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A Construction of Non-Gaussian White Noise Analysis using the Theory of Hypercomplex Systems

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25. T. Hida, *Analysis of Brownian functionals*, in Carleton Math. Lecture Notes **13**, Carleton Univ., Ottawa 1975.

A Construction of Non-Gaussian White Noise Analysis using the Theory of Hypercomplex Systems

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Abstract- In this paper, we present a generalization of white noise analysis to the case of non-Gaussian measures. For this purpose, we use a biorthogonal approach in which instead of the exponentials the characters of commutative hypercomplex systems are employed. Moreover, we construct the elements of Wick calculus in a non-Gaussian setting.

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I. Introduction

White noise analysis appeared in 1975 after the work of T. Hida [25] and then has been actively developed in a series of articles and books. For a sufficiently full bibliography see [26]. As mentioned in [4], Gaussian white noise analysis can be thought of as a theory of generalized functions of infinitely many variables whose pairing with test functions from special spaces is given via the integration by the Gaussian measure. It is well known that there are several approaches to the construction of such theory of generalized functions: the Berezansky-Samoilenko approach [3] and the Hida approach [25]. In the Berezansky-Samoilenko approach, the spaces of test and generalized functions are constructed as infinite tensor products of one-dimensional spaces. The Hida approach consists in a construction of some rigging of a Fock space with subsequent application of the Wiener-Itô-Segal isomorphism to the spaces of this rigging. In most cases, investigations in white noise analysis and its generalizations are based on the Hida approach. So, the Hida approach is more convenient than Berezansky-Samoilenko approach. Recently, some authors as Okb El Bab, Zabel, Ghany and Hyder [15], Ghany [16], Ghany and Hyder [17–19], Ghany and Zakarya [20–22] and Ghany and Qurashi [23], studied some important subjects related to Gaussian white noise analysis. There exist many works investigated to white noise analysis development Works deal with the construction of spaces of test and generalized functions and operators acting in these spaces using the Wiener-Itô-Segal isomorphism and various riggings of the Fock space. For more details, see [4, 25, 28]. Works deal with the so-called Jacobi fields approach to a generalization of white noise analysis [6, 7]. In these works, the role of the Wiener-Itô-Segal isomorphism is

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played as a unitary Fourier transform which is defined by the Jacobi field, i.e., by some family of commuting selfadjoint operators that act in the Fock space and have a Jacobi structure. Works use the biorthogonal approach to a generalization of white noise analysis. In this approach, one replaces the system of Hermite polynomials, that are orthogonal with respect to the Gaussian measure, with a certain biorthogonal system. The biorthogonal approach was inspired by [12], proposed in [1] and developed in [2,8,38]. Note that in [8] was first observed that the biorthogonal approach is deep connected with the theory of hypercomplex systems.

Hypercomplex systems date back to J. Delsarte and B. M. Levitan work during the 1930s and 1940s, but the substantial development had to wait till the 1990s when Berezansky and Kondratiev put hypercomplex systems in the right setting for harmonic analysis [9]. Recently, some authors as Okb El Bab, Zabel and Ghany [37, 40, 41] studied some important subjects in hypercomplex systems.

Generalized translation operators were first introduced by Delsarte [13] as an object that generalizes the idea of translation on a group. They were systematically studied by Levitan [31–35], for some classes of generalized translation operators, he obtained generalizations of harmonic analysis, the Lie theory, the theory of almost periodic functions, the theory of group representations, etc. In fact, each hypercomplex system can be associated with a family of generalized translation operators. This correspondence can be found in [5].

This paper focuses on the connection between white noise analysis and hypercomplex systems. Precisely, we produce a generalization of white noise analysis to the case of non-Gaussian measures. For this aim, we propose a biorthogonal approach in which instead of the exponentials the characters of commutative hypercomplex systems are used. Furthermore, we construct the elements of Wick calculus, namely, we introduce a new Wick product with respect to non-Gaussian measures, the associated Hermite transform and the characterization theorem for the constructed spaces of generalized functions. In Gaussian white noise analysis such calculus has found numerous applications, in particular, in fluid mechanics and financial mathematics, see e.g. [14, 27] for more details.

