obtained from $X^I \Sigma X$ by replacing Σ with its unbiased estimate D which is independently distributed with x.

Notes

Notes

Hotelling's T^2 distribution is a multivariate generalization of Student t-distribution. It is useful in many problems where one will use the t-statistics in the univariate analysis. It can also be used in some situations for which there is no univariate counterpart. The distribution of $\mu = 0$ is called the null distribution of T, while $\mu \neq 0$ is called the non null distribution of T.

Let $x \sim N_p(\mu, \Sigma)$ and, $D \sim W_p(\Sigma, v)$, $D > 0$ then X, D are independent.

Hence,

$$T^2 = VX^I D^{-1} X.$$

Consider

$H_0: \mu = \mu_0$ *vs* $H_1: \mu \neq \mu_0$, and a random sample $\{x_1\ x_2, \ldots, x_n\}$ from $N_p(\mu, \Sigma)$.

Let $\bar{X}$ and S be the sample mean and variance, then

$$s^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^I (X_i - \bar{X})}{n-1}$$

or

$$(n - 1)\, s^2 = \sum_{i=1}^{n}(X_i - \bar{X})^I (X_i - \bar{X})$$

According to Mahalanobis D^2- statistics, the deviation of Hotelling's T^2 for one sample case is as follows

$$\text{Let, Let } t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}, \quad \text{but } \bar{X} - \mu_0 = d. \text{ Hence, } t = \frac{d}{s/\sqrt{n}}; \quad t^2 = \frac{d^2}{s^2/n}.$$

$$\text{Let } D = \frac{d}{s} \text{ and } D^2 = \frac{d^2}{s^2} \Longrightarrow d^2 = D^2 s^2; \text{ then, } t^2 = \frac{D^2 s^2}{s^2/n} = nD^2 = nD^I s^{-1} D$$

$$T^2 = n(\bar{X} - \mu_0)^I s^{-1} (\bar{X} - \mu_0);$$

where s^{-1} is the inverse of the sample covariance matrix, $\bar{X}$ is the mean vector of sample of size n from multivariate normal distribution with mean vector μ_0 and dispersion matrix Σ.

To test the null hypothesis, we compare T^2 with $T^2_{\alpha/2,p,n-p}$ If, $T^2 > T^2_{\alpha/2,p,n-p}$, we reject H_0 and accept if otherwise.

$$\text{Where } T^2_{\alpha/2,p,n-p} \cong \frac{p(n-p)}{n-p} F_{\alpha/2,p,n-p}.$$

For two sample case, let two independent sample $X_i, i = 1,2, \ldots, n_1$ and $Y_j, j = 1,2, \ldots, n_2$ be from $N_p(\mu_1, \Sigma)$ and $N_p(\mu_2, \Sigma)$ respectively. We may wish to test the hypothesis that both samples came from the same distribution; that is $H_0: \mu_1 = \mu_2$ vs $H_1: \mu_1 \neq \mu_2$.

$$s_1 = \frac{\sum_{i=1}^{n_1}(x_i - \bar{x})^I (x_i - \bar{x})}{n_1 - 1}; \quad (n_1 - 1)s_1 = \sum_{i=1}^{n_1}(x_i - \bar{x})^I (x_i - \bar{x})$$

Also,

$$s_2 = \frac{\sum_{j=1}^{n_2}(y_j - \bar{y})^I (y_j - \bar{y})}{n_2 - 1}; \quad (n_2 - 1)s_2 = \sum_{j=1}^{n_2}(y_j - \bar{y})^I (y_j - \bar{y})$$

The pooled estimate S of Σ is $s_p = \frac{(n_1 - 1)s_1 + (n_2 - 1)s_2}{n_1 + n_2 - 2}$

Therefore, the derivation of T^2 for two sample case is as follows ($Mahalanobis\ D^2 - statistic$)

$$t = \frac{\bar{x}-\bar{y}}{s_p\Big/\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}, \text{ let } \bar{x}-\bar{y} = d;\ t = \frac{d}{s_p\Big/\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}};\ t^2 = \frac{d^2}{s_p^2\Big/\frac{1}{n_1}+\frac{1}{n_2}}$$

Let $D = \frac{d}{s_p}$. and $D^2 = \frac{d^2}{s_p^2}$; then, $t^2 = \frac{D^2 s^2}{s_p^2\Big/\frac{1}{n_1}+\frac{1}{n_2}} = \frac{D^2 s_p^2}{s_p^2\left(\frac{n_1+n_2}{n_1 n_2}\right)}$.

Therefore, $T^2 = \frac{n_1 n_2}{n_1+n_2} D^2 = \frac{n_1 n_2}{n_1+n_2} D^I s^{-1} D$

$$= \frac{n_1 n_2}{n_1+n_2} (\bar{x}-\bar{y})^I s_p^{-1} (\bar{x}-\bar{y});$$

The null hypothesis is rejected if $T^2 > T^2_{\alpha/2,p,n_1+n_2-p-1}$ and accepted if otherwise.

Where $T^2_{\alpha/2,p,n_1+n_2-p-1} \cong \frac{p(n_1+n_2-2)}{n_1+n_2-p-1} F_{\alpha/2,p,n_1+n_2-p-1}$.

We can also employ F–statistic given by

$F = \frac{n_1+n_2-p-1}{p(n_1+n_2-2)} T^2$ and compare it with $F_{\alpha/2,p,n_1+n_2-p-1}$.

III. Data Presentation and Analysis

Table 3.1 : The contributions of each sector and their corresponding item (N billion) between 2005 – 2014. Source: National Bureau of Statistics (NBS)

Year	Agricultural Sector				Industrial Sector			
	Crop Production	Live Stock	Forestry	Fishing	Quarrying & Other Mineral	Oil Refining	Cement	Other Manufacturing
2005	192.5	13.7	2.8	7.2	1.4	0.6	0.4	18.5
2006	206.2	14.6	3.0	7.6	1.5	0.7	0.4	20.2
2007	221.6	15.7	3.2	8.1	1.7	0.8	0.4	22.1
2008	237.7	16.7	3.4	8.7	1.9	0.8	0.5	24.2
2009	252.5	17.9	3.6	9.2	2.1	0.9	0.6	26.3
2010	267.2	19.0	3.8	9.8	2.4	1.0	0.6	28.4
2011	282.6	20.3	4.0	10.4	2.7	1.0	0.7	30.5
2012	298.4	21.5	4.2	11.0	3.0	1.1	0.8	32.8
2013	309.6	22.7	4.5	11.7	3.4	1.2	0.8	35.3
2014	324.3	24.0	4.7	12.3	3.8	1.3	1.0	38.0

$$\bar{X} = \begin{bmatrix} 256.26 \\ 18.61 \\ 3.72 \\ 9.60 \end{bmatrix},\ \bar{Y} = \begin{bmatrix} 2.39 \\ 0.94 \\ 0.62 \\ 27.63 \end{bmatrix};\ (\bar{X}-\bar{Y}) = \left(\begin{bmatrix} 256.26 \\ 18.61 \\ 3.72 \\ 9.60 \end{bmatrix} - \begin{bmatrix} 2.39 \\ 0.94 \\ 0.62 \\ 27.63 \end{bmatrix} \right) = \begin{bmatrix} 253.87 \\ 17.67 \\ 3.10 \\ -18.03 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 18105.124 & 1408.244 & 255.878 & 701.32 \\ 1408.244 & 109.949 & 19.978 & 2711.590 \\ 255.878 & 19.978 & 3.636 & 9.690 \\ 701.320 & 2711.590 & 9.960 & 27.320 \end{bmatrix}$$

Notes

Notes

$$S_2 = \begin{bmatrix} 6.049 & 1.614 & 1.482 & 47.953 \\ 1.614 & 0.444 & 0.392 & 12.968 \\ 1.482 & 0.392 & 0.376 & 11.764 \\ 47.953 & 12.968 & 11.764 & 384.601 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 1006.1763 & 78.3254 & 14.2978 & 41.62632 \\ 78.3254 & 6.1329 & 1.1317 & 151.3643 \\ 14.2978 & 1.1317 & 0.2289 & 1.2069 \\ 41.6263 & 151.3643 & 1.2069 & 22.8845 \end{bmatrix}$$

$$s_p^{-1} = \begin{bmatrix} 0.0085 & 0.0020 & -0.5429 & -0.0005 \\ 0.0020 & -0.0003 & -0.1618 & 0.0068 \\ -0.5429 & -0.1618 & 39.1031 & -0.0049 \\ -0.0005 & 0.0068 & -0.0049 & -0.0001 \end{bmatrix}$$

$$T^2 = \frac{n_1 n_2}{n_1 + n_2}(\bar{x} - \bar{y})^I s_p^{-1}(\bar{x} - \bar{y})$$

$$= \frac{100}{20}\begin{bmatrix} 253.87 \\ 17.67 \\ 3.10 \\ -18.03 \end{bmatrix}^I \begin{bmatrix} 0.0085 & 0.0020 & -0.5429 & -0.0005 \\ 0.0020 & -0.0003 & -0.1618 & 0.0068 \\ -0.5429 & -0.1618 & 39.1031 & -0.0049 \\ -0.0005 & 0.0068 & -0.0049 & -0.0001 \end{bmatrix}\begin{bmatrix} 253.87 \\ 17.67 \\ 3.10 \\ -18.03 \end{bmatrix} = 688.90$$

Hypothesis:

H_0: their contributions towards the Nigerian economy are the same

H_1: their contributions towards the Nigerian economy are not the same

$$F = \frac{n_1 + n_2 - p - 1}{p(n_1 + n_2 - 2)}T^2 = \frac{10 + 10 - 2 - 1_1}{2(10 + 10 - 2)}(688.9) = 325.31$$

$$F_{\alpha/2,p,n_1+n_2-p-1} = F_{0.05/2,2,17.} = 3.59$$

Decision Rule: Since $F_{calculated}(325.314) > F_{tabulated}(3.59)$, we reject H_0 and conclude that their contributions are not the same.

a) To Test For Their Level Of Contribution

$t = \frac{\bar{x} - \bar{y}}{s_p \Big/ \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$, which follows Students' t-distribution with $v = n_1 + n_2 - 2$ degree of freedom, where $s_p = \frac{(n_1 - 1)s_1 + (n_2 - 1)s_2}{n_1 + n_2 - 2}$.

Hypothesis

H_0: Agricultural sector contributes more to the Nigerian economy than the industrial sector.

H_1: Industrial sector contributes more to the Nigerian economy than the Agricultural sector

$$s_p = \frac{(10 - 1)(2654.3104) + (10 - 1)(42.7716)}{10 + 10 - 2} = 1348.541$$

$$t = \frac{259.61}{1348.541\Big/\sqrt{\left(\frac{1}{10}+\frac{1}{10}\right)}} = 0.0861$$

$$t_{\alpha,n_1+n_2-2} = t_{0.05,18} = 1.734$$

Decision Rule: Since $t_{calculated}(0.0861) < t_{tabulated}(1.734)$, tcal (0.43047) <ttab (1.734), we do not reject H_0 and conclude that Agricultural sector contributes more to the Nigerian economy than industrial sector.

Notes

IV. Conclusion

Based on the above analysis, it was discovered that the growth of the Nigerian economy is moving on the positive side. It shows that the Agricultural and the Industrial sectors had contributed significantly to the GDP of this nation for the years under study. Also, the Agricultural sector is contributing more to the Nigerian economy than industrial sector.

We therefore recommend that since Agriculture is the major and most certain path to economic growth/development and sustainability, Nigeria Government should give it a priority by developing and exploiting the sector for the upkeep of her teeming populations through the earnings of revenue for development purpose as well as employment for the youths.

The Government should create enabling environment for the industries to succeed and market its industrial produce to the developed and developing countries by maintaining a high level of standard in their industrial products.

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 4 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Dynamical Systems Induced by the Adele Ring

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GJSFR-F Classification : *FOR Code : MSC 2010: 05E15, 11G15, 11R04, 11R09, 11R47, 11R56, 46L10, 46L40, 46L53, 46*

Strictly as per the compliance and regulations of :

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Abstract- In this paper, we construct a dynamical system induced by the Adele ring $\mathbb{A}_{\mathbb{Q}}$; based on the dynamical systems induced by p-adic number fields $\mathbb{Q}_p$; for all primes p: We study fundamental operator-theoretic and operator-algebraic properties of the corresponding crossed product operator algebra generated by such a dynamical system, via free probability.

Keywords: *prime fields (or, p-adic number fields), the adele ring, p-adicvon neumann algebras, adele-ring von neumann algebras, p-adic dynamical systems, adelic dynamical systems.*

I. Introduction

Continued from [10], in this paper, we consider how *primes* (or *prime numbers*) act on operator algebras. In particular, instead of acting each *p-adic number fields* $\mathbb{Q}_p$ to operator algebras, for every prime p, we act *the Adele ring* $\mathbb{A}_{\mathbb{Q}}$ on operator algebras. In [10], we act p-adic number fields $\mathbb{Q}_p$ on a given von Neumann algebra M, and construct a corresponding *dynamical system* generating its *crossed product algebra*. We have studied fundamental properties of such dynamical systems and crossed product W^*-algebras. Also, by applying *free probability*, we considered free-distributional data of certain operators. Here, based on results of [10], we act the Adele ring $\mathbb{A}_{\mathbb{Q}}$ on M.

The relations between primes and operator algebras have been studied in various different approaches. The main purposes of finding such relations are (i) to provide new tools for studying operator algebras, (ii) to apply operator-algebraic techniques (for example, free probability) to study *number theory*, and hence, (iii) to establish bridges between number theory and operator algebra theory. In [4], we studied how primes act "on" certain von Neumann algebras. Also, the primes as *operators* in certain von Neumann algebras have been studied, too, in [5] and [8]. In [6] and [7], we have studied primes as *linear functionals* acting on *arithmetic functions*. i.e., each prime induces a free-probabilistic structure on arithmetic functions. In such a case, one can understand arithmetic functions as *Krein-space operators* (for fixed primes), via certain *representations* (See [11] and [12]). These studies are all motivated by well-known number-theoretic results under free probability techniques.

Arveson studied *histories* as a group of actions induced by *real numbers* $\mathbb{R}$ on (type I subfactors of) $B(H)$, satisfying certain additional conditions, where H is an infinite dimensional separable *Hilbert space* (e.g., [1], [2] and cited papers therein). By understanding the field $\mathbb{R}$ as an additive group $(\mathbb{R}, +)$, he defined an E_0-*group* $\Gamma_{\mathbb{R}}$ of $*$-homomorphisms acting on $B(H)$ indexed by $\mathbb{R}$. By putting additional conditions on $\Gamma_{\mathbb{R}}$, he defined a *history* Γ acting on $B(H)$. We mimic Arveson's con-

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struction to establish our dynamical systems and corresponding crossed product algebras (e.g., [8], [9] and [10]).

In [9], by *framing* (e.g., also see [8]), a group Γ to groupoids generated by partial isometries, we studied possible *distortions* Γ_G *of a history* Γ. It shows that whenever a history Γ acts on H, a family of partial isometries distorts (or reduces, or restricts) the "original" historical property (in the sense of Arveson) of Γ. And such distortions are completely characterized by *groupoid actions*, sometimes called the E_0-*groupoid actions* induced by partial isometries on $B(H)$. The above framed (E_0-)groupoids Γ_G induce corresponding C^*-subalgebras $C^*(\Gamma_G)$ of $B(H)$, investigated by *dynamical system theory* and *free probability* (e.g., [15], [16] and [17]).

Independently, *p-adic analysis* provides a important tool for studying geometry at small distance (e.g., [18]). it is not only interested in various mathematical fields but also in physics (e.g., [3], [4], [5] and [18]). The *p-adic number fields* (or *p-prime fields*) $\mathbb{Q}_p$ and the *Adele ring* $\mathbb{A}_{\mathbb{Q}}$ play key roles in modern *number theory*, *analytic number theory*, *L-function theory*, and *algebraic geometry* (e.g., [3], [13] and [14]). Also, analysis on such Adelic structures gives a way for understanding small-distance-measured geometry (e.g., [18]) and vector analysis under non-Archimedean metric (e.g., [5]). Thus, prime fields and the Adele ring are interesting topics both in mathematics and in other scientific fields.

We attempt to combine the above two topics; dynamical systems and p-adic analysis under Adelic settings; which seem independent from each other.

II. Definitions and Background

In this section, we introduce basic definitions and backgrounds of the paper.

a) p-Adic Number Fields $\mathbb{Q}_p$ and The Adele Ring $\mathbb{A}_{\mathbb{Q}}$. *Fundamental theorem of arithmetic* says that every positive integer in $\mathbb{N}$ except 1 can be expressed as a usual multiplication of *primes* (or prime numbers), equivalently, all positive integers which are not 1 are *prime-factorized* under multiplication. i.e., the primes are the building blocks of all positive integers except for 1. Thus, it is trivial that primes are playing key roles in both classical and advanced *number theory*.

The Adele ring $\mathbb{A}_{\mathbb{Q}}$ is one of the main topics in advanced number theory connected with other mathematical fields like *algebraic geometry* and *L-function theory*, etc.

Throughout this paper, we denote the set of all natural numbers (which are positive integers) by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, and the set of all rational numbers by $\mathbb{Q}$.

Let's fix a prime p. Define *the p-norm* $|.|_p$ on the rational numbers $\mathbb{Q}$ by

$$|q|_p = \left|p^r \tfrac{a}{b}\right|_p \stackrel{def}{=} \tfrac{1}{p^r},$$

whenever $q = p^r \frac{a}{b} \in \mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$, for some $r \in \mathbb{Z}$, with an additional identity:

$$|0|_p \stackrel{def}{=} 0 \text{ (for all primes } p).$$

For example,

$$\left|-\tfrac{24}{5}\right|_2 = \left|2^3 \cdot (-\tfrac{3}{5})\right|_2 = \tfrac{1}{2^3} = \tfrac{1}{8}.$$

It is easy to check that

(i) $|q|_p \geq 0$, for all $q \in \mathbb{Q}$,

(ii) $|q_1 q_2|_p = |q_1|_p \cdot |q_2|_p$, for all $q_1, q_2 \in \mathbb{Q}$

(iii) $|q_1 + q_2|_p \leq \max\{|q_1|_p, |q_2|_p\}$,

for all $q_1, q_2 \in \mathbb{Q}$. In particular, by (iii), we verify that

(iii)$'$ $|q_1 + q_2|_p \leq |q_1|_p + |q_2|_p$,

for all $q_1, q_2 \in \mathbb{Q}$. Thus, by (i), (ii) and (iii)$'$, the p-norm $|.|_p$ is indeed a norm. However, by (iii), this norm is "*non-Archimedean.*"

i. *Definition 2.1.* *We define a set $\mathbb{Q}_p$ by the norm-closure of the normed space $(\mathbb{Q}, |.|_p)$, for all primes p. We call $\mathbb{Q}_p$, the p-adic number field.*

For a fixed prime p, all elements of $\mathbb{Q}_p$ are formed by

$$p^r \left(\sum_{k=0}^{\infty} a_k p^k\right), \text{ for } 0 \leq a_k < p, \tag{2.1.1}$$

for all $k \in \mathbb{N}$, and for all $r \in \mathbb{Z}$. For example,

$$-1 = (p-1)p^0 + (p-1)p + (p-1)p^2 + \cdots.$$

The subset of $\mathbb{Q}_p$, consisting of all elements formed by

$$\sum_{k=0}^{\infty} a_k p^k, \text{ for } 0 \leq a_k < p \text{ in } \mathbb{N},$$

is denoted by $\mathbb{Z}_p$. i.e., for any $x \in \mathbb{Q}_p$, there exist $r \in \mathbb{Z}$, and $x_0 \in \mathbb{Z}_p$, such that

$$x = p^r x_0.$$

Notice that if $x \in \mathbb{Z}_p$, then $|x|_p \leq 1$, and vice versa. i.e.,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}. \tag{2.1.2}$$

So, the subset $\mathbb{Z}_p$ of (2.1.2) is said to be the *unit disk of* $\mathbb{Q}_p$. Remark that

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset p^3\mathbb{Z}_p \supset \cdots.$$

It is not difficult to verify that

$$\mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset p^{-3}\mathbb{Z}_p \subset \cdots,$$

and hence

$$\mathbb{Q}_p = \bigcup_{k=-\infty}^{\infty} p^k \, \mathbb{Z}_p, \text{ set-theoretically.} \tag{2.1.3}$$

Consider the boundary U_p of $\mathbb{Z}_p$. By construction, the boundary U_p of $\mathbb{Z}_p$ is identical to $\mathbb{Z}_p \setminus p\mathbb{Z}_p$, i.e.,

$$U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Z}_p : |x|_p = 1 = p^0\}. \tag{2.1.4}$$

Similarly, the subsets $p^k U_p$ are the boundaries of $p^k\mathbb{Z}_p$ satisfying

$$p^k U_p = p^k\mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}.$$

We call the subset U_p of $\mathbb{Q}_p$ in (2.1.4) the *unit circle of* $\mathbb{Q}_p$. And all elements of U_p are said to be *units of* $\mathbb{Q}_p$.

Therefore, by (2.1.3) and (2.1.4), we obtain that

$$\mathbb{Q}_p = \bigsqcup_{k=-\infty}^{\infty} p^k U_p, \text{ set-theoretically,} \tag{2.1.5}$$

where $\sqcup$ means the disjoint union. By [18], whenever $q \in \mathbb{Q}_p$ is given, there always exist $a \in \mathbb{Q}$, $k \in \mathbb{Z}$, such that

$$q \in a + p^k \, \mathbb{Z}_p, \text{ for } a, k \in \mathbb{Z}.$$

Fact (See [18]) *The p-adic number field $\mathbb{Q}_p$ is a Banach space. And it is locally compact. In particular, the unit disk $\mathbb{Z}_p$ is compact in $\mathbb{Q}_p$.* □

Define now the addition on $\mathbb{Q}_p$ by

$$\left(\sum_{n=-N_1}^{\infty} a_n p^n\right) + \left(\sum_{n=-N_2}^{\infty} b_n p^n\right) = \sum_{n=-\max\{N_1,N_2\}}^{\infty} c_n \, p^n, \tag{2.1.6}$$

for N_1, $N_2 \in \mathbb{N}$, where the summands $c_n p^n$ satisfies that

$$c_n p^n \overset{def}{=} \begin{cases} (a_n + b_n)p^n & \text{if } a_n + b_n < p \\ p^{n+1} & \text{if } a_n + b_n = p \\ s_n p^{n+1} + r_n p^n & \text{if } a_n + b_n = s_n p + r_n, \end{cases}$$

18. V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, p-Adic Analysis and Mathematical Physics, Ser. Soviet & East European Math., vol 1, ISBN: 978-981-02-0880-6,(1994) World Scientific.

for all $n \in \{-\max\{N_1, N_2\}, ..., 0, 1, 2, ...\}$. Clearly, if $N_1 > N_2$ (resp., $N_1 < N_2$), then, for all $j = -N_1, ..., -(N_1 - N_2 + 1)$, (resp., $j = -N_2, ..., -(N_2 - N_1 + 1)$),

$$c_j = a_j \text{ (resp., } c_j = b_j).$$

And define the multiplication "on $\mathbb{Z}_p$" by

$$\left(\sum_{k_1=0}^{\infty} a_{k_1} p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_2} p^{k_2}\right) = \sum_{n=-N}^{\infty} c_n p^n, \tag{2.1.7}$$

where

$$c_n = \sum_{k_1+k_2=n} \Big(r_{k_1,k_2} i_{k_1,k_2} + s_{k_1-1,k_2}\ i^c_{k_1-1,k_2} + s_{k_1,k_2-1} i^c_{k_1,k_2-1} + s_{k_1-1,k_2-1} i^c_{k_1-1,\ k_2-1} \Big),$$

where

$$a_{k_1} b_{k_2} = s_{k_1,k_2}\ p + r_{k_1,k_2},$$

by the division algorithm, and

$$i_{k_1,k_2} = \begin{cases} 1 & \text{if } a_{k_1} b_{k_2} < p \\ 0 & \text{otherwise,} \end{cases}$$

and

$$i^c_{k_1,k_2} = 1 - i_{k_1,k_2},$$

for all $k_1, k_2 \in \mathbb{N}$, and hence, "on $\mathbb{Q}_p$," the multiplication is extended to

$$\left(\sum_{k_1=-N_1}^{\infty} a_{k_1} p^{k_1}\right)\left(\sum_{k_2=-N_2}^{\infty} b_{k_2} p^{k_2}\right)$$

$$= \left(p^{-N_1}\right)\left(p^{-N_2}\right)\left(\sum_{k_1=0}^{\infty} a_{k_1-N_1} p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_1-N_2} p^{k_2}\right). \tag{2.1.7}$$

Then, under the addition (2.1.6) and the multiplication (2.1.7)′, the algebraic triple $(\mathbb{Q}_p, +, \cdot)$ becomes a field, for all primes p. Thus the p-prime fields $\mathbb{Q}_p$ are algebraically fields.

Fact *Every p-acid number field $\mathbb{Q}_p$, with the binary operations* (2.1.6) *and* (2.1.7)′ *is indeed a field.* □

Moreover, the Banach filed $\mathbb{Q}_p$ is also a (unbounded) Haar-measure space $(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \rho_p)$, for all primes p, where $\sigma(\mathbb{Q}_p)$ means the σ-algebra of $\mathbb{Q}_p$, consisting of all measurable subsets of $\mathbb{Q}_p$. Moreover, this measure ρ_p satisfies that

$$\begin{aligned} \rho_p\left(a + p^k \mathbb{Z}_p\right) &= \rho_p\left(p^k \mathbb{Z}_p\right) \\ &= \tfrac{1}{p^k} \\ &= \rho\left(p^k \mathbb{Z}_p^{\times}\right) = \rho\left(a + p^k \mathbb{Z}_p^{\times}\right), \end{aligned} \tag{2.1.8}$$

for all $a \in \mathbb{Q}$, and $k \in \mathbb{Z}$, where $\mathbb{Z}_p^{\times} = \mathbb{Z}_p \setminus \{0\}$. Also, one has

$$\begin{aligned} \rho_p(a + p^k U_p) &= \rho_p\left(p^k U_p\right) = \rho_p\left(p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p\right) \\ &= \rho_p\left(p^k \mathbb{Z}_p\right) - \rho_p\left(p^{k+1} \mathbb{Z}_p\right) \\ &= \tfrac{1}{p^k} - \tfrac{1}{p^{k+1}}, \end{aligned}$$

for all $a \in \mathbb{Q}$. Similarly, we obtain that

$$\rho_p\left(a + p^k U_p\right) = \rho\left(p^k U_p\right) = \tfrac{1}{p^k} - \tfrac{1}{p^{k+1}}, \tag{2.1.9}$$

for all $a \in \mathbb{N}$, and $k \in \mathbb{Z}$ (See Chapter IV of [18]).

Fact *The Banach field $\mathbb{Q}_p$ is an unbounded Haar-measure space, where ρ_p satisfies* (2.1.8) *and* (2.1.9)*, for all primes p.* □

The above three facts show that $\mathbb{Q}_p$ is a unbounded Haar-measured, locally compact Banach field, for all primes p.

ii. *Definition 2.2.* *Let* $\mathcal{P} = \{all\ primes\} \cup \{\infty\}$. *The Adele ring* $\mathbb{A}_{\mathbb{Q}} = (\mathbb{A}_{\mathbb{Q}}, +, \cdot)$ *is defined by the set*

$$\{(x_p)_{p\in\mathcal{P}} : x_p \in \mathbb{Q}_p,\ almost\ all\ x_p \in \mathbb{Z}_p,\ x_\infty \in \mathbb{R}\}, \tag{2.1.10}$$

with identification $\mathbb{Q}_\infty = \mathbb{R}$, *and* $\mathbb{Z}_\infty = [0, 1]$, *the closed interval in* $\mathbb{R}$, *equipped with*

$$(x_p)_p + (y_p)_p = (x_p + y_p)_p, \qquad and \tag{2.1.11}$$

$$(x_p)_p(y_p)_p = (x_p y_p)_p, \tag{2.1.12}$$

for all $(x_p)_p$, $(y_p)_p \in \mathbb{A}_{\mathbb{Q}}$.

Indeed, this algebraic structure $\mathbb{A}_{\mathbb{Q}}$ forms a ring. Also, by the algebraic construction and the product topology, the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is also a locally compact Banach space equipped with the product measure. Set-theoretically,

$$\mathbb{A}_{\mathbb{Q}} \subseteq \prod_{p\in\mathcal{P}} \mathbb{Q}_p = \mathbb{R} \times \left(\prod_{p:prime} \mathbb{Q}_p \right).$$

In fact, the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is a *weak direct product* $\prod'_{p\in\mathcal{P}} \mathbb{Q}_p$ of $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$, i.e.,

$$\mathbb{A}_{\mathbb{Q}} = \prod'_{p\in\mathcal{P}} \mathbb{Q}_p.$$

i.e., whenever $(x_p)_p \in \mathbb{A}_{\mathbb{Q}}$, almost all x_q are in $\mathbb{Z}_q$, for primes q, except for finitely many x_p.

The product measure ρ of the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is given:

$$\rho = \underset{p\in\mathcal{P}}{\times} \rho_p,$$

with identification $\rho_\infty = \rho_{\mathbb{R}}$, the usual distance-measure (induced by $|.|_\infty$) on $\mathbb{R}$.

Fact *The Adele ring* $\mathbb{A}_{\mathbb{Q}}$ *is a unbounded-measured locally compact Banach ring.* □

b) Dynamical Systems Induced by Algebraic Structures: In this section, we briefly discuss about *dynamical systems* induced by algebraic structures. Let X be an arbitrary algebraic structures, i.e., X is a semigroup, or a group, or a groupoid, or an algebra, etc (maybe equipped with topology).

Let M be an algebra over $\mathbb{C}$, and assume there exists a well-defined action α of X acting on M. i.e., $\alpha(x)$ is a well-defined function on X, satisfying that:

$$\alpha(x_1 \cdot x_2) = \alpha(x_1) \circ \alpha(x_2) \text{ on } M,$$

for all x_1, $x_2 \in X$, where $x_1 \cdot x_2$ means the operation on X, and $(\circ)$ means the usual functional composition. For convenience, we denote $\alpha(x)$ simply by α_x, for all $x \in X$.

Then the triple (X, M, α) is called the *dynamical system induced by* X *on* M *via* α. For such a dynamical system (X, M, α), one can define a *crossed product algebra*

$$\mathbb{M}_X = M \times_\alpha X,$$

by the algebra generated by M and $\alpha(X)$, satisfying that:

$$(m_1\alpha_{x_1})\,(m_2\alpha_{x_2}) = (m_1\alpha_{x_1}(m_2))\ \alpha_{x_1x_2} \text{ in } \mathbb{M}_X,$$

where $\alpha_{x_j} = \alpha(x_j)$, for all $m_j\alpha_{x_j} \in \mathbb{M}_X$, for $j = 1, 2$.

If M is a $*$-algebra, then one may have an additional condition;

$$(m\alpha_x)^* = \alpha_x(m^*)\ \alpha_x^* \text{ in } \mathbb{M}_X,$$

for all $m\alpha_x \in \mathbb{M}_X$.

Of course, one can consider the cases where M is equipped with topology. More precisely, in $\mathbb{M}_X$, we may put a topology from the topology on M, making $\alpha(X)$ be continuous.

