Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 16 Issue 1 Version 1.0 Year 2016
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Lyapunov-Type Inequality for Fractional order Difference Equations

By Meihong Cui, Jingxu Xin, Xianmei Huang & Chengmin Houx
Yanbian University, China

Abstract- In this paper, we obtain some Lyapunov-type inequalities for a class of fractional order difference equations with homogeneous boundary value conditions, the results of this paper are new and generalize some early results in the literature.

Keywords: *lyapunov-type inequalities, fractional order difference equations, boundary value problem.*

GJSFR-F Classification : *MSC 2010: 37B25*

Lyapunov-Type Inequality for Fractional order Difference Equations

Meihong Cui [α], Jingxu Xin [σ], Xianmei Huang [ρ] & Chengmin Houx [ω]

Abstract- In this paper, we obtain some Lyapunov-type inequalities for a class of fractional order difference equations with homogeneous boundary value conditions, the results of this paper are new and generalize some early results in the literature.

Keywords: *lyapunov-type inequalities, fractional order difference equations, boundary value problem.*

I. Introduction

Lyapunov inequality plays an important role both in the study of various properties of solutions of differential and difference equations such as oscillation theory, disconjugacy and eigenvalue problems and in the application in many directions of mathematics research areas. There have been many proofs and generalizations as well as improvements in this topic. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey paper [1] by Cheng. For other related references, we refer the reader to the few papers [2-9]. In 1983, Cheng [10] first obtained the following Lyapunov inequality:

$$\mathfrak{F}(b-a)\sum_{n=a}^{b-2} q(n) \geq 4, \tag{1.1}$$

where $a, b \in \mathbf{Z}, q(n)$ is a non-negative function defined on the set $\{a, a+1, \cdots, b\}$ and

$$\mathfrak{F}(m) = \begin{cases} \frac{m^2-1}{m}, & \text{if } m-1 \text{ is even,} \\ m, & \text{if } m-1 \text{ is odd,} \end{cases} \tag{1.2}$$

if the second-order difference equation

$$\Delta^2 x(n) + q(n)x(n+1) = 0, \tag{1.3}$$

has a real solution $x(n)$ such that

$$x(a) = x(b) = 0, x(n) \not\equiv 0, n \in \mathbf{Z}[a, b], \tag{1.4}$$

where and in the sequel, $a, b \in \mathbf{N}, \mathbf{Z}[a, b] = \{a, a+1, a+2, ..., b-1, b\}; c, d \in \mathbf{R}, d = c + N, N \in \mathbf{N}, [c, d]_{\mathbf{N}_c} = \{c, c+1, c+2, ..., d-1, d\}$. The constant 4 in (1.1) cannot be replaced by a larger number, either. For more discrete Lyapunov-type inequalities, we refer the reader to Cheng [11-

Author α σ ρ ω: *Department of Mathematics, Yanbian University, Yanji 133002, P. R. China. e-mail: cmhou@foxmail.com*

Ref

1. S.S. Cheng, *Lyapunov inequalities for differential and difference equations*, Fasc. Math. 23(1991) 25-41.

13], Clark and Hinton [14,15], Guseinov and Kaymakcalan [16], Lin and Yang [17] and Zhang and Tang [18].

In 2012, Zhang and Tang [19] considered the following even order difference equation

$$\Delta^{2k}x(n) + (-1)^{k-1}q(n)x(n+1) = 0, \tag{1.5}$$

under the following boundary conditions:

$$\Delta^{2i}x(a) = \Delta^{2i}x(b) = 0, i = 0, 1, ..., k-1; x(n) \not\equiv 0, n \in \mathbf{Z}[a, b]. \tag{1.6}$$

They obtained the following main result:

