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New Solutions of Radial Teukolsky Equation Via Transformation to Heuns Equation with the Application of Rational Polynomial of at Most Degree 2

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New Solutions of Radial Teukolsky Equation Via Transformation to Heuns Equation with the Application of Rational Polynomial of at Most Degree 2

S. Akinbode [α] & A. Anjorin [σ]

Abstract- The perturbation equation of masseless fields for Kerr-de Sitter geometry are written in form of seperable equations as in[19] called the Radial Teukolsky equation. The Radial Teukolsky equation is converted to General Heun's equation with singularities coinciding through some conuent process of one of five singularities. As in [17], [18] rational polynomials of at most degree two are introduced.

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I. Introduction

Teukolsky equation are the consequences of perturbation equation fer Kerr-de Sitter geometry with the separability of angular and radial parts respectively. Carter[1] was the first to discover that the scalar wave function is separable. Other consideration is the $\frac{1}{2}$ spin electromagnetic field, gravitational perturbations and gravitino for the Kerr-de Sitter class of geometry.

The Teukolsky equation is applicable in he study of black holes in general. The solutions of the equation are in most cases expressed as series solutions of some specialized functions. This approach has been carried out by so many researchers say Teukolsky (1973), Breuer et all (1977), Frackerell and Crossman (1977), Leahy and Unruh (1979), Chakrabarti (1984), Siedel (1989), Suzuki et all (1989) just to mention but few. Although Teukolsky equation has five singular points one irregular with four regular points. By some confluent process, these singular points are reduced to four coinciding with the singular points of Heun's equation.

The objective of this work is to obtain polynomial solutions for the derived Tuekolsky equation through its conversion to Heun's equation through rational polynomials of degree at most 2. New solutions in terms of the rational polynomials are obtained.

The paper is organized as follows; The first section deals with the introduction of Teukolsky equation as described in [19]. The second section deals with the derivation of Teukolsky using the work of [19]. The third

Ref

1. B. Carter, Comm.Math. Phys 10, 280 1968.

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section has to do with the derivation of Radial Teukolsky and its conversion to Heun's equation. The fourth section has to do with Heun's differential equation and its transformation to hypergeometric differential equation via rational polynomials of at most degree two.

II. The Teukolsky Equation [19]

Tekolsky equation was derived using the Kerr(-Newman)-de Sitter geometries.

$$ds^2 = -p^2(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}) - \frac{\Delta_\theta \sin^2\theta}{(1+\alpha)^2 p^2}[adt - (r^2+a^2)d\varphi]^2 + \frac{\Delta_r}{(1+\alpha)^2\rho^2}(dt - a\sin^2\theta d\varphi)^2, \quad (1)$$

where

$$\Delta_r = (r^2+a^2)(1 - \frac{a}{ar^2}r^2) - 2Mr + Q^2 = -\frac{\alpha}{a^2}(r-r_+)(r-r_-)(r-r'_+)(r-r'_-)$$

$$\Delta_\theta = 1 + a\cos^2\theta, \alpha = \frac{\Lambda a^2}{3}, \bar{\rho} = r + ia\cos\theta \text{ and } \rho^2 = \bar{\rho}\bar{\rho}, \quad (2)$$

where Λ is the cosmological constant, M is the mass of the black hole, Mr its radial momentum and Q its charge. The electromagnetic field due to the charge of the black hole was given by

$$A_\mu dx^\mu = -\frac{Qr}{(1+\alpha)^2\rho^2}(dt - a\sin^2\theta d\varphi). \quad (3)$$

In particular, the following vectors were adopted as the null tetrad,

$$\iota^\mu = (\frac{(1+\alpha)(r^2+a^2)}{\Delta_r}, 1, 0, \frac{a(1+\alpha)}{\Delta_r}),$$

$$n^\mu = \frac{1}{2\rho^2}((1+\alpha)(r^2+a^2), -\Delta_r, 0, a(1+\alpha)),$$

$$m^\mu = \frac{1}{\bar{p}\sqrt{2\Delta_\theta}}(ia(1+\alpha)\sin\theta, 0, \Delta_\theta, \frac{i(1+\alpha)}{\sin\theta})\bar{m}^\mu = m^{*\mu}. \quad (4)$$