In this paper, we are interested in cases where given algebras M are *von Neumann algebras*.In such cases, we call the corresponding topological dynamical systems, *W^*-dynamical systems*, and the corresponding crossed product algebra, the *crossed product W^*-algebras*.

c) Free Probability: For more about free probability theory, see [16] and [17]. In this section, we briefly introduce *Speicher*'s combinatorial free probability (e.g., [16]), which is the combinatorial characterization of the original *Voiculescu*'s analytic free probability (e.g., [17]).

Let $B \subset A$ be von Neumann algebras with $1_B = 1_A$ and assume that there exists a *conditional expectation* $E_B : A \to B$ satisfying that:

(i) $E_B(b) = b$, for all $b \in B$,

(ii) $E_B(b\ a\ b') = b\ E_B(a)\ b'$, for all b, $b' \in B$ and $a \in A$,

(iii) E_B is bounded (or continuous), and

(iv) $E_B(a^*) = E_B(a)^*$, for all $a \in A$.

Then the pair (A, E_B) is called a *B-valued (amalgamated) W^*-probability space (with amalgamation over B)*.

For any fixed B-valued random variables a_1, ..., a_s in (A, E_B), we can have the *B-valued free distributional data* of them;

∘ *$(i_1, ..., i_n)$-th B-valued joint $*$-moments*:

$$E_B\left(b_1 a_{i_1}^{r_1}\ b_2 a_{i_2}^{r_2}\ ...\ b_n\ a_{i_n}^{r_n}\right)$$

∘ *$(j_1, ..., j_m)$-th B-valued joint $*$-cumulants*:

$$k_m^B\left(b_1' a_{j_1}^{t_1},\ b_2' a_{j_2}^{t_2},\ ...,\ b_m' a_{j_m}^{t_m}\right),$$

which provide the equivalent B-valued free distributional data of a_1, ..., a_s, for all $(i_1, ..., i_n) \in \{1, ..., s\}^n$, $(j_1, ..., j_m) \in \{1, ..., s\}^m$, for all n, $m \in \mathbb{N}$, where b_1, ..., b_n, b_1', ..., $b_m' \in B$ are arbitrary and r_1, ..., r_n, t_1, ..., $t_m \in \{1, *\}$. By the *Möbius inversion*, indeed, they provide the same, or equivalent, B-valued free distributional data of a_1, ..., a_s. i.e., they satisfy

$$E_B\left(b_1 a_{i_1}^{r_1}\ ...\ b_n a_{i_n}^{r_n}\right) = \sum_{\pi \in NC(n)} k_\pi^B\left(b_1 a_{i_1}^{r_1},\ ...,\ b_n a_{i_n}^{r_n}\right)$$

and

$$k_m^B\left(b_1' a_{j_1}^{r_1},\ ...,\ b_m' a_{j_m}^{t_m}\right) = \sum_{\theta \in NC(m)} E_{B:\theta}\left(b_1' a_{j_1}^{t_1}, ..., b_m' a_{j_m}^{t_m}\right) \mu(\theta, 1_m),$$

where $NC(k)$ is the *lattice of all noncrossing partitions over* $\{1, ..., k\}$, for $k \in \mathbb{N}$, and $k_\pi^B(...)$ and $E_{B:\theta}(...)$ are the *partition-depending cumulant* and the *partition-depending moment*, and where μ is the *Möbius functional* in the *incidence algebra* I_2.

Recall that the *partial ordering on $NC(k)$* is defined by

$$\pi \leq \theta \overset{def}{\Longleftrightarrow} \forall \text{ blocks } V \text{ in } \pi,\ \exists \text{ blocks } B \text{ in } \theta \text{ s.t.} V \subseteq B.,$$

for all $k \in \mathbb{N}$. Under such a partial ordering $\leq$, the set $NC(k)$ is a lattice with its *maximal element* $1_k = \{(1, ..., k)\}$ and its *minimal element* $0_k = \{(1), (2), ..., (k)\}$. The notation $(...)$ inside partitions $\{...\}$ means the *blocks of the partitions*. For example, 1_k is the one-block partition and 0_k is the k-block partition, for $k \in \mathbb{N}$. Also, recall that the *incidence algebra* I_2 is the collection of all functionals

$$\xi : \cup_{k=1}^{\infty}\left(NC(k) \times NC(k)\right) \to \mathbb{C},$$

satisfying $\xi(\pi, \theta) = 0$, whenever $\pi > \theta$, with its usual function addition $(+)$ and its *convolution* $(*)$ defined by

$$\xi_1 * \xi_2(\pi, \theta) \overset{def}{=} \sum_{\pi \leq \sigma \leq \theta} \xi_1(\pi, \sigma)\ \xi_2(\sigma, \theta),$$

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17. D. Voiculescu, K. Dykemma, and A. Nica, Free Random Variables, CRM Monograph Series, vol 1., (1992)

for all $\xi_1, \xi_2 \in I_2$. Then this algebra I_2 has the *zeta functional* ζ, defined by

$$\zeta(\pi, \theta) \overset{def}{=} \begin{cases} 1 & \text{if } \pi \leq \theta \\ 0 & \text{otherwise.} \end{cases}$$

The *Möbius functional* μ is the convolution-inverse of ζ in I_2. So, it satisfies

$$\sum_{\pi \in NC(k)} \mu(\pi, 1_k) = 0, \text{ and } \mu(0_k, 1_k) = (-1)^{k-1} c_{k-1}, \tag{4.1.1}$$

for all $k \in \mathbb{N}$, where $c_m \overset{def}{=} \frac{1}{m+1} \begin{pmatrix} 2m \\ m \end{pmatrix}$ is the *m-th Catalan number*, for all $m \in \mathbb{N}$.

The *amalgamated freeness* is characterized by the amalgamated $*$-cumulants. Let (A, E_B) be given as above. Two W^*-subalgebras A_1 and A_2 of A, having their common W^*-subalgebra B in A, are *free over B in* (A, E_B), if and only if all their "mixed" $*$-cumulants vanish. Two subsets X_1 and X_2 of A are *free over B in* (A, E_B), if $vN(X_1, B)$ and $vN(X_2, B)$ are free over B in (A, E_B), where $vN(S_1, S_2)$ means the von Neumann algebra generated by S_1 and S_2. In particular, two B-valued random variable x_1 and x_2 are *free over B in* (A, E_B), if $\{x_1\}$ and $\{x_2\}$ are free over B in (A, E_B).

Suppose two W^*-subalgebras A_1 and A_2 of A, containing their common W^*-subalgebra B, are free over B in (A, E_B). Then we can construct a W^*-subalgebra $vN(A_1, A_2) = \overline{B[A_1 \cup A_2]}^w$ of A generated by A_1 and A_2. Such W^*-subalgebra of A is denoted by $A_1 *_B A_2$. If there exists a family $\{A_i : i \in I\}$ of W^*-subalgebras of A, containing their common W^*-subalgebra B, satisfying $A = *_{\substack{B \\ i \in I}} A_i$, then we call A, the *B-valued free product algebra of* $\{A_i : i \in I\}$.

Assume now that the W^*-subalgebra B is $*$-isomorphic to $\mathbb{C} = \mathbb{C} \cdot 1_A$. Then the conditional expectation E_B becomes a linear functional on A. By φ, denote E_B. Then, for $a_1, ..., a_n \in (A, \varphi)$,

$$k_n(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, ..., a_n)\, \mu(\pi, 1_n)$$

by the Möbius inversion

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi_V(a_1, ..., a_n) \right) \mu(\pi, 1_n)$$

since the images of φ are in $\mathbb{C}$.

For example, if $\pi = \{(1, 3), (2), (4, 5)\}$ in $NC(5)$, then

$$\begin{aligned} \varphi_\pi(a_1, ..., a_5) &= \varphi(a_1 \varphi(a_2) a_3) \varphi(a_4 a_5) \\ &= \varphi(a_1 a_3) \varphi(a_2) \varphi(a_4 a_5). \end{aligned}$$

Remember here that, if φ is an arbitrary conditional expectation E_B, and if $B \neq \mathbb{C} \cdot 1_A$, then the above second equality does not hold in general.

So, we have

$$k_n(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi_V(a_1, ..., a_n) \mu(0_{|V|}, 1_{|V|}) \right) \tag{4.1.2}$$

by the multiplicativity of μ.

III. p-Adic W^*-Dynamical Systems

In this section, we introduce W^*-dynamical systems induced by p-adic number fields $\mathbb{Q}_p$, for $p \in \mathcal{P}$. They are defined by a certain semigroup(-or-monoidal) dynamical systems induced by semigroups (resp., monoids) $\sigma(\mathbb{Q}_p) = (\sigma(\mathbb{Q}_p), \cap)$. Throughout this section, we fix a von Neumann subalgebra M of $B(H)$, and a prime p.

a) p-Prime von Neumann Algebras $L^\infty(\mathbb{Q}_p)$: As a measure space, the p-adic number field $\mathbb{Q}_p$ has its corresponding *L^2-Hilbert space* H_p, defined by

$$H_p \overset{def}{=} L^2\left(\mathbb{Q}_p,\ \rho_p\right), \text{ for all primes } p. \tag{3.1.1}$$

We call H_p, the *p-prime Hilbert space*. Remark that all elements of H_p are the functions approximated by simple functions

$$\sum_{S\in\sigma(\mathbb{Q}_p)} t_S\ \chi_S$$

(under limit), generated by *characteristic functions* χ_X,

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathbb{Q}_p$, with $t_S \in \mathbb{C}$, for $S \in \sigma(\mathbb{Q}_p)$. So, one can understand each element f of H_p as an expression,

$$f = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\ \chi_S \text{ (possibly an infinite sum).}$$

By definition, the inner product $<,>_p$ on H_p is defined by

$$< f_1,\ f_2 >_p \overset{def}{=} \int_{\mathbb{Q}_p} f_1\ \overline{f_2}\ d\rho_p,$$

for all $f_1,\ f_2 \in H_p$, having the corresponding norm $\|.\|_p$ on H_p,

$$\|f\|_p \overset{def}{=} \sqrt{< f,\ f >_p} = \sqrt{\int_{\mathbb{Q}_p} |f|^2\ d\rho_p},$$

for all $f \in H_p$. Thus, if $f = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\chi_S$ in H_p, then

$$\int_{\mathbb{Q}_p} f\ d\rho_p = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\ \rho_p(S).$$

Let's fix a function $g \in L^\infty\left(\mathbb{Q}_p,\ \rho_p\right)$, which is an essential-norm bounded function. Similar to H_p-case, one can / may understand g as the approximation of simple functions. Then

$$gf \in H_p, \text{ too, for all } f \in H_p.$$

Definition 3.1. The von Neumann subalgebras $\mathfrak{M}_p = L^\infty\left(\mathbb{Q}_p,\ \rho_p\right)$ of $B(H_p)$ are called the p-prime von Neumann algebras, for all $p \in \mathcal{P}$.

By locally compactness, and Hausdorff property of $\mathbb{Q}_p$, for any $x \in \mathbb{Q}_p$, there exist $a \in \mathbb{Q}$, and $n \in \mathbb{Z}$, such that $x \in a + p^n U_p$ (See [18]). Therefore, we obtain the following lemma.

Lemma 3.1. Let $X \in \sigma(\mathbb{Q}_p)$ be a measurable subset. Then there exists $N \in \mathbb{N} \cup \{\infty\}$, such that: (i) there are corresponding $a_1, ..., a_N \in \mathbb{Q}$, and $n_1, ..., n_N \in \mathbb{Z}$, and (ii) X is covered by the unions of $a_k + p^{n_k}U_p$, for $k = 1, ..., N$, i.e.,

$$X \subseteq \bigcup_{k=1}^{N} \left(a_k + p^{n_k}U_p\right), \tag{3.1.2}$$

where U_p is the unit circle of $\mathbb{Q}_p$, which is the boundary of the unit disk $\mathbb{Z}_p$. □

In (3.1.2), we show that every measurable subset X of $\mathbb{Q}_p$ is covered by a union of transformed boundaries $a + p^k\ U_p$ of $a\ + p^k\mathbb{Z}_p$ ($a \in \mathbb{Q}$, $k \in \mathbb{Z}$).

Lemma 3.2. Let X be a measurable subset of the unit circle U_p in $\mathbb{Q}_p$, for primes p. Then there exists

$$0 \leq r_X \leq 1 \text{ in } \mathbb{R}, \tag{3.1.3}$$

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18. V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, p-Adic Analysis and Mathematical Physics, Ser. Soviet & East European Math., vol 1, ISBN: 978-981-02-0880-6,(1994) World Scientific.

such that

$$\rho_p(X) = r_X\left(1 - \frac{1}{p}\right).$$

□

By (3.1.3), we can obtain the following theorem.

Theorem 3.3. *(See [10]) Let χ_S be a characteristic function for $S \in \sigma(\mathbb{Q}_p)$. Then there exist $N \in \mathbb{N} \cup \{\infty\}$, and $k_1, \ldots, k_N \in \mathbb{Z}$, $r_1, \ldots, r_N \in \mathbb{R}$, such that*

$$\int_{\mathbb{Q}_p} \chi_S \, d\rho_p = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right). \tag{3.1.4}$$

□

The above formula (3.1.4) characterizes the identically-distributedness under the integral in $\mathfrak{M}_p$. By (3.1.4), one can obtain the following corollary.

Corollary 3.4. *Let $g = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S$ be an element of the p-prime von Neumann algebra $\mathfrak{M}_p$. Then there exist*

$$r_j \in [0, 1] \text{ in } \mathbb{R}, \; k_j \in \mathbb{Z}, \text{ and } t_j \in \mathbb{C},$$

and

$$h = \sum_{j=-\infty}^{\infty} (t_j r_j p^{k_j}) \chi_{U_p} \tag{3.1.5}$$

such that g and h are identically distributed under the integral $\int_{\mathbb{Q}_p} \bullet \, d\rho_p$. □

b) p- Adic Semigroup W^*- Dynamical Systems: Now, let M be a fixed von Neumann algebra in the operator algebra $B(H)$ on the Hilbert space H, and $\mathbb{Q}_p$, a fixed p-adic number field, and let $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ be the p-prime von Neumann algebra in the sense of Section 3.1.

Let $\mathcal{H}_p$ be the *tensor product* Hilbert space $H \otimes H_p$ of the p-prime Hilbert space H_p and the Hilbert space H, where $\otimes$ means the topological tensor product of Hilbert spaces. i.e.,

$$\mathcal{H}_p = H \otimes H_p.$$

Understand the σ-algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$ as a monoid $(\sigma(\mathbb{Q}_p), \cap)$. It is not difficult to check indeed $\sigma(\mathbb{Q}_p)$ is a semigroup under the intersection $(\cap)$, with $(\cap)$-identity $\mathbb{Q}_p \in \sigma(\mathbb{Q}_p)$, i.e., it is a well-defined monoid.

Define an action α of the monoid $\sigma(\mathbb{Q}_p)$, acting on the von Neumann algebra M in $B(\mathcal{H}_p)$ by

$$\alpha(S)(m) \overset{def}{=} \chi_S \, m \, \chi_S^* = \chi_S \, m \, \chi_S, \tag{3.2.1}$$

for all $S \in \sigma(\mathbb{Q}_p)$, and $m \in M$, in $B(\mathcal{H}_p)$, by understanding

$$\chi_S = \chi_S \otimes 1_M, \text{ and } m = 1_{\mathfrak{M}_p} \otimes m \text{ in } B(\mathcal{H}_p),$$

where $1_{\mathbb{Q}_p}$ is the identity map $\chi_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, and 1_M is the identity element of M.

Lemma 3.5. *(See [10]) The action α of $\sigma(\mathbb{Q}_p)$ in the sense of (3.2.1) acting on a von Neumann algebra M is a monoid action, and hence, the triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ forms a monoidal dynamical system.* □

Indeed, the morphism α of (3.2.1) satisfies that:

$$\alpha(S_1 \cap S_2) = \alpha(S_1) \circ \alpha(S_2) \text{ on } M,$$

for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$.

Remark that all elements f of the p-prime von Neumann algebra $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ is generated by the σ-algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$, in the sense that: every element $f \in \mathfrak{M}_p$ has its expression, $\sum_{S \in Supp(f)} t_S \chi_S$. So, the action α of (3.2.1) is extended to the linear morphism, also denoted by α, from $\mathfrak{M}_p$ into $B(\mathcal{H}_p)$, acting on M, with

$$\alpha(f)(m) = \alpha\left(\sum_{S\in Supp(f)} t_S\chi_S\right)(m) \tag{3.2.2}$$

$$\stackrel{def}{=} \sum_{S\in Supp(f)} t_S\ \alpha(S)(m) = \sum_{S\in Supp(f)} t_S\ (\chi_S m\chi_S),$$

for all $f \in \mathfrak{M}_p$.

Definition 3.2. Let $\sigma(\mathbb{Q}_p)$ be the σ-algebra of the p-adic number field $\mathbb{Q}_p$, understood as a monoid $(\sigma(\mathbb{Q}_p), \cap)$, and let α be the action of $\sigma(\mathbb{Q}_p)$ on a von Neumann algebra M in the sense of (3.2.1). Then the mathematical triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ is called the p-adic (monoidal) W^*-dynamical system. For this p-adic W^*-dynamical system, define the crossed product W^*-algebra

$$\mathcal{M}_p \stackrel{def}{=} M \times_\alpha \sigma(\mathbb{Q}_p) \tag{3.2.5}$$

by the von Neumann subalgebra of $B(\mathcal{H}_p)$ generated by M and $\chi(\sigma(\mathbb{Q}_p))$ satisfying (3.2.2) (See Section 2.2 above).

The von Neumann algebra $\mathcal{M}_p$ is called the p-adic dynamical W^*-algebra induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$.

Note that, all elements of the p-adic dynamical W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ induced by the p-adic W^*-dynamical system $\mathcal{Q}(M, p)$ have their expressions,

$$\sum_{S\in\sigma\mathbb{Q}_p)} m_S\ \chi_S, \text{ with } m_S \in M$$

(possibly infinite sums under topology). Define the *support* $Supp(T)$ of a fixed element $T = \sum_{S\in\sigma(\mathbb{Q}_p)} m_S\ \chi_S$ in $\mathcal{M}_p$ by

$$Supp(T) \stackrel{def}{=} \{S \in \alpha(\mathbb{Q}_p) : m_S \neq 0_M\}.$$

Now, let $m_1\chi_{S_1}$, $m_2\chi_{S_2} \in \mathcal{M}_p$, with m_1, $m_2 \in M$, S_1, $S_2 \in \sigma(\mathbb{Q}_p)$. Then

$$\begin{aligned}(m_1\chi_{S_1})(m_2\chi_{S_2}) &= m_1\chi_{S_1}m_2\chi_{S_1}\chi_{S_2}\\ &= m_1\chi_{S_1}m_2\chi_{S_1}^2\chi_{S_2} = m_1\chi_{S_1}m_2\chi_{S_1}\chi_{S_1}\chi_{S_2}\end{aligned}$$

since $\chi_S = 1_M \otimes \chi_S$ (in $B(\mathcal{H}_p)$) are projections ($\chi_S^2 = \chi_S = \chi_S^*$), for all $S \in \sigma(\mathbb{Q}_p)$

$$= m_1\ \alpha_{S_1}(m_2)\ \chi_{S_1}\chi_{S_2} = m_1\ \alpha_{S_1}(m_2)\ \chi_{S_1\cap S_2}.$$

Notation For convenience, if there is no confusion, we denote $\alpha_S(m)$ by m^S, for all $S \in \sigma(\mathbb{Q}_p)$, and $m \in M$. □

More generally, one has that:

(3.2.3)

$$\begin{aligned}\prod_{j=1}^{N}(m_j\chi_{S_j}) &= m_1m_2^{S_1}m_3^{S_1\cap S_2}...m_N^{S_1\cap...\cap S_{N-1}}\chi_{S_1\cap...\cap S_N}\\ &= \left(\prod_{j=1}^{N} m_j^{\bigcap_{i=0}^{j-1}S_i}\right)\left(\chi_{\bigcap_{j=1}^{N}S_j}\right)\end{aligned}$$

for all $N \in \mathbb{N}$. Also, we obtain that

$$(m\chi_S)^* = \chi_S m^*\ \chi_S\chi_S = (m^*)^S\ \chi_S, \tag{3.2.4}$$

for all $m\ \chi_S \in \mathcal{M}_p$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

So, let

$$T_k = \sum_{S_k\in Supp(T_k)} m_{S_k}\ \chi_{S_k} \in \mathcal{M}_p, \text{ for } k = 1, 2.$$

Notes

Then

$$T_1T_2 = \sum_{(S_1,S_2)\in Supp(T_1)\times Supp(T_2)} m_{S_1}\chi_{S_1}m_{S_2}\chi_{S_2} = \sum_{(S_1,S_2)\in Supp(T_1)\times Supp(T_2)} m_{S_1}m_{S_2}^{S_1}\ \chi_{S_1\cap S_2}, \tag{3.2.5}$$

by (3.2.3).

Also, if $T = \sum_{S\in Supp(T)} m_S\chi_S$ in $\mathcal{M}_p$, then

(3.2.6)

$$T^* = \sum_{S\in Supp(T)} (m_S^*)^S\chi_S,$$

by (3.2.4).

So, one can have that if

$$T_k = \sum_{S_k\in Supp(T_k)} m_{S_k}\chi_{S_k} \in \mathcal{M}_p, \text{ for } k = 1, ..., n,$$

for $n \in \mathbb{N}$, then

$$T_1^{r_1}T_2^{r_2}\cdots T_n^{r_n} = \prod_{j=1}^{n}\ \sum_{S_j\in Supp(T_j)} [m_{S_j}^{r_j}]^{S_j}\chi_{S_j}\Bigg)$$

where

(3.2.7)

$$[m_{S_j}^{r_j}]^{S_j} \overset{def}{=} \begin{cases} m_{S_j} & \text{if } r_j = 1 \\ (m_{S_j}^*)^{S_j} & \text{if } r_j = *, \end{cases}$$

for $j = 1, ..., n$

$$= \sum_{(S_1,...,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\chi_{S_j}\right)\right)$$

(3.2.8)

$$= \sum_{(S_1,...,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1}S_i\right)}\right)\ \chi_{\bigcap_{j=1}^{n}S_j}\right)\Bigg),$$

for all $(r_1, ..., r_n) \in \{1, *\}^n$.

Lemma 3.6. *Let $T_k = \sum_{S_k\in Supp(T_k)} m_{S_k}\chi_{S_k}$ be elements of the p-adic semigroup W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ in $B(\mathcal{H}_p)$, for $k = 1, ..., n$, for $n \in \mathbb{N}$. Then*

(3.2.9)

$$\prod_{j=1}^{n} T_j^{r_j} = \sum_{(S_1,...,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1}S_i\right)}\right)\ \chi_{\bigcap_{j=1}^{n}S_j}\right)\Bigg),$$

for all $r_1, ..., r_n \in \{1, *\}$, where $[m_{S_j}^{r_j}]^{S_j}$ are in the sense of (3.2.7).

Proof. The proof of (3.2.9) is done by (3.2.8) with (3.2.7). ∎

c) Structure Theorems of $M \times_\alpha \sigma(\mathbb{Q}_p)$. Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic W^*-algebra induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$. In this section, we consider a structure theorem for this crossed product von Neumann algebra $\mathcal{M}_p$.

Define the usual tensor product W^*-subalgebra

$$\mathcal{M}_0 = M \otimes_{\mathbb{C}} \mathfrak{M}_p \text{ of } B(\mathcal{H}_p),$$

where $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ is the p-prime von Neumann algebra in the sense of Section 3.1, and where $\otimes_{\mathbb{C}}$ is the topological tensor product of topological operator algebras over $\mathbb{C}$. By definition, clearly, one can verify that $\mathcal{M}_p$ is a W^*-subalgebra of $\mathcal{M}_0$ in $B(\mathcal{H}_p)$, i.e.,

$$\mathcal{M}_p \overset{\text{Subalgebra}}{\subseteq} \mathcal{M}_0.$$

Now, define the "*conditional*" *tensor product* W^**-algebra*

$$\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p,$$

induced by an action α of $\mathfrak{M}_p$ acting on M (in the sense of (3.2.4)), by a W^*-subalgebra of $\mathcal{M}_0$ dictated by the α*-relations*:

$$(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1 m_2^{S_1}) \otimes \chi_{S_1}\chi_{S_2}, \tag{3.3.1}$$

and

$$(m \otimes \chi_S)^* = (m^*)^S \otimes \chi_S^*, \tag{3.3.2}$$

for all $m_1, m_2, m \in M$, and $S_1, S_2, S \in \sigma(\mathbb{Q}_p)$. i.e., the W^*-subalgebra $\mathcal{M}_0^p$ of $\mathcal{M}_0$ satisfying the α-relations (3.3.1) and (3.3.2) is the conditional tensor product W^*-algebra $M \otimes_\alpha \mathfrak{M}_p$.

Theorem 3.7. *(See [10]) Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic W^*-algebra induced by the p-adic W^*-dynamical system $\mathcal{Q}(M, p)$, and let $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$ be the conditional tensor product W^*-algebra of M and the p-prime von Neumann algebra $\mathfrak{M}_p$ satisfying the α-relations (3.3.1) and (3.3.2). Then these von Neumann algebras $\mathcal{M}_p$ and $\mathcal{M}_0^p$ are $*$-isomorphic in $B(\mathcal{H}_p)$, i.e.,*

(3.3.3)

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \overset{*\text{-iso}}{=} M \otimes_\alpha \mathfrak{M}_p = \mathcal{M}_0^p,$$

in $B(\mathcal{H}_p)$. □

IV. Free Probability on p-Adic Dynamical W^*-Algebras

In this section, we consider free probability on the p-adic dynamical W^*-algebra

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$$

induced by the p-adic W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$.

By Section 3.3, the von Neumann subalgebra $\mathcal{M}_p$ is $*$-isomorphic to the conditional tensor product W^*-algebra $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$ of a fixed von Neumann subalgebra M of $B(H)$ and the p-prime von Neumann algebra $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$, in $B(\mathcal{H}_p)$, for $p \in \mathcal{P}$. So, throughout this section, we understand $\mathcal{M}_p$ and $\mathcal{M}_0^p$, alternatively.

By understanding $\mathcal{M}_p$ as $\mathcal{M}_0^p$, we construct a well-defined conditional expectation

(4.1)

$$E_p : \mathcal{M}_0^p \overset{*\text{-iso}}{=} \mathcal{M}_p \to M_p,$$

where

$$M_p = M \otimes_\alpha \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p),\ S \subseteq U_p\}\right],$$

where U_p is the unit circle of $\mathbb{Q}_p$, which is the boundary $\mathbb{Z}_p - p\mathbb{Z}_p$ of the unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$, satisfying that:

$$E_p(m\chi_S) = E_p(m \otimes \chi_S) \overset{def}{=} m\ \chi_{S \cap U_p},$$

for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

Define now a morphism

$$F_p : M_p \to M_p$$

by a linear transformation satisfying that:

$$F_p\left(m\chi_S\right) = m\left(r_S \chi_{U_p}\right), \tag{4.1}'$$

for all $S \in \sigma(\mathbb{Q}_p)$, where $r_S \in [0, 1]$ in $\mathbb{R}$, making

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10. I. Cho, On Dynamical Systems Induced by p-Adic Number Fields, Opuscula Math., (2015) To Appear.

$$\int_{\mathbb{Q}_p}\left(\chi_{S\cap U_p}\right)d\rho_p = r_S\int_{\mathbb{Q}_p}\left(\chi_{U_p}\right)d\rho_p$$

$$\Longleftrightarrow$$

$$\rho_p\left(r_S\chi_{U_p}\right) = r_S\left(1-\tfrac{1}{p}\right) = \rho_p\left(\chi_{S\cap U_p}\right),$$

by the identically distributedness (3.1.4) (and (3.1.5)), i.e.,

$$\rho_p(S\cap U_p) \overset{def}{=} r_S\left(1-\tfrac{1}{p}\right) = r_S\rho_p(U_p). \qquad (4.1)''$$

Define now a linear functional

$$\gamma : M_p \to \mathbb{C}$$

by

$$\gamma \overset{def}{=} \left(\otimes\int_{\mathbb{Q}_p} \bullet\, d\rho_p\right)\circ F_p, \qquad (4.2)$$

where F_p is in the sense of $(4.1)'$. More precisely, it satisfies that:

$$\begin{aligned}\gamma\left(m \otimes \chi_S\right) &\overset{def}{=} (m)\int_{\mathbb{Q}_p}\left(r_S\chi_{U_p}\right)d\rho_p\\ &= r_S\ (m)\left(1-\tfrac{1}{p}\right).\end{aligned}$$

And then define a linear functional

$$\gamma_p : \mathcal{M}_p \overset{*\text{-iso}}{=} \mathcal{M}_p^0 \to \mathbb{C}$$

by

$$\gamma_p = \gamma\circ E_p, \qquad (4.3)$$

where γ and E_p are in the sense of (4.2) and (4.1), respectively. i.e., for all $m\in M$, and $S\in\sigma(\mathbb{Q}_p)$,

$$\begin{aligned}\gamma_p\left(m\ \chi_S\right) &= \gamma\left(E_p(m\ \chi_S)\right)\\ &= \gamma\left(m^{Y\cap U_p}\ (r\chi_{U_p})\right) = \ (m)\int_{\mathbb{Q}_p}\left(r\ \chi_{U_p}\right)d\rho_p\\ &= r\ \psi(m)\left(1-\tfrac{1}{p}\right),\end{aligned}$$

for some $r\in[0,\,1]$, satisfying $(4.1)''$.

Then the pair $(\mathcal{M}_p,\,\gamma_p)$ is a W^*-probability space in the sense of Section 2.3. We consider the free distributional data of certain elements of $(\mathcal{M}_p,\,\gamma_p)$.

Let $\mathcal{M}_p = M\times_\alpha\sigma(\mathbb{Q}_p)$ be the p-adic dynamical W^*-algebra in $B(\mathcal{H}_p)$, understood also as its $*$-isomorphic von Neumann algebra, $\mathcal{M}_0^p = M\otimes_\alpha\mathfrak{M}_p$. Let $\gamma_p = \gamma\circ E_p$ be the linear functional in the sense of (4.3) on $\mathcal{M}_0^p = \mathcal{M}_p$, where γ is in the sense of (4.2) and E_p is in the sense of (4.1), with $(4.1)''$. i.e., γ_p is a linear functional on $\mathcal{M}_p$, satisfying that:

$$\gamma_p(m\chi_S) = \gamma\left(E_p(m\chi_S)\right) = \gamma\left(m(r\chi_{U_p})\right) = r\psi(m)\left(1-\tfrac{1}{p}\right),$$

for some $r\in[0,\,1]$, satisfying $(4.1)''$, for all $m\in M$, and $S\in\sigma(\mathbb{Q}_p)$.

By [10], the morphism $\gamma_p = \gamma\circ E_p : \mathcal{M}_p\to\mathbb{C}$ of (4.3) is indeed a well-defined bounded linear functional on $\mathcal{M}_p \overset{*\text{-iso}}{=} \mathcal{M}_p^0$.

Definition 4.1. *The pair $(\mathcal{M}_p,\,\gamma_p)$ is called the p-adic dynamical W^*-probability space.*

The following lemmas are obtained by the straightforward computations.

