Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 16 Issue 4 Version 1.0 Year 2016
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Multi-Criteria Optimization Problems with Non-Smooth Functions

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Abstract- In this paper, we deal with non-smooth multi criteria optimization problem. Throughout the text we consider the functions which are non-differentiable. A few special derivatives have been deployed to obtain first order optimality conditions. We presented the generalized form of optimality conditions for multi-objective optimization problem when classical derivatives failed to apply. We illustrate these optimality conditions by means of suitable examples.

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GJSFR-F Classification : *MSC 2010: 46N10*

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I. Introduction

We intend the term "non-smooth" to refer to certain situations in which smoothness (differentiability) of the data is not necessarily postulated. If we want to use the first order necessary condition, we have to require that the function under discussion actually has a derivative. Recent research is concerned with problems that do not have this property. So we need several alternative concepts of generalized directional derivatives that allow to establish an analysis for non-smooth functions. First order non-smooth necessary optimality conditions for single and multi-objective optimization problems, have been provided by many authors. Among those we may refer to Bigi [1], Clarke [2], Stein [3], Yang [4,5] etc. Of these, Preda [6] established the necessary conditions for semi differentiable function, where the Lagrange multipliers associated with each of the objective function are positive.

In section 3, we review and prove first order optimality conditions for non-smooth optimization problems. We have extended to multi objective non-smooth optimization the approach introduced by O. Stein [3] for scalar objective optimization. To deal with multi objective optimization a number of approaches have been proposed to develop a necessary optimality conditions with example. Also have discussed some review result bases on G. Bigi's [1] work.

II. Preliminaries

In this section, we introduce some notations and definitions, which are used throughout the paper. Let E_n be n-dimensional Euclidean space.

For $\mathbf{x}, \mathbf{y} \in E_n$, we use the following conventions.

$$\mathbf{x} \geqq \mathbf{y}, \quad \text{iff } x_i \geqq y_i, \quad i=1,\ldots,n,$$

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$$\mathbf{x} \geq \mathbf{y}, \quad \text{iff}\, \mathbf{x} \geqq \mathbf{y} \quad \text{and} \quad \mathbf{x} \neq \mathbf{y},$$

$$\mathbf{x} > \mathbf{y}, \quad \text{iff}\, x_i > y_i \; \text{i}=1,\ldots,\text{n},$$

At first, we consider the following multi objective optimization problem **P**:
$\min \mathbf{f}(\mathbf{x})$, subject to the conditions that the minimizing point (or vector) $\bar{\mathbf{x}}$ should lie in the set X:

$$\bar{\mathbf{x}} \in \mathrm{X} = \{\mathbf{x} \in \mathrm{E}_n | \mathbf{g}(\mathbf{x}) \leqq 0, \mathbf{h}(\mathbf{x}) = 0\}$$

Let, $f:\mathrm{E}_\mathrm{n} \to \mathrm{E}_1$, $g:\mathrm{E}_\mathrm{n} \to \mathrm{E}_\mathrm{m}$ and $h:\mathrm{E}_\mathrm{n} \to \mathrm{E}_\mathrm{p}$ be vector-valued functions defined by $\mathbf{f}(\mathbf{x}) \equiv (f_1(\mathbf{x}), f_2(\mathbf{x}),\ldots, f_l(\mathbf{x}))$, $\mathbf{g}(\mathbf{x}) \equiv (g_1(\mathbf{x}), g_2(\mathbf{x}),\ldots, g_m(\mathbf{x}))$ and $\mathbf{h}(\mathbf{x}) \equiv (h_1(\mathbf{x}), h_2(\mathbf{x}),\ldots, h_p(\mathbf{x}))$ where $f_i:\mathrm{E}_\mathrm{n} \to \mathrm{E}_1$ for i=1,..,l, $g_j:\mathrm{E}_\mathrm{n} \to \mathrm{E}_1$ for j=1,…,m and $h_k:\mathrm{E}_\mathrm{n} \to \mathrm{E}_1$ for k=1,…,p. Assume that $I(\bar{\mathbf{x}}) = \{j : g_j(\bar{\mathbf{x}}) = 0\}$ for j=1,…,m.