It was assumed that the time and azimuthal dependence of the fields has the form $e^{-i(\omega t - m\varphi)}$,the tetrad components of derivatives and the electromagnetic field were

$$\iota^\mu = D_0, \ n^\mu\partial_\mu = \frac{\Delta_r}{2\bar{\rho}}D_0^\dagger, \ m^\mu\partial_\mu = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\bar{\rho}}L_0^\dagger,$$

$$m^\mu\partial_\mu = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\bar{p}^*}L_0, \ \iota^\mu A_\mu = -\frac{Qr}{\Delta_r}, \ n^\mu A_\mu = -\frac{Qr}{2\rho^2},$$

$$m^\mu A_\mu = \bar{m}^\mu A_\mu = 0, \quad (5)$$

where

$$D_n = \partial_r - \frac{\imath(1+\alpha)K}{\Delta_r} + n\frac{\partial_r\Delta_r}{\Delta_r}, \ D_n^\dagger = \partial_r + \frac{\imath(1+\alpha)K}{\Delta_r} + n\frac{\partial_r\Delta_r}{\Delta_r},$$

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$$L_n = \partial_\theta + \frac{\imath(1+\alpha)H}{\Delta_\theta} + n\frac{\partial_\theta(\sqrt{\Delta_\theta}\sin\theta)}{\sqrt{\Delta_\theta}\sin\theta},$$

$$L_n^\dagger = \partial_\theta - \frac{\imath(1+\alpha)H}{\Delta_\theta} + n\frac{\partial_\theta(\sqrt{\Delta_\theta}\sin\theta)}{\sqrt{\Delta_\theta}\sin\theta}, \tag{6}$$

with $K = \omega(r^2+a^2) - am$ and $H = -a\omega\sin\theta + \frac{m}{\sin\theta}$.

Using the Newman-Penrose formalism it was shown that perturbation equation in the Kerr-de sitter geometry are separable for massless spin 0, $\frac{1}{2}$,1,$\frac{3}{2}$ and 2 fields. Similarly in the Kerr-Newman-de sitter space those for spin 0,$\frac{1}{2}$ fields are also separable. The separated equations for fields with spin s and charge e were given by

$$[\sqrt{\Delta_\theta}L_{1-s}^\dagger\sqrt{\Delta_\theta}L_s$$
$$-2(1+\alpha)(2s-1)a\omega\cos\theta - 2\alpha(s-1)(2s-1)\cos^2\theta + \lambda]S_s(\theta) = 0$$
$$[\Delta_r D_1 D_s^\dagger + 2(1+\alpha)(2s-1)\imath\omega - \frac{2\alpha}{a^2}(s-1)(2s-1)$$
$$+\frac{-2(1+\alpha)eQKr + \imath seQr\partial_r\Delta_r + e^2Q^2r^2}{\Delta_r} - 2\imath seQ - \lambda]R_s(r) = 0. \tag{7}$$

III. Transformation of Teukolsky Equation to Heun's Equation [19]

It was shown in [19] that the Teukolsky equations can be transformed to the Heun's equation by factoring out a single regular singularity.

a) Radial Teukolsky equation

From (7), the radial teukolsky equation is explictly written by

$$\Big\{\Delta_r^{-s}\frac{d}{dr}\Delta_r^{s+1}\frac{d}{dr} + \frac{1}{\Delta_r}\Big[(1+\alpha)^2\Big(K - \frac{eQr}{1+\alpha}\Big)^2\Big(K - \frac{eQr}{1+\alpha}\Big)\frac{d\Delta_r}{dr}\Big]$$
$$+4\imath s(1+\alpha)\omega r - \frac{2\alpha}{a^2}(s+1)(2s+1)r^2 + 2s(1-\alpha) - s\imath seQ - \lambda\Big\}R = 0, \tag{8}$$