Lemma 4.1. *(See [10]) Let $m\chi_S$ be a free random variable in the p-adic dynamical W^*-probability space $(\mathcal{M}_p,\,\gamma_p)$, with $m\in M$, and $S\in\sigma(\mathbb{Q}_p)$. Then*

$$\gamma_p\left((m\chi_S)^n\right) = r_S\,\psi\big(m(m^S)^{n-1}\big)\left(1-\tfrac{1}{p}\right), \qquad (4.4)$$

for all $n \in \mathbb{N}$, *where* $r_S \in [0, 1]$ *satisfies (4.1)″*. □

Lemma 4.2. *(See [10]) Let* $m_1\chi_{S_1}$, ..., $m_n\chi_{S_n}$ *be free random variables in the p-adic dynamical* W^**-probability space* $(\mathcal{M}_p, \gamma_p)$, *with* $m_k \in M$, $S_k \in \sigma(\mathbb{Q}_p)$, *for* $k = 1, ..., n$, *for* $n \in \mathbb{N}$. *Then there exists* $r_0 \in [0, 1]$, *such that:*

$$\gamma_p\left(\prod_{j=1}^{n} m_j\chi_{S_j}\right) = r_0\left(\psi\left(\prod_{j=1}^{N} m_j^{\bigcap_{i=0}^{j-1} S_i}\right)\right)\left(1-\frac{1}{p}\right). \tag{4.5}$$

□

By (4.4) and (4.5), we obtain the following free distributional data of free random variables of the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$.

Theorem 4.3. *(See [10]) Let* $(\mathcal{M}_p, \gamma_p)$ *be the p-adic dynamical* W^**-probability space, and let*

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k}\chi_{S_k}, \text{ for } k = 1, ..., n,$$

be free random variables in $(\mathcal{M}_p, \gamma_p)$, *for* $n \in \mathbb{N}$. *Then*

$$\gamma_p\left(\prod_{j=1}^{n} T_j\right) = \sum_{(S_1,...,S_n)\in\prod_{j=1}^{n} Supp(T_j)} r_{(S_1,...,S_n)}\left(\psi\left(\prod_{j=1}^{n}\left(m_{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\left(1-\frac{1}{p}\right). \tag{4.6}$$

□

So, by the Möbius inversion of Section 2.3, one can obtain that:

$$\begin{aligned} &k_n\left(\left(m_1\chi_{S_1}\right)^{r_1}, ..., \left(m_n\chi_{S_n}\right)^{r_n}\right) \\ &= \sum_{\pi\in NC(n)} (\gamma_p)_\pi\left([m_1^{r_1}]^{S_1}\chi_{S_1}, ..., [m_n^{r_n}]^{S_n}\chi_{S_n}\right)\mu(\pi, 1_n) \\ &= \sum_{\pi\in NC(n)}\left(\prod_{V\in\pi}(\gamma_p)_V\left([m_1^{r_1}]^{S_1}\chi_{S_1}, ..., [m_n^{r_n}]^{S_n}\chi_{S_n}\right)\mu\left(0_{|V|}, 1_{|V|}\right)\right) \end{aligned}$$

by the Möbius inversion (See Section 4.1)

$$= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,...,i_k)\in\pi}\gamma_p\left([m_{i_1}^{r_{i_1}}]^{S_{i_1}}\chi_{S_{i_1}} \cdots [m_{i_k}^{r_{i_k}}]^{S_{i_k}}\chi_{S_{i_k}}\right)\mu\left(0_k, 1_k\right)\right) \tag{4.7}$$

$$= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,...,i_k)\in\pi}\left(r_V\left(\psi\left(\prod_{t=1}^{k}\left([m_{i_t}^{r_{i_t}}]^{S_{i_t}}\right)^{\bigcap_{t=1}^{k-1} S_{i_t}}\right)\right)\left(1-\frac{1}{p}\right)\right)\mu\left(0_k, 1_k\right)\right),$$

by (4.6), where $r_V \in [0, 1]$ satisfy (4.1)″.

By (4.7), we obtain the following inner free structure of the p-adic dynamical W^*-algebra $\mathcal{M}_p$, with respect to γ_p.

Theorem 4.4. *(See [10]) Let* $m_1\chi_S$, *and* $m_2\chi_S$ *be free random variables in the p-adic dynamical* W^**-probability space* $(\mathcal{M}_p, \gamma_p)$, *with* $m_1, m_2 \in M$, *and* $S \in \sigma(\mathbb{Q}_p) \setminus \{\varnothing\}$. *Also, assume that* S *is not a measure-zero element in* $\sigma(\mathbb{Q}_p)$. *Then* $\{m_1, m_1^S\}$ *and* $\{m_2, m_2^S\}$ *are free in the* W^**-probability space* (M, ψ), *if and only if* $m_1\chi_S$ *and* $m_2\chi_S$ *are free in* $(\mathcal{M}_p, \gamma_p)$. □

Now, let $m_1\ \chi_S$ and $m_2\ \chi_{U_p} \in \mathcal{M}_p$, with $m_1, m_2 \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Assume that $S \cap U_p$ is empty. Since $S \cap U_p = \varnothing$, all mixed cumulants of $m_1\chi_S$ and $m_2\chi_{U_p}$ have $r_V = 0$, for some $V \in \pi$ in (4.7), for all $\pi \in NC(n)$. Therefore, one obtains the following inner freeness condition of $(\mathcal{M}_p, \gamma_p)$.

Theorem 4.5. *(See [10]) Let* $S_1 \neq S_2 \in \sigma(\mathbb{Q}_p)$ *such that* $S_1 \cap S_2 = \varnothing$. *Then the subsets*

$\{m\chi_{S_1} : m \in M\}$ *and* $\{a\chi_{S_2} : a \in M\}$

are free in $(\mathcal{M}_p, \gamma_p)$. □

One may do the same process by fixing p^kU_p instead of fixing U_p, for $k \in \mathbb{Z}$. Recall that p^kU_p are the boundaries $p^k\mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p$ of $p^k\mathbb{Z}_p$, for all $k \in \mathbb{Z}$ (See Section 2.1). i.e., for a fixed $k \in \mathbb{Z}$, define

(4.8)

$$M_{p:k} \overset{def}{=} M \otimes_\alpha \mathbb{C}\left[\{\chi_{p^kU_p}\}\right] \overset{*\text{-iso}}{=} M,$$

Then $M_p = M \otimes_\alpha \mathbb{C}\left[\{\chi_{U_p}\}\right]$ of (4.1) is identical to $M_{p:0}$ in the sense of (4.8).

Similar to (4.1), construct a conditional expectation

$$E_{p:k} : \mathcal{M}_p = \mathcal{M}_p^0 \to M_{p:k}$$

by a linear morphism satisfying that:

(4.9)

$$E_{p:k}\left(m\chi_S\right) = m\ \chi^0_{S\cap p^kU_p},$$

with

$$\chi^0_{S\cap p^kU_p} = r\ \chi_{p^kU_p},$$

where $r \in [0,\ 1]$ satisfying

(4.9)′

$$\int_{\mathbb{Q}_p} \chi^0_{S\cap p^kU_p}\ d\rho_p = r \int_{\mathbb{Q}_p} \chi_{p^kU_p}\ d\rho_p = r\left(\tfrac{1}{p^k} - \tfrac{1}{p^{k+1}}\right).$$

Then, just like (4.1), $E_{p:k}$ is a well-defined conditional expectation from $\mathcal{M}_p$ onto $M_{p:k} = M$.

And then, for $k \in \mathbb{Z}$, define a linear functional

$$\gamma_k : M_{p:k} \to \mathbb{C}$$

by

$$\gamma_k\left(m\ \chi_{p^kU_p}\right) \overset{def}{=} \psi(m)\int_{\mathbb{Q}_p}\left(\chi_{p^kU_p}\right)d\rho_p \tag{4.10}$$

$$= \psi(m)\left(\tfrac{1}{p^k} - \tfrac{1}{p^{k+1}}\right).$$

Then one has a well-defined linear functional

$$\gamma_{p:k} : \mathcal{M}_p \to \mathbb{C}$$

defined by

$$\gamma_{p:k} \overset{def}{=} \gamma_k \circ E_{p:k}, \text{ for all } k \in \mathbb{Z}. \tag{4.11}$$

Note that our linear functional γ_p in the sense of (4.3) is identified with $\gamma_{p:0}$ of (4.11).

Observation 4.1 *Let's replace $M_p = M_{p:0}$ of (4.1) to $M_{p:k}$, for $k \in \mathbb{Z}$. Then the formulae (4.4), (4.5), (4.6) and (4.7) can be re-obtained by replacing factors $\left(1 - \frac{1}{p}\right)$ to $\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)$. So, the freeness of the above two theorems are same under $(\mathcal{M}_p,\ \gamma_{p:k})$-settings.*

For instance, if $m_j\chi_{S_j} \in (\mathcal{M}_p,\ \gamma_{p:k})$, for $j = 1,\ ...,\ n$, for $n \in \mathbb{N}$, then

$$\gamma_{p:k}\left(\prod_{j=1}^{n} m_j\chi_{S_j}\right) = r_0\left(\psi\left(\prod_{j=1}^{N} m_j^{\bigcap_{i=0}^{j-1} S_i}\right)\right)\left(\tfrac{1}{p^k} - \tfrac{1}{p^{k+1}}\right),$$

for some $r_0 \in [0,\ 1]$, satisfying (4.9)′. □

The above **Observation 4.1** shows that we have systems of W^*-probability spaces

$$\left\{(\mathcal{M}_p,\ \gamma_{p:k})\right\}_{k\in\mathbb{Z}},$$

sharing similar free probability with $(\mathcal{M}_p,\ \gamma_p = \gamma_{p:0})$.

V Adelic W^*-Dynamical Systems

In this section, we consider a W^*-dynamical system induced by the σ-algebra $\sigma(\mathbb{A}_{\mathbb{Q}})$ of the Adele ring $\mathbb{A}_{\mathbb{Q}}$. Similar to the p-adic cases of Sections 3 and 4, one may understand $\sigma(\mathbb{A}_{\mathbb{Q}})$ as a monoid

$$\sigma(\mathbb{A}_{\mathbb{Q}}) = (\sigma(\mathbb{A}_{\mathbb{Q}}), \cap),$$

equipped with its binary operation $\cap$, the set intersection. Like in Section 3, we fix a von Neumann algebra M embedded in an operator algebra $B(H)$ on a Hilbert space H.

We will define a suitable action, also denoted by α, of the monoid $\sigma(\mathbb{A}_{\mathbb{Q}})$ acting on M in $B(\mathcal{H}_{\mathbb{Q}})$, where

$$\mathcal{H}_{\mathbb{Q}} = H \otimes H_{\mathbb{Q}}.$$

Before proceeding, we introduce weak tensor product structures in Section 5.1.

a) Weak Tensor Product Structures: Let X_i be arbitrary sets, for $i \in \Lambda$, where Λ means any countable index set. Let

$$g_i : X_i \to X_i \tag{5.1.1}$$

be well-defined functions, for all $i \in \Lambda$.

Now, let X be the *Cartesian product* $\prod\limits_{i\in\Lambda} X_i$ *of* $\{X_i\}_{i\in\Lambda}$. Define the subset $\mathcal{X}$ of X by

$$\mathcal{X} = \left\{ (x_i)_{i\in\Lambda} \in X \;\middle|\; \begin{array}{c} \text{finitely many } x_i \in X_i, \text{ and} \\ \text{almost of all other } x_j \in g_j(X_j), \\ \text{for } i,\ j \in \Lambda \end{array} \right\}, \tag{5.1.2}$$

determined by a system $g = \{g_i\}_{i\in\Lambda}$ of (5.1.1). We denote this subset $\mathcal{X}$ of (5.1.2) in X by

$$\mathcal{X} = \prod_{i\in\Lambda,\, g} X_i.$$

It is clear that $\mathcal{X}$ is a subset of X. If g_i are bijections, for all $i \in \Lambda$, then $\mathcal{X}$ is *equipotent* (or *bijective*) to X. However, in general, $\mathcal{X}$ is a subset of X.

Definition 5.1. *The subset* $\mathcal{X} = \prod\limits_{i\in\Lambda,\, g} X_i$ *of* $X = \prod\limits_{i\in\Lambda} X_i$, *in the sense of (5.1.2), is called the weak tensor product set of* $\{X_i\}_{i\in\Lambda}$ *induced by a system* $g = \{g_i\}_{i\in\Lambda}$ *of functions* g_i.

Let $\mathbb{Q}_p$ be our p-adic number fields, for all $p \in \mathcal{P}$. Define a function

$$g_p : \mathbb{Q}_p \to \mathbb{Q}_p$$

by

$$g_p\left(p^{-N}\left(\sum_{j=0}^{\infty} a_j p^j\right)\right) \overset{def}{=} \sum_{j=0}^{\infty} a_j p^j, \tag{5.1.3}$$

for all $p^{-N}\sum_{j=0}^{\infty} a_j p^j \in \mathbb{Q}_p$ (with $N \in \mathbb{N} \cup \{0\}$), for all $p \in \mathcal{P}$. Then the image $g_p(\mathbb{Q}_p)$ is identical to the compact subset $\mathbb{Z}_p$, the unit disk of $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. Therefore, the Adele ring

$$\mathbb{A}_{\mathbb{Q}} = \prod_{p\in\mathcal{P}}{}' \mathbb{Q}_p$$

is identified with

$$\mathbb{A}_{\mathbb{Q}} = \prod_{p\in\mathcal{P},\, g} \mathbb{Q}_p,$$

by (2.1.10) and (5.1.2), where $g = \{g_p\}_{p\in\mathcal{P}}$ is the system of functions g_p of (5.1.3).

Remark here that, for example, if we have real number r in $\mathbb{R} = \mathbb{Q}_\infty$, with its decimal notation

$$|r| = \sum_{k\in\mathbb{Z}} t_k \cdot 10^{-k} = \cdots t_{-2}t_{-1}t_0.\ t_1t_2t_3\cdots$$

with $0 \leq t_k < 10$ in $\mathbb{N}$, then

$$g_\infty(r) = 0.\ t_1t_2t_3\ \cdots, \tag{5.1.4}$$

with identification $g_\infty(\pm 1) = 1$.

Traditionally, we simply write $\mathbb{A}_\mathbb{Q} = \prod'_{p\in\mathcal{P}} \mathbb{Q}_p$ as before, if there is no confusion.

Remark also that, X_i's of (5.1.1) and (5.1.2) may / can be algebraic structures (e.g., semigroups, or groups, or monoids, or groupoids, or vector spaces, etc), or topological spaces (e.g., Hilbert spaces, or Banach spaces, etc). One may put product topology on the weak tensor product, with continuity on $\{g_i\}_{i\in\Lambda}$. Similarly, if X_i's are topological algebras (e.g., Banach algebras, or C^*-algebras, or von Neumann algebras, etc), then we may have suitable product topology, with bounded (or continuous) linearity on the system $\{g_i\}_{i\in\Lambda}$.

Notation In topological-$*$-algebraic case, to distinguish with other situations, we use the notation $\underset{i\in\Lambda}{\otimes_\Phi}$, instead of using $\underset{i\in\Lambda}{\Pi_\Phi}$, for a system Φ of functions. □

b) The Adele-Ring von Neumann Algebra $\mathfrak{M}$. In this section, we establish a von Neumann algebra $\mathfrak{M}$ generated by the Adele ring $\mathbb{A}_\mathbb{Q}$. Recall that the Adele ring $\mathbb{A}_\mathbb{Q}$ is a unbounded-measured product topological ring induced by $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$. In particular, it is a weak tensor product of $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$, i.e.,

$$\mathbb{A}_\mathbb{Q} = \prod'_{p\in\mathcal{P}} \mathbb{Q}_p = \prod_{p\in} {}_g\mathbb{Q}_p,$$

where $g = \{g_p\}_{p\in\mathcal{P}}$ is the system of functions (5.1.3) satisfying (5.1.4).

By understanding $\mathbb{A}_\mathbb{Q}$ as a measure space $(\mathbb{A}_\mathbb{Q}, \sigma(\mathbb{A}_\mathbb{Q}), \rho)$ (e.g., see Section 2.1), we have the L^2-Hilbert space $H_\mathbb{Q}$, defined by

(5.2.1)

$$H_\mathbb{Q} \overset{def}{=} L^2(\mathbb{A}_\mathbb{Q}, \rho).$$

It has its inner product $<,>$, defined by

(5.2.1)′

$$< F_1, F_2 > \overset{def}{=} \int_{\mathbb{A}_\mathbb{Q}} F_1\ \overline{F_2}\ d\rho,$$

for all $F_1, F_2 \in H_\mathbb{Q}$. And similar to Section 3.1, the von Neumann algebra $\mathfrak{M}$ is defined by

$$\mathfrak{M} \overset{def}{=} L^\infty(\mathbb{A}_\mathbb{Q},\ \rho). \tag{5.2.2}$$

Definition 5.2. We call the Hilbert space $H_\mathbb{Q}$ of (5.2.1) the Adele-ring Hilbert space. The von Neumann algebra $\mathfrak{M}$ of (5.2.2) is said to be the Adele-ring von Neumann algebra.

Let $F = \sum_{Y\in\sigma(\mathbb{A}_\mathbb{Q})} t_Y\ \chi_Y$ be an element of the Adele-ring von Neumann algebra $\mathfrak{M}$. Then

$$\begin{aligned} \int_{\mathbb{A}_\mathbb{Q}} F\ d\rho &= \int_{\mathbb{A}_\mathbb{Q}} \left(\sum_{Y\in\sigma(\mathbb{A}_\mathbb{Q})} t_Y\ \chi_Y \right) d\rho \\ &= \sum_{Y\in\sigma(\mathbb{A}_\mathbb{Q})} t_Y\ \rho(Y). \end{aligned} \tag{5.2.3}$$

By construction, if Y is a subset of the Adele ring $\mathbb{A}_\mathbb{Q}$, then

$$Y = \prod_{p\in\mathcal{P}} Y_p, \tag{5.2.4}$$

where Y_p's are subsets of $\mathbb{Q}_p$, for $p \in \mathcal{P}$. So, by (2.1.10), (5.2.3) and (5.2.4), one has

$$\rho(Y) = \rho\left(\prod_{p\in\mathcal{P}} Y_p\right) = \prod_{p\in\mathcal{P}} \rho_p(Y_p), \tag{5.2.5}$$

by identifying ρ_∞ with the usual distance measure on $\mathbb{Q}_\infty = \mathbb{R}$.

As we discussed in (3.1.3) and (3.1.4), if $S \in \sigma(\mathbb{Q}_p)$, for a prime p, then the element χ_S is identically distributed with

$$\sum_{j=1}^{N} r_j \ \chi_{p^{k_j}U_p},$$

for some $N \in \mathbb{N} \cup \{\infty\}$, $r_j \in [0, 1]$ in $\mathbb{R}$, and $k_j \in \mathbb{Z}$, for $j = 1, ..., N$. Therefore, we obtain the following theorem.

Theorem 5.1. Let $Y \in \sigma(\mathbb{A}_\mathbb{Q})$ and let χ_Y be a generating element of the Adele-ring von Neumann algebra $\mathfrak{M}$. Then there exist $N_p \in \mathbb{N} \cup \{\infty\}$, $r_{p:j} \in [0, 1]$ in $\mathbb{R}$, and $k_{p:j} \in \mathbb{Z}$, for $j = 1, ..., N_p$, for $p \in \mathcal{P}$, such that

$$\int_{\mathbb{A}_\mathbb{Q}} \chi_Y \ d\rho = \prod_{p\in\mathcal{P}} \left(\sum_{j=1}^{N_p} \ r_{p:j}\left(\frac{1}{p^{k_{p:j}}} - \frac{1}{p^{k_{p:j}+1}}\right)\right). \tag{5.2.6}$$

Proof. Let $Y \in \sigma(\mathbb{A}_\mathbb{Q})$. Then, by (5.2.4), there exist $Y_p \in \sigma(\mathbb{Q}_p)$, for all $p \in \mathcal{P}$, such that $Y = \prod_{p\in\mathcal{P}} Y_p$. By Section 3.1, for each $p \in \mathcal{P}$, the ρ_p-measurable subsets Y_p has $N_p \in \mathbb{N} \cup \{\infty\}$, and $k_{p:1}$, ..., $k_{p:N_p} \in \mathbb{Z}$, and $r_{p:1}$, ..., $r_{p:N_p} \in [0, 1]$, such that:

$$\rho_p(Y_p) = \sum_{j=1}^{N_p} \ r_{p:j}\left(\frac{1}{p^{k_{p:j}}} - \frac{1}{p^{k_{p:j}+1}}\right) = \int_{\mathbb{Q}_p} \chi_{Y_p} \ d\rho_p.$$

Therefore, by the product measure $\rho = \times_{p\in\mathcal{P}} \rho_p$ on the Adele ring $\mathbb{A}_\mathbb{Q}$, we have that:

$$\begin{aligned}\int_{\mathbb{A}_\mathbb{Q}} \chi_Y \ d\rho = \rho(Y) &= \left(\times_{p\in\mathcal{P}} \rho_p\right)\left(\prod_{p\in\mathcal{P}} Y_p\right)\\ &= \prod_{p\in\mathcal{P}} \rho_p(Y_p)\\ &= \prod_{p\in\mathcal{P}} \left(\sum_{j=1}^{N_p} \ r_{p:j}\left(\frac{1}{p^{k_{p:j}}} - \frac{1}{p^{k_{p:j}+1}}\right)\right).\end{aligned}$$

Therefore, the formula (5.2.6) holds. ∎

The formula (5.2.6) characterizes the identically-distributedness on elements of the Adele-ring von Neumann algebra $\mathfrak{M}$.

The following theorem provides a structure theorem of the Adele-ring von Neumann algebra $\mathfrak{M}$ in terms of the p-prime von Neumann algebras $\{\mathfrak{M}_p\}_{p\in\mathcal{P}}$.

Theorem 5.2. Let $\mathfrak{M} = L^\infty(\mathbb{A}_\mathbb{Q}, \rho)$ be the Adele-ring von Neumann algebra, and let $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ be the p-prime von Neumann algebras, for $p \in \mathcal{P}$. Then $\mathfrak{M}$ is $*$-isomorphic to the weak tensor product von Neumann algebra $\otimes_{\varphi\atop p\in\mathcal{P}} \mathfrak{M}_p$ of $\{\mathfrak{M}_p\}_{p\in\mathcal{P}}$, induced by the system of functions $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$, i.e.,

$$\mathfrak{M} \overset{*\text{-}iso}{=} \otimes_{\varphi\atop p\in\mathcal{P}} \mathfrak{M}_p, \text{ with } \mathfrak{M}_\infty = L^\infty(\mathbb{R}), \tag{5.2.8}$$

where the weak tensor product $\otimes_\varphi$ is not only algebraic, but also topological, satisfying

$$\varphi_p\left(\sum_{X\in\sigma(\mathbb{Q}_p)} t_X \ \chi_X\right) \overset{def}{=} \sum_{X\in\sigma(\mathbb{Q}_p)} t_X \ \chi_{X\cap\mathbb{Z}_p}, \tag{5.2.9}$$

for all $p \in \mathcal{P}$.

Proof. By the construction of the Adele ring $\mathbb{A}_{\mathbb{Q}}$, it is the weak tensor product $\prod'_{p\in\mathcal{P}} \mathbb{Q}_p$ of $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$ as we discussed in Sections 2.1 and 5.1. Therefore,

$$\mathfrak{M} \overset{def}{=} L^\infty(\mathbb{A}_{\mathbb{Q}},\ \rho) \overset{*\text{-iso}}{=} L^\infty\left(\prod'_{p\in\mathcal{P}} \mathbb{Q}_p,\ \underset{p\in\mathcal{P}}{\times} \rho_p\right)$$

$$\overset{*\text{-iso}}{=} \underset{p\in\mathcal{P}}{\otimes_\varphi} L^\infty\left(\mathbb{Q}_p,\ \rho_p\right) = \underset{p\in\mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p,$$

where $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ is the family of $*$-homomorphisms $\varphi_p : \mathbb{Q}_p \to \mathbb{Z}_p$ of (5.2.9).

Now, it suffices to show that φ_p are $*$-homomorphisms, for all $p \in \mathcal{P}$. Trivially φ_p are linear and bounded, by the very definition. Also, it satisfies that

$$\begin{aligned}\varphi_p\left(\chi_{S_1}\chi_{S_2}\right) &= \varphi_p\left(\chi_{S_1\cap S_2}\right) = \chi_{S_1\cap S_2\cap\mathbb{Z}_p}\\ &= \chi_{(S_1\cap\mathbb{Z}_p)\cap(S_2\cap\mathbb{Z}_p)} = \chi_{S_1\cap\mathbb{Z}_p}\ \chi_{S_2\cap\mathbb{Z}_p}\\ &= \varphi_p\left(S_1\right)\ \varphi_p\left(S_2\right),\end{aligned}$$

for all $S_1,\ S_2 \in \sigma(\mathbb{Q}_p)$. So, for any $g_1,\ g_2 \in \mathfrak{M}_p$,

$$\varphi_p(g_1g_2) = \varphi_p(g_1)\ \varphi_p(g_2).$$

Now, observe that

$$\begin{aligned}\varphi_p\left((t\ \chi_S)^*\right) &= \varphi_p\left(\bar{t}\ \chi_S^*\right) = \bar{t}\ \varphi_p(\chi_S)\\ &= \bar{t}\ \chi_{S\cap\mathbb{Z}_p} = \bar{t}\ \overline{\chi_{S\cap\mathbb{Z}_p}} = \bar{t}\ \left(\chi_{S\cap\mathbb{Z}_p}\right)^*\\ &= \left(t\chi_{S\cap\mathbb{Z}_p}\right)^* = \left(\varphi_p(t\chi_S)\right)^*,\end{aligned}$$

for all $S \in \sigma(\mathbb{Q}_p)$, and $t \in \mathbb{C}$. Therefore, for $g \in \mathfrak{M}_p$,

$$\varphi_p(g^*) = \left(\varphi_p(g)\right)^*.$$

Therefore, φ_p are well-defined $*$-homomorphisms, for all $p \in \mathcal{P}$. So, the family $\{\varphi_p\}_{p\in\mathcal{P}}$ is a system of $*$-homomorphisms.

Therefore, indeed, $\mathfrak{M}$ is $*$-isomorphic to the weak tensor product $\underset{p\in\mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p$, as a well-defined W^*-subalgebra of the usual tensor product W^*-algebra $\underset{p\in\mathcal{P}}{\otimes_{\mathbb{C}}} \mathfrak{M}_p$. ∎

The above theorem shows that, to study the Adele-ring von Neumann algebra $\mathfrak{M}$, we can investigate the system of conditional summands $\mathfrak{M}_p$, the p-prime von Neumann algebras, for $p \in \mathcal{P}$.

c) Adele W^*-Dynamical Systems

Let's fix an arbitrary von Neumann algebra M in an operator algebra $B(H)$, and let $\mathfrak{M} = L^\infty(\mathbb{A}_{\mathbb{Q}}, \rho)$ be the Adele-ring von Neumann algebra, which is $*$-isomorphic to the weak tensor product W^*-algebra $\underset{p\in\mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p$ of p-prime von Neumann algebras $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$, where $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ is in the sense of (5.2.9). Thus, in this section, we understand $\mathfrak{M}$ and $\underset{p\in\mathcal{P}}{\otimes_\varphi} \mathfrak{M}_p$, alternatively.

Consider the σ-algebra $\sigma(\mathbb{A}_{\mathbb{Q}})$ of the Adele ring $\mathbb{A}_{\mathbb{Q}}$ as a monoid $(\sigma(\mathbb{A}_{\mathbb{Q}}), \cap)$, with its identity $\mathbb{A}_{\mathbb{Q}}$. Define a monoidal action α of $\sigma(\mathbb{A}_{\mathbb{Q}})$ acting on M in $B(\mathcal{H}_{\mathbb{Q}})$ by

$$\alpha_S(m) \overset{def}{=} \chi_S\ m\ \chi_S^* = \chi_S\ m\ \chi_S, \tag{5.3.1}$$

for all $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, and $m \in M$, where

$$\mathcal{H}_{\mathbb{Q}} = H \otimes H_{\mathbb{Q}}.$$

Then the action α of $\sigma(\mathbb{A}_{\mathbb{Q}})$ is indeed a well-defined monoidal action, since

$$\begin{aligned}\alpha_{S_1\cap S_2}(m) &= \chi_{S_1\cap S_2}\ m\ \chi_{S_1\cap S_2} = \chi_{S_1\cap S_2}\ m\ \chi_{S_2\cap S_1}\\ &= \chi_{S_1}\ \chi_{S_2}\ m\ \chi_{S_2}\ \chi_{S_1} = \chi_{S_1}\left(\alpha_{S_2}(m)\right)\chi_{S_1}\\ &= \alpha_{S_1}\left(\alpha_{S_2}(m)\right) = \left(\alpha_{S_1}\circ\alpha_{S_2}\right)(m),\end{aligned}$$

and

$$\left(\alpha_{S_1}(m)\right)^* = \left(\chi_{S_1}m\chi_{S_1}\right)^* = \chi_{S_1}\ m^*\ \chi_{S_1} = \alpha_{S_1}(m^*),$$

for all S_1, $S_2 \in \sigma(\mathbb{A}_{\mathbb{Q}})$, and $m \in M$. Thus,

$$\left(\alpha_{S_1\cap S_2}\right)(m) = \left(\alpha_{S_1}\circ\alpha_{S_2}\right)(m), \text{ and } \left(\alpha_{S_1}(m)\right)^* = \alpha_{S_1}(m^*), \qquad (5.3.2)$$

for all S_1, $S_2 \in \sigma(\mathbb{A}_{\mathbb{Q}})$, for all $m \in M$.

The action α compresses operators of M in $B(\mathcal{H}_{\mathbb{Q}})$, and hence it is bounded. So, α is a well-defined monoidal action of $\sigma(\mathbb{A}_{\mathbb{Q}})$ acting on M in $B(\mathcal{H}_{\mathbb{Q}})$, by (5.3.2).

Notation Similar to Section 3.2, we denote $\alpha_S(m)$ simply by m^S, for all $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$ and $m \in M$. □

Notice that there exists an action χ of the monoid $\sigma(\mathbb{A}_{\mathbb{Q}})$ acting on $H_{\mathbb{Q}} = L^2(\mathbb{A}_{\mathbb{Q}}, \rho)$, such that

$$\chi(S) = \chi_S, \text{ the characteristic function of } S, \qquad (5.3.3)$$

for all $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. By construction,

$$H_{\mathbb{Q}} = \overline{\text{linear span of } \chi(\sigma(\mathbb{A}_{\mathbb{Q}}))}^{<,>},$$

under the Hilbert space topology induced by $<,>$ of (5.2.1)′.