This paper is organized as follows: Section 2 is devoted to provide the basic topics of hypercomplex systems with locally compact basis. In Section 3, we construct spaces of test and generalized functions by means of generalized Delsarte characters. In Section 4, we introduce the elements of Wick calculus.

II. Hypercomplex Systems with Locally Compact Basis

Let Q be a complete separable locally compact metric space of points $p, q, r, \ldots; \mathcal{B}(Q)$ is the σ-algebra of Borel subsets, and $\mathcal{B}_0(Q)$ is the subring of $\mathcal{B}(Q)$, which consists of sets with compact closure. We shall consider the Borel measures; i.e., positive regular measures on $\mathcal{B}(Q)$, finite on compact sets. We denote by $C(Q)$ the space of continuous functions on $Q, C_b(Q), C_\infty(Q)$ and $C_0(Q)$ consists respectively of bounded, tending to zero at infinity and compactly supported functions from $C(Q)$.

A hypercomplex system with the basis Q is defined by its structure measure $c(A, B, r)(A, B \in B(Q); r \in Q)$. A structure measure $c(A, B, r)$ is a Borel measure in A (respectively B) if we fix B, r (respectively A, r) which satisfies the following properties:

(I) $\forall A, B \in \mathcal{B}_0(Q)$, the function $c(A, B, r) \in C_0(Q)$,

(II) $\forall A, B \in \mathcal{B}_0(Q)$ and $s, r \in Q$, the following associativity relation holds

Ref

13. J. Delsarte, Sur certaines transformation fonctionelles relative aux equations linearies aux derivees partiels du seconde ordre, *C. R. Acad. Sci., Ser.* **17** (1938), 178-182.

$$\int_Q c(A,B,r)d_r c(E_r,C,s) = \int_Q c(B,C,r)d_r c(A,E_r,s), \quad C \in \mathcal{B}(Q). \tag{2.1}$$

(III) The structure measure is said to be commutative if

$$c(A,B,r) = c(B,A,r), \quad (A,B \in \mathcal{B}_0(Q)). \tag{2.2}$$

A measure m is said to be a multiplicative measure if

$$\int_Q c(A,B,r)dm(r) = m(A)m(B); \quad A,B \in \mathcal{B}_0(Q). \tag{2.3}$$

(IV) We will suppose the existence of a multiplicative measure. Under certain restrictions imposed on the commutative structure measure, multiplicative measure exists. (See [30]).

Consider the space $L_1(Q,dm(x))$ of functions on Q with respect to the multiplicative measure m. For any $\Phi,\Psi \in L_1(Q,dm(x))$, the convolution

$$\begin{aligned}(\Phi * \Psi)(r) &= \int_Q \Phi(p)dp \int_Q \Psi(q)dqc(E_p,E_q,r)\\ &= \int_Q\int_Q \Phi(p)\Psi(q)c(p,q,r)dm(p)dm(q)\\ &= \int_Q\int_Q \Phi(p)\Psi(q)dm_r(p,q)\end{aligned} \tag{2.4}$$

is well defined. (See [5]).

The space $L_1(Q,dm(x))$ with the convolution (2.4) is a Banach algebra which is commutative if (**III**) holds. This Banach algebra is called the hypercomplex system with the basis Q.

It is obvious that $c(A,B,r) = (K_A * K_B)(r); A,B \in \mathcal{B}_0(Q)$ and K_A is the characteristic function of the set A.

A hypercomplex system may or may not have a unity. If a unity not included in $L_1(Q,dm(x))$, then it is convenient to join it formally to $L_1(Q,dm(x))$.