This equation has five regular singularities at $r^\pm, r'_\pm$ and ∞ which are assigned such that $r_\pm \to M \pm \sqrt{M^2-a^2-Q^2} = r_\pm^0$ and $r'_\pm \to \pm\frac{a}{\sqrt{\alpha}}$ in the limit $\alpha \to 0(\Lambda \to 0)$. By using the new vriable

$$z = \Big(\frac{r_+ - r'_-}{r_+ - r_-}\Big)\Big(\frac{r - r_-}{r - r'_-}\Big),$$

equation (8) becomes an equation which has regular singularities at $0, 1, z_r, z_\infty$ and ∞,

$$z_r = \Big(\frac{r_+ - r'_-}{r_+ - r_-}\Big)\Big(\frac{r'_+ - r_-}{r'_+ - r'_-}\Big),$$

$$z_\infty = \frac{r_+ - r'_-}{r + - r_-}.$$

Again we can factor out the singularity at $z = z_\infty$ by the transformation as

$$Ra(z) = z^{B_1}(z-1)^{B_2}(z-z_r)^{B_3}(z-z_\infty)^{2s+1}g(z)$$
$$B_1 = \frac{1}{2}\Big\{-s \pm \imath\Big[\frac{2(1+\alpha)a^2(\omega(r_-^2+a^2) - am - \frac{eQr_-}{1+\alpha})}{\alpha(r'_+ - r_-)(r'_- - r_-)(r_+ - r_-)} - \imath s\Big]\Big\}$$

Ref

19. Hisao Suzuki, Ehuchi Takasugi and Hiroshi Umetsu (1998), "Pertubations of Kerr-de Sitter Black Hole and Heun's Equation. EPHOU 98005 OU-HET-296.

$$B_2 = \frac{1}{2}\Big\{ - s \pm \imath\Big[\frac{2(1+\alpha)a^2(\omega(r_+^2+a^2) - am - \frac{eQr_+}{1+\alpha})}{\alpha(r'_+ - r_+)(r'_- - r_+)(r_- - r_+)} - \imath s\Big]\Big\}$$

$$B_3 = \frac{1}{2}\Big\{ - s \pm \imath\Big[\frac{2(1+\alpha)a^2(\omega(r_+^2+a^2) - am - \frac{eQr'_+}{1+\alpha})}{\alpha(r'_- - r_+)(r'_- - r_+)(r_+ - r_+)} - \imath s\Big]\Big\}. \quad (9)$$

Then $g(z)$satisfies the Heun's equation as

$$\Big\{\frac{d^2}{dz^2} + \Big[\frac{2B_1+s+1}{z} + \frac{2B_2+s+1}{z-1} + \frac{2B_3+s+1}{z-z_r}\Big]\frac{d}{dz}$$

$$+\frac{\sigma_+\sigma_- z + \nu}{z(z-1)(z-z_r)}\Big\}g(z) = 0, \quad (10)$$

where

$$\sigma_\pm = B_1 + B_2 + B_3 + 2s$$

$$+\frac{1}{2}\Big\{ - s \pm \imath\Big[\frac{2(1+\alpha)a^2\Big(\omega(r_-'^2+a^2) - am - \frac{eQr'_-}{1+\alpha}\Big)}{(r_+ - r'_-)(r_- - r'_-)(r'+ - r'_-)} - \imath s\Big]\Big\}$$

$$\nu = \frac{2a^4(1+\alpha)^2(r_+ - r_+)^2(r_+ - r'_-)^2(r_- - r'_-)(r'_+ - r'_-)}{\alpha^2 D(r_+ - r\_)}$$

$$\Big\{ - \omega^2 r_-^3(r_+r_- - 2r_+r'_+ + r_-r'_+) + 2a\omega(a\omega - m)r_-(r_+r'_+ - r_-^2)$$

$$-a^2(a\omega - m)^2(2r_- - r_+ - r'_+)$$

$$+\frac{eQ}{1+\alpha}[\omega r_-^2(r_+r_- + r_-^2 - 3r_+r'_+ + r_-r'_+)$$

$$-a(a\omega - m)(r_+r_- - 3r_-^2 + r_+r'_+ + r_-r'_+)]$$

$$+\Big(\frac{eQ}{1+\alpha}\Big)^2 r_-(-r_-^2 + r_+r'_+)\Big\}$$

$$+\frac{2\imath sa^2(1+\alpha)\Big[\omega(r_-r'_- + a^2) - am - \frac{eQ}{1+\alpha}\frac{r_- + r'_-}{2}\Big]}{(r_+ - r_-)(r'_+ - r'_-)(r_- - r'-)}$$