Definition 5.3. *The triple $A_M = (M, \sigma(\mathbb{A}_{\mathbb{Q}}), \alpha)$ of a fixed von Neumann algebra M in $B(H)$, the σ-algebra $\sigma(\mathbb{A}_{\mathbb{Q}})$ of the Adele ring $\mathbb{A}_{\mathbb{Q}}$, understood as a monoid equipped with $(\cap)$, and the monoidal action α of $\sigma(\mathbb{A}_{\mathbb{Q}})$ in the sense of (5.3.1), is called an Adele W^*-dynamical system in $B(\mathcal{H}_{\mathbb{Q}})$. For an Adele W^*-dynamical system A_M, define the corresponding crossed product W^*-algebra*

$$\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$$

by the W^*-subalgebra of $B(\mathcal{H}_{\mathbb{Q}})$ generated by M and $\alpha\left(\chi\left(\sigma(\mathbb{A}_{\mathbb{Q}})\right)\right)$, where χ is in the sense of (5.3.3), consisting of all elements

$$\sum_{S\in\sigma(\mathbb{A}_{\mathbb{Q}})} m_S\ \chi_S \text{ with } m_S \in M.$$

This W^*-subalgebra $\mathbb{M}_{\mathbb{Q}}$ of $B(\mathcal{H}_{\mathbb{Q}})$ is said to be the Adele dynamical W^*-algebra induced by A_M.

Let $A_M = (M, \sigma(\mathbb{A}_{\mathbb{Q}}), \alpha)$ be an Adele W^*-dynamical system, and let $\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$ be the Adele dynamical W^*-algebra induced by A_M. Let $m_j\chi_{S_j}$ be elements of $\mathbb{M}_{\mathbb{Q}}$, with $m_j \in M$, and $S_j \in \sigma(\mathbb{A}_{\mathbb{Q}})$, for $j = 1, ..., n$, for $n \in \mathbb{N}$. Then one can obtain that

$$\prod_{j=1}^{n}\left(m_j\ \chi_{S_j}\right) = \left(m\left(\prod_{j=2}^{n} m^{\bigcap_{i=1}^{j-1} S_i}\right)\right)\left(\chi_{\bigcap_{j=1}^{n} S_j}\right), \qquad (5.3.4)$$

since

$$\begin{aligned}(m_1\chi_{S_1})(m_2\chi_{S_2}) &= m_1\chi_{S_1}m_2\chi_{S_1}^2\chi_{S_2}\\ &= m_1\chi_{S_1}m_2\chi_{S_1}\chi_{S_1\cap S_2}\\ &= m_1m_2^{S_1}\chi_{S_1\cap S_2}.\end{aligned}$$

Also, we have

$$\begin{aligned}(m\chi_S)^* &= \chi_S^*m^* = \chi_Sm^*\\ &= \chi_Sm^*\chi_S^2 = \chi_Sm^*\chi_S\chi_S\\ &= (m^*)^S\chi_S = (m^*)^S\chi_S^*,\end{aligned} \qquad (5.3.5)$$

for all $m\,\chi_S \in \mathbb{M}_{\mathbb{Q}}$, with $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$.

So, the Adele dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$ is a W^*-subalgebra of $B(\mathcal{H}_{\mathbb{Q}})$ generated by M and $\chi\left(\sigma(\mathbb{A}_{\mathbb{Q}})\right)$, satisfying the conditions (5.3.4) and (5.3.5).

Let $\mathfrak{M} = L^{\infty}(\mathbb{A}_{\mathbb{Q}},\, \rho)$ be the Adele-ring von Neumann algebra. For a fixed von Neumann algebra M, construct the tensor product W^*-algebra

$$\mathcal{M}_0 = M \otimes_{\mathbb{C}} \mathfrak{M},$$

which is a W^*-subalgebra of $B(\mathcal{H}_{\mathbb{Q}})$. Define now a W^*-subalgebra $\mathcal{M}_{\mathbb{Q}}$ of $\mathcal{M}_0$ by the "conditional" tensor product W^*-algebra

(5.3.6)

$$\mathcal{M}_{\mathbb{Q}} \overset{def}{=} M \otimes_{\alpha} \mathfrak{M},$$

satisfying the following α-*relations* (5.3.7) and (5.3.8);

(5.3.7)

$$(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1 m_2^{S_1}) \otimes \chi_{S_1}\chi_{S_2},$$

and

(5.3.8)

$$(m \otimes \chi_S)^* = (m^*)^S \otimes \chi_S^*,$$

for all $m_1, m_2, m \in M$, and $S_1, S_2, S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. Of course, the α-relations; (5.3.7) and (5.3.8); are determined under linearity.

Similar to Section 3.3, we obtain the following structure theorem for $\mathbb{M}_{\mathbb{Q}}$.

Theorem 5.3. *Let $\mathbb{M}_{\mathbb{Q}} = M \times_{\alpha} \sigma(\mathbb{A}_{\mathbb{Q}})$ be the Adele dynamical W^*-algebra in $B(\mathcal{H}_{\mathbb{Q}})$ induced by an Adele W^*-dynamical system A_M, and let $\mathfrak{M}$ be the Adele-ring von Neumann algebra. Then $\mathbb{M}_{\mathbb{Q}}$ and the conditional tensor product W^*-algebra $M \otimes_{\alpha} \mathfrak{M}$ of (5.3.6) are $*$-isomorphic, i.e.,*

(5.3.9)

$$\mathbb{M}_{\mathbb{Q}} = M \times_{\alpha} \sigma(\mathbb{A}_{\mathbb{Q}}) \overset{*\text{-}iso}{=} M \otimes_{\alpha} \mathfrak{M} = \mathcal{M}_{\mathbb{Q}}.$$

Proof. Let $\mathbb{M}_{\mathbb{Q}}$ be the Adele dynamical W^*-algebra $M \times_{\alpha} \sigma(\mathbb{A}_{\mathbb{Q}})$ induced by an Adele W^*-dynamical system A_M, and let $\mathcal{M}_{\mathbb{Q}} = M \otimes_{\alpha} \mathfrak{M}$ be the conditional tensor product W^*-algebra (5.3.6) of a fixed von Neumann algebra M, and the Adele-ring von Neumann algebra $\mathfrak{M}$ in $B(\mathcal{H}_{\mathbb{Q}})$, satisfying the α-relations (5.3.7) and (5.3.8).

Define now a morphism

$$\Phi : \mathcal{M}_{\mathbb{Q}} \to \mathbb{M}_{\mathbb{Q}}$$

by a linear transformation satisfying

$$\Phi\left(m \otimes \chi_S\right) = m\chi_S, \tag{5.3.10}$$

for all $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. Then it is generator-preserving, and hence, it is bijective and bounded. Also, it satisfies that

$$\begin{aligned}\Phi\left((m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2})\right) &= \Phi\left((m_1 m_2^{S_1}) \otimes \chi_{S_1}\chi_{S_2}\right)\\ &= (m_1 m_2^{S_1})\ \chi_{S_1 \cap S_2} = (m_1\chi_{S_1})(m_2\chi_{S_2})\\ &= \Phi(m_1 \otimes \chi_{S_1})\ \Phi(m_2 \otimes \chi_{S_2}),\end{aligned}$$

for all $m_1, m_2 \in M$, and $S_1, S_2 \in \sigma(\mathbb{A}_{\mathbb{Q}})$.

Thus, for any $T_1, T_2 \in \mathcal{M}_{\mathbb{Q}}$, we have

$$\Phi(T_1T_2) = \Phi(T_1)\ \Phi(T_2) \text{ in } \mathbb{M}_{\mathbb{Q}}, \tag{5.3.11}$$

by the linearity of Φ.Furthermore,

$$\begin{aligned}\Phi\left((m \otimes \chi_S)^*\right) &= \Phi\left((m^*)^S \otimes \chi_S^*\right)\\ &= (m^*)^S\ \chi_S = (m\ \chi_S)^* = \left(\Phi(m \otimes \chi_S)\right)^*,\end{aligned}$$

for all $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$.

So, for any $T \in \mathcal{M}_{\mathbb{Q}}$,

$$\Phi(T^*) = \Phi(T)^* \text{ in } \mathbb{M}_{\mathbb{Q}}. \tag{5.3.12}$$

Therefore, by (5.3.11) and (5.3.12), the bijective linear transformation Φ of (5.3.10) is a $*$-isomorphism from $\mathcal{M}_{\mathbb{Q}}$ onto $\mathbb{M}_{\mathbb{Q}}$. ∎

The above structure theorem (5.3.9) shows that, just like the p-adic cases, Adelic dynamical W^*-algebras $M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$ are understood as conditional tensor product W^*-algebras $M \otimes_\alpha \mathfrak{M}$. As in Section 3, we handle von Neumann algebras $\mathbb{M}_{\mathbb{Q}}$ and $\mathcal{M}_{\mathbb{Q}}$, alternatively.

One of the most interesting results of the above structure theorem (5.3.9) is the following structure theorem.

Theorem 5.4. *Let $\mathbb{M}_{\mathbb{Q}}$ be the Adele dynamical W^*-algebra induced by an Adele W^*-dynamical system A_M. Then $\mathbb{M}_{\mathbb{Q}}$ is $*$-isomorphic to the weak tensor product W^*-algebra $\otimes_{\varphi_M}\mathcal{M}_p$ of the p-adic dynamical W^*-algebras $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ in the sense of (3.2.5), for $p \in \mathcal{P}$, i.e.,*

$$\mathbb{M}_{\mathbb{Q}} \overset{*\text{-}iso}{=} \underset{p\in\mathcal{P}}{\otimes_{\varphi_M}} \mathcal{M}_p, \tag{5.3.13}$$

with the system φ_M,

$$\varphi_M \overset{def}{=} 1_M \otimes \varphi = \{1_M \otimes \varphi_p\}_{p\in\mathcal{P}}, \tag{5.3.14}$$

where $\varphi = \{\varphi_p\}_{p\in\mathcal{P}}$ is in the sense of (5.2.9).

Proof. By (5.3.9), the given Adele dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$ is $*$-isomorphic to $\mathcal{M}_{\mathbb{Q}} = M \otimes_\alpha \mathfrak{M}$;

$$\mathbb{M}_{\mathbb{Q}} \overset{*\text{-iso}}{=} \mathcal{M}_{\mathbb{Q}}.$$

Also, by (5.2.8), the Adele-ring von Neumann algebra $\mathfrak{M}$ is $*$-isomorphic to $\underset{p\in\mathcal{P}}{\otimes_\varphi}\mathfrak{M}_p$, where φ is in the sense of (5.2.9);

$$\mathfrak{M} \overset{*\text{-iso}}{=} \underset{p\in\mathcal{P}}{\otimes_\varphi}\mathfrak{M}_p,$$

where $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ are p-prime von Neumann algebras, for $p \in \mathcal{P}$.

Thus, one can have that

$$\begin{aligned}\mathbb{M}_{\mathbb{Q}} &\overset{*\text{-iso}}{=} M \otimes_\alpha \mathfrak{M} \overset{*\text{-iso}}{=} M \otimes_\alpha \left(\underset{p\in\mathcal{P}}{\otimes_\varphi}\mathfrak{M}_p\right)\\ &\overset{*\text{-iso}}{=} \underset{p\in\mathcal{P}}{\otimes_{\varphi_M}} (M \otimes_\alpha \mathfrak{M}_p)\end{aligned}$$

where $\varphi_M = \{1_M \otimes \varphi_p\}_{p\in\mathcal{P}}$

$$\overset{*\text{-iso}}{=} \underset{p\in\mathcal{P}}{\otimes_{\varphi_M}} \mathcal{M}_p,$$

by (3.3.3). ∎

The structure theorem (5.3.13) provides a useful tool for studying our Adelic dynamical W^*-algebras $\mathbb{M}_{\mathbb{Q}}$ in terms of p-adic dynamical W^*-algebras $\mathcal{M}_p$'s.

VI. Adelic Dynamical W^*-Algebras

Let M be a fixed von Neumann algebra in $B(H)$, and let

$$\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$$

be the Adele dynamical W^*-algebra induced by an Adele W^*-dynamical system

$$A_M = (M, \sigma(\mathbb{A}_{\mathbb{Q}}), \alpha) \text{ in } B(\mathcal{H}_{\mathbb{Q}}).$$

In Section 5, we showed that $\mathbb{M}_{\mathbb{Q}}$ is $*$-isomorphic to the conditional tensor product W^*-algebra

$$\mathcal{M}_{\mathbb{Q}} = M \otimes_{\alpha} \mathfrak{M}$$

of M and the Adele von Neumann algebra $\mathfrak{M} = L^{\infty}(\mathbb{A}_{\mathbb{Q}}, \rho)$ by (5.3.9). And hence, it is $*$-isomorphic to the weak tensor product von Neumann algebra

$$\mathbf{M}_{\mathbb{Q}} = \underset{p\in\mathcal{P}}{\otimes_{\varphi_M}} \mathcal{M}_p$$

of p-adic dynamical W^*-algebras $\mathcal{M}_p = M \times_{\alpha} \sigma(\mathbb{Q}_p)$, for $p \in \mathcal{P}$, by (5.3.13).

We understand these three von Neumann algebras $\mathbb{M}_{\mathbb{Q}}$, $\mathcal{M}_{\mathbb{Q}}$ and $\mathbf{M}_{\mathbb{Q}}$, as the same von Neumann algebras $\mathbb{M}_{\mathbb{Q}}$. Especially, case-by-case, we use a suitable one among $\{\mathbb{M}_{\mathbb{Q}}, \mathcal{M}_{\mathbb{Q}}, \mathbf{M}_{\mathbb{Q}}\}$ as $\mathbb{M}_{\mathbb{Q}}$.

First, recall that, if $Y \in \sigma(\mathbb{A}_{\mathbb{Q}})$, then there exist $Y_p \in \sigma(\mathbb{Q}_p)$, for all $p \in \mathcal{P}$, such that

$$Y = \prod_{p\in\mathcal{P}} Y_p,$$

where most of Y_q's are identical to $Y_q \cap \mathbb{Z}_q$ (i.e., $Y_q \subseteq \mathbb{Z}_q$), for $q \in \mathcal{P}$.

For instance, the subset U of $\mathbb{A}_{\mathbb{Q}}$,

$$U = \prod_{p\in\mathcal{P}} U_p \tag{6.0.1}$$

is a well-determined element of $\sigma(\mathbb{A}_{\mathbb{Q}})$, where U_p are the unit circles of $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. We call U, the *unit circle of the Adele ring* $\mathbb{A}_{\mathbb{Q}}$. Indeed, for any element $(u_p)_{p\in\mathcal{P}} \in U$, we have

$$|(u_p)_{p\in\mathcal{P}}|_{\mathbb{Q}} = \prod_{p\in\mathcal{P}} |u_p|_p = 1,$$

where $|.|_{\mathbb{Q}}$ is the non-Archimedean norm on $\mathbb{A}_{\mathbb{Q}}$ induced by the p-norms $\{|.|_p\}_{p\in\mathcal{P}}$ (e.g., see [18]).

Define now a conditional expectation

$$E : \mathbb{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} \to M \otimes_{\alpha} \mathbb{C}[\{\chi_U\}] \overset{*\text{-iso}}{=} M$$

by a linear morphism satisfying that:

$$E(m\chi_Y) = m\,(r\chi_U), \tag{6.0.2}$$

where $r \in [0, 1]$ satisfies that:

$$\int_{\mathbb{A}_{\mathbb{Q}}} \chi_{Y\cap U}\, d\rho = r \int_{\mathbb{A}_{\mathbb{Q}}} \chi_U\, d\rho = r \left(\prod_{p\in\mathcal{P}} \left(1 - \tfrac{1}{p}\right)\right). \tag{6.0.3}$$

Remark here that the quantity $\prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)$ on the right-hand side of (6.0.2)$'$ satisfies that:

$$\begin{aligned}\prod_{p\in\mathcal{P}} \left(1 - \tfrac{1}{p}\right) &= \left(1 - \tfrac{1}{\infty}\right) \left(\prod_{p:primes} \left(1 - \tfrac{1}{p}\right)\right) \\ &= \prod_{p:primes} \left(1 - \tfrac{1}{p}\right) = \tfrac{1}{\zeta(1)},\end{aligned}$$

where

$$\zeta(s) \overset{def}{=} \sum_{n=1}^{\infty} \tfrac{1}{n^s} = \prod_{p:primes} \tfrac{1}{1-p^{-s}} = \tfrac{1}{\prod_{p:primes}\left(1-\frac{1}{p^s}\right)}$$

is the *Riemann zeta function*, satisfying that:

$$\tfrac{1}{\zeta(s)} = \prod_{p:prime} \left(1 - \tfrac{1}{p^s}\right), \text{ for } s \in \mathbb{C}.$$

By definition, it is clear that

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

and hence,

$$\frac{1}{\zeta(1)} = 0.$$

Thus, one can verify that the formula (6.0.3) becomes 0, for all $m\chi_Y \in \mathbb{M}_{\mathbb{Q}}$, with $m \in (M, \psi)$ and $Y \in \sigma(\mathbb{A}_{\mathbb{Q}})$. In other words, we cannot directly mimic the p-adic dynamical free-probabilistic approaches as in Section 4.

Therefore, we consider a new, but similar approach to establish a suitable free probability model on our Adelic dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$.

a) Adelic Dynamical W^*-Probability Spaces $\{(\mathbb{M}_{\mathbb{Q}}, \varphi_P)\}_{P \subset \mathcal{P}}$**.** As we have discussed at the beginning of this section, we cannot directly mimic the free-probabilistic settings from the p-adic dynamical W^*-probability spaces to our Adelic W^*-probability settings. So, we construct suitable linear functionals differently from those of Section 4 (and those of [10]).

Take first a "finite" subset P of $\mathcal{P}$, say

$$P = \{p_1, ..., p_n\}, \tag{6.1.1}$$

for some $n \in \mathbb{N}$, in particular, suppose all p_1, ..., p_n of P are primes (not ∞) in $\mathcal{P}$. We call such subsets P of $\mathcal{P}$, *finite prime (sub)sets of* $\mathcal{P}$.

Let P be a finite prime set (6.1.1) of $\mathcal{P}$. Define an element U_P of $\sigma(\mathbb{A}_{\mathbb{Q}})$ by

$$U_P \stackrel{def}{=} \left(\prod_{p \in P} U_p \right) \times \left(\prod_{q \in \mathcal{P} \setminus P} \mathbb{Z}_q \right), \tag{6.1.2}$$

under possible re-arrangement. i.e., for all p in P, take the unit circle U_p of $\mathbb{Q}_p$, and for almost all other q in $\mathcal{P}$, take $\mathbb{Z}_q$ of $\mathbb{Q}_q$, and then product them to construct a ρ-measurable subset U_P in the Adele ring $\mathbb{A}_{\mathbb{Q}}$.

Then define a subalgebra M_P of $\mathcal{M}_{\mathbb{Q}} = \mathbb{M}_{\mathbb{Q}} = \mathbf{M}_{\mathbb{Q}}$ by

$$M_P \stackrel{def}{=} M \otimes_{\alpha} \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{A}_{\mathbb{Q}}),\ S \subseteq U_P\}\right]. \tag{6.1.3}$$

Define now a conditional expectation

$$E_P : \mathbb{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} \to M_P$$

by a linear morphism satisfying that:

$$E_P\left(m\chi_Y\right) = m\ \chi_{Y \cap U_P}. \tag{6.1.4}$$

Now, let's check the morphism E_P of (6.1.4) is indeed a conditional expectation:
(6.1.5) For any $m\ \chi_S \in M_P$, one has

$$E_P\left(m\chi_S\right) = m\chi_{S \cap U_P} = m\chi_S,$$

since $S \subseteq U_P$, and hence, for any $x \in M_P$, we have

$$E_P(x) = x, \text{ under linearity.}$$

(6.1.6) For $m_j\chi_{S_j} \in M_P$, for $j = 1, 2$, and $m\ \chi_Y \in \mathbb{M}_{\mathbb{Q}}$, we have

$$\begin{aligned} E_P\left((m_1\chi_{S_1})(m\chi_Y)(m_2\chi_{S_2})\right) &= E_P\left(m_1 m^{S_1} m_2^{S_1 \cap Y}\ \chi_{S_1 \cap Y \cap S_2}\right) \\ &= E_P\left(m_1 m^{S_1} m_2^{S_1 \cap Y}\ \chi_{S_1 \cap Y \cap S_2}\right) \\ &= \left(m_1 m^{S_1} m_2^{S_2 \cap Y}\right) \chi_{S_1 \cap Y \cap S_2}, \end{aligned}$$

and

$$\begin{aligned} \left(m_1\chi_{S_1}\right)\left(E_P(m\chi_Y)\right)\left(m_2\chi_{S_2}\right) &= \left(m_1\chi_{S_1}\right)\left(m\chi_{Y \cap U_P}\right)\left(m_2\chi_{S_2}\right) \\ &= \left(m_1 m^{S_1} m_2^{S_1 \cap Y \cap U_P}\right) \chi_{S_1 \cap Y \cap S_2}, \\ &= \left(m_1 m^{S_1} m_2^{S_1 \cap Y}\right) \chi_{S_1 \cap Y \cap S_2}, \end{aligned}$$

because $S_1 \cap U_P = S_1$, so, $S_1 \cap Y \cap U_P = S_1 \cap Y$, and hence,

$$E_P\left((m_1\chi_{U_P})(m\chi_Y)(m_2\chi_{U_P})\right) = \left(m_1\chi_{U_P}\right)\left(E_P(m\chi_Y)\right)\left(m_2\chi_{U_P}\right).$$

Thus, under linearity, we have that:

$$E_P\left(x_1 y x_2\right) = x_1\ E_P(y)\ x_2,$$

for all $x_1, x_2 \in M_P$ and $y \in \mathbb{M}_{\mathbb{Q}}$.

(6.1.7) Also, one has that:

$$\begin{aligned} E_P\left((m\chi_Y)^*\right) &= E_P\left((m^*)^Y\chi_Y\right) \\ &= (m^*)^Y(\chi_{Y\cap U_P}) = \left(E_P(m\chi_Y)\right)^*, \end{aligned}$$

and hence, for all $y \in \mathbb{M}_{\mathbb{Q}}$, we have

$$E_P(y^*) = E_P(y)^*.$$

Proposition 6.1. *The morphism E_P of (6.1.4) is a well-defined conditional expectation from $\mathbb{M}_{\mathbb{Q}}$ onto M_P, for any finite prime set P of $\mathcal{P}$.*

Proof. By definition, the morphism E_P of (6.1.4) is bounded and linear. So, it is a conditional expectation because of (6.1.5), (6.1.6) and (6.1.7). ∎

Define now a morphism $F_P : M_P \to M_P$ by a linear morphism satisfying that:

$$F_P\left(m\chi_Y\right) = m\left(r_Y\chi_{U_P}\right), \text{ for some } r_Y \in [0,\ 1]. \tag{6.1.8}$$

In particular, the quantity r_Y in (6.1.8) is determined as follows in $[0,\ 1]$ of $\mathbb{R}$:

$$\begin{aligned} \int_{\mathbb{A}_{\mathbb{Q}}} \chi_{Y\cap U_P}\ d\rho &= \rho\left(Y \cap U_P\right) \\ &= \rho\left(\left(\prod_{p\in\mathcal{P}} Y_p\right) \cap \left(\prod_{p\in\mathcal{P}} V_p\right)\right) \end{aligned}$$

where $U_p = \prod\limits_{p\in\mathcal{P}} V_p$ satisfies (6.1.2) (under possible re-arrangement)

$$\begin{aligned} &= \left(\underset{p\in\mathcal{P}}{\times}\rho_p\right)\left(\prod_{p\in\mathcal{P}}\left(Y_p \cap V_p\right)\right) \\ &= \prod_{p\in\mathcal{P}} \rho_p\left(Y_p \cap V_p\right) \\ &= \left(\prod_{p\in P}\rho_p\left(Y_p \cap U_p\right)\right)\left(\prod_{q\in\mathcal{P}\setminus P}\rho_p\left(Y_q \cap \mathbb{Z}_q\right)\right) \end{aligned}$$

by (6.1.2)

$$= \left(\prod_{p\in P} r_p\left(1 - \tfrac{1}{p}\right)\right)\left(\prod_{q\in\mathcal{P}\setminus P} r_q \cdot 1\right)$$

for $r_w \in [0,\ 1]$, since

$$\rho_w(U_w) = 1 - \tfrac{1}{w}, \text{ and } \rho_w(\mathbb{Z}_w) = 1$$

for all $w \in \mathcal{P}$, and hence, we have

$$\int_{\mathbb{A}_{\mathbb{Q}}} \chi_{Y\cap U_P} d\rho = \left(\prod_{q\in\mathcal{P}} r_q\right)\left(\prod_{p\in P}\left(1 - \tfrac{1}{p}\right)\right). \tag{6.1.9}$$

Define r_Y in $[0,\ 1]$ by

$$r_Y = \prod_{q\in\mathcal{P}} r_q, \tag{6.1.10}$$

where the quantity of the right-hand side of (6.1.10) is from (6.1.9).

i.e., the morphism F_P on M_P satisfies

$$F_P\left(m\chi_Y\right) = m\ \left(r_Y\chi_{U_P}\right),$$

where $r_Y \in [0, 1]$ satisfy (6.1.10), for all $m\chi_Y \in M_P$.

By Section 5.2, without loss of generality, one can verify that: if

$$Y = \prod_{p\in\mathcal{P}} Y_p \in \sigma(\mathbb{A}_{\mathbb{Q}}), \text{ with } Y_p \in \sigma(\mathbb{Q}_p),$$

then almost all Y_q's are identical to $\mathbb{Z}_q$.

Assumption If we take $Y = \prod_{p\in\mathcal{P}} Y_p$ in $\sigma(\mathbb{A}_{\mathbb{Q}})$, with $Y_p \in \sigma(\mathbb{Q}_p)$, then we assume almost all Y_q's are identical to $\mathbb{Z}_q$. □

Define now a linear functional

$$\gamma_0 : M_P \to \mathbb{C}$$

by

$$\gamma_0 \overset{def}{=} \left(\otimes \int_{\mathbb{A}_{\mathbb{Q}}} \bullet \, d\rho \right) \circ F_P \tag{6.1.11}$$

i.e., γ_0 is a linear morphism satisfying that:

$$\begin{aligned} \gamma_0 (m\ \chi_Y) &\overset{def}{=} (m) \left(\int_{\mathbb{A}_{\mathbb{Q}}} r_Y \chi_{U_P} d\rho \right) \\ &= r_Y\ \ (m) \left(\prod_{p\in P} \left(1 - \tfrac{1}{p} \right) \right), \end{aligned} \tag{6.1.12}$$

for all $m \in (M, \psi)$ and $Y \in \sigma(\mathbb{A}_{\mathbb{Q}})$, where $r_Y \in [0, 1]$ is in the sense of (6.1.10). The linear morphism γ_0 of (6.1.12) is indeed a well-defined linear functional on M_P.

Define now a linear functional γ_P on $\mathbb{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} = \mathbf{M}_{\mathbb{Q}}$ by

$$\gamma_P \overset{def}{=} \gamma_0 \circ E_P, \tag{6.1.13}$$

for any fixed finite prime sets P of $\mathcal{P}$.

Since γ_0 is a bounded linear functional, and E_P is a bounded conditional expectation, γ_P of (6.1.13) is indeed a well-defined linear functional on the Adelic dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$.

Definition 6.1. Let $\mathbb{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}} = \mathbf{M}_{\mathbb{Q}}$ be an Adelic dynamical W^*-algebra over a W^*-probability space (M, ψ). Let P be a finite prime set of $\mathcal{P}$, and γ_P, the corresponding linear functional in the sense of (6.1.13). Then the pair $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$ is called the Adelic dynamical W^*-probability space induced by a finite prime set P of $\mathcal{P}$.

By definition, for any $m\ \chi_S \in \mathbb{M}_{\mathbb{Q}}$, one has that:

$$\begin{aligned} \gamma_P (m\chi_S) &= \gamma_0 (E_P(m\chi_S)) \\ &= \gamma_0 \left(m\chi_{S\cap U_P} \right) \\ &= r_{S\cap U_P}\ \ (m) \left(\prod_{p\in P} \left(1 - \tfrac{1}{p} \right) \right), \end{aligned} \tag{6.1.14}$$

where $r_{S\cap U_P} \in [0, 1]$ satisfies (6.1.10) and (6.1.12).

Notice now that

$$U_p = \left(\prod_{p\in P} U_p \right) \times \left(\prod_{q\in\mathcal{P}\setminus P} \mathbb{Z}_q \right),$$

under possible re-arrangement. Like in Section 4, if we replace U_p's to $p^k U_p$, for some $k \in \mathbb{Z}$, i.e., if we define

$$U_{P:k} \overset{def}{=} \left(\prod_{p\in P} U_{p:k} \right) \times \left(\prod_{q\in\mathcal{P}\setminus P} \mathbb{Z}_q \right),$$

where $U_{p:k} = p^k U_p$, as in (4.8), then we have similar structures, for all $k \in \mathbb{Z}$, with identity:

$$U_P = U_{P:0}.$$

However, in such cases, the formula (6.1.9) will be replaced by

$$\int_{\mathbb{A}_{\mathbb{Q}}} U_{P:k}\, d\rho = \prod_{p\in P}\left(\tfrac{1}{p^k} - \tfrac{1}{p^{k+1}}\right).$$

In this paper, we only consider the case where we have $U_P = U_{P:0}$.

b) Free Structure of $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$**.** Let $\mathbb{M}_{\mathbb{Q}} = M \times_\alpha \sigma(\mathbb{A}_{\mathbb{Q}})$ be the Adelic dynamical W^*-algebra in $B(\mathcal{H}_{\mathbb{Q}})$ induced by an Adele W^*-dynamical system $A_M = (M,\ \sigma(\mathbb{A}_{\mathbb{Q}}),\ \alpha)$. As before, we understand $\mathbb{M}_{\mathbb{Q}}$ as its $*$-isomorphic von Neumann algebras $\mathcal{M}_{\mathbb{Q}} = M \otimes_\alpha \mathfrak{M}$, and $\mathbf{M}_{\mathbb{Q}} = \otimes_{\varphi_M}\mathcal{M}_p$, case-by-case, and let γ_P be the linear functional in the sense of (6.1.13), satisfying that:

$$\gamma(m\chi_S) = r_S \quad (m)\left(\prod_{p\in P}\left(1 - \tfrac{1}{p}\right)\right),$$

where $r_S \in [0,\ 1]$ in $\mathbb{R}$ satisfying (6.1.10) and (6.1.12), for all $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, for all finite primes sets P of $\mathcal{P}$.

Throughout this section, we fix a finite prime set P of $\mathcal{P}$, and concentrate on free probabilistic structure on $\mathbb{M}_{\mathbb{Q}}$ in terms of γ_P of (6.1.13). The following lemma is obtained by the straightforward computations.

Lemma 6.2. *Let $m\chi_S$ be a free random variable in the Adelic dynamical W^*-probability space $(\mathbb{M}_{\mathbb{Q}},\ \gamma)$, with $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. Then*

$$\gamma_P\left((m\chi_S)^n\right) = r_{S\cap U_P}\left(\quad (m(m^S)^{n-1})\right)\left(\prod_{p\in P}\left(1 - \tfrac{1}{p}\right)\right), \qquad (6.2.1)$$

for all $n \in \mathbb{N}$, where $r_{S\cap U_P} \in [0,\ 1]$ satisfies (6.1.10) and (6.1.12).