Theorem A. Assume that $k \in \mathbf{N}$ *and* $q(n)$ *is a real-valued function on* $\mathbf{Z}$*. If (1.5) has a solution* $x(n) \not\equiv 0$ *satisfying the boundary value conditions (1.6), then*

$$\sum_{n=a}^{b-1}[|q(n)|(n-a+1)(b-n-1)] \geq \frac{2^{3(k-1)}}{(b-a)^{2k-3}}. \tag{1.7}$$

In this paper, we will consider the following fractional order difference equation

$$\Delta^{\nu}_{\nu-2k+a}x(n) + (-1)^{k-1}q(n)x(n+\nu-2k+1) = 0, \tag{1.8}$$

where $k \in \mathbf{N}, n \in \mathbf{Z}, \nu \in \mathbf{R}^+$ satisfy $2k-1 < \nu \leq 2k$ and $q(n)$ is a real-valued function defined on $\mathbf{Z}$. In this work, we establish two discrete Lyapunov-type inequalities for (1.8) under the following boundary conditions:

$$x(a+\nu-2k) = x(b+\nu-2k) = 0, \tag{1.9}$$

$$\Delta^{\nu-2i}_{a+\nu-2k}x(n)|_{n=a} = \Delta^{\nu-2i}_{a+\nu-2k}x(n)|_{n=b} = 0, i = 1, ..., k-1; x(n+\nu-2k) \not\equiv 0, n \in \mathbf{Z}[a, b]. \tag{1.10}$$

When $\nu = 2, k = 1$, (1.8)(1.9) reduce to (1.3)(1.4). When $\nu = 2k$, (1.8)(1.9) (1.10) reduce to (1.5)(1.6).

II. Preliminaries

In this section, we collect some basic definitions and lemmas for manipulating discrete fractional operators. These can be found in the reference [20].

For any integer β, let $\mathbf{N}_\beta = \{\beta, \beta+1, \beta+2, ...\}$. We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Deffinition 2.1. The ν-th fractional sum of f for $\nu > 0$ is defined by

$$\Delta_a^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s),$$

for $t \in \mathbf{N}_{a+\nu}$. Also, we define the trivial sum by $\Delta_a^{-0}f(t) := f(t)$, for $t \in N_a$. We also define the ν-th fractional difference for $\nu > 0$ by $\Delta_a^{\nu}f(t) := \Delta^N\Delta_a^{\nu-N}f(t)$,where $t \in \mathbf{N}_{a+N-\nu}$ and $N \in \mathbf{N}$ is chosen so that $0 \leq N-1 < \nu \leq N$.

Lemma 2.2. Let $f : \mathbf{N}_a \longrightarrow \mathbf{R}$ *be given and suppose* $k \in \mathbf{N}_0$ *and* $\nu > 0$*. Then for* $t \in \mathbf{N}_{a+\nu}$*,*

$$\Delta_a^{-\nu}\Delta^k f(t) = \Delta_a^{k-\nu}f(t) - \sum_{j=0}^{k-1}\frac{\Delta^j f(a)}{\Gamma(\nu-k+j+1)}(t-a)^{\underline{\nu-k+j}}.$$

Global Journal of Science Frontier Research (F) Volume XVI Issue I Version I Year 2016
2

Ref

19. Q. Zhang, X.H. Tang, *Lyapunov-type inequalities for even order difference equations*, Appl. Math. Lett. 25 (2012) 1830-1834.

Moreover, if $\mu > 0$ *with* $M-1 < \mu \leq M$*, then for* $t \in \mathbf{N}_{a+M-\mu+\nu}$*,*

$$\Delta_{a+M-\mu}^{-\nu}\Delta_a^{\mu}f(t) = \Delta_a^{\mu-\nu}f(t) - \sum_{j=0}^{M-1}\frac{\Delta_a^{j-M+\mu}f(a+M-\mu)}{\Gamma(\nu-M+j+1)}(t-a-M+\mu)^{\underline{\nu-M+j}}.$$