$$+(s+1)(2s+1)\Big[\frac{2r_-'^2}{(r_+ - r_-)(r'_+ - r'_-)} - z_\infty\Big]$$

$$-2B_1(z_rB_2 + B_3) - (s+1)[(1+z_r)B_1 + z_rB_2 + B_3]$$

$$-\frac{a^2}{\alpha(r_+ - r_-)(r'_+ - r'_-)}[-\lambda - 2\imath seQ + 2s(1-\alpha)]. \quad (11)$$

Here D is the discriminant of

$$\Delta_r = 0$$

$$D = (r_+ - r_-)^2(r_+ - r'_+)^2(r_+ - r'_-)^2(r_- - r'_+)^2(r_- - r'_-)^2(r'_+ - r'_-)^2$$

$$= \frac{16a^{10}}{\alpha^5}\Big\{(1-\alpha)^3[M^2 - (1-\alpha)(a^2+Q^2)]$$

$$+\frac{\alpha}{a^2}[-27M^4 + 36(1-\alpha)M^2(a^2+Q^2)$$

Notes

$$-8(1-\alpha)^2(a^2+Q^2)^2] - \frac{16\alpha^2}{a^4}(a^2+Q^2)^3\Big\}. \tag{12}$$

The sign ambiguity in B_2 or B_3 are related to the boundary condition at the horizon or at the de Sitter horizon, respectively. We can either one of signs of B_1

IV. Heun's Equation to Hypergeometric Via Rational Polynomial Transformations

In this section, we transform the Heun's equation derived above to hypergeometric differential equation with three singularities and back again to the Heun's solutions with polynomial terms.

The hypergeometric equation has three regular singular points. Heun's equation has four regular points. The problem of conversion from Heun's equation to hypergeometric equation has been treated in the works of K.Kuiken[17]. The purpose of this work is to derive some forms solution to the Heun's equation via some rational transformation as stated earlier. The steps taken shall be conversion of Heun's function to the hypergeometric function then taken the derivatives, and through a push and pull back process we arrive back to a new Heun;s function different from the original Heun's function.

Every homogenous linear second order differential equation with four regular singularities can be transformed into (10) with the assumption that $2B_1+s+1=\gamma$, $2B_2+s+1=\delta$, $2B_3+s+1=\epsilon$, $\rho_{\pm}=\alpha\beta$, $\nu=q$, $z=t$ and $z_r=d$ as defined above, and read as

$$\frac{d^2u}{dt^2}+(\frac{\gamma}{t}+\frac{\delta}{t-1}+\frac{\epsilon}{t-1})\frac{du}{dt}+\frac{\alpha\beta t-q}{t(t-1)(t-d)}u=0, \tag{13}$$

where $\{\alpha,\beta,\gamma,\delta,\epsilon,d,q\}(d\neq 0,1)$ are parameters, generally complex and arbitrary, linked by FUSCHAIN constraint $\alpha+\beta+1=\gamma+\delta+\epsilon$. This equation has four regular singular points at $\{0,1,a,\infty\}$, with the exponents of these singular being respectively, $\{0,1,-\gamma\},\{0,1,-\epsilon\}$ and $\{\alpha,\beta\}$. The equation (13) is called Heun's equation.

The Hypergeometric equation

$$z(1-z)\frac{d^2u}{dt^2}+[c-(a+b+1)z]\frac{dy}{dz}-aby=0, \tag{14}$$

has three regular singular points. in the above (13), it has been shown that these two equation above can be transformed to one another via six rational polynomial $z=R(t)$, where $R(t)=t^2, 1-t^2, (t-1)^2, 2t-t^2(2t-1)^2, 4t(1-t)$. The following parameter relations were deduced.