Proof. If $m\chi_S \in \mathbb{M}_{\mathbb{Q}}$, with $m \in M$, and $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, then

$$\begin{aligned}(m\chi_S)^n &= mm^S m^{S\cap S} \ldots m^{S\cap S\cap\ldots\cap S}\ \chi_{S\cap\ldots\cap S}\\ &= m\ m^S\ m^S \ldots m^S\ \chi_S = m\ (m^S)^{n-1}\ \chi_S,\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, one can have that

$$\begin{aligned}\gamma_P\left((m\chi_S)^n\right) &= \gamma_P\left(m(m^S)^{n-1}\chi_S\right)\\ &= r_{S\cap U_P} \quad \left(m(m^S)^{n-1}\right)\left(\prod_{p\in P}\left(1 - \tfrac{1}{p}\right)\right)\end{aligned}$$

for all $n \in \mathbb{N}$, by (6.1.14), where $r_{S\cap U_P} \in [0,\ 1]$ satisfies (6.1.10) and (6.1.12). ∎

More general to (6.2.1), we obtain the following lemma.

Lemma 6.3. *Let $m_1\chi_{S_1}$, ..., $m_n\chi_{S_n}$ be free random variables in an Adelic dynamical W^*-probability space $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$, with $m_k \in M$, $S_k \in \sigma(\mathbb{Q}_p)$, for $k = 1,\ \ldots,\ n$, for $n \in \mathbb{N}$. Then*

$$\gamma\left(\prod_{j=1}^{n} m_j\chi_{S_j}\right) = r_{\left(\bigcap_{j=1}^{n} S_j\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{N} m_j^{\bigcap_{i=0}^{j-1} S_i}\right)\right)\left(\rho\left(\bigcap_{j=1}^{n} S_j\right)\right), \qquad (6.2.2)$$

where $r_{\left(\bigcap_{j=1}^{n} S_j\right)\cap U_P} \in [0,\ 1]$, satisfying (6.1.10) and (6.1.12).

Proof. If $m_k\ \chi_{S_k} \in (\mathcal{M}_p,\ \gamma_p)$ are given as above, for $k = 1,\ \ldots,\ n$, then

$$\begin{aligned}\prod_{j=1}^{n}(m_j\chi_{S_j}) &= m_1 m_2^{S_1} m_3^{S_1\cap S_2}\ldots m_N^{S_1\cap\ldots\cap S_{n-1}}\chi_{S_1\cap\ldots\cap S_n}\\ &= \left(\prod_{j=1}^{n} m_j^{\bigcap_{i=0}^{j-1} S_i}\right)\left(\chi_{\bigcap_{j=1}^{n} S_j}\right),\end{aligned}$$

in $\mathbb{M}_{\mathbb{Q}}$, for all $n \in \mathbb{N}$.

Thus, one has that:

$$\gamma_P\left(\prod_{j=1}^{n}(m_j\chi_{S_j})\right) = \gamma_P\left(\left(\left(\prod_{j=1}^{n} m_j^{\bigcap_{i=0}^{j-1}S_i}\right)\left(\chi_{\bigcap_{j=1}^{n}S_j}\right)\right)\right)$$
$$= r_{\left(\bigcap_{j=1}^{n}S_j\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n} m_j^{\bigcap_{i=0}^{j-1}S_i}\right)\right)\left(\prod_{p\in P}\left(1-\frac{1}{p}\right)\right),$$

by (6.1.14), where $r_{\left(\bigcap_{j=1}^{n}S_j\right)\cap U_P} \in [0,\,1]$ satisfies (6.1.10) and (6.1.12). ∎

Notation 6.2 *In the following, we denote* $\prod_{p\in P}\left(1-\frac{1}{p}\right)$ *by* ζ_P^-, *for convenience.* □

By (6.2.1) and (6.2.2), we obtain the following free-distributional data of free random variables of $(\mathbb{M}_{\mathbb{Q}},\,\gamma_P)$.

Theorem 6.4. *Let* $(\mathbb{M}_{\mathbb{Q}},\,\gamma_P)$ *be an Adelic dynamical* W^**-probability space determined by a finite prime set* P *of* $\mathcal{P}$, *and let*

$$T_k = \sum_{S_k\in Supp(T_k)} m_{S_k}\chi_{S_k},\ \textit{for}\ k = 1,\,...,\,n,$$

be free random variables, for $n\in\mathbb{N}$. *Then*
(6.2.3)

$$\gamma_P\left(\prod_{j=1}^{n} T_j\right) =$$
$$\zeta_P^-\left(\sum_{(S_1,...,S_n)\in\prod_{j=1}^{n}Supp(T_j)} r_{\left(\bigcap_{i=1}^{n}S_i\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\bigcap_{i=1}^{j-1}S_i\right)}\right)\right)\right),$$

where ζ_P^- *is in the sense of* ***Notation 6.2****, and where* $r_{r_{\left(\bigcap_{i=1}^{j-1}S_i\right)\cap U_P}} \in [0,\,1]$ *satisfy (6.1.10) and (6.1.12).*

Proof. Inductively, one can get that

$$\prod_{j=1}^{n} T_j = \sum_{(S_1,...,S_n)\in\prod_{j=1}^{n}Supp(T_j)}\left(\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\bigcap_{i=1}^{j-1}S_i\right)}\right)\left(\chi_{\bigcap_{j=1}^{n}S_j}\right)\right),$$

for all $j = 1,\,...,\,n$. So,

$$\gamma_P\left(\prod_{j=1}^{n} T_j\right) = \gamma_P\left(T_1T_2\,...\,T_n\right)$$
$$= \gamma_P\left(\sum_{(S_1,...,S_n)\in\prod_{j=1}^{n}Supp(T_j)}\left(\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\bigcap_{i=1}^{j-1}S_i\right)}\right)\left(\chi_{\bigcap_{j=1}^{n}S_j}\right)\right)\right)$$
$$= \sum_{(S_1,...,S_n)\in\prod_{j=1}^{n}Supp(T_j)}\left(\gamma_P\left(\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\bigcap_{i=1}^{j-1}S_i\right)}\right)\left(\chi_{\bigcap_{j=1}^{n}S_j}\right)\right)\right)$$
$$= \sum_{(S_1,...,S_n)\in\prod_{j=1}^{n}Supp(T_j)}\left(r_{\left(\bigcap_{i=1}^{n}S_i\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\bigcap_{i=1}^{j-1}S_i\right)}\right)\right)(\zeta_P^-)\right)$$

by (6.2.2), where $r_{\left(\bigcap_{i=1}^{n} S_i\right)\cap U_P} \in [0,\ 1]$ satisfy (6.1.10) and (6.1.12), and where ζ_P^- is in the sense of **Notation 6.2**. ∎

Thanks to (6.2.3), we obtain the following corollary.

Corollary 6.5. *Let* $T = \sum_{S\in Supp(T)} m_S\,\chi_S$ *be a free random variable in* $(\mathcal{M}_p,\ \gamma_p)$. *Then*

$$\gamma_P\left(T^n\right) = \zeta_P^-\left(\sum_{(S_1,\ldots,S_n)\in Supp(T)^n}\left(r_{\left(\bigcap_{j=1}^{n} S_j\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}\left(m_{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\right)\right), \quad (6.2.4)$$

$$\gamma_P\left((T^*)^n\right) = \zeta_P^-\left(\sum_{(S_1,\ldots,S_n)\in Supp(T)^n}\left(r_{\left(\bigcap_{j=1}^{n} S_j\right)\cap U_P}\left(\psi\left(\prod_{j=1}^{n}\left((m_{S_j}^*)^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\right)\right), \quad (6.2.5)$$

for all $n \in \mathbb{N}$, *where* $r_{\left(\bigcap_{j=1}^{n} S_j\right)\cap U_P} \in [0,\ 1]$ *satisfy (6.1.10) and (6.1.12).* □

Let $m_1\chi_{S_1}$, ..., $m_n\chi_{S_n}$ be free random variables in $(\mathbb{M}_{\mathbb{Q}},\ \gamma)$, for $n \in \mathbb{N}$, where m_1, ..., $m_n \in M$, and S_1, ..., $S_n \in \sigma(\mathbb{A}_{\mathbb{Q}})$. Then, by (6.2.3), one can obtain that:

$$k_n^P\left(m_1\chi_{S_1},\ \ldots,\ m_n\chi_{S_n}\right)$$

$$= \sum_{\pi\in NC(n)} (\gamma_P)_\pi\left(m_1\chi_{S_1},\ \ldots,\ m_n\chi_{S_n}\right)\mu(\pi,\ 1_n)$$

$$= \sum_{\pi\in NC(n)}\left(\prod_{V\in\pi}(\gamma_P)_V\left(m_1\chi_{S_1},\ \ldots,\ m_n\chi_{S_n}\right)\mu\left(0_{|V|},\ 1_{|V|}\right)\right)$$

by the Möbius inversion (See Section 2.3)

$$= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,\ldots,i_k)\in\pi}\gamma_P\left(m_{i_1}\chi_{S_{i_1}}\ \cdots\ m_{i_k}\chi_{S_{i_k}}\right)\mu\left(0_k,\ 1_k\right)\right) \quad (6.2.7)$$

$$= \sum_{\pi\in NC(n)}\left(\prod_{V=(i_1,\ldots,i_k)\in\pi}\left(\zeta_P^-\left(r_V\psi\left(\prod_{t=1}^{k}\left(m_{i_t}\right)^{\left(\bigcap_{t=1}^{k-1} S_{i_t}\right)}\right)\right)\right)\mu\left(0_k,\ 1_k\right)\right),$$

where $k_n^P(\ldots)$ mean free cumulants induced by γ_V in the sense of Section 2.3.

By (6.2.7), we obtain the following inner free structure of the given Adelic dynamical W^*-algebra $\mathbb{M}_{\mathbb{Q}}$, with respect to γ_P.

Theorem 6.6. *Let* $m_1\chi_S$, *and* $m_2\chi_S$ *be free random variables in an Adelic dynamical* W^**-probability space* $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$, *with* $m_1,\ m_2 \in M$, *and* $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, *with* $\rho(S) \neq 0$. *Moreover, assume that* S *contains* U_P. *i.e., suppose*

$$S = \prod_{q\in\mathcal{P}} S_q \text{ in } \mathbb{A}_{\mathbb{Q}}, \text{ and } S_p \supseteq U_p, \text{ for all } p \in P. \quad (6.2.8)$$

Then $\{m_1,\ m_1^S\}$ *and* $\{m_2,\ m_2^S\}$ *are free in the* W^**-probability space* $(M,\ \psi)$, *if and only if* $m_1\chi_S$ *and* $m_2\chi_S$ *are free in* $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$. *i.e.,*

$$\{m_1,\ m_1^S\} \text{ and } \{m_2,\ m_2^S\} \text{ are free in } (M,\ \psi) \quad (6.2.9)$$

$$\Longleftrightarrow$$

$$m_1\chi_S \text{ and } m_2\chi_S \text{ are free in } (\mathbb{M}_{\mathbb{Q}},\ \gamma_P),$$

under the condition (6.2.8).

Proof. ($\Rightarrow$) Assume that $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in (M, ψ). Then, by definition, all mixed free $*$-cumulants of them (with respect to the linear functional ψ) vanish (See Section 2.3, or [16]). i.e.,

$$k_n^{\psi}\left(u_{i_1}^{r_1},\ ...,\ u_{i_n}^{r_n}\right) = 0 \text{ in } \mathbb{C},$$

for all $n \in \mathbb{N} \setminus \{1\}$, where $(u_{i_1}, ..., u_{i_n}) \in \{m_1, m_2, m_1^S, m_2^S\}$ are "mixed," and $(i_1, ..., i_n) \in \{1, 2\}^n$, and $(r_1, ..., r_n) \in \{1, *\}^n$, where $k_n(...)$ mean free cumulants induced by ψ.

Consider mixed free $*$-cumulants of $m_1\chi_S$ and $m_2\chi_S$ in $(\mathbb{M}_{\mathbb{Q}}, \gamma)$, for a fixed nonzero ρ-measurable set $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$. By (6.2.7), one has that

$k_n^P\left((m_{i_1}\chi_S)^{r_1},\ ...,\ (m_{i_n}\chi_S)^{r_n}\right)$

$$= \left(\zeta_P^-\right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} r_V\, \psi\left(\prod_{t=1}^{k} \left([m_{j_t}^{r_{j_t}}]^{S_{i_t}}\right)^{\bigcap_{t=1}^{k-1} S_{j_t}} \right) \mu\left(0_k,\ 1_k\right) \right)$$

where all S_{j_t} are identical to S, and r_V satisfy (6.1.10) and (6.1.12), and where

$$\left([m_j^r]^S\right)^Y \overset{def}{=} \begin{cases} m_j^Y & \text{if } r = 1 \\ (m_j^*)^{S\cap Y} & \text{if } r = *, \end{cases}$$

for all j, $r \in \{1, *\}$ and S, $Y \in \sigma(\mathbb{A}_{\mathbb{Q}})$, and hence,

$$= \left(\zeta_P^-\right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} r_V \quad \left(\prod_{t=1}^{k} \left([m_{j_t}^{r_{j_t}}]^S\right)^S \right) \mu\left(0_k,\ 1_k\right) \right)$$

$$= \left(\zeta_P^-\right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} \quad \left(\prod_{t=1}^{k} \left([m_{j_t}^{r_{j_t}}]^S\right)^S \right) \mu\left(0_k,\ 1_k\right) \right)$$

by the condition (6.2.8) (Since the assumption (6.2.8) holds, $r_V = 1$, for all $V \in \pi$, for all $\pi \in NC(n)$, for all $n \in \mathbb{N}$)

$$= \left(\zeta_P^-\right) \left(k_n(u_{i_1}^{r_1},\ ...,\ u_{i_n}^{r_n})\right) = \left(\zeta_P^-\right) \cdot 0$$
$$= 0,$$

for all $n \in \mathbb{N} \setminus \{1\}$. It shows that, if $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in $(M,$), then $\{m_1\chi_S, m_2\chi_S\}$ are free in $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$, under the condition (6.2.8).

($\Leftarrow$) Assume now that two free random variables $m_1\chi_S$ and $m_2\chi_S$ are free in $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$, where S satisfies $\rho(S) \neq 0$ and the condition (6.2.8), i.e.,

$k_n^P\left((m_{i_1}\chi_S)^{r_1},\ ...,\ (m_{i_n}\chi_S)^{r_n}\right)$

(6.2.10)

$$= \left(\zeta_P^-\right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,...,j_n)\in\pi} \left(r_V \quad \left(\prod_{t=1}^{k} \left([m_{j_t}^{r_{j_t}}]^S\right)^S \right) \right) \mu\left(0_k,\ 1_k\right) \right)$$

$$= 0,$$

whenever $(i_1, ..., i_n)$ are "mixed" in $\{1, 2\}^n$, for $(r_1, ..., r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N} \setminus \{1\}$.

The formula (6.2.10) is identical to

$$\left(\zeta_P^-\right) \left(k_n(u_{i_1}^{r_1},\ ...,\ u_{i_n}^{r_n})\right),$$

since $r_V = 1$, by (6.2.8), for the mixed n-tuple $(u_{i_1}, ..., u_{i_n})$ of $\{m_1, m_1^S\} \cup \{m_2,$ $m_2^S\}$.

Since $\rho(S) \neq 0$, and since the condition (6.2.8) is assumed, $\rho(S \cap U_P) \neq 0$, and hence,

$$\left(\zeta_P^-\right)\left(k_n^{\psi}(u_{i_1}^{r_1},\ ...,\ u_{i_n}^{r_n})\right) = 0,$$

as in (6.2.10), equivalently,

$$k_n^{\psi}(u_{i_1}^{r_1},\ ...,\ u_{i_n}^{r_n}) = 0,$$

for all mixed n-tuple $(u_{i_1},\ ...,\ u_{i_n}) \in \{m_1, m_1^S, m_2, m_2^S\}$. Equivalently, $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in $(M,\ \psi)$. ∎

Notes

The above theorem shows that, the freeness of $(M,\ \psi)$ acts like a certain kind of free-filterizations for the inner freeness of $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$, under the assumption (6.2.8).

The following corollary is a direct consequence of the above theorem.

Corollary 6.7. *Let M_1 and M_2 be W^*-subalgebras of M in $B(H)$, and assume that the subsets $\{M_1,\ \alpha_S(M_1)\}$ and $\{M_2,\ \alpha_S(M_2)\}$ are free in $(M,\ \psi)$, for $S \in \sigma(\mathbb{A}_{\mathbb{Q}})$, with $\rho(S \cap U_P) \neq 0$, satisfying the condition (6.2.8). Then two subsets*

$$M_1 \otimes_\alpha \{\chi_S\} \text{ and } M_2 \otimes_\alpha \{\chi_S\} \text{ of } \mathcal{M}_{\mathbb{Q}} = \mathbb{M}_{\mathbb{Q}},$$

are free in $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$, for a fixed finite prime set P of $\mathcal{P}$.

Conversely, if $M_1 \otimes \{\chi_S\}$ and $M_2 \otimes \{\chi_S\}$ are free in $(\mathcal{M}_p,\ \gamma_p)$, where S satisfies (6.2.8), then $\{M_1,\ \alpha_S(M_1)\}$ and $\{M_2,\ \alpha_S(M_2)\}$ are free in $(M,\ \psi)$, too. □

Let U_P be in the sense of (6.1.2) for a fixed finite prime set P of $\mathcal{P}$. Assume now that $S_1,\ S_2 \in \sigma(\mathbb{A}_{\mathbb{Q}})$ satisfies

$$S_1 \cap U_P \neq \varnothing \text{ and } S_2 \cap U_P = \varnothing. \tag{6.2.11}$$

For example, "$S_2 \cap U_P = \varnothing$" means that, if $S_2 = \prod\limits_{q \in \mathcal{P}} S_2^q$, with $S_2^q \in \sigma(\mathbb{Q}_q)$, then

$$S_2^p \cap U_p = \varnothing, \text{ for all } p \in P,$$

and

$$S_2^q \cap \mathbb{Z}_q = \varnothing, \text{ for all } q \in \mathcal{P} \setminus P.$$

By (6.2.11), it is clear that

$$(S_1 \cap U_P) \cap (S_2 \cap U_P) = \varnothing, \tag{6.2.12}$$

even though $S_1 \cap S_2 \neq \varnothing$.

Theorem 6.8. *Let $m_1\chi_{S_1},\ m_2\chi_{S_2}$ be free random variables in an Adelic dynamical W^*-probability space $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$. If S_1 and S_2 satisfy the condition (6.2.11) in $\sigma(\mathbb{A}_{\mathbb{Q}})$, then they are free in $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$. i.e.,*

$$S_1 \text{ and } S_2 \text{ satisfy (6.2.11)} \tag{6.2.13}$$

$\Longrightarrow$

$$M \otimes_\alpha \mathbb{C}\left[\{\chi_{S_1}\}\right] \text{ and } M \otimes_\alpha \mathbb{C}[\{\chi_{S_2}\}] \text{ are free}$$

in $(\mathbb{M}_{\mathbb{Q}},\ \gamma_P)$.

Proof. Suppose $S_1,\ S_2 \in \sigma(\mathbb{A}_{\mathbb{Q}})$ satisfy the condition (6.2.11). Then, with respect to U_P of (6.1.2), they also satisfy the condition (6.2.12). Therefore, one has that:

$$k_n^P\left((m_{i_1}\chi_S)^{r_1},\ ...,\ (m_{i_n}\chi_S)^{r_n}\right) \tag{6.1.14}$$

$$= \left(\zeta_P^-\right) \sum_{\pi \in NC(n)} \left(\prod_{V=(i_1,...,i_k)\in\pi} r_{\left(\bigcap\limits_{t=1}^{k} S_{i_t}\right)\cap U_P} \left(\prod_{t=1}^{k} (m_{i_t})^{\left(\bigcap\limits_{t=1}^{k-1} S_{i_t}\right)} \right) \mu\left(0_k,\ 1_k\right) \right)$$

by (6.2.7), where $\left(\bigcap\limits_{t=1}^{k-1} S_{i_t}\right) \cap U_P \in [0,\ 1]$ satisfy (6.1.10) and (6.1.12), and the elements $\left([m_j^r]^S\right)^Y$ are in the sense of the proof of the above Theorem .

Assume that a block $V = (i_1,\ ...,\ i_k)$ of π in (6.1.13) is mixed in $\{1,\ 2\}^k$. Then the corresponding quantity

$$r_{\left(\bigcap_{t=1}^{k} S_{i_t}\right)\cap U_P} = 0 \text{ in } [0, 1],$$

by (6.2.12). Therefore, whenever a noncrossing partition π of $NC(n)$ contains at least one mixed block, then the corresponding summand vanishes. Even though a noncrossing partition θ of $NC(n)$ does not contain a mixed block, since it contains a block corresponding S_2, one obtains the quantity

$$r_{S_2\cap S_2\cap\ldots\cap S_2\cap U_P} = r_{S_2\cap U_P} = 0 \text{ in } [0, 1],$$

for at least one block of θ. Thus, even though θ does not contain a mixed block, the corresponding partition-depending free moment vanishes.

i.e., whenever $(i_1, \ldots, i_n) \in \{1, 2\}^n$ are mixed for $n \in \mathbb{N} \setminus \{1\}$, then the free cumulants (6.1.13) vanish. Equivalently, $m_1\chi_{S_1}$ and $m_2\chi_{S_2}$ are free in $(\mathbb{M}_{\mathbb{Q}}, \gamma_P)$. ∎

With a freeness characterization (6.2.9) (under (6.2.8)), the above freeness necessary condition (6.2.13) provide inner free structures of the Adelic W^*-algebra $\mathbb{M}_{\mathbb{Q}}$ in terns of linear functionals γ_P, for finite prime sets P of $\mathcal{P}$.

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 4 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

The Modified Simple Equation Method and its Applications in Mathematical Physics and Biology

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Keywords: the system of shallow water wave equations; modified benjamin-bona-mahony equation; nonlinear dynamics of microtubules; the modified simple equation method; traveling wave solutions, solitary wave solutions.

GJSFR-F Classification : FOR Code : MSC 2010: 35Q20 - 35K99 - 35P05.

Strictly as per the compliance and regulations of :

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I. Introduction

No one can deny the important role which played by the nonlinear partial differential equations in the description of many and a wide variety of phenomena not only in physical phenomena, but also in plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry phenomena. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. For example, tanh - sech method [2]-[4], extended tanh - method [5]-[7], sine - cosine method [8]-[10], homogeneous balance method [11], the $\exp(\varphi(\xi))$-expansion Method [12], Jacobi elliptic function method [13]-[16], F-expansion method [17]-[19], exp-function method [20] and [21], trigonometric function series method [22], $(\frac{G'}{G})-$expansion method [23]-[26], the modified simple equation method [27]-[32] and so on.

The objective of this article is to apply the modified simple equation method for finding the exact traveling wave solution of some nonlinear partial differential equations, namely the system of shallow water wave equations [33], modified Benjamin-Bona-Mahony equation [34] and nonlinear dynamics of microtubules-A new model [35], which play an important role in mathematical physics.

The rest of this paper is organized as follows: In section 2, we give the description of the modified simple equation method. In section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In section 5, conclusions are given.

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II. Description of the Modified Simple Equation Method

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}....) = 0, \tag{2.1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [27]-[32]:

Step 1. We use the wave transformation

$$u(x,y,t) = u(\xi), \qquad \xi = (x + y - ct), \tag{2.2}$$

where c is a nonzero constant, to reduce Eq.(2.1) to the following ODE:

$$P(u, u', u'', u''', ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}$.

Step 2. Suppose that the solution of Eq.(2.3) has the formal solution:

$$u(\xi) = \sum_{k=0}^{N} A_k \left[\frac{'(\xi)}{(\xi)}\right]^k, \tag{2.4}$$

where A_k are arbitrary constants to be determined, such that $A_N \neq 0$, while the function $\psi(\xi)$ is an unknown function to be determined later, such that $\psi' \neq 0$.

Step 3. Determined the positive integer N in Eq.(2.4) by considering the homogenous balance between the highest order derivatives and the nonlinear terms in Eq.(2.3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = m$, which gives rise to degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q, \; D\left(u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n+q).$$

Step 4. Substitute Eq.(2.4) into Eq.(2.3), we calculate all the necessary derivative $u', u'', ...$ of the function $u(\xi)$ and we account the function $\psi(\xi)$. As a result of this substitution, we get a polynomial of $\psi^{-j}(j = 0, 1, 2, ...)$. In this polynomial, we gather all terms of the same power of $\psi^{-j}(j = 0, 1, 2, ...)$, and we equate with zero all coe˙cient of this polynomial. This operation yields a system of equations which can be solved to find A_k and $\psi(\xi)$. Consequently, we can get the exact solution of Eq.(2.1).

III. Application

Here, we will apply the modified simple equation method described in sec.2 to find the exact traveling wave solutions and then the solitary wave solutions for the following nonlinear systems of evolution equations.

a) Example 1: The system of shallow water wave equations

We first consider the system of the shallow water wave equation[33]

$$\begin{cases} u_t + (uv)_x + v_{xxx} = 0, \\ v_t + u_x + vv_x = 0. \end{cases} \tag{3.1}$$

Using the wave transformation $u(x,t) = u(\xi), \xi = (x - ct)$ carries the partial differential equation (3.1) into the ordinary differential equaion:

$$\begin{cases} -cu' + vu' + uv' + v''' = 0, \\ u' - cv' + vv' = 0, \end{cases} \tag{3.2}$$

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Integrating once the second ordinary differential equation with zero constant of integration, we get

$$u = cv - \frac{v^2}{2}. \tag{3.3}$$

Substituting Eq.(3.3) into the ˙rst equation of Eq.(3.2) we obtain

$$v''' + (3cv - \frac{3v^2}{2} - c^2)v' = 0. \tag{3.4}$$

Notes

Integrating Eq.(3.4) and neglecting the constant of integration, we obtain

$$v'' + \frac{3}{2}cv^2 - \frac{1}{2}v^3 - c^2v = 0. \tag{3.5}$$

Balancing $v''$and v^3 in Eq.(3.5) yields, $(N + 2 = 3N) \Longrightarrow (N = 1)$. So that, by using Eq.(2.4) we get the formal solution of Eq.(3.5)

$$v = A_0 + A_1 \left(\frac{\psi'}{\psi}\right). \tag{3.6}$$

Substituting Eq.(3.6) and its derivative into Eq.(3.5) and collecting all term with the same power of ψ^{-3}, ψ^{-2}, ψ^{-1}, ψ^0 we get:

$$\psi^{-3}: \ A_1\psi'^3\left(1 - \frac{1}{2}A_1^2\right) = 0, \tag{3.7}$$

$$\psi^{-2}: \ A_1\psi'\left[-3\psi'' + \frac{3}{2}A_1\psi'(c - A_0)\right] = 0, \tag{3.8}$$

$$\psi^{-1}: \ A_1\left[\psi''' + \psi'\left(3cA_0 - \frac{3}{2}A_0^2 - c^2\right)\right] = 0, \tag{3.9}$$

$$\psi^{0}: \ A_0\left[\frac{-1}{2}A_0^2 + \frac{3}{2}cA_0 - c^2\right] = 0. \tag{3.10}$$

From Eqs.(3.7) and (3.10), we deduce that

$$A_1 = \pm 2, \ A_0 = 0, \ A_0 = c \ and \ A_0 = 2c.$$

Let us discuss the following cases.

Case 1. If $A_0 \neq 0$.