A non zero measurable and bounded almost everywhere function $Q \ni r \mapsto \chi(r) \in \mathbb{C}$ is said to be a character of the hypercomplex system $L_1(Q,dm(x))$ if $\forall A,B \in \mathcal{B}_0(Q)$

$$\int_Q c(A,B,r)\chi(r)dm(r) = \chi(A)\chi(B), \tag{2.5}$$

$$\int_C \chi(r)dm(r) = \chi(C), \quad C \in \mathcal{B}_0(Q). \tag{2.6}$$

(V) A hypercomplex system is said to be normal, if there exists an involution homomorphism $Q \ni r \mapsto r^* \in Q$ such that $m(A) = m(A^*)$ and $c(A,B,C) = c(C,B^*,A), c(A,B,C) = c(A^*,C,B), (A,B \in \mathcal{B}_0(Q))$, where

$$c(A,B,C) = \int_C c(A,B,r)dm(r) \tag{2.7}$$

Ref

30. A. A. Kalyuzhnyi, A theorem on the existence of multiplicative measure, *Ukr. Math. Zh.*, **35**(1983), 369-371.

(VI) A normal hypercomplex system possesses a basis unity if there exists a point $e \in Q$ such that $e^* = e$ and

$$c(A, B, e) = m(A^* \cap B), \quad A, B \in \mathcal{B}(Q). \tag{2.8}$$

We should remark that, for a normal hypercomplex system, the mapping

$$L_1(Q, dm(x)) \ni \Phi(r) \mapsto \Phi^*(r) \in L_1(Q, dm(x))$$

is an involution in the Banach algebra $L_1(Q, dm(x))$, the multiplicative measure is unique and the characters of such a system are continuous. (See [5]). A character χ of a normal hypercomplex system is said to be Hermitian if

$$\chi(r^*) = \overline{\chi(r)}, \quad (r \in Q), \tag{2.9}$$

and ordinary if it is bounded. The function $\chi(r) = 1$, $r \in Q$ is always a character; for all χ, $\chi(e) = 1$.

Denote the set of all bounded Hermitian characters by X_h, i.e.,

$$X_h = \{\chi \in C_b(Q) : \chi \neq 0, \int_Q c(A, B, r)\chi(r)dm(r) = \chi(A)\chi(B), \overline{\chi(r)} = \chi(r^*)\}. \tag{2.10}$$

Let $L_1(Q, dm(x))$ be a hypercomplex system with basis Q and let Ω be a space of complex valued functions on Q. Assume that an operator valued function $Q \ni p \mapsto L_p : \Omega \to \Omega$ is given such that the function $g(p) = (L_p f)(q)$ belongs to Ω for any $f \in \Omega$ and any fixed $q \in Q$. For such hypercomplex system it is possible to introduce generalized translation operators L_p:

$$Q \times Q \ni \langle p, q \rangle \mapsto (L_p \Phi)(q) \in \mathbb{C}, \quad \Phi \in C(Q), \tag{2.11}$$

where $(L_p\Phi)(q) = \Phi(q + p)$ in case of the usual translation on $Q = \mathbb{R}$. We will suppose, in addition, that this function is separately continuous. By using the operators L_p one can rewrite the convolution (2.4) as follows:

$$\big(\Phi * \Psi\big)(p) = \int_Q \Phi(q)\big(L_{q^*}\Psi\big)(p)\, dm(q) \quad \Phi, \Psi \in L_1(Q, dm(x)). \tag{2.12}$$

A generalized character of a hypercomplex system $L_1(Q, dm(x))$ is defined to be a function $\chi \in C(Q)$ for which

$$(L_p\chi)(q) = \chi(p)\chi(q), \qquad p, q \in Q. \tag{2.13}$$

Now, we give Some examples to illustrative the concept of hypercomplex systems

Example 2.1. Let $Q = G$ be commutative, locally compact group. with unite e. Consider G is groups algebra, i.e., a set $L_1(G, m)$ of functions defined on the groups G with respect to the measure m is denoted by Haar measure $dm(p)$. Then, we can define the involution with the form

$$G \ni p \longmapsto p^* = p^{-1}, \tag{2.14}$$

where

$$\big(L_p\Phi\big)(q) = \Phi(pq), \quad (p, q \in G), \tag{2.15}$$

Ref

5. Yu. M. Berezansky and A. A. Kalyuzhnyi, *Harmonic Analysis in Hypercomplex Systems*, Naukova Dumka,Kyiv, Ukrania, 1992. (in Russian; English transl.: Kluwer: Dordrecht-Boston-London, 1996).