For the polynomial $R(t)=t^2$

- $\alpha+\beta=2(a+b), \alpha\beta=4ab, \gamma=-1+2c, \delta=1+a+b-c, \delta\epsilon=\delta, q=0$ and $d=-2$.

  For the polynomial $R(t)=1-t^2$

- $\alpha+\beta=2(a+b), \alpha\beta=4ab, \gamma=-1-2c+2a+2b, \delta=c, \epsilon=\delta, q=0$ and $d=12$.

  For the polynomial $R(t)=2t-t^2$

Ref

17. K. Kuiken, *Heun's equations and the Hypergeometric equations*, S. I. A. M. J. Math. Anal. 10(3), 655-657 (1979).

- $\alpha+\beta = 2(a+b), \alpha\beta = 4ab, \gamma = c, \delta = 1-2c+2a+2b, \epsilon = \delta = c, q = 4ab$ and $d = 2$.

  For the polynomial $R(t) = (2t-1)^2$

- $\alpha+\beta = 2(a+b), \alpha\beta = 4ab, \gamma = -1+a+b-c, \delta = \gamma, \delta = \epsilon = -1, q = 4ab$ and $d = \frac{1}{2}$.

  For the polynomial $R(t) = 4t(1-t)^2$

- $\alpha+\beta = 2(a+b), \alpha\beta = 4ab, \gamma = c, \delta = \gamma, \delta = 1-2c+2a+2b, q = 2ab$ and $d = \frac{1}{2}$.

Assuming $H(d, q, \alpha, \beta, \gamma, \delta, \epsilon; t) = R_s(t);\ \ s = 1 \ldots 14$ are solutions of the Radial Teukolsky in terms of Heun's with polynomial factor and ${}_2F_1(a, b; c; z = R(t))$ are representative forms of the solutions of (13) and (14) respective, together with parameters above relations can be established between these two forms via the polynomials data given above. We provide an answer to this in this paper. Indeed, we provide that the derivative of the solution of Heun's can be expressed in terms of another Heun's solution giving rise to new solutions of Teukolsky Radial equation.

V. Main Results

a) *New Derived Solutions of Radial Teukolsky Equation*

In this section we shall apply the relation above in obtaining the derive solutions via these polynomial transformations. let $D = \frac{d}{dt}$ be a differential operator. Since $D{}_2F_1(a, b; c; z = R(t)) = R'(t)\frac{ab}{c}{}_2F_1(a+1, b+1; c+1; z = R(t))$ and through a push and pull back processes we have the following possible solutions for the Teukolsky Radial equation;

1. For polynomial $R(t) = t^2$.

   [a] Using $c = \frac{(\gamma+1)}{2}$, we get

$$DH(-1, 0; \alpha, \beta, \gamma, \delta, \epsilon; t)$$

$$= \frac{\alpha\beta t}{\gamma+1}{}_2F_1(\frac{\beta+2}{2}, \frac{\alpha+2}{2}, \frac{\gamma+3}{2}; R(t) = t^2)$$

$$= \frac{\alpha\beta t}{\gamma+1}H(-1, 0; \alpha+2, \beta+2, \gamma+2, \frac{\alpha+\beta-\gamma+3}{2}, \frac{\alpha+\beta-\gamma+3}{2}; t)$$

$$= R_1(t). \tag{15}$$

   [b] Using $c = 1-\delta+a+b$, we get

$$DH(-1, 0; \alpha, \beta, \gamma, \delta, \epsilon; t)$$

$$= \frac{\alpha\beta t}{\alpha+\beta+2(1-\delta)}{}_2F_1(\frac{\beta+2}{2}, \frac{\alpha+2}{2}, \alpha+\beta-2(2-\delta); R(t) = t^2)$$

$$= \frac{\alpha\beta t}{\gamma+1}\times$$

$$H(-1, 0; \alpha+2, \beta+2, \gamma+2, \frac{\alpha+\beta-\gamma+3}{2}, \frac{\alpha+\beta-\gamma+3}{2}; t)$$

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$$= \frac{\alpha\beta t}{\alpha+\beta+2(1-\delta)}$$

$$\times H(-1,0;\alpha+2,\beta+2,\alpha+\beta-2\delta+1,1+\delta,1+\delta;t) = R_2(t). \quad (16)$$

By changing δ to ϵ, similar expression can be obtained.