III. First Order Optimality Conditions

Extensions of optimality conditions to non-smooth optimization problems can be found in the literature under different assumptions on the directional differentiability of the functions involved. Preda and Yang [4,6] use directional differentiability in the Gateaux differentiable sense; G. Bigi and Castalenni and Papalardo [1] use directional derivatives in the Dini-Hadamard sense; Kuntz and Scholtes [7,8] use quasi differentiability i.e. Dini directional differentiability where the directional derivative can be written as the difference of two sub linear functions; Clarke [2] assumes Clarke directional differentiability.

In contrast to these approaches, the chapter does not make any directional differentiability assumptions, but we just present a few component-wise optimality criteria based on Dini-Hadamard derivatives.

Definition 3.1 An efficient point $\bar{\mathbf{x}}$ for **P** is said to be of order one, if a bounded open neighbourhood N can be chosen such that there exists a positive constant c with

$$\mathbf{f}(\mathbf{x}) \geqq \mathbf{f}(\bar{\mathbf{x}}) + c\|\mathbf{x_n} - \bar{\mathbf{x}}\|, \text{ for all } \mathbf{x} \in \mathrm{X} \cap \mathrm{N}.$$

Also we define the inner tangent cone.

Definition3.2

$$\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}}) \equiv \left\{ \begin{array}{l} \mathbf{d} \in \mathrm{E}_n | \exists \bar{t} > 0 \text{ such that } \bar{\mathbf{x}} + \mathbf{t}\bar{\mathbf{d}} \in \mathrm{X}, \\ \text{for all } \mathrm{t} \in (0,\bar{\mathrm{t}}), \bar{\mathbf{d}} \in N(\mathbf{d}) \end{array} \right\}$$

The following proposition cites some of the Laurent results [3,9,10] about the basic properties of these tangent cones, which will be important in the sequel.

Proposition 3.1 [See 3, 11, 12]. Let $\bar{\mathbf{x}} \in \mathrm{X} \subset \mathrm{E}_\mathrm{n}$. Then

i) $\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}})$ and $\mathrm{T}(\mathrm{X};\bar{\mathrm{x}})$ are open and closed cones, respectively

ii) $\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}}) \subset \mathrm{T}(\mathrm{X};\bar{\mathrm{x}})$

iii) $\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}})^c = \mathrm{T}(\mathrm{X}^c;\bar{\mathrm{x}})$

Thus, we start recalling the following classical definitions of upper and lower directional derivatives of f at $\bar{\mathbf{x}}$ in the direction $\boldsymbol{d}$ in the Hadamard sense (see [2,3])

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Definitions 3.3 The upper Hadamard derivative of $f:\mathrm{E}_n \to \mathrm{E}_1$ at $\bar{\mathbf{x}}$ in the direction $\boldsymbol{d}$ is

$$F^+ f(\bar{\mathbf{x}},\mathbf{d}) = \lim_{d\to d}\sup_{t\to 0^+} \frac{f(\bar{\mathbf{x}}+t\mathbf{d}) - f(\bar{\mathbf{x}})}{t}$$

Definitions 3.4 The lower Hadamard derivative of $f:\mathrm{E}_n \to \mathrm{E}_1$ at $\bar{\mathbf{x}}$ in the direction $\boldsymbol{d}$ is

Notes

$$F^- f(\bar{\mathbf{x}},\mathbf{d}) = \lim_{d\to d}\inf_{t\to 0^+} \frac{f(\bar{\mathbf{x}}+t\mathbf{d}) - f(\bar{\mathbf{x}})}{t}$$

It is easy to check that, for each $\mathbf{d}\neq\mathbf{0}$, we have $F^+ f(\bar{\mathbf{x}},\mathbf{d}) = F^- f(\bar{\mathbf{x}},\mathbf{d}) = \nabla f(\bar{\mathbf{x}})^{\mathbf{T}}\mathbf{d}$, whenever f is differentiable at $\bar{\mathbf{x}}$.
In this case, we write

$$Ff(\bar{\mathbf{x}},\mathbf{d}) = \lim_{d\to d, t\to 0^+} \frac{f(\bar{\mathbf{x}}+t\mathbf{d}) - f(\bar{\mathbf{x}})}{t}$$

Thus we can rely on the Hadamard derivatives of the components of the objective function f to study optimality for problem **P**.