then the convolution in (2.12) becomes in the form

$$\big(\Phi * \Psi\big)(p) = \int\limits_Q \Phi(pq)\Psi(q^*)d(q), \tag{2.16}$$

and we can define the structure measure as the following

$$C\big(A, B, r\big) = m\big(A^{-1}r \cap B\big), \tag{2.17}$$

where $A \in \mathcal{B}(G), r \in G$. Thus, we obtain a commutative hypercomplex System $L_1(G, m)$ with basis unity e.

Example 2.2. Consider the sturm-Liouvvlle equation in the quadratic axis $Q = \mathbb{R}_+ = [0, \infty)$:

$$(Mu)(x) = -u'' + g(x)u = \lambda u, \tag{2.18}$$

where $g \in C(\mathbb{R}_+)$ is a non-negative and non-increasing potential. connect with this equation the following hyperbolic equation on $\mathbb{R}^2 : M_{(x)}u = M_{(y)}v$, i.e.,

$$-\frac{\partial^2 v}{\partial x^2} + g(x)v = -\frac{\partial^2 v}{\partial y^2} + g(y)v, \quad v = v(x, y), \tag{2.19}$$

where $x, y \in \mathbb{R}$, g is the even extension of the potential to the whole axis. Equation (2.19) gives rise to a generalized translation operator $L_{(x)}$, $x \in \mathbb{R}_+$: by definition

$$\big(L_{(x)}f\big)(y) = \frac{v\big(x, y\big)}{v(x)v(y)}, \quad x, y \in \mathbb{R}_+, \tag{2.20}$$

where $v(x, y)$ is the solution of the Cauchy problem for (2.19) with initial conditions $v(x, 0) = f(x)v(x)$, $\frac{\partial v}{\partial y}(x, 0) = 0$, and $v(x) = \phi(x, 0)$, where $\phi(x, \lambda)$ is the solution of the following problem on $\mathbb{R}_+ : M\phi = \lambda\phi$, $\phi(0, \lambda) = 1$, $\phi'(0, \lambda) = 0$ $\big($both functions $f(x)$ and $v(x)$ being evenly extended to $\mathbb{R}\big)$. Now, for any $x \in \mathbb{R}_+$, $x^* = x, dm(x) = v^2(x)d(x)$, and $\chi(x) = \phi\big(x, \lambda\big)$, where $\lambda \in \mathbb{C}$ is arbitrary, the convolution for hypercomplex system with basis $Q = \mathbb{R}_+$ is constructed by (2.12) or (2.4) with some structure measure c.

Example 2.3. Consider the special case Example 2.2, when $g = 0$. Then, define generalized translation operators as following

$$(L_{(x)}f)(y) = \frac{1}{2}\Big(f\big(|y - x|\big) + f\big(y + x\big)\Big), \quad x, y \in Q = \mathbb{R}_+, \tag{2.21}$$

and

$$\phi(x, \lambda) = \cos\Big(\sqrt{\lambda}x\Big), \quad v(x) = 1, \quad x \in \mathbb{R}_+, \ \lambda \in \mathbb{C}. \tag{2.22}$$

In fact, there are many examples of hypercomplex systems. For more details (see [5]).