2. For the polynomial $R(t) = 1-t^2$.

[a] Using $\delta = c$, we get

$$DH(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t)$$
$$= \frac{\alpha\beta t}{2\delta}{}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\delta+1;R(t)=t^2)$$
$$= \frac{\alpha\beta t}{2\delta}H(-1,0,\alpha+2,\beta+2,\alpha+\beta,-2\delta+3,\delta+1,\delta+1;t)$$
$$= R_3(t). \quad (17)$$

[b]
$$DH(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t)$$
$$= \frac{\alpha\beta t}{2\delta}{}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\epsilon+1;R(t)=1-t^2)$$
$$= \frac{\alpha\beta t}{2\delta}H(-1,0,\alpha+2,\beta+2,\alpha+\beta,-2\epsilon+3,\epsilon+1,\epsilon+1;t)$$
$$= R_4(t). \quad (18)$$

[c] Using $c = 1-\gamma+2a+2b$ we obtain

$$DH(-1,0;\alpha,\beta,\gamma,\delta,\epsilon;t) =$$
$$-\frac{\alpha\beta t}{\alpha+\beta+2(1-\delta)}\times$$
$${}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2},\frac{\alpha+\beta-\gamma+3}{2};R(t)=1-t^2)$$
$$= \frac{\alpha\beta t}{\alpha+\beta+2(1-\delta)}\times$$
$$\mathrm{H}(-1,0;\alpha+2,\beta+2,\gamma+2,\frac{\alpha+\beta-\gamma+3}{2},\frac{\alpha+\beta-\gamma+3}{2})$$
$$= R_5(t). \quad (19)$$

3. For the polynomial $R(t) = 2t-2t^2$.

[a] Using $c = 1-a+b-\gamma$ we obtain

$$DH(2,\alpha\beta,\alpha,\gamma,\delta,\epsilon;t) =$$
$$\frac{\alpha\beta(1-t)}{2\gamma}{}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\gamma+1;R(t)=2t-t^2)$$
$$= \frac{\alpha\beta(1-t)}{2\gamma}\times$$
$$H(2,(\beta+2)(\alpha+2);\alpha+2,\beta+2,\gamma+1,\alpha+\beta-2\gamma+3,\eta;t)$$
$$= R_6(t)$$

$$\text{where} \quad \eta = \frac{\alpha+\beta-2\gamma+3}{2}. \tag{20}$$

$[b]$ $DH(2,\alpha\beta,\alpha,\gamma,\delta,\epsilon;t)$

$$= \frac{\alpha\beta(1-t)}{2\gamma} {}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\frac{\alpha+\beta+2(2-\gamma)}{2};R(t)=2t-t^2)$$

$$= \frac{\alpha\beta(1-t)}{2\gamma}\times$$

$$H(2,(\alpha+2)(\beta+2);\alpha+2,\beta+2,\gamma+1,\alpha+\beta-2\gamma+3,\gamma+1;t)$$

$$= R_7(t). \tag{21}$$

By changing γ to ϵ, we can obtain similar expression.

[c] Using $c = \frac{(1+\delta)}{2}$, we get

$$DH(2,\alpha\beta,\alpha,\gamma,\delta,\epsilon;t)$$

$$= \frac{\alpha\beta(1-t)}{2\gamma} {}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2},\frac{\delta+2}{2};R(t)=2t-t^2)$$

$$= \frac{\alpha\beta(1-t)}{2\gamma}$$

$$\times(2,(\alpha+2)(\beta+2);\alpha+2,\beta+2,\zeta,\delta+2,\zeta;t)$$

$$= R_8(t).$$

$$\text{where} \quad \zeta = \frac{\alpha+\beta-\delta+3}{2} \tag{22}$$

4. For polynomial $R(t) = (1-t^2)$

[a]. Using $c = \frac{(1+\delta)}{2}$, we obtain

$$DH(2,\alpha\beta;\alpha,\beta,\gamma,\delta,\epsilon;t)$$

$$= \frac{\alpha\beta(t-1)}{\delta+1} {}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\frac{\delta+3}{2};R(t)=(t-1)^2)$$