Notes

III. The Main Results

Theorem 3.1. *Let* $\nu \in (1,2]$*. Assume that* $x(n+\nu-2)$ *is a real valued function on* $\mathbf{Z}[a,b]$ *satisfy (1.8)(1.9) and* $x(n+\nu-2) \not\equiv 0$ *for* $n \in \mathbf{Z}[a,b]$*. Then,*

$$\widetilde{\mathfrak{F}}(b-a)\sum_{n=a}^{b-2}|q(n)| \geq 1, \tag{3.1}$$

where $a, b \in \mathbf{Z}$ *and*

$$\widetilde{\mathfrak{F}}(m) = \frac{1}{\Gamma(\nu)}\begin{cases} \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}(\frac{m-1}{2}+\nu-2)^{\underline{\nu-1}}}{(m+\nu-2)^{\underline{\nu-1}}}, & \text{if } m-1 \text{ is even,} \\ \frac{((\frac{m}{2}+\nu-2)^{\underline{\nu-1}})^2}{(m+\nu-2)^{\underline{\nu-1}}}, & \text{if } m-1 \text{ is odd.} \end{cases} \tag{3.2}$$

Proof. By Lemma 2.2, we have

$$x(n) = -\Delta_a^{-\nu}q(n)x(n+\nu-1) + c_1(n-a)^{\underline{\nu-1}} + c_2(n-a)^{\underline{\nu-2}}.$$

Since $x(a+\nu-2) = x(b+\nu-2) = 0$, we get

$$x(n) = \sum_{s=a}^{b-2}G(n,s)q(s)x(s+\nu-1) = -\sum_{s=a}^{b-2}G(n,s)\Delta_{a+\nu-2}^{\nu}x(s), \tag{3.3}$$

where

$$G(n,s) = \frac{1}{\Gamma(\nu)}\begin{cases} \frac{(b+\nu-s-3)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}} - (n-s-1)^{\underline{\nu-1}}, & a \leq s < n-\nu+1 \leq b-2, \\ \frac{(b+\nu-s-3)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}}, & a \leq n-\nu+1 \leq s \leq b-2. \end{cases} \tag{3.4}$$

One can see that $\Delta_s G(n,s) < 0$, for $a \leq n-\nu+1 \leq s \leq b-2$ and $\Delta_s G(n,s) > 0$, for $a \leq s < n-\nu+1 \leq b-2$. Indeed, for $a \leq s < n-\nu+1 \leq b-2$, we have $\Delta_s G(n,s) = -(\nu-1)\frac{(b+\nu-s-4)^{\underline{\nu-2}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}} + (\nu-1)(n-s-2)^{\underline{\nu-2}}$.

Thus, $\Delta_s G(n,s) > 0$, if and only if

$$\frac{(n-s-2)^{\underline{\nu-2}}(b+\nu-a-2)^{\underline{\nu-1}}}{(n-a)^{\underline{\nu-1}}(b+\nu-s-4)^{\underline{\nu-2}}} > 1.$$

The inequality follows from the fact that $t^{\underline{\alpha}}$ is increasing and $t^{\underline{-\alpha}}$ is decreasing if $0 < \alpha \leq 1$. Note that $G(n, b-2) > 0$ and

$$\begin{aligned} G(n,a) &= \frac{1}{\Gamma(\nu)}\left(\frac{(b+\nu-a-3)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}} - (n-a-1)^{\underline{\nu-1}}\right) \\ &= \frac{1}{\Gamma(\nu)}\frac{\Gamma(n-a)}{\Gamma(n-a-\nu+1)}\left(\frac{(b-a-1)(n-a)}{(b-a+\nu-2)(n-a+1-\nu)} - 1\right) \\ &= \frac{1}{\Gamma(\nu)}\frac{\Gamma(n-a)}{\Gamma(n-a-\nu+1)}\frac{(\nu-1)(b-n+\nu-2)}{(b-a+\nu-2)(n-a+1-\nu)} > 0, \end{aligned}$$

we can get

$$G(n,s)>0,\ \text{for}\ (n,s)\in[a+\nu-1,b+\nu-3]_{\mathbf{N}_{a+\nu-1}}\times\mathbf{Z}[a,b-2].$$