III. Spaces of Test and Generalized Functions

The classical approach to white noise analysis [4, 25, 26] consists in the construction of some rigging of the Fock space $F(H)$ and in the consequent application to the spaces of this rigging the Wiener-Itô- Segal transform I. This operator transfers the space $F(H)$ to the space $L_2(\Omega', d\gamma(x))$ and the spaces of rigging to some spaces of test and generalized functions. The pairing between test and generalized functions is given by the scalar product in $L_2(\Omega', d\gamma(x))$. Here, Ω' is the dual space of a real nuclear space Ω and γ is a Gaussian measure on Ω'.

Ref

4. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral methods in infinite dimensional analysis*, vol. 1, 2, Kluwer, Dordrecht 1995.

In the model one-dimensional case the space $F(H)$ is equal to the space l_2 of complex-valued sequences $\varphi = (\varphi_0, \varphi_1, ...), \Omega' = \mathbb{R}$, and the Wiener-Itô- Segal transform has the form

$$l_2 \ni \varphi = (\varphi_0, \varphi_1, ...) \mapsto (I\varphi)(x) = \sum_{n=0}^{\infty} \varphi_n H_n(x) \in L_2(\mathbb{R}, d\gamma(x)), \ x \in \mathbb{R}, \tag{3.1}$$

where $\big(H_n(x)\big)_{n=0}^{\infty}$ is the sequence of orthonormal Hermite polynomials with respect to γ. The operator I transfers the rigging of l_2 to that of $L_2(\mathbb{R}, d\gamma(x))$, which gives a model of white noise analysis.

The generalization of white noise analysis proposed in [1], consists in the following. Let ρ be a probability measure on $\mathbb{R}$ (instead of the Gaussian measure γ) and let $(P_n(x), Q_n(x))_{n=0}^{\infty}$ $(x \in \mathbb{R})$ be the biorthogonal system of functions constructed in some canonical way. The biorthogonality means

$$\int_{\mathbb{R}} P_n(x) Q_m(x) d\rho(x) = \delta_{n,m}.\ n!, \quad , \quad m, n \in \mathbb{Z}_+. \tag{3.2}$$

Instead of the functions P_n, we take the Appell polynomials connected with the measure ρ, i.e. the coefficients of the power expansion with respect to $\lambda \in \mathbb{C}$

$$\widehat{\rho}(\lambda) = \int_{\mathbb{R}} e^{\lambda x} d\rho(x), \qquad \frac{e^{\lambda x}}{\widehat{\rho}(\lambda)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} P_n(x), \quad x \in \mathbb{R}. \tag{3.3}$$

The functions Q_n are defined by

$$Q_n(x) = \left(\left(\left(\frac{d}{dx} \right)^* \right)^n 1 \right)(x), \tag{3.4}$$

where the adjoint operator is considered in the $L_2(\mathbb{R}, d\rho(x))$ sense.

The operator I is defined now just as above, but with changing H_n by P_n. This operator transfers the rigging of l_2 to a rigging of $L_2(\mathbb{R}, d\rho(x))$, which gives some theory of generalized functions on $\mathbb{R}$ with pairing determinated by $L_2(\mathbb{R}, d\rho(x))$.

In this section, we suggest a generalization of the above mentioned construction in which instead of the function $e^{\lambda x}$ the character of some pretty arbitrary, commutative hypercomplex system is used. Such a generalization gives the possibility of constructing a lot of spaces of generalized functions connected with different examples of hypercomplex system.

Consider a subclass of the above hypercomplex systems for which the set of generalized characters is in one-to-one correspondence with the complex plane $\mathbb{C}$: $\chi \longleftrightarrow \lambda \in \mathbb{C}$; denote this character by $\chi(x, \lambda)$. We assume that $\chi(x, 0) = 1$ $(x \in Q)$, i.e., the unit character corresponds to $\lambda = 0$. We suppose the function $Q \times \mathbb{C} \ni \langle x, \lambda \rangle \mapsto \chi(x, \lambda) \in \mathbb{C}$ is continuous, and the function $\mathbb{C} \ni \lambda \mapsto \chi(x, \lambda) \in \mathbb{C}$ is an entire for each $x \in Q$. Thus, for each $x \in Q$ the following expansion holds

$$\chi(x, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \chi_n(x), \quad \lambda \in \mathbb{C}, \tag{3.5}$$

where the coefficients $\chi_n \in C(Q)$ called the Delsarte characters [6]. It is possible, of course, to give a direct definition of the Delsarte characters if we will rewrite Eq.(2.13) in terms of χ_n.