$$\frac{\alpha\beta(t-1)}{\delta+1}\times H(2,(\alpha+2)(\beta+2);\alpha+2,\beta+2,\zeta,\delta+2,\zeta;t)$$

$$= R_9(t),$$

$$\text{where} \quad \zeta = \frac{\alpha+\beta-\delta+3}{2}. \tag{23}$$

$[b]$. Using $c = \frac{(1-\gamma+2a+2b)}{2}$, we get

$$DH(2,\alpha\beta;\alpha,\beta,\gamma,\delta,\epsilon;t)$$

$$= \frac{\alpha\beta(t-1)}{\alpha+\beta-\gamma+1} {}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\frac{\alpha+\beta-\gamma+3}{2}R(t)=(t-1)^2)$$

$$= \frac{\alpha\beta(t-1)}{\alpha+\beta-\gamma+1}$$

Notes

$$\times H(2,(\beta+2)(\alpha+2);\alpha+2,\beta+2,\frac{\gamma+3}{2},\alpha+\beta-2\gamma+3,\frac{\gamma+3}{2};t)$$

$$=R_{10}(t). \tag{24}$$

$[c]$ By changing γ to ϵ in above, similar relation can be obtained.

5. For polynomial $R(t)=(2t-1)^2$.

$[a]$. Using $c=\frac{1+\epsilon}{2}=\frac{1+\delta}{2}$

$$DH(\frac{1}{2},\alpha\beta;\alpha,\beta,\gamma,\delta,\epsilon;t)$$

$$=\frac{2(2t-1)\alpha\beta}{(\epsilon+1)}\,{}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\frac{\epsilon+3}{2}R(t)=(2t-1)^2)$$

$$=\frac{2(2t-1)\alpha\beta}{(\epsilon+1)}$$

$$\times(\frac{1}{2},\frac{(\alpha+2)(\beta+2)}{2};\alpha+2,\beta+2,\frac{\alpha+\beta-\epsilon-1}{2},\epsilon+2;t)$$

$$=R_{11}(t). \tag{25}$$

By changing ϵ to δ a similar expression can be obtained.

[b] Using $c=-1+a+b-\gamma$, we obtained

$$DH(\frac{1}{2},\alpha\beta;\alpha,\beta,\gamma,\delta,\epsilon;t)$$

$$=\frac{2(2t-1)\alpha\beta}{(\epsilon+1)}\,{}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\frac{\alpha+\beta-2\gamma}{2}R(t)=(2t-1)^2)$$

$$=\frac{2(2t-1)\alpha\beta}{(\epsilon+1)}$$

$$\times(\frac{1}{2},\frac{(\alpha+2)(\beta+2)}{2};\alpha+2,\beta+2,\mu,\mu,\tau;t)$$

$$=R_{12}(t). \tag{26}$$

where $\tau=\alpha+\beta-2(\gamma-\frac{1}{2})$ and $\mu=\frac{\alpha+\beta+2(1-\gamma)}{2}$

6. For the polynomial $R(t)=4t(1-t)$

$[a]$ Using $c=\gamma$,we get

$$DH(\frac{1}{2},\alpha\beta;\alpha,\beta,\gamma,\delta,\epsilon;t)$$

$$=\frac{(1-2t)\alpha\beta}{\gamma}\,{}_2F_1(\frac{\beta+2}{2},\frac{\alpha+2}{2};\gamma+1;R(t)=4t(1-t))$$

$$=\frac{(1-2t)\alpha\beta}{\gamma}$$

$$\times H(\frac{1}{2},\frac{(\alpha+2)(\beta+2)}{2};\alpha+2,\beta+2,\gamma+1,\alpha+\beta-2\gamma+3;t)$$

$$= R_{13}(t). \quad (27)$$

$[b]$ Using $c = \frac{(1-\epsilon+2a+2b)}{2}$

$$DH(\frac{1}{2}, \frac{\alpha\beta}{2}; \beta, \alpha, \gamma, \delta, \epsilon, ; t)$$