Thus

$$\max_{s\in\mathbf{Z}[a,b-2]}G(n,s)=G(n,n-\nu+1),n\in[a+\nu-1,b+\nu-3]_{\mathbf{N}_{a+\nu-1}}.$$

$$|x(n)|\leq\frac{(b+2\nu-n-4)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=a}^{b-2}|\Delta^{\nu}_{a+\nu-2}x(s)|,n\in[a+\nu-1,b+\nu-3]_{\mathbf{N}_{a+\nu-1}}$$

and

$$|x(n+\nu-1)|\leq\frac{(b+\nu-n-3)^{\underline{\nu-1}}(n+\nu-a-1)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=a}^{b-2}|\Delta^{\nu}_{a+\nu-2}x(s)|,n\in\mathbf{Z}[a,b-2].\quad(3.5)$$

Next, we denote $g(n)=(b+\nu-n-3)^{\underline{\nu-1}}(n+\nu-a-1)^{\underline{\nu-1}}$, then by computing, we have

$$\Delta g(n)=\frac{\Gamma(b+\nu-n-3)\Gamma(n+\nu-a)}{\Gamma(b-n-1)\Gamma(n-a+2)}(\nu-1)(b-2n+a-3).$$

If $b-a-1$ is even, let $\Delta g(n)=0$, then $n=\frac{b-a-1}{2}+a-1$.

$$\max_{n\in\mathbf{Z}[a,b-2]}g(n)=g(\frac{b-a-1}{2}+a-1)=(\frac{b-a+1+\nu-2}{2})^{\underline{\nu-1}}(\frac{b-a-1+\nu-2}{2})^{\underline{\nu-1}}.\quad(3.6)$$

If $b-a-1$ is odd, we obtain $\Delta g(n)>0$, for $n<\frac{b-a-1}{2}+a-1$; $\Delta g(n)<0$, for $n>\frac{b-a-1}{2}+a-1$. Therefore,

$$\max_{n\in\mathbf{Z}[a,b-2]}g(n)=\max\{g(\frac{b-a}{2}+a-1),g(\frac{b-a}{2}+a-2)\}.$$

Since

$$g(\frac{b-a}{2}+a-1)=g(\frac{b+a}{2}-1)=((\frac{b-a}{2}+\nu-2)^{\underline{\nu-1}})^2,$$

$$g(\frac{b-a}{2}+a-2)=g(\frac{b+a}{2}-2)=(\frac{b-a}{2}+\nu-3)^{\underline{\nu-1}}(\frac{b-a}{2}+\nu-1)^{\underline{\nu-1}},$$

$$\begin{aligned}g(\frac{b-a}{2}+a-1)-g(\frac{b-a}{2}+a-2)&=((x+\nu-2)^{\underline{\nu-1}})^2-(x+\nu-3)^{\underline{\nu-1}}(x+\nu-1)^{\underline{\nu-1}}\\&=\frac{(x+\nu-2)\Gamma^2(x+\nu-2)}{x(x-1)^2\Gamma(x-1)}(\nu-1)>0,\end{aligned}$$

where, $x=\frac{b-a}{2}$, we see that

$$\max_{n\in\mathbf{Z}[a,b-2]}g(n)=g(\frac{b-a}{2}+a-1)=((\frac{b-a}{2}+\nu-2)^{\underline{\nu-1}})^2.\quad(3.7)$$

Note that (3.5) and by (3.6)(3.7), we obtain

$$|x(n+\nu-1)|\leq\widetilde{\mathfrak{F}}(b-a)\sum_{s=a}^{b-2}|\Delta^{\nu}_{a}x(s)|=\widetilde{\mathfrak{F}}(b-a)\sum_{s=a}^{b-2}|q(s)||x(s+\nu-1)|.$$

Notes

Furthermore,

$$\sum_{n=a}^{b-2}|q(n)||x(n+\nu-1)| \le \widetilde{\mathfrak{F}}(b-a)\sum_{s=a}^{b-2}|q(n)|\sum_{s=a}^{b-2}|q(s)||x(s+\nu-1)|,$$

it shows that (3.1) holds.