Theorem 3.1

i) If $\bar{\mathbf{x}}\in\mathrm{X}$ is an efficient solution of P then for any direction $\mathbf{d}\in\mathrm{T}(\mathrm{X};\bar{\mathrm{x}})$

the system $$F^+ f_i(\bar{\mathbf{x}})^{\mathbf{T}}\mathbf{d} < 0, \mathrm{i}=1,2,\ldots,l \quad \ldots\ \ldots\ \ldots \quad (3.1)$$

has no solution $\mathbf{d}\in\mathrm{E}_n$.

ii) If $\bar{\mathbf{x}}\in\mathrm{X}$ is an efficient solution of order one for **P** then for any direction $\mathbf{d}\in\mathrm{T}(\mathrm{X};\bar{\mathrm{x}})\setminus\{\mathbf{0}\}$ the system

$$F^+ f_i(\bar{\mathbf{x}})^{\mathbf{T}}\mathbf{d} \leqq 0, \mathrm{i}=1,2,\ldots,l \quad \ldots\ \ldots\ \ldots \quad (3.2)$$

has no solution $\mathbf{d}\in\mathrm{E}_n$.

iii) If $\bar{\mathbf{x}}\in\mathrm{X}$ is an efficient solution of order one for **P** then for any direction $\mathbf{d}\in\mathrm{T}(\mathrm{X};\bar{\mathrm{x}})\setminus\{\mathbf{0}\}$ the system $$F^- f_i(\bar{\mathbf{x}})^{\mathbf{T}}\mathbf{d} \leqq 0, \mathrm{i}=1,2,\ldots,l \quad \ldots\ \ldots\ \ldots \quad (3.3)$$

has no solution $\mathbf{d}\in\mathrm{E}_n$.

iv) Let $\boldsymbol{f}$ be directionally differentiable. If $\bar{\mathbf{x}}\in\mathrm{X}$ is an efficient solution of order one for **P** then for any direction $\mathbf{d}\in\mathrm{T}(\mathrm{X};\bar{\mathrm{x}})\setminus\{\mathbf{0}\}$

the system $$Ff_i(\bar{\mathbf{x}})^{\mathbf{T}}\mathbf{d} \leqq 0, \mathrm{i}=1,2,\ldots,l \quad \ldots\ \ldots\ \ldots \quad (3.4)$$

has no solution $\mathbf{d}\in\mathrm{E}_n$.

i) Let $\mathbf{d}\in\mathrm{T}(\mathrm{X};\bar{\mathrm{x}})$, that is $\mathbf{d}=\lim_{n\to\infty} t_n(\mathbf{x_n}-\bar{\mathbf{x}})$, where $t_n>0$, $\mathbf{x}_n\in\mathrm{X}$ for each n, and $\lim_{n\to\infty}\mathbf{x_n}=\bar{\mathbf{x}}$.

Since $\bar{\mathbf{x}}$ is an efficient solution, so there is no point $\mathbf{x}_n\in\mathrm{X}$, where $\mathbf{f}(\mathbf{x}_n)\le\mathbf{f}(\bar{\mathbf{x}})$
Then for all n we have

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$$f_i(\mathbf{x}_n) = f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \le f_i(\bar{\mathbf{x}}),\ i = 1,2,\ldots,l$$

for sufficiently large n. Consequently, it holds that

$$\lim_{d \to \bar{d}} \sup_{\frac{1}{t_n} \to 0^+} \frac{f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d}\right) - f_i(\bar{\mathbf{x}})}{\frac{1}{t_n}} \le 0,\quad i = 1,2,\ldots,l$$