$$= \frac{(1-2t)\alpha\beta}{\alpha+\beta-\epsilon+1}{}_2F_1(\frac{\beta+2}{2}, \frac{\alpha+2}{2}; \frac{\alpha+\beta-\epsilon+3}{2} R(t) = 4t(1-t))$$

$$= \frac{(1-2t)\alpha\beta}{\alpha+\beta-\epsilon+1}$$

$$\times(\frac{1}{2}, \frac{(\alpha+2)(\beta+2)}{2}; \alpha+2, \beta+2, \gamma, \omega, \omega, \epsilon+2; t)$$

$$= R_{14}(t), \quad (28)$$

where

$$\omega = \frac{\alpha+\beta-\epsilon+3}{2}.$$

IV. Concluding Remarks and Suggestions

In this paper, we have shown that the solutions of the derived Radial Teukol-sky equation transformed to Heun's equation could be obtained in form of Heun's functions via polynomials of at most degree two transformations. The Heun's equation was initially compare with the hypergeometric differential equation with three singularities via the giving polynomials. Another results could be obtained if we apply the integral operator instead of the differential operator. Polynomials of higher degrees are being consider.

Notes

References Références Referencias

1. B. Carter, Comm.Math. Phys 10, 280 1968.
2. A. Ronveaux, *Heun's Differential equation* (Oxford University press, Oxford, 1995).
3. A. O. Smirnov, *Elliptic solutions and Heun's Equations*, C. R. M. Proceedings and Lecture notes 32, 287-305 (2002).
4. P. A. Clarkson and P. J. Oliver, J. Diff. Equations 124, 225-246(1996).
5. R. S. Maier, *Heun-to-hypergeometric transformations*, contribution to the conference of Foundations of Com-putational Mathematics 02 (2002); downloadable from http:// www.math.umn.edu/~focm/c_/Maier.pdf
6. N. H. Christ and T. D. Lee, *Phys. Rev. D* 12 1606 (1975);
7. A. Ishkhanyan and K. A. Souminen, J. Phys. A: Math. Gen. 36, L81-L85 (2003).
8. G. Valent, *Heun functions versus elliptic functions*, International Conference on Differential Equations, Special Functions and Applications, Munich, 2005; [e-print math-ph/0512006].
9. V. Stanley Grossman, *Multivariate calculus, Linear Algebra, and differential equation* (Saunder college publishing, New York, 1995).
10. S. P. Tsarev, *An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator*, Proceedings of the international symposium on Symbolic and algebraic computation, pp 226-231 (Switzerland, 1996).
11. M. Van Hoeij, Journal of Symbolic Computation 24 n 5, 537-561(1997).

Notes

12. R. K. Bhadari, A. Khare, J. Law, M. V. N. Murthy and D. Sen, J. Phys. A: Math. Gen. 30, 2557-2260 (1997).
13. M. Suzuki, E. Takasugi and H. Umetsu, Prog. Theor. Phys. 100, 491-505 (1998).
14. K. Takemura, Commun. Math. Phys. 235, 467-494 (2003); J. Non-linear Math. Phys. 11, 480-498 (2004).
15. P. Dorey, J. Suzuki and R. Tateo, J. Phys. A: Math. Gen. 37, 2047-2061 (2004).
16. M. N. Hounkonnou, A. Ronveaux and A. Anjorin, *Derivatives of Heun's function from some properties of hypergeometric equation*, Proceeding of International Workshop on Special Functions, Mar-seille (2007), *in press*; preprint; ICMPA-MPA/2007/21.
17. K. Kuiken, *Heun's equations and the Hypergeometric equations*, S. I. A. M. J. Math. Anal. 10(3), 655-657 (1979).
18. Anjorin A (2009). "On Derivative of solutions to Heun's Differential equation via Rational polynomial of Degree 2; 3; 4; 5; 6. Journal of the Association of Nigerian Mathematical Physics. Vol. 14, pp 289-292. Nigeria.
19. Hisao Suzuki, Ehuchi Takasugi and Hiroshi Umetsu (1998),"Pertubations of Kerr-de Sitter Black Hole and Heun's Equation. EPHOU 98005 OU-HET-296.