Notes

Remark 3.2. The inequality (3.1) is sharp. Indeed, since

$$\begin{aligned}
\Delta\widetilde{\mathfrak{F}}(m) &= \frac{1}{\Gamma(\nu)}\left(\frac{((\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}})^2}{(m+\nu-1)^{\underline{\nu-1}}}-\frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}(\frac{m-1}{2}+\nu-2)^{\underline{\nu-1}}}{(m+\nu-2)^{\underline{\nu-1}}}\right)\\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\\
&\quad\times\left((\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-2)^{\underline{\nu-1}}-(\frac{m-1}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}\right)\\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\\
&\quad\times\left(\frac{\Gamma(\frac{m+1}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m+1}{2})\Gamma(m)}-\frac{\Gamma(\frac{m-1}{2}+\nu-1)\Gamma(m+\nu)}{\Gamma(\frac{m-1}{2})\Gamma(m+1)}\right)\\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\frac{\Gamma(\frac{m-1}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m-1}{2})\Gamma(m)}(\frac{2(\nu-1)}{m-1}-\frac{\nu-1}{m})>0
\end{aligned}$$

for $m-1$ is even,

$$\begin{aligned}
\Delta\widetilde{\mathfrak{F}}(m) &= \frac{1}{\Gamma(\nu)}\left(\frac{(\frac{m+2}{2}+\nu-2)^{\underline{\nu-1}}(\frac{m}{2}+\nu-2)^{\underline{\nu-1}}}{(m+\nu-1)^{\underline{\nu-1}}}-\frac{((\frac{m}{2}+\nu-2)^{\underline{\nu-1}})^2}{(m+\nu-2)^{\underline{\nu-1}}}\right)\\
&= \frac{(\frac{m}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\\
&\quad\times\left((\frac{m+2}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-2)^{\underline{\nu-1}}-(\frac{m}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}\right)\\
&= \frac{(\frac{m}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\left(\frac{\Gamma(\frac{m+2}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m+2}{2})\Gamma(m)}-\frac{\Gamma(\frac{m}{2}+\nu-1)\Gamma(m+\nu)}{\Gamma(\frac{m}{2})\Gamma(m+1)}\right)\\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\frac{\Gamma(\frac{m}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m}{2})\Gamma(m)}\frac{\nu-1}{m}>0
\end{aligned}$$

for $m-1$ is odd. We see that $\widetilde{\mathfrak{F}}$ is a strictly increasing function of m. If

$$\widetilde{\mathfrak{F}}(b-a)\sum_{n=a}^{b-2}|q(n)|<1,$$

then (1.8) cannot have a nontrivial solution $x(n+\nu-1)$ defined on the set $\mathbf{Z}[c,d]$ which satisfies $x(c+\nu-2)=x(d+\nu-2)=0$, where $a-1\le c-1<d+1\le b+1, \nu\in(1,2]$. Otherwise,

$$\sum_{n=a}^{b-2}|q(n)|\ge\sum_{n=c}^{d-2}|q(n)|\ge\frac{1}{\widetilde{\mathfrak{F}}(d-c)}\ge\frac{1}{\widetilde{\mathfrak{F}}(b-a)}>\sum_{n=a}^{b-2}|q(n)|$$

which is a contradiction.

Remark 3.3. In the Theorem 3.1, if $\nu = 2$ and $q(n)$ is a non-negative function defined on the set $\mathbf{Z}[a,b]$, then $\widetilde{\mathfrak{F}}(m) = \mathfrak{F}(m)$, i.e., (1.1) holds. Therefore Theorem 3.1 contains the result of reference [10].