Our purpose is now to construct the expansions of functions on Q using the Delsarte characters and to introduce the corresponding spaces of test and generalized functions.

Ref

6. Yu. M. Berezansky, A generalization of white noise analysis by means of theory of hypergroups, *Reports on Mathematical Physics* **38** (1996), 289-300.

Let ρ be a fixed Borel probability measure ($\rho(Q) = 1$), positive on open sets, which is not connected directly with the multiplicative measure m . We suppose that χ_n's belong to $L_2(Q, d\rho(x))$, satisfy the estimate

$$\exists C > 0 : \|\chi_n\|_{L_2(Q,d\rho(x))} = \left(\int_Q |\chi_n(x)|^2 d\rho(x) \right)^2 \leq C^n n!, \quad n \in \mathbb{Z}_+. \tag{3.6}$$

(explaining that always $\|\chi_n\|_{L_2(Q,d\rho(x))} = 1$), are linearly independent and the system $(\chi_n)_n^\infty$ is total in $L_2(Q, d\rho(x))$.

Note that for a large class of hypercomplex systems (the Taylor-Delsarte hypercomplex system) vectors χ_n are automatically linearly independent. If in the estimate (3.6) instead of $n!$ the quantity $(n!)^\epsilon$ with fixed ($\epsilon < 1$) stands or if supp ρ is a compact set, then also automatically the system $(\chi_n)_n^\infty$ is total. In general we will understand the above mentioned conditions as some conditions on the measure ρ.

Now, we use the simple constructions connected with Hilbert rigging [4].

Let us consider a rigging of a Hilbert space H_0 with positive and negative spaces H_+ and H_- as following

$$H_- \supseteq H_0 \supseteq H_+. \tag{3.7}$$

Let $V : H_- \longrightarrow H_+$ be the canonical isometry transferring H_- onto H_+ [4]. A biorthogonal basis $(p_n, q_n)_{n=0}^\infty$ in the space H_0 is to be understood as sequences $(p_n)_{n=0}^\infty \subset H_+$ and $(q_n = V^{-1}p_n)_{n=0}^\infty \subset H_-$, where the first one is an orthogonal basis in H_+ and therefore, the second one is an orthogonal basis in H_-.

Note that these systems of vectors p_n and q_n are biorthogonal (with respect to H_o):

$$(p_n, q_n)_{H_0} = \delta_{n,m} h_n, \quad h_n = \|p_n\|^2_{H_+} = \|q_n\|^2_{H_-}, \quad n, m \in \mathbb{Z}_+, \tag{3.8}$$

$$\forall \varphi \in H_+, \quad \varphi = \sum_{n=0}^\infty \varphi_n p_n, \quad \varphi_n = (\varphi, q_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^\infty |\varphi_n|^2 h_n = \|\varphi\|^2_{H_+} < \infty,$$

$$\forall \xi \in H_-, \quad \xi = \sum_{n=0}^\infty \xi_n q_n, \quad \xi_n = (\xi, p_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^\infty |\xi_n|^2 h_n = \|\xi\|^2_{H_-} < \infty,$$

$$(\xi, \varphi)_{H_0} = \sum_{n=0}^\infty \xi_n \overline{\varphi_n} h_n. \tag{3.9}$$

This definition is justified by the following.