$\Rightarrow F^+ f_i(\bar{\mathbf{x}}, \mathbf{d}) \le 0,\quad i = 1,2,\ldots,l$

which implies (3.1) has no solution.

ii) Let $\mathbf{d} \in \mathrm{T}(\mathrm{X}; \bar{\mathrm{x}}) \setminus \{\mathbf{0}\}$, that is $\mathbf{d} = \lim_{n\to\infty} t_n(\mathbf{x_n} - \bar{\mathbf{x}})$, where $t_n > 0$, $\mathbf{x}_n \in \mathrm{X}$ for each n, and $\lim_{n\to\infty} \mathbf{x_n} = \bar{\mathbf{x}}$ with $\mathbf{d} \ne \mathbf{0}$. We find a c such that

$$t_n f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{- } f_i(\bar{\mathbf{x}}) \geqq c t_n \|\mathbf{x_n} - \bar{\mathbf{x}}\|$$

for sufficiently large $n \in \mathrm{N}$, since $\bar{\mathbf{x}}$ is an efficient solution of order one, we have $\limsup_{n\to\infty} \frac{1}{1/t_n} f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{- } f_i(\bar{\mathbf{x}}) \geqq c\|\mathbf{d}\| > 0$, as $\mathbf{d} \ne \mathbf{0}$. Thus, we have

$$F^+ f_i(\bar{\mathbf{x}})^{\mathrm{T}} \mathbf{d} > 0,\quad \mathrm{i} = 1, 2, \ldots, l$$

which implies (3.2) has no solution.

iii) Suppose that $\bar{\mathbf{x}}$ is not an efficient solution of order one. Let $\mathbf{d} \in \mathrm{T}(\mathrm{X}; \bar{\mathrm{x}}) \setminus \{\mathbf{0}\}$, that is $\mathbf{d} = \lim_{n\to\infty} t_n(\mathbf{x_n} - \bar{\mathbf{x}})$, where $t_n > 0$, $\mathbf{x}_n \in \mathrm{X}$ for each n, and $\lim_{n\to\infty} \mathbf{x_n} = \bar{\mathbf{x}}$ with $\mathbf{d} \ne \mathbf{0}$ and $\|\mathbf{d}\| = 1$.

Since $\bar{\mathbf{x}}$ is not an efficient solution of order one. Then, there exist sequences $\{c_n\}$ with $c_n \to 0$.

$$\frac{1}{1/t_n} f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{- } f_i(\bar{\mathbf{x}}) < c_n t_n \|\mathbf{x_n} - \bar{\mathbf{x}}\|$$

For sufficiently large $n \in \mathrm{N}$,

$\liminf_{n\to\infty} \frac{1}{1/t_n} f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{- } f_i(\bar{\mathbf{x}}) < 0$, as $\|\mathbf{d}\| = 1$ and $c_n \to 0$. Thus, we have

$$F^- f_i(\bar{\mathbf{x}})^{\mathrm{T}} \mathbf{d} < 0,\quad \mathrm{i} = 1, 2, \ldots, l$$

which implies (3.3) has no solution.

This follows directly from parts (ii) and (iii).

G. Bigi has achieved similar result of 3.1(i) by deriving the following Theorem 3.2.

Theorem 3.2 If $\bar{\mathbf{x}} \in \mathrm{X}$ is an efficient solution of $\mathbf{P}$ then

Notes

$$\max_{\forall i} F^+ f_i(\bar{\mathbf{x}}, \mathbf{d}) \geq 0 \qquad \dots \quad \dots \quad (3.5)$$

holds for any $\mathbf{d} \in T(X; \bar{\mathbf{x}})$

Proof: see [1].

It would be reasonable to try to replace the upper Dini-Hadamard derivatives with the lower ones for at least some components of *f* since a stronger condition would be achieved; G. Bigi [1] has given the following example and shows that it is not possible to substitute even one of them with the corresponding lower derivative.