Theorem 3.4. Let $\nu \in (1,2]$. Assume that $x(n+\nu-2)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.8)(1.9) and $x(n+\nu-2) \not\equiv 0$ for $\mathbf{Z}[a,b]$. Then,

$$|x(n+\nu-1)| \le \frac{(b+\nu-n-3)^{\underline{\nu-1}}(n+\nu-a-1)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=a}^{b-2} |\Delta^{\nu}_{a+\nu-2} x(s)|, \tag{3.8}$$

$$\sum_{n=a}^{b-2} |x(n+\nu-1)| \le \sum_{n=a}^{b-2} \frac{(b+\nu-n-3)^{\underline{\nu-1}})(n+\nu-a-1)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=a}^{b-2} |\Delta^{\nu}_{a+\nu-2} x(s)|, \tag{3.9}$$

and

$$\sum_{n=a}^{b-2} |x(n+\nu-1)|^2 \le \sum_{n=a}^{b-2}\sum_{s=a}^{b-2} [G(n+\nu-1,s)]^2 \sum_{s=a}^{b-2} |\Delta^{\nu}_{a+\nu-2} x(s)|^2. \tag{3.10}$$

Proof. By (3.5), we can immediately get (3.8) and (3.9). Using Cauchy inequality and by (3.3), we may easily see that (3.10) holds.

Corollary 3.5. Let $\nu = 2$. Assume that $x(n)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.3)(1.4) and $x(n) \not\equiv 0$ for $\mathbf{Z}[a,b]$. Then,

$$|x(n)| \le \frac{(b-n)(n-a)}{(b-a)} \sum_{s=a}^{b-2} |\Delta^2 x(s)|, \tag{3.11}$$

$$\sum_{n=a}^{b-2} |x(n)| \le \sum_{n=a}^{b-2}\sum_{s=a}^{b-2} G(n,s)|\Delta^2 x(s)| \le \frac{(b-a)^2}{8} \sum_{s=a}^{b-2} |\Delta^2 x(s)|. \tag{3.12}$$

Proof.

Since

$$G(n,s) = \frac{1}{b-a}\begin{cases} (s+1-a)(b-n), & a \le s < n-1 \le b-2, \\ (b-s-1)(n-a), & a \le n-1 \le s \le b-2, \end{cases}$$

it follows from above form that

$$|x(n)| \le \sum_{s=a}^{b-2} G(n,s)|\Delta^2 x(s)| \le \frac{(n-a)(b-n)}{b-a} \sum_{s=a}^{b-2} |\Delta^2 x(s)|.$$

Furthermore

$$\begin{aligned} \sum_{n=a}^{b-2} |x(n)| &\le \sum_{n=a}^{b-2}\sum_{s=a}^{b-2} G(n,s)|\Delta^2 x(s)| \\ &\le \sum_{s=a}^{b-2}\sum_{n=a}^{b-1} G(n,s)|\Delta^2 x(s)| \\ &\le \sum_{s=a}^{b-2}\Big(\sum_{n=a}^{s+1} G(n,s) + \sum_{n=s+2}^{b-1} G(n,s)\Big)|\Delta^2 x(s)| \\ &= \frac{1}{b-a}\sum_{s=a}^{b-2}\big((b-s-1)(1+2+\cdots+(s-a+1)\big) \end{aligned}$$

Ref

10. S. S. Cheng, *A discrete analogue of the inequality of Lyapunov*, Hokkaido Math. J. 12 (1983) 105-112.

$$+(s+1-a)(1+2+\cdots+(b-s-2)))|\Delta^2 x(s)|$$

$$= \frac{1}{b-a}\sum_{s=a}^{b-2}((b-s-1)(1+2+\cdots+s-a)$$

$$+(s+1-a)(1+2+\cdots+b-s-1))|\Delta^2 x(s)|$$

$$= \frac{1}{2}\sum_{s=a}^{b-2}(b-s-1)(s+1-a)|\Delta^2 x(s)|$$

$$\leq \frac{(b-a)^2}{8}\sum_{s=a}^{b-2}|\Delta^2 x(s)|.$$

Notes

Theorem 3.6. *Let $\nu \in (2k-1, 2k]$. Assume that $x(n+\nu-2)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.8)-(1.10) and $x(n+\nu-2) \not\equiv 0$ for $n \in \mathbf{Z}[a,b]$. Then,*