In this case, we deduce from Eqs.(3.8) and (3.9) that :

$$\psi' = \frac{1}{c^2 + \frac{3}{2}A_0^2 - 3cA_0}\psi''', \tag{3.11}$$

and

$$\psi' = \frac{1}{\frac{c}{2}A_1 - \frac{1}{2}A_0A_1}\psi''. \tag{3.12}$$

Eqs.(3.11) and (3.12) yield

$$\frac{\psi'''}{\psi''} = E_0, \tag{3.13}$$

where $\left(E_0 = \frac{c^2 + \frac{3}{2}A_0^2 - 3cA_0}{\frac{c}{2}A_1 - \frac{1}{2}A_0A_1} \neq 0\right) \Rightarrow$ consequently, when $(A_0 = c\ )$ is rejected since it make $(E_0 = 0)$ whilst, when $(\ A_0 = 2c\ )$ it make $(E_0 = \mp c \neq 0\,)$ Integrating (3.13) and using (3.12), we deduce that

$$\psi' = c_2 exp\left(\mp c\,\xi\right), \tag{3.14}$$

where $c_2 = \frac{c_1}{\frac{c}{2}A_1 - \frac{1}{2}A_0A_1} = \frac{c_1}{\mp c}$, and consequently, we get

$$\psi = \frac{c_2}{\mp c} exp\left(\mp c\,\xi\right) + c_3, \tag{3.15}$$

where c_1, c_2 and c_3 are arbitrary constants.
Substituting (3.14) and (3.15) into Eq.(3.6), we have the exact traveling wave solution:

$$v = 2c \mp 2c\left[\frac{exp\left(\mp c\,\xi\right)}{exp\left(\mp c\,\xi\right) + c_3}\right], \tag{3.16}$$

when $c_1 = 1$, we obtain $c_2 =$ E0. So that we get the solitary wave solutions

- If $c_3 = 1$ and $c > 0$

$$v_{(1,2)} = 2c \mp c\left[1 \mp tanh\left(\frac{c}{2}\xi\right)\right], \tag{3.17}$$

- while, if $c_3 = 1$ and $c < 0$

$$v_{(3,4)} = 2c \mp c\left[1 \pm tanh\left(\frac{c}{2}\xi\right)\right], \tag{3.18}$$

- If $c_3 = -1$, and $c > 0$

$$v_{(5,6)} = 2c \mp c\left[1 \mp coth\left(\frac{c}{2}\xi\right)\right], \tag{3.19}$$

- while, if $c_3 = -1$, and c $<$ 0

$$v_{(7,8)} = 2c \mp c\left[1 \pm coth\left(\frac{c}{2}\xi\right)\right]. \tag{3.20}$$

Case 2. If $A_0 = 0$.
In this case, we deduce from Eqs.(3.8) and (3.9) that :

$$\psi' = \frac{1}{c^2}\,\psi''', \tag{3.21}$$

and

$$\psi' = \frac{2}{cA_1}\psi''. \tag{3.22}$$

Eqs.(3.21) and (3.22) yield

$$\frac{\psi'''}{\psi''} = E_1, \tag{3.23}$$

where $\left(E_1 = \frac{2c}{A_1} = \pm c \neq 0\right)$ integrating (3.23) and using (3.22), we deduce that

$$\psi' = c_5 exp\left(\pm c\,\xi\right), \tag{3.24}$$

where $c_5 = \frac{2c_4}{cA_1} = \frac{\pm c_4}{c}$, and consequently, we get

$$\psi = \frac{\pm c_2}{c} exp\left(\pm c\,\xi\right) + c_6, \tag{3.25}$$

where c_4, c_5 and c_6 are arbitrary constants.
Substituting (3.24) and (3.25) into Eq.(3.6), we have the exact traveling wave solution:

$$v = \pm 2c\left[\frac{exp\left(\pm c\,\xi\right)}{exp\left(\pm c\,\xi\right) + c_6}\right], \tag{3.26}$$

when $c_4 = c^2$, we obtain $c_5 = E_1$. So that we get the solitary wave solutions

- If $c_6 = 1$, $c > 0$, we get

$$v_{(9,10)} = \pm c\left[1 \pm tanh\left(\frac{c}{2}\xi\right)\right], \tag{3.27}$$

Notes

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- If $c_6 = 1,\ c < 0$, we get

$$v_{(11,12)} = \pm c\left[1 \mp tanh\left(\frac{c}{2}\xi\right)\right]. \tag{3.28}$$

- If $c_6 = 1,\ c > 0$

$$v_{(13,14)} = \pm c\left[1 \pm coth\left(\frac{c}{2}\xi\right)\right], \tag{3.29}$$

- While, if $c_6 = 1,\ c < 0$

$$v_{(15,16)} = \pm c\left[1 \mp coth\left(\frac{c}{2}\xi\right)\right]. \tag{3.30}$$

b) Example 2: Modified Benjamin-Bona-Mahony equation

The modified Benjamin-Bona-Mahony(MBBM)equation [34] is in the form,

$$u_t + u_x + au^2u_x + bu_{xxt} = 0, \tag{3.31}$$

where a and b are positive constants. Using the transformation $u(x,t) = u(\xi);\ (\xi = x + kt)$ to reduce Eq.(3.31) to the following ordinary di˙erential equation

$$ku' + u' + au^2u' + bku''' = 0, \tag{3.32}$$

Integrating Eq.(3.32) with zero constant of integration we obtain

$$(k+1)u + \frac{a}{3}u^3 + bku'' = 0, \tag{3.33}$$

hence, Eq.(3.33) take the form:

$$u - \alpha u^3 + \beta u''' = 0, \tag{3.34}$$

when, $\left(\alpha = \frac{-a}{3(k+1)}\right)$ and $\left(\beta = \frac{bk}{(k+1)}\right)$. Balancing u'' with u^3 in Eq.(3.34) yield, $(N+2 = 3N) \Rightarrow (N = 1)$. So that, we have the same formal solution of Eq.(3.5). Substituting Eq.(3.6) and its derivative into Eq.(3.34) and collecting all term with the same power of $\psi^{-3}, \psi^{-2}, \psi^{-1}, \psi^0$ we get:

$$\psi^{-3}:\ A_1\psi'^3\left[-\alpha A_1^2 + 2\beta\right] = 0, \tag{3.35}$$

$$\psi^{-2}:\ -3A_1\psi'\left[\alpha A_0 A_1\psi' + \beta\psi''\right] = 0, \tag{3.36}$$

$$\psi^{-1}:\ A_1\left[\psi'\left(1 - 3\alpha A_0^2\right) + \beta\psi'''\right] = 0, \tag{3.37}$$

$$\psi^0:\ A_0\left[1 - \alpha A_0^2\right] = 0. \tag{3.38}$$

From Eqs.(3.35) and (3.38), we deduce that

$$A_1 = \pm\sqrt{\frac{2\beta}{\alpha}},\ A_0 = \pm\sqrt{\frac{1}{\alpha}},\ and\ A_0 = 0.$$

Let us discuss the following cases.

case 1. when $A_0 \neq 0$.
in this case, we deduce from Eqs.(3.36) and (3.37) that

$$\psi' = \frac{-\beta}{\alpha A_0 A_1}\psi'', \tag{3.39}$$

and

$$\psi' = \frac{\beta}{3\alpha A_0^2 - 1}, \tag{3.40}$$

Eqs.(3.39) and (3.40) yield.

$$\frac{\psi'''}{\psi''} = E_2, \tag{3.41}$$

where $\left(E_2 = \frac{1-3\alpha A_0^2}{\alpha A_0 A_1} = \frac{-2}{\sqrt{2\beta}} \neq 0\right)$ and $(\beta > 0)$. Integrating Eq.(3.41) and using Eq.(3.39), We deduce that

$$\psi' = c_8 exp\left(\frac{-2}{\sqrt{2\beta}}\xi\right), \tag{3.42}$$

where $\left(c_8 = \frac{-\beta c_7}{\alpha A_0 A_1} = \frac{-c_7\sqrt{2\beta}}{2}\right)$ and consequently, we get

$$\psi = \frac{-c_8\sqrt{2\beta}}{2} exp\left(\frac{-2}{\sqrt{2\beta}}\xi\right) + c_9, \tag{3.43}$$

where c_7, c_8 and c_9 are arbitrary constant of integration.
Substituting Eq.(3.42) and (3.43) into Eq.(3.6) we have the exact traveling wave solution

$$u = \pm\sqrt{\frac{2\beta}{\alpha}} \mp \frac{2}{\sqrt{2\alpha\beta}}\left[\frac{exp\left(\frac{-2}{\sqrt{2\beta}}\xi\right)}{exp\left(\frac{-2}{\sqrt{2\beta}}\xi\right) + c_9}\right], \tag{3.44}$$

when $c_7 = \frac{2}{\beta}$, we obtain $c_8 = E_2$. So that we get the solitary wave solutions

- If $c_9 = 1$,

$$u_{(1,2)} = \pm\sqrt{\frac{2\beta}{\alpha}} \mp \frac{1}{\sqrt{2\alpha\beta}}\left[1 - tanh\left(\frac{1}{\sqrt{2\beta}}\xi\right)\right], \tag{3.45}$$

- If $c_9 = -1$,

$$u_{(3,4)} = \pm\sqrt{\frac{2\beta}{\alpha}} \mp \frac{1}{\sqrt{2\alpha\beta}}\left[1 - coth\left(\frac{1}{\sqrt{2\beta}}\xi\right)\right], \tag{3.46}$$

case 2. when $A_0 = 0$
In this case, we deduce from Eqs.(3.36) and (3.37) that $\psi' = 0$, and hence this case will be rejected.

c) Example 3: Nonlinear dynamics of microtubules- A new model
We consider the nonlinear dynamical equation of motion [35]

$$m\frac{\partial^2 z}{\partial t^2} - kl^2\frac{\partial^2 z}{\partial x^2} - qE - Az + Bz^3 + \gamma\frac{\partial z}{\partial t} = 0, \tag{3.47}$$

where *m, k, l, q, A, B* and γ are arbitrary constants to be determined later. It is well known that, for a given wave equation, a traveling wave $z(\xi)$ is a solution which depends upon x and t only through a unified variable $(\xi = \kappa x - \omega t)$, where κ and ω are constants. This allows us to obtain the final dimensionless ordinary differential equation

$$\alpha u'' - \rho u' - u + u^3 - \sigma = 0, \tag{3.48}$$

where

$$u' = \frac{du}{d\xi},\ \alpha = \frac{m\omega^2 - kl^2\kappa^2}{A},\ z = \sqrt{\frac{A}{B}}u,\ \rho = \frac{\gamma\omega}{A}\ and\ \sigma = \frac{qE}{A\sqrt{\frac{A}{B}}}.$$

Balancing between u'' and u^3 yield, $(N + 2 = 3N) \Rightarrow (N = 1)$. So that, we have the same formal solution of Eq.(3.5). Substituting Eq.(3.6) and its derivative into Eq.(3.48) and collecting all of the term with the same power of ψ^{-3}, ψ^{-2}, ψ^{-1}, ψ^0 we get:

$$\psi^{-3}: \ A_1\psi'^3\left[2\alpha + A_1^2\right] = 0, \tag{3.49}$$

$$\psi^{-2}: \ A_1\psi'\left[-3\alpha\psi'' + \psi'\left(\rho + 3A_0A_1\right)\right] = 0, \tag{3.50}$$

$$\psi^{-1}: \ A_1\left[\alpha\psi''' - \rho\psi'' - \psi'\left(1 - 3A_0^2\right)\right] = 0, \tag{3.51}$$

$$\psi^{0}: \ A_0^3 - A_0 - \sigma = 0. \tag{3.52}$$

Notes

From Eqs. (3.49) and (3.52), we deduce that

$$A_1 = \pm\sqrt{-2\alpha}, \ \ where\ \alpha < 0,$$

$$A_0 = \frac{1}{6}\sqrt[3]{108\,\sigma + 12\sqrt{-12 + 81\,\sigma^2}} + 2\,\frac{1}{\sqrt[3]{108\,\sigma + 12\sqrt{-12 + 81\,\sigma^2}}}$$

and

$$A_0 = \frac{-1}{12}\sqrt[3]{108\,\sigma + 12\sqrt{-12 + 81\,\sigma^2}} - \frac{1}{\sqrt[3]{108\,\sigma + 12\sqrt{-12 + 81\,\sigma^2}}}$$

$$\pm\frac{\sqrt{3}}{2}\left(\frac{1}{6}\sqrt[3]{108\,\sigma + 12\sqrt{-12 + 81\,\sigma^2}} - 2\,\frac{1}{\sqrt[3]{108\,\sigma + 12\sqrt{-12 + 81\,\sigma^2}}}\right).$$

where $\left(-12 + 81\,\sigma^2\right) > 0$.

So that, we deduce from Eqs.(3.50) and (3.51) that

$$\psi' = \frac{3\alpha}{\rho + A_0A_1}\psi'', \tag{3.53}$$

and

$$\alpha\psi''' - \rho\psi'' - \left(1 - 3A_0^2\right)\psi' = 0, \tag{3.54}$$

and hence, Eqs. (3.53) and (3.54) yield

$$\frac{\psi'''}{\psi''} = E_3, \tag{3.55}$$

where $E_3 = \left(\frac{\rho}{\alpha} + \frac{3\left(1-3A_0^2\right)}{\rho+3A_0A_1}\right)$.

Integrating Eq.(3.55) and using (3.53), we deduce that

$$\psi' = c_{11}exp\left(E_3\xi\right), \tag{3.56}$$

where $c_{11} = \frac{3\alpha c_{10}}{\rho+3A_0A_1}$, and consequently, we get

$$\psi = \frac{c_{11}}{E_3}exp\left(E_3\xi\right) + c_{12} \tag{3.57}$$

where c_{10}, c_{11} and c_{12} are arbitrary constants of integration.
Substituting Eqs.(3.56) and (3.57) into (3.6), we have the exact solution:

$$u = A_0 \pm E_3\sqrt{-2\alpha}\left[\frac{exp\left(E_3\xi\right)}{exp\left(E_3\xi\right) + c_{12}}\right], \tag{3.58}$$

when $c_{11} = E_3$, we have the solitary wave solutions.

- If $c_{12} = 1$, $E_3 > 0$

$$u_1 = A_0 \pm \frac{E_3\sqrt{-2\alpha}}{2}\left[1 + tanh\left(\frac{E_3}{2}\xi\right)\right]. \tag{3.59}$$

- While, if $c_{12} = 1$, $E_3 < 0$

$$u_2 = A_0 \pm \frac{E_3\sqrt{-2\alpha}}{2}\left[1 - tanh\left(\frac{E_3}{2}\xi\right)\right]. \tag{3.60}$$

- If $c_{12} = -1$, $E_3 > 0$

$$u_3 = A_0 \pm \frac{E_3\sqrt{-2\alpha}}{2}\left[1 + coth\left(\frac{E_3}{2}\xi\right)\right]. \tag{3.61}$$

- ·While, if $c_{12} = -1$, $E_3 < 0$

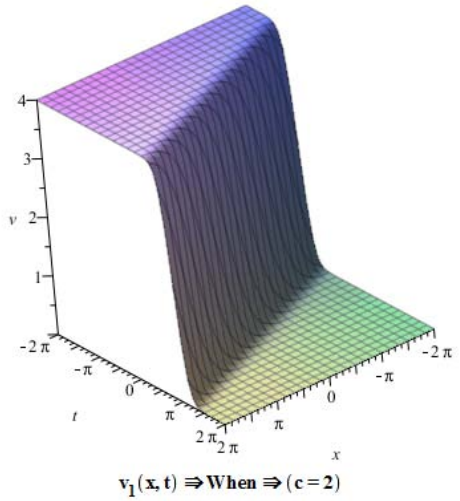
$$u_4 = A_0 \pm \frac{E_3\sqrt{-2\alpha}}{2}\left[1 - coth\left(\frac{E_3}{2}\xi\right)\right]. \tag{3.62}$$

- Note that:

All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.

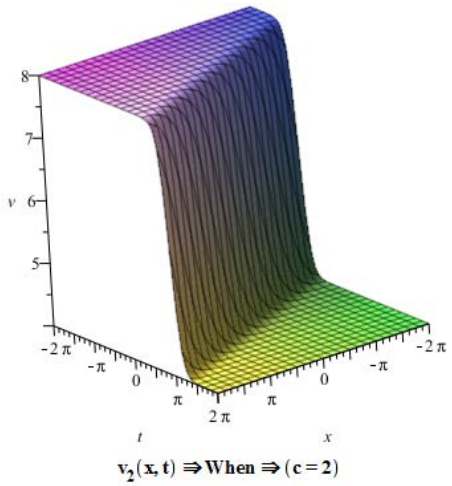
IV. Physical Interpretations of the Solutions

In this section, we depict the graph and signify the obtained solutions to each of the system of shallow water wave equations, modified Benjamin-Bona-Mahony equation and nonlinear dynamics o microtubules-a new model. Now, we will discuss all possible physical signi¨cances for parameter. For this value of parameter c $>$ 0 the solution $v_{(1)}$ and $v_{(2)}$ in Eq.(3.17) represent kink shape soliton solution also when parameter c $<$ 0 the solution $v_{(3)}$ and $v_{(4)}$ in Eq.(3.18) represent kink shape soliton solution, when parameter c $>$ 0the solution $v_{(5)}$ and $v_{(6)}$ in Eq.(3.19) represent singular soliton solution while, when parameter c $<$ 0the solution $v_{(7)}$ and $v_{(8)}$ in Eq.(3.20) represent dark singular soliton solution, when parameter c $>$ 0the solution $v_{(9)}$ and $v_{(10)}$ in Eq.(3.27) represent kink shape soliton solution also when parameter c $<$ 0the solution $v_{(11)}$ and $v_{(12)}$ in Eq.(3.28) represent kink shape soliton solution, when parameter c $>$ 0the solution $v_{(13)}$ and $v_{(14)}$ in Eq.(3.29) represent singular soliton solution also when parameter c $<$ 0the solution $v_{(15)}$ and $v_{(16)}$ in Eq.(3.30) represent dark singular soliton solution, when parameter $\beta = 2, \alpha = 4, k = 1$ the solution $u_{(1)}$ and $u_{(2)}$ in Eq.(3.45) represent kink shape soliton solution, when parameter $\beta = 2$, $\alpha = 4, k = 1$ the solution $u_{(3)}$ in Eq.(3.46) represent dark singular soliton solution and $u_{(4)}$ in Eq.(3.46) represent bell singular soliton solution, when parameter $\alpha = -2$, $A_1 = 2$, $\sigma = 1$, $A_0 = 1.32$, $\rho = -10$, $E_3 = 10.87$ the solution $u_{(1)}$ and $u_{(2)}$ in Eq.(3.59) represent kink shape soliton solution, when parameter $\alpha = -2$, $A_1 = 2$, $\sigma = 1$, $A_0 = 1.32$, $\rho = 4$, $E_3 = -3.07$ the solution $u_{(3)}$ and u(4) in Eq.(3.60) represent kink shape soliton solution, when parameter $\alpha = -2$, $A_1 = 2$, $\sigma = 1$, $A_0 = 1.32$, $\rho = -10$, $E_3 = 10.87$ the solution $u_{(5)}$ in Eq.(3.61) represent dark singular shape soliton solution and $u_{(6)}$ in Eq.(3.61) represent singular bell shape soliton solution, and when parameter $\alpha = -2$, $A_1 = 2$, $\sigma = 1$, $A_0 = 1.32$ $= 4$, $E_3 = -3.07$ the solution $u_{(7)}$ in Eq.(3.62) represent dark singular shape soliton solution and $u_{(8)}$ in Eq.(3.62) represent bell singular shape soliton solution.



(a)

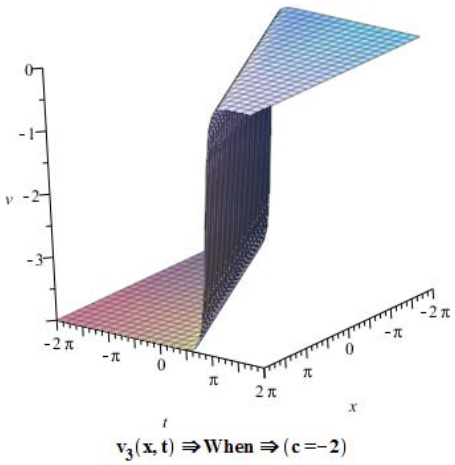
Eq.(3.17)



(b)

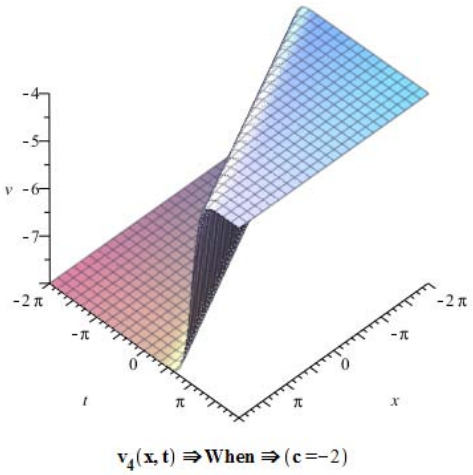
Eq.(3.17)

Figure 1 : The Solitary wave solution of Eqs.(3.17)



(a)

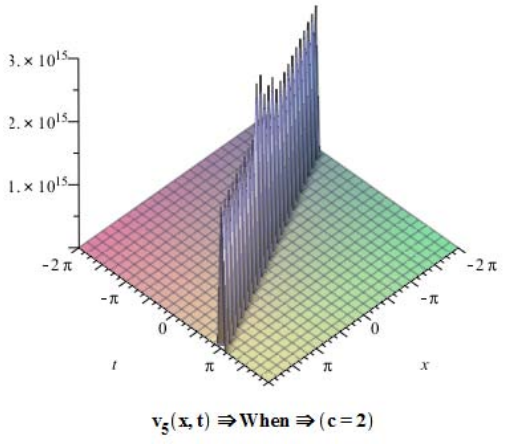
Eq.(3.18)



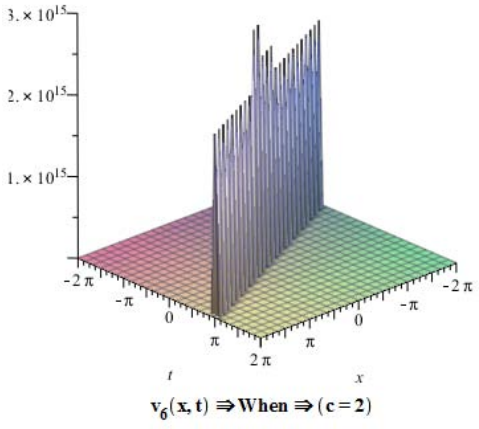
(b)

Eq.(3.18)

Figure 2 : The Solitary wave solution of Eqs.(3.18)

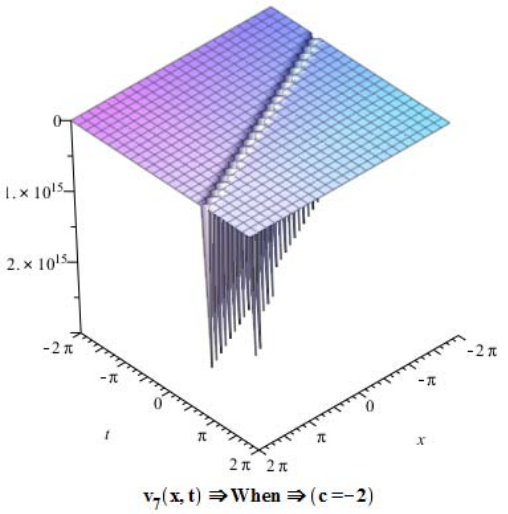


(a)
Eq.(3.19)

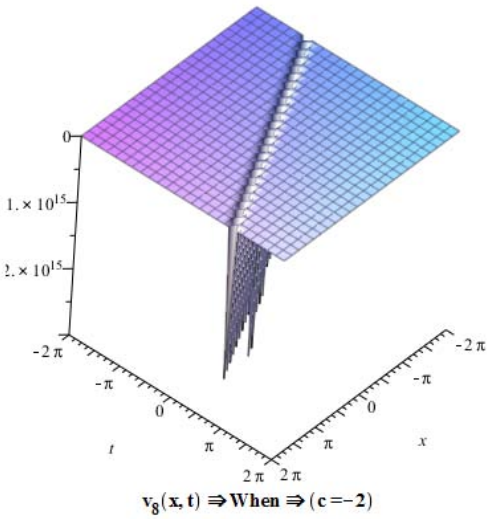


(b)
Eq.(3.19)

Figure 3 : The Solitary wave solution of Eqs.(3.19)

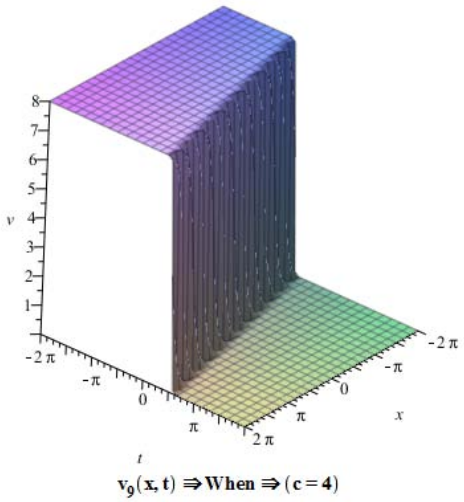


(a)
Eq.(3.20)

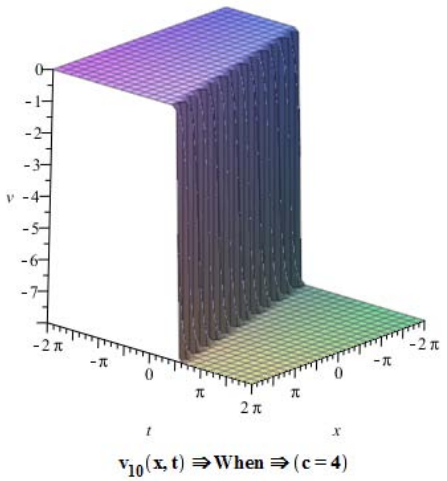


(b)
Eq.(3.20)

Figure 4 : The Solitary wave solution of Eqs.(3.20)

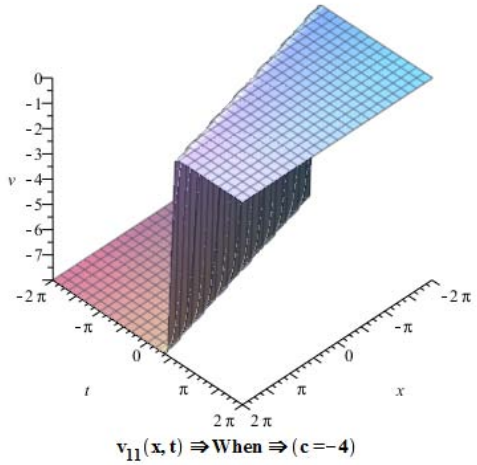


(a)
Eq.(3.27)

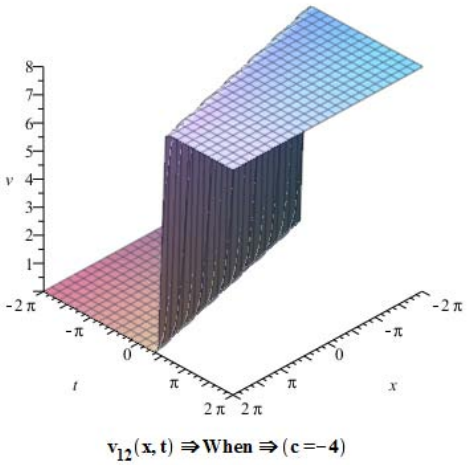


(b)
Eq.(3.27)

Figure 5 : The Solitary wave solution of Eqs.(3.27)

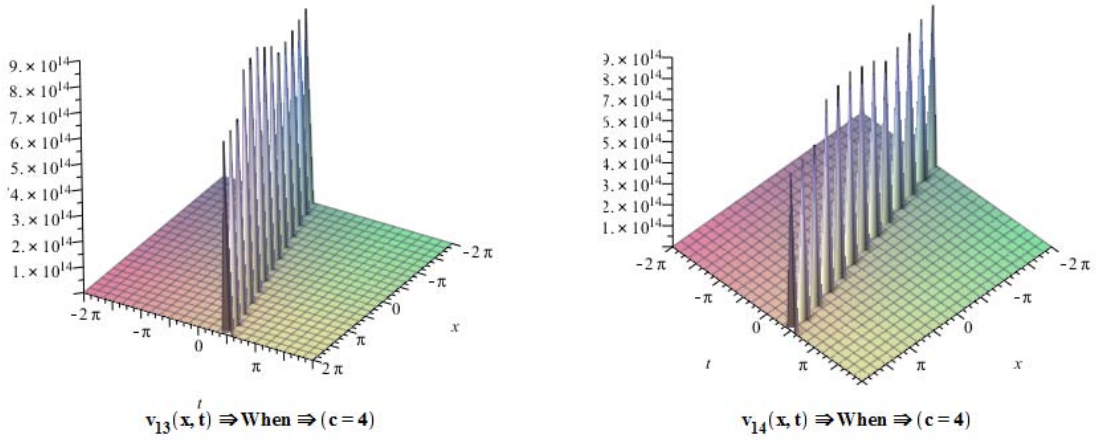


(a)
Eq.(3.28)



(b)
Eq.(3.28)

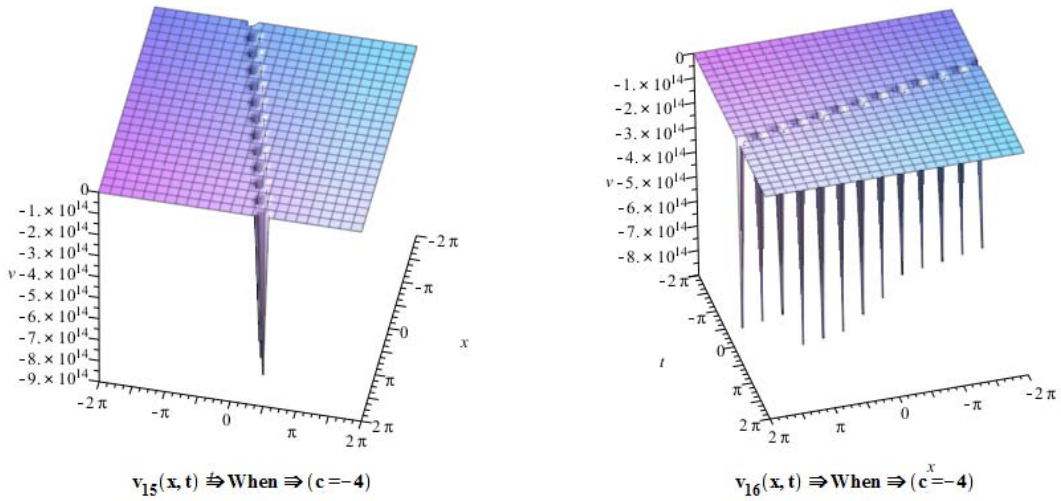
Figure 6 : The Solitary wave solution of Eqs.(3.28)



(a) Eq.(3.29)

(b) Eq.(3.29)

Figure 7 : The Solitary wave solution of Eqs.(3.29)

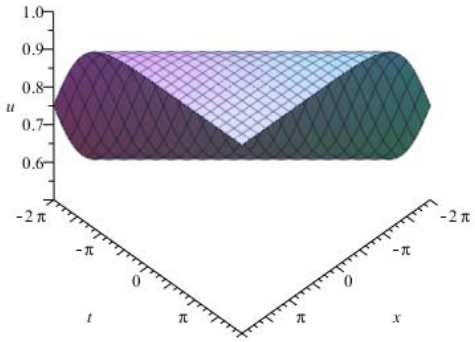


(a) Eq.(3.30)

(b) Eq.(3.30)

Figure 8 : The Solitary wave solution of Eqs.(3.30)

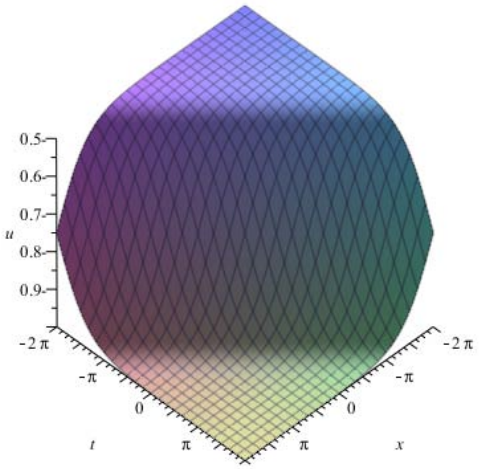
Notes



$u_1(x,t) \Rightarrow$ When $\Rightarrow (\alpha = 4, \beta = 2, k = 1)$

(a)

Eq.(3.45)

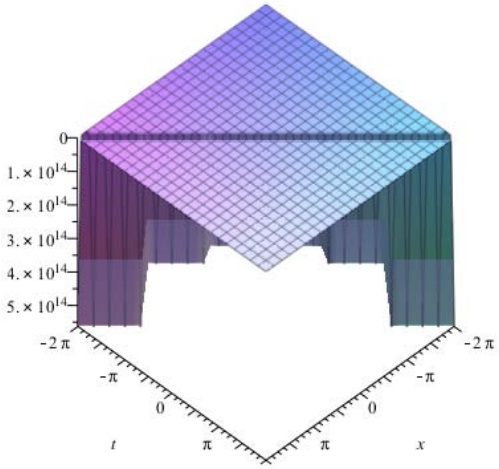


$u_2(x,t) \Rightarrow$ When $\Rightarrow (\alpha = 4, \beta = 2, k = 1)$

(b)

Eq.(3.45)

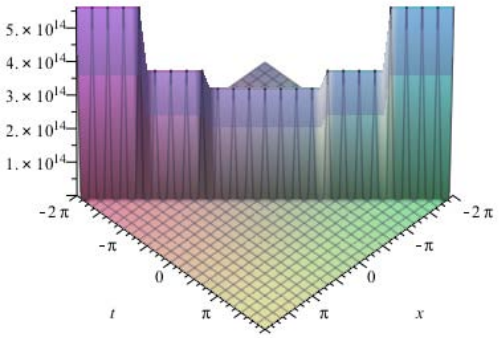
Figure 9 : The Solitary wave solution of Eqs.(3.45)



$u_3(x,t) \Rightarrow$ When $\Rightarrow (\alpha = 4, \beta = 2, k = 1)$

(a)

Eq.(3.46)

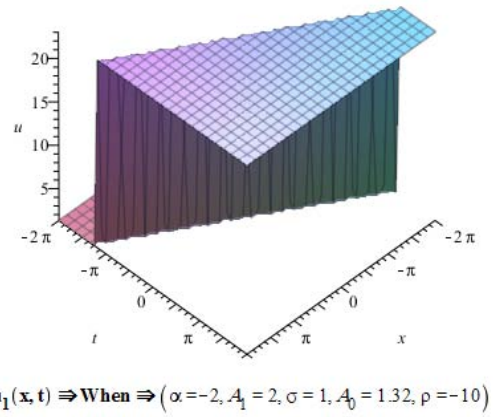


$u_4(x,t) \Rightarrow$ When $\Rightarrow (\alpha = 4, \beta = 2, k = 1)$

(b)