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Example 3.1 (G. Bigi [1]) Consider problem

$X = \{x \in E_2^+ : x_1 \geq x_2^2 \geq x_1^4\}$ and $f(x_1, x_2) = (x_2 - x_1, x_1 - x_2)$ if $\mathbf{x} \in X$ and $f(x_1, x_2) = (-x_1 - x_2, -x_1 - x_2)$ if $\mathbf{x} \notin X$.

Thus, $\bar{x} = (0,0)$ is a vector minimum point and $F^+ f_1(\bar{x}; w) = w_2 - w_1 = -F^+ f_2(\bar{x}; w)$ so that (3.5) holds.

Since $F^- f_i(\bar{x}; w) = -w_1$ whenever $w_2 = 0$ and $F^- f_i(\bar{x}; w) = -w_2$ whenever $w_1 = 0$, then we have

$$\max\{F^+ f_1(\bar{x}; w), F^- f_2(\bar{x}; w)\} = -w_1 \langle 0$$

For any nonzero $(w_1, 0) \in T(X, \bar{x}) = E_2^+$ and $\max\{F^- f_1(\bar{x}; w), F^+ f_2(\bar{x}; w)\} = -w_2 \langle 0$ for any nonzero $(0, w_2) \in T(X, \bar{x})$.

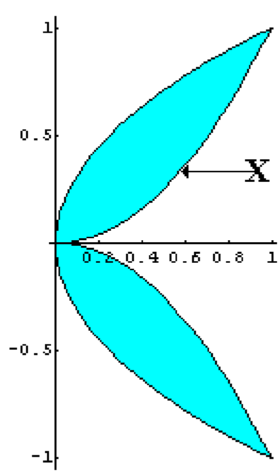

Fig. 1

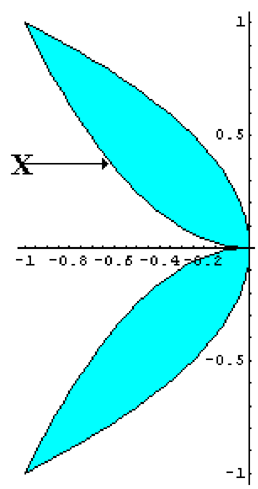

Fig. 2

Now we deduce the following necessary optimality condition where we use lower derivative.

Theorem 3.3 If $\bar{\mathbf{x}} \in X$ is an efficient solution of $\mathbf{P}$ then for any direction $\mathbf{d} \in T^*(X; \bar{x})$, the system

$$F^- f_i(\bar{\mathbf{x}})^T \mathbf{d} < 0, i = 1, 2, \dots, l \qquad \dots \quad \dots \quad \dots \quad (3.6)$$

has no solution $\mathbf{d} \in E_n$.

Now we shall see the sufficiency of Theorem 3.1(i).

Theorem 3.4 Let $\overline{\mathbf{x}} \in X$. If for any direction $\mathbf{d} \in T(X;\overline{x})$ the system

$$F^{+} f_i(\overline{\mathbf{x}})^{\mathbf{T}}\mathbf{d} < 0, i = 1, 2, \ldots, l \qquad \ldots \ldots \ldots \quad (3.7)$$

has no solution $\mathbf{d} \in E_n$ then $\overline{\mathbf{x}}$ is an efficient solution of **P**.

The above Theorem is no longer true, as the following example shows.

Example 3.2 Consider the problem

$$\min\{x_1, x_2\} \quad \text{and} \quad X = \{x \in E_2 \mid x_1^4 \le x_2^2 \le -x_1\}$$

It is easily verified that:

i) $F = \{F^{+} f_i(\overline{\mathbf{x}})^{\mathbf{T}}\mathbf{d} < 0, i = 1, 2\}$

$= \{(d_1, d_2)^T \in E_2 \mid d_1 < 0, d_2 < 0\}$.

ii) Clearly $F \cap T \neq \phi$

iii) $x_0 = (0,1)$ is not an efficient solution to the problem.

iv) Figure 2.

IV. Conclusion

Combining the result of [6], we generalized first order optimality conditions for non-differentiable functions. We illustrated the obtained results by means of two suitable examples. The results can be developed for the optimization problem where the functions are Lipschitz continuous. We leave it as our future work.

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