$$\sum_{n=a}^{b-2}|q(n)|(b+\nu-2k-n-1)^{\underline{\nu-2k+1}}(n+\nu-2k+1-a)^{\underline{\nu-2k+1}} \geq (\frac{2^{3(k-1)}}{(b-a)^2})^{k-1}(b+\nu-2k-a)^{\underline{\nu-2k+1}}. \quad (3.13)$$

Proof. Assume that $x(n+\nu-2)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.8)-(1.10) and $x(n+\nu-2) \not\equiv 0$ for $n \in \mathbf{Z}[a,b]$. Since conditions (1.10), by Corollary 3.3, we have

$$\begin{aligned}
\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu-2(k-1)}x(s)| &\leq \frac{(b-a)^2}{8}\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu-2(k-1)+2}x(s)| \\
&\leq (\frac{(b-a)^2}{8})^2\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu-2(k-3)}x(s)| \\
&\leq \cdots \\
&\leq (\frac{(b-a)^2}{8})^{k-1}\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu}x(s)|. \qquad (3.14)
\end{aligned}$$

Denote $\beta = \nu-2(k-1)$, then $\beta \in (1,2]$, $\Delta_{\nu-2k+a}^{\nu-2(k-1)}x(n) = \Delta_{\beta-2+a}^{\beta}x(n)$. By Theorem 3.4, we have

$$|x(n+\beta-1)| \leq \frac{(b+\beta-s-3)^{\underline{\beta-1}}(n+\beta-a-1)^{\underline{\beta-1}}}{(b+\beta-a-2)^{\underline{\beta-1}}}\sum_{s=a}^{b-2}|\Delta_{a+\beta-2}^{\beta}x(s)|,$$

combine (3.14), we get

$$\begin{aligned}
|x(n+\nu-2k+1)| &\leq (\frac{(b-a)^2}{8})^{k-1}\frac{(b+\beta-s-3)^{\underline{\beta-1}}(n+\beta-a-1)^{\underline{\beta-1}}}{(b+\beta-a-2)^{\underline{\beta-1}}}\sum_{s=a}^{b-2}|\Delta_{a+\nu-2k}^{\nu}x(s)| \\
&= (\frac{(b-a)^2}{8})^{k-1}\frac{(b+\nu-2k-n-1)^{\underline{\nu-2k+1}}(n+\nu-2k-a+1)^{\underline{\nu-2k+1}}}{(b+\nu-2k-a)^{\underline{\nu-2k+1}}} \\
&\quad \times\sum_{s=a}^{b-2}|q(s)x(s+\nu-2k+1)|.
\end{aligned}$$

$$\sum_{n=a}^{b-2} q(n)|x(n+\nu-2k+1)| \le \sum_{n=a}^{b-2} q(n)(\frac{(b-a)^2}{8})^{k-1} \frac{(b+\nu-2k-n-1)^{\underline{\nu-2k+1}}(n+\nu-2k-a+1)^{\underline{\nu-2k+1}}}{(b+\nu-2k-a)^{\underline{\nu-2k+1}}}$$

$$\times \sum_{s=a}^{b-2} |q(s)x(s+\nu-2k+1)|.$$

it follows that (3.13) holds.

Corollary 3.7. *Let* $\nu = 2k$. *Assume that* $x(n)$ *is a real valued function on* $\mathbf{Z}[a,b]$ *satisfy (1.8)-(1.10) and* $x(n) \not\equiv 0$ for $\mathbf{Z}[a,b]$. *Then,*

$$\sum_{n=a}^{b-2} |q(n)|(b-n-1)(n+1-a) \ge \frac{2^{3(k-1)}}{(b-a)^{2k-3}}.$$

Remark 3.8. The Corollary 3.7 is the Theorem A. Therefore our results contains the main results of reference [19].