Let $(p_n)_{n=0}^\infty$ be an arbitrary total sequence of vectors p_n of a Hilbert space H_0. It is easy to prove that such sequence $(h_n)_{n=0}^\infty$ of positive numbers h_n exists for which the set of test functions

$$H_+ = \left\{ \varphi = \sum_{n=0}^\infty \varphi_n \, p_n \mid \varphi_n \in \mathbb{C} : \|\varphi\|^2_{H_+} = \sum_{n=0}^\infty |\varphi_n|^2 \, h_n < \infty \right\}, \tag{3.10}$$

with the corresponding scalar product is the positive space with respect to H_0 and therefore the chain (3.7) exists. Note that it is necessary to assume in addition the fulfilment of the following necessary and sufficient condition on $(p_n)_{n=0}^\infty$: an arbitrary sequence $(\varphi^{(i)})_{i=0}^\infty$ of vectors $\varphi^{(i)} \in H_+$ with finite sequences of coordinates $\varphi_n^{(i)}$ which is fundamental in H_+ and converges to 0 in H_0 must converge to 0 in H_+. This condition will always be fulfilled in our case.

Ref

4. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral methods in infinite dimensional analysis*, vol. 1, 2, Kluwer, Dordrecht 1995.

Similarly for negative space H_- by replacement p_n by q_n, we have the set of generalized functions as follows

$$H_- = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n \, q_n \mid \xi_n \in \mathbb{C} : \|\xi\|_{H_-}^2 = \sum_{n=0}^{\infty} |\xi_n|^2 \, h_n < \infty \right\}. \tag{3.11}$$

Therefore, we can construct a rigging of the space $H_0 := L_2(Q, d\rho(x))$ of the form (3.7), (3.10) and (3.11) in which the Delsarte characters $\chi_n(x)$ will play the role of the vectors p_n. This result can be stated as follows:

Theorem 3.1. There exists a quasinuclear rigging (3.7) such that $H_0 = L_2(Q, d\rho(x))$,

$$H_{-1}^{\chi} \supseteq H_0 \supseteq H_1^{\chi}, \tag{3.12}$$

and the space $H_1^{\chi} = H_+$ is of the form (3.10), where

$$p_n = \chi_n, \quad h_n = (n!)^2 K^n, \quad n \in \mathbb{Z}_+, \tag{3.13}$$

($K > 1$ is a fixed sufficiently large number), and consists of continuous functions on Q.

The system $(\chi_n, q_n^{\chi})_{n=0}^{\infty}$, where $q_n^{\chi} = V^{-1}\chi_n$ (V is connected with (3.12)), is a biorthogonal basis of the space H_0. For the vectors q_n^{χ}, the following representation holds:

$$q_n^{\chi} = (n!) K^n \Theta_n, \quad \Theta_n = (D^+)^n \delta_e \in H_{-1}^{\chi}, \quad n \in \mathbb{Z}_+, \tag{3.14}$$

where Θ_n are "Delsarte co-characters" connected with the hypercomplex system $L_1(Q, d\rho(x))$, and the measure ρ. Here D is a continuous operator acting on the space H_1^{χ} and is defined by the expression

$$D\chi_n = n\chi_{n-1}, \quad n \in \mathbb{N}, \quad D\chi_0 = 0, \tag{3.15}$$

$D^+ : H_{-1}^{\chi} \longrightarrow H_{-1}^{\chi}$ is its dual operator with respect to the rigging (3.12). The space H_1^{χ} consists of continuous functions, therefore the δ-function δ_x, lumped at the point $x \in Q$, exist as an element of the negative space H_{-1}^{χ}.