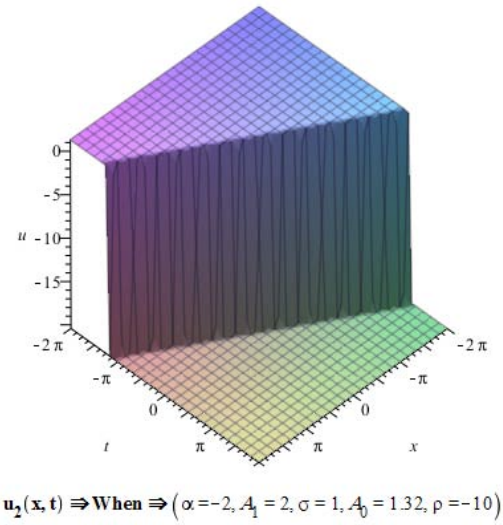
Eq.(3.46)

Figure 10 : The Solitary wave solution of Eqs.(3.46)



(a)

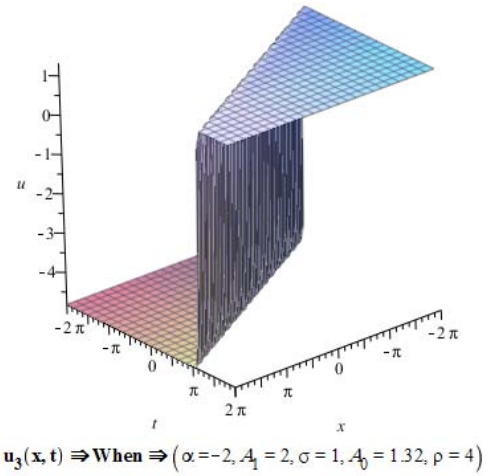
Eq.(3.59)



(b)

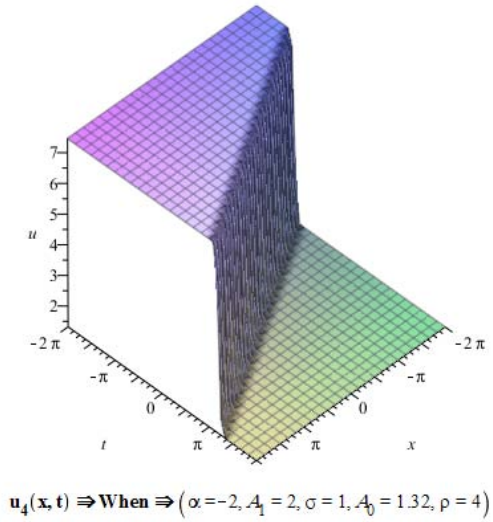
Eq.(3.59)

Figure 11 : The Solitary wave solution of Eqs.(3.59)



(a)

Eq.(3.60)

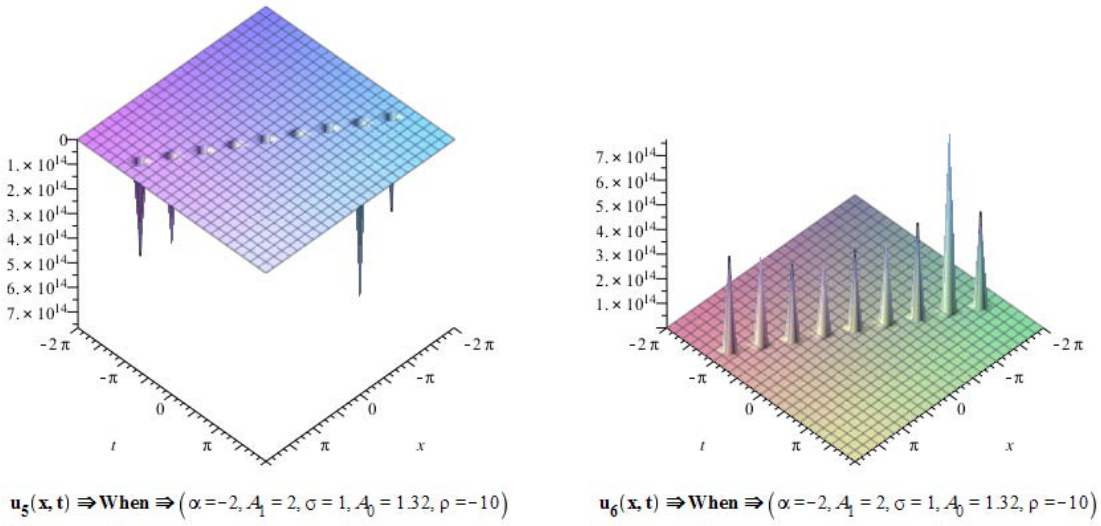


(b)

Eq.(3.60)

Figure 12 : The Solitary wave solution of Eqs.(3.60)

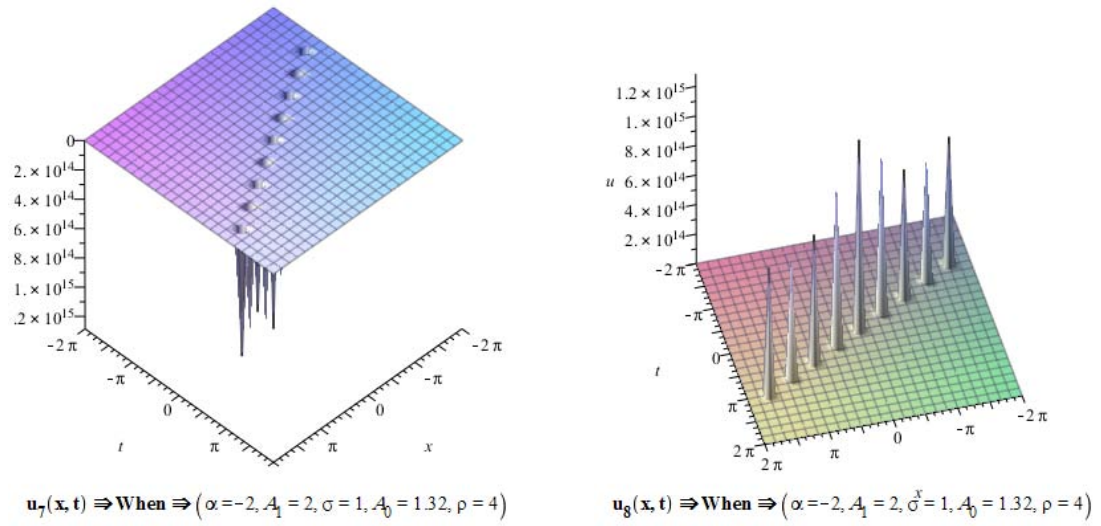
Notes



(a) Eq.(3.61)

(b) Eq.(3.61)

Figure 13 : The Solitary wave solution of Eqs.(3.61)



(a) Eq.(3.62)

(b) Eq.(3.62)

Figure 14 : The Solitary wave solution of Eqs.(3.62)

V. Conclusion

The modified simple equation method has been successfully used to find the exact traveling wave solutions of some nonlinear evolution equations. As an application, the traveling wave solutions for the system of shallow water wave equations, modified Benjamin-Bona-Mahony equation and nonlinear dynamics of microtubules-a new model which have been constructed using the modified simple equation method. Let us compare

between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the system of shallow water wave equations, modified Benjamin-Bona-Mahony equation and nonlinear dynamics of microtubules-A new model are new and different from those obtained in [[33]; [36] and [37]], [[34]; [38] and [39]] and [[35] and [40]]. and also we can see [39] which is considered a special case of modified Benjamin-Bona-Mahony equation when $a = 1$. It can be concluded that this method is reliable and propose a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations. Figs.[1−14] represent the solitary traveling wave solution for the system of shallow water wave equations, modified Benjamin-Bona-Mahony equation and nonlinear dynamics of microtubules A new model.

Competing interests

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

Author's contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied.

VI. Acknowledgment

The author thanks the referees for their suggestions and comments.

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 4 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Some Relationships between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities

By M. P. Chaudhary, Diriba Kejela Geleta & Gedefa Negassa Feyissa
Madawalabu University, Ethiopia

Abstract- The author establish a set of four presumbaly new results which depict the inter-relationship between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities. Several closely related *q*-identities such as (for example) Jacobi's triple-product identity are also considered.

Keywords: infinite product identities, jacobi's triple-product identity, continued-fraction identities, combinatorial partition identity, q-product identities.

GJSFR-F Classification : FOR Code : MSC 2010: Primary 05A30, 11F27; Secondary 05A17, 11P83.

Strictly as per the compliance and regulations of :

Some Relationships between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities

M. P. Chaudhary[α], Diriba Kejela Geleta[σ] & Gedefa Negassa Feyissa[ρ]

Abstract- The author establish a set of four presumbaly new results which depict the inter-relationship between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities. Several closely related q-identities such as (for example) Jacobi's triple-product identity are also considered.

Keywords: infinite product identities, jacobi's triple-product identity, continued-fraction identities, combinatorial partition identity, q-product identities.

I. Introduction

For $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n \tag{1.1}$$

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{(n-1)} \tag{1.2}$$

$$(a_1, a_2, a_3, ..., a_k; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty...(a_k;q)_\infty \tag{1.3}$$

Ramanujan has defined general theta function, as

$$f(a,b) = \sum_{-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad ; \quad |ab| < 1, \tag{1.4}$$

In [1], Jacobi's triple product identity is given, as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty \tag{1.5}$$

Special cases of Jacobi's triple products identity are given, as

$$\Phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty \tag{1.6}$$

$$\Psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{1.7}$$

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$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{1.8}$$

Equation (1.8) is known as Euler's pentagonal number theorem. Euler's another well known identity is as

$$(q;q^2)-1_\infty = (-q;q)_\infty \tag{1.9}$$

In [1], Roger-Ramanujan identities are given as

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.10}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \frac{(q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.11}$$

Roger-Ramanujan function is given by

$$R(q) = q^{\frac{1}{5}} \frac{H(q)}{G(q)} = q^{\frac{1}{5}} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \tag{1.12}$$

Throughout this paper we use the following representations

$$(q^a;q^n)_\infty (q^b;q^n)_\infty (q^c;q^n)_\infty \cdots (q^t;q^n)_\infty = (q^a, q^b, q^c \cdots q^t; q^n)_\infty \tag{1.13}$$

$$(q^a;q^n)_\infty (q^a;q^n)_\infty (q^c;q^n)_\infty \cdots (q^t;q^n)_\infty = (q^a, q^a, q^c \cdots q^t; q^n)_\infty \tag{1.14}$$

In [1, p-28(2.2)], following identity is given

$$(q^2;q^2)_\infty (-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

$$= \frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-\cdots}\ ; (|q|<1) \tag{1.15}$$

In [1, p-27(2.1)], following identity is given

$$\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \frac{1}{1+}\ \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^4}{1+}\ \frac{q^5}{1+}\ \frac{q^6}{1+\cdots}\ ; (|q|<1) \tag{1.16}$$

In [1, p-28(2.4)], following identity is given

$$C(q) = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = 1 + \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^4}{1+}\ \frac{q^5}{1+}\ \frac{q^6}{1+\cdots}\ ; (|q|<1) \tag{1.17}$$

Lastly, we turn to the recent investigation by Andrews et. al. [2], involving combinatorial partition identities associated with the following general family

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l,u,v,w;n) \tag{1.18}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{\left[\frac{n}{u}\right]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj} (q^{uv};q^{uv})_j} \tag{1.19}$$

Ref
2. George Andrews, Kathrin Bringman, and Karl Mahlburg : *Double series Representations for Schur's partition function and related identities*, pre-print.

In particular, we recall the following combinatorial partition identities [2, p.106, Th.3]

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty \tag{1.20}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty \tag{1.21}$$

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty} \tag{1.22}$$

Computation of q-product identities

Chaudhary[1], has computed several q-product identities. Here we are giving some identities from [1], and some new identities have been computed, are useful for next section of this paper, as given below

$$\begin{aligned}(q^2;q^2)_\infty &= \prod_{n=0}^{\infty}(1-q^{2n+2}) \\ &= \prod_{n=0}^{\infty}(1-q^{2(4n)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+1)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+2)+2}) \times \prod_{n=0}^{\infty}(1-q^{2(4n+3)+2}) \\ &= \prod_{n=0}^{\infty}(1-q^{8n+2}) \times \prod_{n=0}^{\infty}(1-q^{8n+4}) \times \prod_{n=0}^{\infty}(1-q^{8n+6}) \times \prod_{n=0}^{\infty}(1-q^{8n+8}) \\ &= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty = (q^2,q^4,q^6,q^8;q^8)_\infty \end{aligned} \tag{1.23}$$

$$\begin{aligned}(q^4;q^4)_\infty &= \prod_{n=0}^{\infty}(1-q^{4n+4}) \\ &= \prod_{n=0}^{\infty}(1-q^{4(3n)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+1)+4}) \times \prod_{n=0}^{\infty}(1-q^{4(3n+2)+4}) \\ &= \prod_{n=0}^{\infty}(1-q^{12n+4}) \times \prod_{n=0}^{\infty}(1-q^{12n+8}) \times \prod_{n=0}^{\infty}(1-q^{12n+12}) \\ &= (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty = (q^4,q^8,q^{12};q^{12})_\infty \end{aligned} \tag{1.24}$$

Similarly we can compute following, as

$$(q^1;q^1)_\infty = (q^1;q^2)_\infty(q^2;q^2)_\infty = (q^1,q^2;q^2)_\infty \tag{1.25}$$

$$(q^2;q^2)_\infty = (q^2;q^4)_\infty(q^4;q^4)_\infty = (q^2,q^4;q^4)_\infty \tag{1.26}$$

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^8)_\infty(q^4;q^8)_\infty(q^6;q^8)_\infty(q^8;q^8)_\infty \\ &= (q^2,q^4,q^6,q^8;q^8)_\infty \end{aligned} \tag{1.27}$$

$$\begin{aligned}(q^2;q^2)_\infty &= (q^2;q^{12})_\infty(q^4;q^{12})_\infty(q^6;q^{12})_\infty(q^8;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty \\ &= (q^2,q^4,q^6,q^8,q^{10},q^{12};q^{12})_\infty \end{aligned} \tag{1.28}$$

Ref

1. M. P. Chaudhary : *Generalization for character formulas in terms of continued fraction identities*, Malay J. Mat. 1(1)(2014) 24-34.

$$(q^2;q^2)_\infty = (q^2;q^{16})_\infty(q^4;q^{16})_\infty(q^6;q^{16})_\infty(q^8;q^{16})_\infty(q^{10};q^{16})_\infty\times$$
$$\times(q^{12};q^{16})_\infty(q^{14};q^{16})_\infty(q^{16};q^{16})_\infty$$
$$= (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16};q^{16})_\infty \tag{1.29}$$

$$(q^2;q^2)_\infty = (q^2;q^{20})_\infty(q^4;q^{20})_\infty(q^6;q^{20})_\infty(q^8;q^{20})_\infty(q^{10};q^{20})_\infty(q^{12};q^{20})_\infty\times$$
$$\times(q^{14};q^{20})_\infty(q^{16};q^{20})_\infty(q^{18};q^{20})_\infty(q^{20};q^{20})_\infty$$
$$= (q^2,q^4,q^6,q^8,q^{10},q^{12},q^{14},q^{16},q^{18},q^{20};q^{20})_\infty \tag{1.30}$$

$$(q^3;q^3)_\infty = (q^3;q^6)_\infty(q^6;q^6)_\infty = (q^3,q^6;q^6)_\infty \tag{1.31}$$

$$(q^4;q^4)_\infty = (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty = (q^4,q^8,q^{12};q^{12})_\infty \tag{1.32}$$

$$(q^4;q^4)_\infty = (q^4;q^{16})_\infty(q^8;q^{16})_\infty(q^{12};q^{16})_\infty(q^{16};q^{16})_\infty$$
$$= (q^4,q^8,q^{12},q^{16};q^{16})_\infty \tag{1.33}$$

$$(q^4;q^4)_\infty = (q^4;q^{20})_\infty(q^8;q^{20})_\infty(q^{12};q^{20})_\infty(q^{16};q^{20})_\infty(q^{20};q^{20})_\infty$$
$$= (q^4,q^8,q^{12},q^{16},q^{20};q^{20})_\infty \tag{1.34}$$

$$(q^4;q^4)_\infty = (q^4;q^{24})_\infty(q^8;q^{24})_\infty(q^{12};q^{24})_\infty(q^{16};q^{24})_\infty(q^{20};q^{24})_\infty(q^{24};q^{24})_\infty$$
$$= (q^4,q^8,q^{12},q^{16},q^{20},q^{24};q^{24})_\infty \tag{1.35}$$

$$(q^4;q^{12})_\infty = (q^4;q^{60})_\infty(q^{16};q^{60})_\infty(q^{28};q^{60})_\infty(q^{40};q^{60})_\infty(q^{52};q^{60})_\infty$$
$$= (q^4,q^{16},q^{28},q^{40},q^{52};q^{60})_\infty \tag{1.36}$$

$$(q^6;q^6)_\infty = (q^6;q^{12})_\infty(q^{12};q^{12})_\infty = (q^6,q^{12};q^{12})_\infty \tag{1.37}$$

$$(q^6;q^6)_\infty = (q^6;q^{24})_\infty(q^{12};q^{24})_\infty(q^{18};q^{24})_\infty(q^{24};q^{24})_\infty$$
$$= (q^6,q^{12},q^{18},q^{24};q^{24})_\infty \tag{1.38}$$

$$(q^6;q^{12})_\infty = (q^6;q^{60})_\infty(q^{18};q^{60})_\infty(q^{30};q^{60})_\infty(q^{42};q^{60})_\infty(q^{54};q^{60})_\infty$$
$$= (q^6,q^{18},q^{30},q^{42},q^{54};q^{60})_\infty \tag{1.39}$$

$$(q^8;q^8)_\infty = (q^8;q^{24})_\infty(q^{16};q^{24})_\infty(q^{24};q^{24})_\infty = (q^8,q^{16},q^{24};q^{24})_\infty \tag{1.40}$$

$$(q^8;q^8)_\infty = (q^8;q^{48})_\infty(q^{16};q^{48})_\infty(q^{24};q^{48})_\infty(q^{32};q^{48})_\infty(q^{40};q^{48})_\infty(q^{48};q^{48})_\infty$$
$$= (q^8,q^{16},q^{24},q^{32},q^{40},q^{48};q^{48})_\infty \tag{1.41}$$

$$(q^8;q^{12})_\infty = (q^8;q^{60})_\infty(q^{20};q^{60})_\infty(q^{32};q^{60})_\infty(q^{44};q^{60})_\infty(q^{56};q^{60})_\infty$$
$$= (q^8,q^{20},q^{32},q^{44},q^{56};q^{60})_\infty \tag{1.42}$$

$$(q^8;q^{16})_\infty = (q^8;q^{48})_\infty(q^{24};q^{48})_\infty(q^{40};q^{48})_\infty = (q^8,q^{24},q^{40};q^{48})_\infty \tag{1.43}$$

$$(q^{10};q^{20})_\infty = (q^{10};q^{60})_\infty(q^{30};q^{60})_\infty(q^{50};q^{60})_\infty = (q^{10},q^{30},q^{50};q^{60})_\infty \tag{1.44}$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{24})_\infty(q^{24};q^{24})_\infty = (q^{12},q^{24};q^{24})_\infty \tag{1.45}$$

$$(q^{12};q^{12})_\infty = (q^{12};q^{60})_\infty(q^{24};q^{60})_\infty(q^{36};q^{60})_\infty(q^{48};q^{60})_\infty(q^{60};q^{60})_\infty$$
$$= (q^{12},q^{24},q^{36},q^{48},q^{60};q^{60})_\infty \tag{1.46}$$

Notes

$$(q^{16};q^{16})_\infty = (q^{16};q^{48})_\infty (q^{32};q^{48})_\infty (q^{48};q^{48})_\infty = (q^{16},q^{32},q^{48};q^{48})_\infty \tag{1.47}$$

$$(q^{20};q^{20})_\infty = (q^{20};q^{60})_\infty (q^{40};q^{60})_\infty (q^{60};q^{60})_\infty = (q^{20},q^{40},q^{60};q^{60})_\infty \tag{1.48}$$

The outline of this paper is as follows. In sections 2, we record a set of known results which are found to be useful in the paper. In section 3, we state and prove our main results, associated with the families given in (1.15)-(1.17) and (1.22), which depict the inter-relationships between Infinite Product Identities, Continued-Fraction Identities and Combinatorial Partition Identities.

Notes

II. Preliminaries

In [3], following identities are given

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5,q^8;q^8)_\infty}{(q,q^4,q^7;q^8)_\infty} \tag{2.1}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\left(\begin{matrix} q^3,q^5,q^8 \\ q,q^4,q^7 \end{matrix} ; q^8 \right) \tag{2.2}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q,q^7,q^8;q^8)_\infty}{(q^3,q^4,q^5;q^8)_\infty} \tag{2.3}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\left(\begin{matrix} q,q^7,q^8 \\ q^3,q^4,q^5 \end{matrix} ; q^8 \right) \tag{2.4}$$

In [4; Theorem 3], following identities are given

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{5n^2} = 2\left(\begin{matrix} q^2,q^8,q^{10},q^{12},q^{18},q^{20} \\ q,q^4,q^9,q^{11},q^{16},q^{19} \end{matrix} ; q^{20} \right) \tag{2.5}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q\left(\begin{matrix} q^4,q^6,q^{10},q^{14},q^{16},q^{20} \\ q^3,q^7,q^8,q^{12},q^{13},q^{17} \end{matrix} ; q^{20} \right) \tag{2.6}$$

III. Main Results

We have the following identities

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5,q^8;q^8)_\infty}{(q,q^4,q^7;q^8)_\infty} = 2\left(\begin{matrix} q^3,q^5,q^8 \\ q,q^4,q^7 \end{matrix} ; q^8 \right)$$

$$= 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty} \cdot \left(\frac{1}{1-} \; \frac{q^4}{1+} \; \frac{q^4(1-q^4)}{1-} \; \frac{q^{12}}{1+} \; \frac{q^8(1-q^8)}{1-} \; \frac{q^{20}}{1+} \; \frac{q^{12}(1-q^{12})}{1-\cdots} \right) ; (|q|<1)$$

$$= 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty} R(4,4,1,1,1,2) \tag{3.1}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q,q^7,q^8;q^8)_\infty}{(q^3,q^4,q^5;q^8)_\infty} = 2q\left(\begin{matrix} q,q^7,q^8 \\ q^3,q^4,q^5 \end{matrix}; q^8\right)$$

$$= 2q\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}.\left(\frac{1}{1-}\ \frac{q^4}{1+}\ \frac{q^4(1-q^4)}{1-}\ \frac{q^{12}}{1+}\ \frac{q^8(1-q^8)}{1-}\ \frac{q^{20}}{1+}\ \frac{q^{12}(1-q^{12})}{1-\cdots}\right);(|q|<1)$$

$$= 2q\frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty}R(4,4,1,1,1,2) \tag{3.2}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{5n^2} = 2\left(\begin{matrix} q^2,q^{10},q^{18},q^{20} \\ q,q^9,q^{11},q^{19} \end{matrix}; q^{20}\right)\times$$

$$\times\left(1+\frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+\cdots}\right);(|q|<1) \tag{3.3}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q\left(\begin{matrix} q^6,q^{10},q^{14},q^{20} \\ q^3,q^7,q^{13},q^{17} \end{matrix}; q^{20}\right)\times$$

$$\times\left(\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+\cdots}\right);(|q|<1) \tag{3.4}$$

Proof of (3.1): From (2.1) and (2.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5,q^8;q^8)_\infty}{(q,q^4,q^7;q^8)_\infty} = 2\left(\begin{matrix} q^3,q^5,q^8 \\ q,q^4,q^7 \end{matrix}; q^8\right) \tag{3.1.1}$$

Which can be represented as

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}\times\frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \tag{3.1.2}$$

Applying (1.15) for $q = q^4$, in (3.1.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2}$$

$$= 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}.\left(\frac{1}{1-}\ \frac{q^4}{1+}\ \frac{q^4(1-q^4)}{1-}\ \frac{q^{12}}{1+}\ \frac{q^8(1-q^8)}{1-}\ \frac{q^{20}}{1+}\ \frac{q^{12}(1-q^{12})}{1-\cdots}\right);(|q|<1) \tag{3.1.3}$$

Applying (1.18) for $m = 4$, in (3.1.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{2n^2} = 2\frac{(q^3,q^5;q^8)_\infty}{(q,q^7;q^8)_\infty}R(4,4,1,1,1, \tag{3.1.4}$$

Joining (3.1.1), (3.1.3) and (3.1.4) togather, we get the required result (3.1).
Proof of (3.2): From (2.3) and (2.4), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q, q^7, q^8; q^8)_\infty}{(q^3, q^4, q^5; q^8)_\infty} = 2q \left(\begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5 \end{matrix} ; q^8 \right) \tag{3.2.1}$$

Which can be represented as

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty} \times \frac{(q^8; q^8)_\infty}{(q^4; q^8)_\infty} \tag{3.2.2}$$

Applying (1.15) for $q = q^4$, in (3.2.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} = 2q\frac{(q, q^7, q^8; q^8)_\infty}{(q^3, q^4, q^5; q^8)_\infty} = 2q \left(\begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5 \end{matrix} ; q^8 \right)$$

$$= 2q\frac{(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty} \cdot \left(\frac{1}{1-} \; \frac{q^4}{1+} \; \frac{q^4(1-q^4)}{1-} \; \frac{q^{12}}{1+} \; \frac{q^8(1-q^8)}{1-} \; \frac{q^{20}}{1+} \; \frac{q^{12}(1-q^{12})}{1-\cdots} \right) \cdots \; (|q| < 1) \tag{3.2.3}$$

Applying (1.18) for $m = 4$, in (3.2.2), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{2n^2} == 2q\frac{(q, q^7; q^8)_\infty}{(q^3, q^5; q^8)_\infty} R(4, 4, 1, 1, 1, 2) \tag{3.2.4}$$

Joining (3.2.1), (3.2.3) and (3.2.4) togather, we get required result (3.2). Proof of (3.3): From (2.5), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} q^{5n^2} = 2 \left(\begin{matrix} q^2, q^{10}, q^{18}, q^{20} \\ q, q^9, q^{11}, q^{19} \end{matrix} ; q^{20} \right) \times \frac{(q^8; q^{20})_\infty (q^{12}; q^{20})_\infty}{(q^4; q^{20})_\infty (q^{16}; q^{20})_\infty} \tag{3.3.1}$$

Applying (1.17) for $q = q^4$, in (3.3.1), we get required result (3.3). Proof of (3.4): From (2.6), we get

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q \left(\begin{matrix} q^6, q^{10}, q^{14}, q^{20} \\ q^3, q^7, q^{13}, q^{17} \end{matrix} ; q^{20} \right) \times \frac{(q^4; q^{20})_\infty (q^{16}; q^{20})_\infty}{(q^8; q^{20})_\infty (q^{12}; q^{20})_\infty} \tag{3.4.1}$$

Applying (1.16) for $q = q^4$, in (3.4.1), we get required result (3.4).

IV. Acknowledgement

First author [M. P. Chaudhary] is thankful to the Department of Mathematics, University of Hildesheim (Stiftung Universitat Hildesheim), GERMANY for providing hospitality during July -August, 2014.

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Notes

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 4 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A Summation Formula of Half Argument in the Light of Hypergeometric Function and Involving Contiguous Relation

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Abstract- In this paper we have developed a summation formula involving contiguous relation. We have developed this formula using the contiguous relation and involving the formula derived by Salahuddin et. al.[7, p.78-84(8)].

Keywords: *gauss second summation theorem, recurrence relation, prudnikov .*

GJSFR-F Classification : *FOR Code : 05A30, 11F27; 05A17, 11P83.*

Strictly as per the compliance and regulations of :

Notes

A Summation Formula of Half Argument in the Light of Hypergeometric Function and Involving Contiguous Relation

Salahuddin [α], Upendra Kumar Pandit [σ] & M. P. Chaudhary [ρ]

Abstract- In this paper we have developed a summation formula involving contiguous relation. We have developed this formula using the contiguous relation and involving the formula derived by Salahuddin et. al.[7, p.78-84(8)].

Keywords: gauss second summation theorem, recurrence relation, prudnikov .

I. Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A & ; \\ & \\ b_1, b_2, \cdots, b_B & ; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are nonnegative integers and $|z| = 1$

Contiguous Relation is defined by

[1,p.558(15.2.14)]

$$ (a-b)\ {}_2F_1\left[\begin{array}{ll} a, & b\ ; \\ c & ; \end{array} z\right] = a\ {}_2F_1\left[\begin{array}{ll} a+1, & b\ ; \\ c & ; \end{array} z\right] - b\ {}_2F_1\left[\begin{array}{ll} a, & b+1\ ; \\ c\ ; & \end{array} z\right] \tag{2} $$

Gauss second summation theorem is defined by [Prudnikov., 491(7.3.7.5)]

$$ {}_2F_1\left[\begin{array}{ll} a, & b\ ; \\ \frac{a+b+1}{2} & ; \end{array} \frac{1}{2}\right] = \frac{\Gamma(\frac{a+b+1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \tag{3} $$

$$ = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(b)\ \Gamma(\frac{a+1}{2})} \tag{4} $$

In a monograph of Prudnikov et al., a summation theorem is given in the form [Prudnikov., p.491(7.3.7.8)]

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$$ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b-1}{2} \end{matrix};\ \frac{1}{2}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\,\Gamma(\frac{b+1}{2})} + \frac{2\,\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\,\Gamma(b)}\right] \tag{5} $$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above theorem can be written in the form

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b-1}{2} \end{matrix};\ \frac{1}{2}\right] = \frac{2^{(b-1)}\,\Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\,\Gamma(\frac{a}{2})\,\Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right] \tag{6} $$

Recurrence relation is defined by

$$ \Gamma(\zeta+1) = \zeta\,\Gamma(\zeta) \tag{7} $$

II. Main Summation Formula

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b+53}{2} \end{matrix};\ \frac{1}{2}\right] = \frac{2^b\,\Gamma(\frac{a+b+53}{2})}{(a-b)\,\Gamma(b)\Big[\prod_{\varrho=1}^{26}\{a-b-(2\varrho-1)\}\Big]\Big[\prod_{\varsigma=1}^{26}\{a-b+(2\varsigma-1)\}\Big]} $$

$$ \left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a+1}{2})}\Big\{999998548762017902008617761038663680000000a\right. $$

$$ -26108303036483264021084841366647734272000 0a^2 $$

$$ +27961800964681455688642771850983636992000 0a^3 $$

$$ -17187288099765515806914788984840886681600 0a^4 $$

$$ +69892599463741434031897360198231130112000a^5 $$

$$ -20340875557500786200866985453858605498368a^6 $$

$$ +44516117837598716938511048583082074439 68a^7 $$

$$ -75766617323753642755736059802372014080 0a^8 $$

$$ +10271907576563083688357297219363143680 0a^9 - 11288918038327853438226939343599042560a^{10} $$

$$ +10188910678525880338556550974642585 60a^{11} - 76251472850207440069588780109004800a^{12} $$

$$ +47644982074606594341709920862208 00a^{13} - 24972485311261345491308257476608 0a^{14} $$

$$ +11008905953427255161153210286080a^{15} - 40851776544370126793820078080 0a^{16} $$

$$ +12746421102946663778798796800a^{17} - 33336504521651088389046272 0a^{18} $$

$$ +72683664738896594547507 20a^{19} - 13101667792025896878080 0a^{20} $$

$$ +19290929431857594368 00a^{21} - 22804003140392714240a^{22} + 21104875613978624 0a^{23} $$

$$ -1472113462476800a^{24} + 7273761996800a^{25} - 22682796032a^{26} + 33554432a^{27} $$

$$ -999998548762017902008617761038663680000000b $$

$$ +40406621367876653426867548216425971712 0000a^2b $$

$$ -54982239427333265456776951146162526617 6000a^3b $$

$$ +38107115929381279353275237898951422115 8400a^4b $$

$$ -15793903738857755039197040490885214843 6992a^5b $$

Notes

$$+48872928892597015728913280637383234027520a^6b$$
$$-10445491030661761020123603312798943150080a^7b$$
$$+18775689318067468347167773810452804403 20a^8b$$
$$-24314159185286916142477707981975715840 0a^9b$$
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$$-23996448211514572312720762741771468 80a^{11}b+19204913695808536025563575596613632 0a^{12}b$$
$$-11059270429497680692117080878612480a^{13}b+627006473640717784800458356817920a^{14}b$$
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$$-15182660351552902167265280a^{19}b+313927312305717740830720a^{20}b$$
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$$+3219668043038720a^{24}b-9617505517568a^{25}b+44459622400a^{26}b$$
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$$-38107115929381279353275237898951422115840 0ab^4$$
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Notes

$$-898940203136759663781763447584194560 0a^9b^5$$
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Notes

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$$+27921301580965478400a^2b^{22} - 17336553823514132480a^3b^{22} + 15464828653430374 40a^4b^{22}$$
$$-505580936746762240a^5b^{22} - 211048756139786240b^{23} + 363397445981306880ab^{23}$$
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$$-\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\Big\{ - 3103140809483669577922589142054390988800 00a$$
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Notes

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$$-2855638326498128880153314669363 20a^6b^{12} + 1954718015100927273956354188902 4a^7b^{12}$$

$$-2686379768320812013946604093440a^8b^{12} + 867388197292412281729811742 72a^9b^{12}$$
$$-7146708382512041317841764352a^{10}b^{12} + 76288801336508014026293248a^{11}b^{12}$$
$$-5857033140402566830489 6a^{13}b^{12} + 836719020057509547212 8a^{14}b^{12}$$
$$+257665539286557852805193306996736 0b^{13} - 519057560199763190736018466668544 0ab^{13}$$
$$+310214012793641333681297703095500 8a^2b^{13} - 174920893588553435891396047903129 6a^3b^{13}$$
$$+357950824573904251331631137161216a^4b^{13} - 104220584730912563853529437962240a^5b^{13}$$
$$+961322459612055584307782510182 4a^6b^{13} - 172274269737143275555352530124 8a^7b^{13}$$
$$+756832423737381933181021716 48a^8b^{13} - 886395838006407815652966400 0a^9b^{13}$$
$$+168640988401952358114787328a^{10}b^{13} - 12159956545687393587429376a^{11}b^{13}$$
$$+585703314040256683048 96a^{12}b^{13} - 142609340940634067820065534771200b^{14}$$
$$+203054958013020479395466104012800ab^{14} - 187795264949971795332713334964224a^2b^{14}$$
$$+592055381148534113800753439047 68a^3b^{14} - 235044888585698395837123610542 08a^4b^{14}$$
$$+297531100275534975622893797376 0a^5b^{14} - 685795695238085936055972790272a^6b^{14}$$
$$+396743705522516058116550819 84a^7b^{14} - 595386094261269211520591462 4a^8b^{14}$$
$$+152042642318901839690465280a^9b^{14} - 154139838056271791039447 04a^{10}b^{14}$$
$$+131175949596112787079168a^{11}b^{14} - 836719020057509547212 8a^{12}b^{14}$$
$$+495869154325763864489292201984 0b^{15} - 100640213192089904876877774848 00ab^{15}$$
$$+558298727454469533626913521664 0a^2b^{15} - 320547799462517163841397194752 0a^3b^{15}$$
$$+582141530495391051207388692480a^4b^{15} - 174900292183819738556368158720a^5b^{15}$$
$$+134557645089812244725130854 40a^6b^{15} - 253814880450316989886365696 0a^7b^{15}$$
$$+838836714316643767104307 20a^8b^{15} - 107477907673193645709721 60a^9b^{15}$$
$$+121459212588993321369600a^{10}b^{15} - 107963744523549618995 20a^{11}b^{15}$$
$$-199418501102557740088919326720b^{16} + 266335249474739060996843765760ab^{16}$$
$$-248679648338502677328769843200a^2b^{16} + 699409336688355631966873190 40a^3b^{16}$$
$$-283266091261535794120517222 40a^4b^{16} + 298780194608682145048166400 0a^5b^{16}$$
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$$-483965721945818523500544 0a^8b^{16} + 698390472386711597875 20a^9b^{16}$$
$$-775989413763012886528 0a^{10}b^{16} + 475627823532679498926391296 0b^{17}$$
$$-967584784330748288292618240 0ab^{17} + 483755945471354405564448768 0a^2b^{17}$$
$$-280909646330381508450189312 0a^3b^{17} + 427374230669356228222648320a^4b^{17}$$
$$-131219075021515078710067200a^5b^{17} + 749086166756227534553088 0a^6b^{17}$$
$$-146322924947158408888320 0a^7b^{17} + 269807447996774114918 40a^8b^{17}$$
$$-365171488829653123072 0a^9b^{17} - 13866023976351561108422656 0b^{18}$$
$$+169571332713142044862709760ab^{18} - 158942424126743343309783040a^2b^{18}$$
$$+378371090819780636678553 60a^3b^{18} - 155231868783261013088665 60a^4b^{18}$$

Notes

$$+122188641246308946935808 0a^5b^{18} - 29801685755366113017856 0a^6b^{18}$$
$$+7140203435432198799360a^7b^{18} - 1172003089149224550400a^8b^{18}$$
$$+221199090733470411915264 0b^{19} - 44859656403912040539750 40ab^{19}$$
$$+19285483631491411869696 00a^2b^{19} - 11256797306580139979571 20a^3b^{19}$$
$$+12918585794180897832960 0a^4b^{19} - 40236592507791228272640a^5b^{19}$$
$$+12890529483736272076 80a^6b^{19} - 25812537671950532608 0a^7b^{19} - 45919802420941155205120b^{20}$$
$$+49436825533319668039680ab^{20} - 46230448188340222033920a^2b^{20}$$
$$+84378760808085690777 60a^3b^{20} - 34835944913710664908 80a^4b^{20} + 15565487301470453760 0a^5b^{20}$$
$$-38561412985535856640a^6b^{20} + 46352970256383737856 0b^{21} - 93062004024528601088 0ab^{21}$$
$$+31418081214858264576 0a^2b^{21} - 18306712386993651712 0a^3b^{21} + 12133942481922293760a^4b^{21}$$
$$-3806727053152092160a^5b^{21} - 65760836142392934 40b^{22} + 57675181847897702 40ab^{22}$$
$$-5336938763727667200a^2b^{22} + 57708362241933312 0a^3b^{22} - 23792044082200576 0a^4b^{22}$$
$$+3737489667784704 0b^{23} - 7348514023538688 0ab^{23} + 1528073665118208 0a^2b^{23}$$
$$-8803716589158400a^3b^{23} - 332006340689920b^{24} + 191425350205440ab^{24}$$
$$-172930147287040a^2b^{24} + 785173708800b^{25} - 1479616233472ab^{25} - 3489660928b^{26}\Big\}\Big] \qquad (8)$$

III. Derivation of the Summation Formula

Putting $c = \frac{a+b+53}{2}$ and $z = 12$ in equation (2), we get

$$(a-b)\ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{a+b+53}{2} \end{matrix};\ \frac{1}{2}\right] = a\ {}_2F_1\left[\begin{matrix} a+1,\ b \\ \frac{a+b+53}{2} \end{matrix};\ \frac{1}{2}\right] - b\ {}_2F_1\left[\begin{matrix} a,\ b+1 \\ \frac{a+b+53}{2} \end{matrix};\ \frac{1}{2}\right]$$

Now involving the formula derived by Salahuddin et. al.[7, p.78-84(8)], the summation formula is obtained.

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Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 15 Issue 4 Version 1.0 Year 2015
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Bayesian Regression Method with Gaussian and Binomial Links for the Analysis of Nigerian Children Nutritional Status (Stunting)

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Abstract- Children's nutritional status is a reflection of their overall health. Malnutrition is associated with more than half of all children deaths worldwide. A study into geographical variability of nutritional status of children in Nigeria was observed from geo statistical mapping and a continuous covariates stunting (height for age) that exhibit pronounced non-linear relationships with the response variable was analysed. To properly account for stunting effects on child's age, sex, their place of resident, mothers' educational levels, parents' wealth index, regions and state of the child, kriging and additive models were merged using modified Cox model. The resulting Generalized Additive Mixed Model (GAMM) representation for the geo additive model allows for fitting and analysis using BayesX software. The Multiple Indicator Cluster Survey 3 (MICS3) data set contains several variables. Only those that are believed to be related to nutritional status were selected. All categorical covariates are effect coded. The child's age is assumed to be nonlinear; the state is spatial effect while other variables are parametric in nature.

Keywords: binomial, bayesian, gaussian, stunting, and geostatistical.

GJSFR-F Classification : FOR Code : MSC 2010: 60G15, 05A10

1. Food and Agriculture Organization (FAO)of the United Nations, Under Nourishment around the world. In the State of Food Insecurity in the World. Rome; 2004.

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Lasisi, K. E. [α], Nwaosu, S. C. [σ] & Abdulhamid, B. M. [ρ]

Abstract- Children's nutritional status is a reflection of their overall health. Malnutrition is associated with more than half of all children deaths worldwide. A study into geographical variability of nutritional status of children in Nigeria was observed from geo statistical mapping and a continuous covariates stunting (height for age) that exhibit pronounced non-linear relationships with the response variable was analysed. To properly account for stunting effects on child's age, sex, their place of resident, mothers' educational levels, parents' wealth index, regions and state of the child, kriging and additive models were merged using modified Cox model. The resulting Generalized Additive Mixed Model (GAMM) representation for the geo additive model allows for fitting and analysis using BayesX software. The Multiple Indicator Cluster Survey 3 (MICS3) data set contains several variables. Only those that are believed to be related to nutritional status were selected. All categorical covariates are effect coded. The child's age is assumed to be nonlinear; the state is spatial effect while other variables are parametric in nature.

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I. Introduction

Child mortality reflects a country's socio-economic development and quality of life. In developing countries, mortality rates are not only influenced by socio-economic, demographic and health variables but they also vary considerably across regions. Worldwide, an estimated 852million people are undernourished with about 815millions living in developing countries [1]. The Millennium Development Goal is to reduce by half the proportion of people who suffer from hunger between 1990 and 2015. The World Fit for Children goal is to reduce the prevalence of malnutrition among children under five years of age by at least one-third (between 2000 and 2010), with special attention to children under 2 years of age. A reduction in the prevalence of malnutrition will assist in the goal to reduce child mortality. [2].

Geo statistics is concerned with the problem of producing a map of quantity of interest over a particular geographical region based on, usually noisy, measurements taken at set of locations in the region. Classical parametric regression models for analyzing child mortality or survival have severe problems with estimating small area effects and simultaneously adjusting for the covariates, in

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particular when some of the covariates are nonlinear and time-varying. Usually a very high number of parameters will be needed for modeling purposes, resulting in rather unstable estimates with high variance. Therefore, flexible semi-parametric approaches are needed which allow one to incorporate small area spatial effects, nonlinear and time-varying effects of covariates and usual linear effects in a joint model. [3]

The Cox proportional hazards model is a commonly used method when analyzing the impact of covariates on continuous survival times. In its classical form, the Cox model was introduced in the setting of right-censored observations. However, in practice other sampling schemes are frequently encountered and therefore extensions allowing for interval and left censoring or left truncation are clearly desired. Furthermore, many applications require a more flexible modeling of covariate information than the usual linear predictor. Further extensions should allow for time-varying effects of covariates or covariates that are themselves time-varying. Such models relax the assumption of proportional hazards. [4]

II. Materials and Methods

The analysis was carried out using BayesX software package, which permits Bayesian inference based on MCMC simulation techniques. The statistical significance of apparent associations between potential risk factors and the stunting components was used to evaluate the significance of the posterior mean determined for the fixed effects or the categorical data, while non-linear effects and spatial effects were analysed using the estimation of spatial effects based on Markov random fields, stationary Gaussian random fields, and two-dimensional extensions of penalized splines properties of the programme and viewing the map through GS view 4.9 software. We also run a sensitivity analysis for the choice of priors. Standard choices for the hyper-parameters are a = b = 0:001, with 25000 iteration and burn-in period of 5000, there are 17093 observations.

a) Gaussian Processes

The empirical work of [5] has demonstrated that Gaussian process models have better predictive performance than several other nonparametric regression methods over a range of tasks with varying characteristics. The conceptual simplicity, flexibility, and good performance of Gaussian process models make them very attractive for a wide range of problems. Hence, the process was modified to fit into the Generalized Additive Mixed Model (GAMM) of Bayesian method. Furthermore, the response variables of interest are defined for Gaussian process as:

$$y \sim N(\mu, \Sigma), \text{ and } y \sim f(\gamma),$$

$$\text{where } \gamma = \beta_o + \beta_i X_i + \ldots + \beta_k X_k + f(Z)$$

Where y is the regression response for stunting with respect to Gaussian regressions. And γ is the geoadditive predictor which can be specified for a particular child *i*. The β_0, $\beta_i X_i$ and f(Z)represent the estimates of the unknown nonlinear smoothing effects of the metrical covariates child's age (cage), a vector of the fixed effect parameters and the spatial effect respectively. To enhance identifiability, functions are centred about zero, thus the fixed effect parameters automatically include an intercept term γ_0. Stunting = HAZ (Normal regression), where: HAZ - Height for Age Z-score.

Ref

3. Adebayo, SBA and Fahrmeir, L. Analyzing Child Mortality in Nigeria with Geo additive Survival Models. Hieronymus, Munich; 2002.

b) Binomial regression

[6] used it to investigate the joint contribution of individual and aggregate (population-based) socioeconomic factors to mortality in Florence. They illustrated how an individual-level analysis that ignored the multilevel structure could produce biased results. Hence, the need to consider the multilevel analysis as against the individual level analysis used with the Gaussian process, therefore the Binary regression was modified to fit into the Generalized Additive Mixed Model (GAMM) of Bayesian method. Furthermore, the response variables of interest are defined for Binomial process as:

$$y \sim B(n, p), \text{ and } y \sim f(\eta),$$

$$\text{where } \eta = f(\text{cage}_i) + f_{spat}(s_i) + v_i'\gamma$$

Where y is the regression response for stunting with respect to Binomial regressions. And where ηis the geoadditive predictor which can be specified for a particular child *i.* The $f(\text{cage}_i)$, $f_{spat}(s_i)$ and γ represent the estimates of the unknown nonlinear smoothing effects of the metrical covariates cage(child's age), the spatial effect and a vector of the fixed effect parameters. To enhance identifiability, functions are centred about zero, thus the fixed effect parameters automatically include an intercept term γ_0.

$$\text{Stunting} \begin{cases} 1 \text{ if } HAZ_i\text{-}2 \\ \\ 0 \text{ otherwise} \end{cases}$$

III. The Models

The Cox proportional hazards model assumes the multiplicative structuresuch that the influences of covariates on survival times are commonly described by a regression model for the hazard rate. [7]

$$\lambda(t, v) = \lambda_0(t) \exp(v'\gamma), \tag{1}$$

where $\lambda_0(t)$ is an unspecified smooth baseline hazard rate and $v'\gamma$ is a linear predictor form of covariates v and regression coefficientsγ. On the line of additive regression models, the Cox model can be extended to

$$\lambda_i(t) = \exp(\eta_i(t)), \qquad i = 1, \ldots, n, \tag{2}$$

where i is an observation index and $\eta_i(t)$ is a geoadditive predictor of the form

$$\eta_i(t) = v_i'\gamma + g_0(t) + \sum_{l=1}^{L} g_l(t)u_{il} + \sum_{j=1}^{J} f_j(x_{ij}) + f_{spat}(S_i) \tag{3}$$

Here $g_0(t) = \log(\lambda_0(t))$ is the log-baseline hazard, $g_l(t)$ represents time-varying effects of covariates u_{il}, $f_j(x_{ij})$ are nonlinear effects of continuous covariates, $f_{spat}(s_i)$ is a spatial effect, and $v_i'\gamma$ corresponds to covariate effects that are modeled in the usual parametric way. Nonparametric effects f_j as well as time-varying effects $g_0(t)$ and $g_l(t)$ are estimated based on penalized splines.

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IV. Results and Discussion

Nigerian children nutritional data was analyzed with the aim of assessing the influence of some covariates on the response variable (malnutrition). Since the Multiple Indicator Cluster Survey3 (MICS3) data set contains several variables, only those that are believed to be related to nutritional status were selected. All categorical covariates are effect coded. The child's age is assumed to be nonlinear; the state is special effect while other variables are parametric in nature.

a) Stunting Gaussian Regression

>f.regress stunting =state_rec(spatial, map=m, lambda=0.1) + CAGE(psplinerw2) + urban + WIndex2 + WIndex3 + WIndex4 + WIndex5 + primary + secondary + non_stdcur + UF11 + male + NEast + NWest + SEast + SSouth + SWest, iterations=25000 burnin=5000 step=20 family=gaussian predict using d

b) Stunting Binomial Regression

>f. regress stuntbin = state_rec(spatial, map=m, lambda=0.1) + CAGE(psplinerw2) + urban + WIndex2 + WIndex3 + WIndex4 + WIndex5 + primary + secondary + non_stdcur + UF11 + male + NEast + NWest + SEast + SSouth + SWest, iterations=25000 burnin=5000 step=20 family=binomial predict using d

Table 1 : Stunting: Gaussian and Binomial Regression Analysis

Variable	**Gaussian Stunting** Odds ratio	95% Confidence Interval		**Binomial Stunting** Odds ratio	95% Confidence Interval	
		lower limit Odds	upper limit Odds		lower limit Odds	upper limit Odds
Urban	0.7313	0.6507	0.8231	1.1649	0.2507	0.9289
Wealth Index2	1.0145	0.8902	1.1557	0.9582	1.0277	1.3277
Wealth Index3	1.0090	0.8845	1.1684	0.9251	0.8488	1.0843
Wealth Index4	1.0891	0.9241	1.2761	0.8877	0.8060	1.0542
Wealth Index5	1.4260	1.1748	1.7209	0.7994	0.7615	1.0475
Primary	0.9967	0.8837	1.1181	1.1058	0.6563	0.9801
Secondary	1.1416	0.9977	1.3041	0.9276	0.9824	1.2436
Non-std. curriculum	1.0194	0.7421	1.4111	0.8942	0.8046	1.0578
Male	0.8650	0.8018	0.9451	1.1086	0.6721	1.1862
Northeast	0.6835	0.5100	0.9094	1.2018	1.0211	1.2049
Northwest	0.2414	0.1441	0.3818	1.5622	0.8518	1.6277
Southeast	1.2425	0.7142	2.2780	0.6944	0.8092	2.9716
South south	1.0883	0.6385	1.9342	0.5603	0.3141	1.7379
Southwest	0.9666	0.5416	1.7729	0.7288	0.2386	1.2764

The above table shows that at 95% Confidence Interval, the prevalence of stunting (Gaussian) was higher among children living in the rural area with 27%more, while severe stunting (Binomial) was about 16.5% higher in children living in urban area. When comparing the two situations, we discovered that stunting which is a reflection of chronic malnutrition as a result of failure to receive adequate nutrition over a long period and recurrent or chronic illness is prevalence in children living in rural area than their counterpart in the urban region. As observed by [8], with reference to the province of residence, the lowest prevalence of stunting is observed in Kinshasa, the capital-city, whereas the highest is observed in provinces under war during the survey (Equateur, Orientale, Nord-Kivu, Sud-Kivu and Maniema). The risk for a child living in these provinces to experience stunting is double of a child living in Kinshasa.

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Stunting in relation to the parent wealth index, the wealth index of the parents are grouped as Poorest, which is the reference (Wealth Index1), second (Wealth Index2), middle (Wealth Index3), fourth (Wealth Index4) and Richest (Wealth Index5). Wealth of the parents has negative relationship with the children stunting (Gaussian) in the sense that the richest parents have more stunting children of about 42% higher than the poorest parent children, the fourth have 8%, the middle and the second rich parents have 1% more stunting children than the poorest. While the richer the parents the less severely stunting the child, as the richest parent has 20%less severely stunting children, as well as the fourth with 11%less, the middle 7%less and the second rich parents with 4%less severely stunting children. Hence, severe stunted children is prevalence with poor parents as observed by [8], that stunting is linearly associated with socio-economic status of the household (higher among children from the poorest household, followed by children from poor, middle or rich households but lower among children from richest households: 49.8, 48.0,45.5, 43.9 versus 28.7 percent)

Mother education inversely influence the moderate stunting status of their children, as children from mothers with primary education have almost equal chance with children from mothers with no education. While mother with secondary school education and above have 14% more of moderate stunted children than none educated mothers, this was supported by the findings of [8], that there is no significant association between maternal education and the prevalence of stunting among children under the age of 5 years in the DRC. On the other hand, mother education has positive effect on severely stunted children, as mother with secondary education and above has 7%less of severely stunted children than children from non-educated mothers, with 11% more for children with primary education mothers. Therefore, the more educated the mothers the less severely stunted their children as reported by [9], that severe stunting is linearly associated with maternal education (higher among children from non-educated mother, followed by children from mothers with primary education but lower among children from mothers with secondary or higher education 49.8, 47.0 versus 35.2 percent).

Male children are 13%less stunted moderately than their female counterpart, while they (male) are more severely stunted by 11% than female. Previous research in this direction shows that the prevalence of stunting was higher among boyscompared to girls (46.1 versus 41.7 percent). And that stunting has aninverse linear association with the age of the child(higher in the age groups ranging from 4 years, followedby 3 years, 2 years, 1 years but lower in the younger age(0 year): 55.1, 49.4, 48.5, 46.5 versus 23.1 percent).

On the regional effect, the northern regions have less prevalence of moderate stunting children with North East and North West having 32% and 66% less respectively compared with the North Central, while the Southern regions have more prevalence of moderate stunted children with South East and South-South having 24% and 9% more respectively when compared with the North Central, South West is 0.3% less of moderate stunted children. On the other hand, the prevalence of severe stunted is pronounced in the Northern regions where the North East and North West were having 20% and 56% more of severe stunted children, while the Southern regions have less prevalence with South East, South-south and South West have 31%, 44% and 27% less of severe stunted children compared with the North Central.

The nonlinear effect of child's age in the Stunting Gaussian processis displayed in Figure 1a. The graph shows that the nutritional status of the child followed a downward slope from left to right, which implies that as the child grows the nutritional

Ref

9. Emina J.B, Kandala NB, Inung J and E.YYazoume.Maternal Education and Child Nutritional Status in DRC.Journal of Public Health. 2011;(12): 576-592.

status is declining. That is, more children become stunted after two years of age. Hence the age of the child influences his nutritional status. Figures 1b, is map of Nigeria showing the posterior probabilities of significance estimates of the spatial effects. In Figure 1b, the colour white is associated with positively significant states, the colour black with negatively significant states, and the colour grey with non-significant states. The posterior means within 95% credible interval showing that Kano, Niger, Kwara, and Oyo states are states with more stunting children, while Gombe, Adamawa, Taraba, Plateau, and FCT Abuja having less stunting children, with the remaining states are not statistically significant in children with stunting. (Figure1)

While the Stunting Binomial regression shows that the age of the child (figure 2a) indicates an upward trend, although somehow irregular, which implies that the severely stunting children get improved as they grow. On the state performance (figure 2b) regarding severe stunting, the 95% confidence interval shows that only Zamfara, Gombe, Taraba and Benue states have more pronounced severely stunted children, while Sokoto, Kebbi and Jigawa has less with the other state having non-significant effect of severe stunting children. (Figure 4)

In comparing the Gaussian and Binomial analysis, one important thing to note is that, Gaussian regression analyses assume a normally distributed data, the properties of a normal distribution holds. This implies that the Gaussian analysis result is for moderate or global nutritional deficiency status, while the Binomial analysis result is for severe cases of nutritional deficiency. Hence, the only condition for comparison is to see which of the determinants is moderate or severe with respect to which of the factors. For this reason, it means that the bases of their comparison would not be to infer that one method of the analysis is more suitable than the other, since the parameters are assessed with different perspectives.

V. Conclusion

The aim of site-specific province analysis is to accelerate policy interventions, optimise inputs (unobserved factors such as distal ones: food security and prices policies, environmental), improve child nutrition by taking into account the environmental impact and reduce the timescale to attain the Millennium Development Goals (MDGs). It is an approach that deals with multiple groups of factors input to improve child nutritional status in order to satisfy the actual needs of parts of the provinces rather than average needs of the whole country. This research work study childhood malnutrition which had been viewed with respect to stunting and further grouped into moderate and severe condition with a view to have a thorough understanding of the specific nutritional status and to determine the effects of various factors such as place of residence, parents wealth index, mothers educational status, sex of the child and geographical location (the geo-political zones), while the child's age and states were considered as spatial effects.

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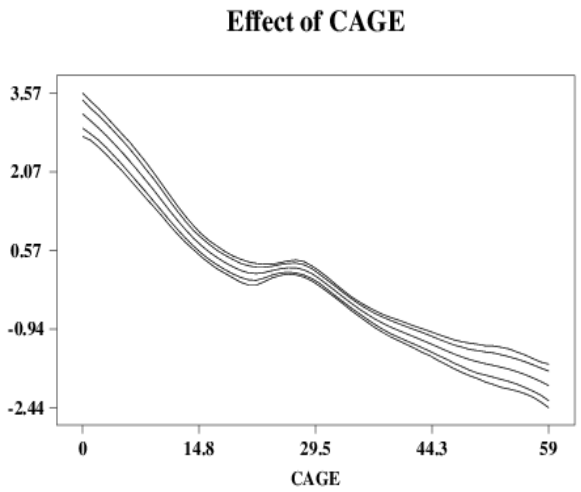

Fig.1a : Effects of Child Age (in Months) on Stunting (Gaussian Analysis)

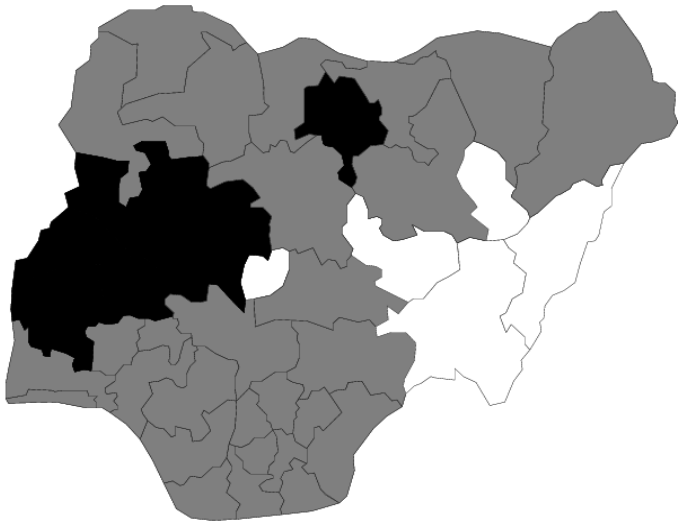

Fig.1b : State Effect on Stunting (Gaussian on Stunting (Gaussian Analysis)

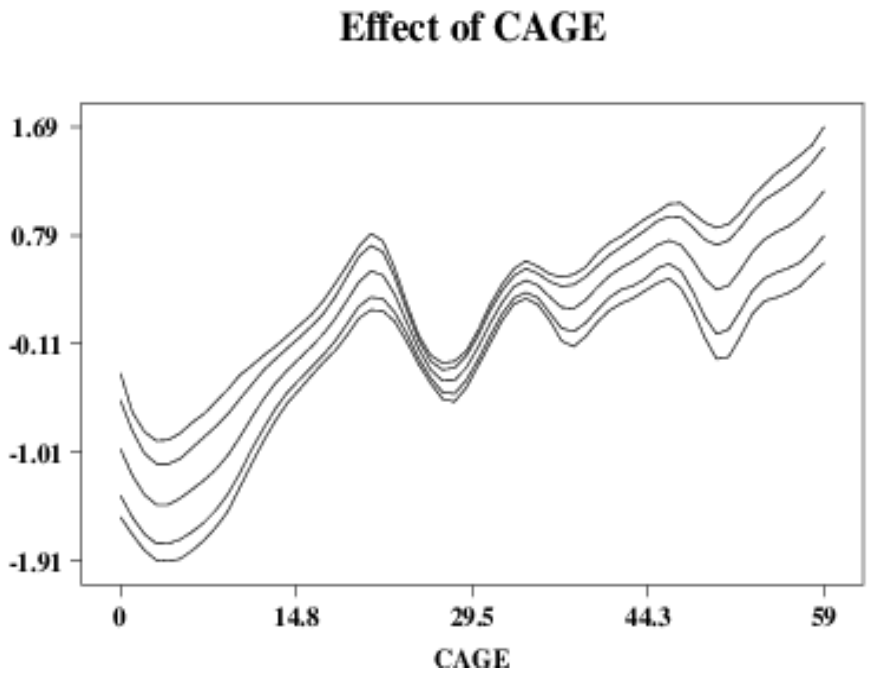

Fig.2a : Effects of Child Age (in months)on Stunting (Binomial Analysis)

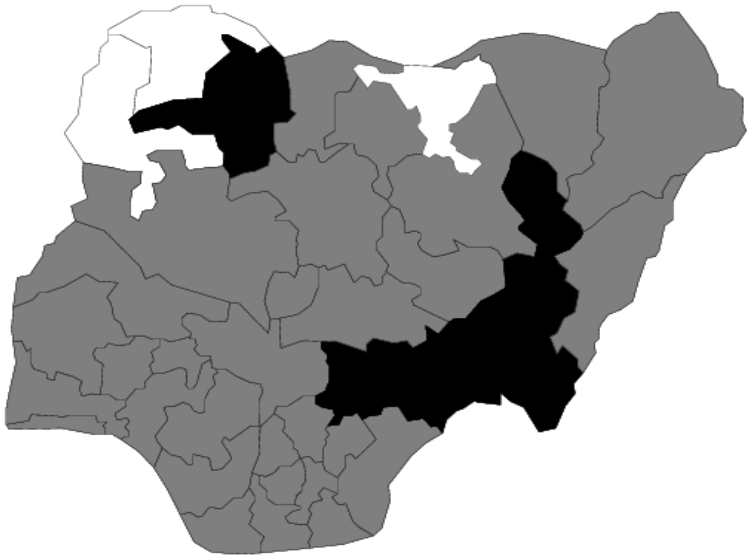

Fig.2b : State Effect on Stunting (Binomial Analysis) at 95% CI

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Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
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References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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ISSN 9755896