References Références Referencias

1. S.S. Cheng, *Lyapunov inequalities for differential and difference equations*, Fasc. Math. 23(1991) 25-41.
2. A.M. Liapunov, *Problme gnral de la stabilit du mouvement*, Ann. Fac. Sci. Univ. Toulouse 2 (1907) 203-407.
3. D. Cakmak, *Lyapunov-type integral inequalities for certain higher order differential equations,* Appl. Math. Comput. 216 (2010) 368-373.
4. X. He, X.H. Tang, *Lyapunov-type inequalities for even order differential equations,* Commun Pure Appl. Anal. 11 (2) (2012) 465-473.
5. L.Q. Jiang, Z. Zhou, *Lyapunov inequality for linear Hamiltonian systems on time scales,* J. Math. Anal. Appl. 310 (2005) 579-593.
6. X. Wang, *Stability criteria for linear periodic Hamiltonian systems,* J. Math. Anal. Appl. 367 (2010) 329-336.
7. X. Yang, *On Liapunov-type inequality for certain higher-order differential equations,* Appl. Math. Comput. 134 (2-3) (2003) 307-317.
8. X. Yang, *On inequalities of Lyapunov type,* Appl. Math. Comput. 134 (2-3) (2003) 293-300.
9. X. Yang, K. Lo, *Lyapunov-type inequality for a class of even-order differential equations,* Appl. Math. Comput. 215 (2010) 3884-3890.
10. S. S. Cheng, *A discrete analogue of the inequality of Lyapunov,* Hokkaido Math. J. 12 (1983) 105-112.
11. S.S. Cheng, L.Y. Hsieh, *On discrete analogue of Lyapunovs inequality: addendum,* Tamkang J. Math. 20 (1989) 333-340.
12. S.S. Cheng, *A sharp condition for the ground state of difference equation,* Appl. Anal. 34 (1989) 105-109.
13. S.S. Cheng, L.Y. Hsieh, D. Chao, *Discrete Lyapunov inequality conditions for partial difference equations,* Hokkaido Math. J. 19 (1990) 229-239.
14. S. Clark, D.B. Hinton, *Discrete Lyapunov inequalities for linear Hamiltonian systems,* Math. Inequal. Appl. 1 (1998) 201-209.
15. S. Clark, D.B. Hinton, *Discrete Lyapunov inequalities,* Dynam. Systems Appl. 8 (1999) 369-380.
16. G.Sh. Guseinov, B. Kaymakcalan, *Lyapunov inequalities for discrete linear Hamiltonian systems,* Comput. Math. Appl. 45 (2003) 1399-1416.

Ref

19. Q. Zhang, X.H. Tang, *Lyapunov-type inequalities for even order difference equations,* Appl. Math. Lett. 25 (2012) 1830-1834.

17. S.H. Lin, G.S. Yang, *On discrete analogue of Lyapunovs inequality,* Tamkang J. Math. 20(1989) 169-186.
18. Q. Zhang, X.H. Tang, *Lyapunov inequalities and stability for discrete linear Hamiltonian system,* Appl. Math. Comput. 218 (2011) 574-582.
19. Q. Zhang, X.H. Tang, *Lyapunov-type inequalities for even order difference equations,* Appl. Math. Lett. 25 (2012) 1830-1834.
20. M.T. Holm, *The Theory of Discrete Fractional Calculus: Development and Application,* DigitalCommons@University of Nebraska-Lincoln, 2011.
21. M. Unal, D. Cakmak, A. Tiryaki, *A discrete analogue of Lyapunov-type inequalities for nonlinear systems,* Comput. Math. Appl. 55 (2008) 2631-2642.

Notes