It is essential to introduce the rigging of the space H_0 by means of projective and inductive limits of Hilbert spaces which are constructed by rules of type (3.10), (3.12) and (3.13). Namely, for every $q \in \mathbb{N}$ we introduce the Hilbert space of type (3.10):

$$H_q^{\chi} = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \chi_n \in H_0 : \|\varphi\|_{H_q^{\chi}}^2 = \sum_{n=0}^{\infty} |\varphi_n|^2 (n!)^2 K^{qn} < \infty \right\}. \tag{3.16}$$

Then we have the nuclear rigging:

$$(\Psi^{\chi})' \supseteq H_{-q}^{\chi} \supseteq H_0 \supseteq H_q^{\chi} \supseteq \Psi^{\chi}, \tag{3.17}$$

$$\Psi^{\chi} = \operatorname{pr}\lim_{q \in \mathbb{N}} H_q^{\chi} = \bigcap_{q \in \mathbb{N}} H_q^{\chi}, \qquad (\Psi^{\chi})' = \operatorname{ind}\lim_{q \in \mathbb{N}} H_{-q}^{\chi} = \bigcup_{q \in \mathbb{N}} H_{-q}^{\chi},$$

$$H_{-q}^{\chi} = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} : \|\xi\|_{H_{-q}^{\chi}}^2 = \sum_{n=0}^{\infty} |\xi_n|^2 (n!)^2 K^{-qn} < \infty \right\}, \tag{3.18}$$

Notes

with the action

$$(\xi,\varphi)_{H_0}=\sum_{n=0}^{\infty}\xi_n\overline{\varphi_n}(n!)^2K^{qn},\quad \varphi\in H_q^{\chi},\ \ \xi\in H_{-q}^{\chi}.$$

Note that due to the first equality in (3.8) for Delsarte characters and co-characters the following biorthogonality is true:

$$(\Theta_n,\chi_n)_{H_0}=\delta_{n,m}.n!,\quad n,m\in\mathbb{Z}_+. \tag{3.19}$$

To illustrate the above result, we give the following example

Example 3.1. Let $H_0=L_2(\mathbb{R},dx)$, with respect to the Lebesgue measure dx and ordinary convolution given in Eq.(2.12) the generalized character $\chi(x,\lambda)=e^{\lambda x}$ $(\lambda\in\mathbb{C})$ and $\chi_n(x)=x^n$ $(x\in\mathbb{R},\ n\in\mathbb{Z}_+)$. Therefore, the space (3.16) consists of entire functions $\varphi(x)$ and $\varphi_n(x)$ are the Taylor coefficients of $\varphi(x)$. Formula (3.8) gives their representation as the Fourier coefficients using the scalar product $(\xi,\varphi)_{H_0}$, $(\xi\in H_{-1}^{\chi},\ \varphi\in H_1^{\chi})$. The operator D now is equal d/dx.

IV. Elements of Wick Calculus

The Wick product was first introduced by Wick [39] and used as a tool to renormalize certain infinite quantities in quantum field theory. Later on, the Wick product was considered, in a stochastic setting, by Hida and Ikeda [24]. In [11], Dobroshin and Minlos were comprehensively treated this subject both in mathematical physics and probability theory. Currently, the Wick product provides a useful concept for various applications, for example, it is important in the study of stochastic ordinary and partial differential equations (see, e.g., [27]).

In this section, under the assumption that $\|\chi\|_{H_0}^2\le C^n$ for some $C>0$, we define a new Wick product, called χ-Wick product, on the space H_{-q}^{χ}, which is constructed in Section 3. Then, we give the definition of the χ-Hermite transform and apply it to establish a characterization theorem for the space H_{-q}^{χ}.

Definition 4.1. Let $\xi=\sum_{m=0}^{\infty}\xi_m q_m^{\chi}$, $\eta=\sum_{n=0}^{\infty}\eta_n q_n^{\chi}\in H_{-q}^{\chi}$ with $\xi_m,\eta_n\in\mathbb{C}$. The χ-Wick product of ξ, η, denoted by $\xi\diamond_{\chi}\eta$, is defined by the formula

$$\xi\diamond_{\chi}\eta=\sum_{m,n=0}^{\infty}\xi_m\eta_n q_{m+n}^{\chi}. \tag{4.1}$$

It is important to show that the spaces $H_{-q}^{\chi}, H_q^{\chi}$ are closed under χ-Wick product.

Lemma 4.1. If $\xi,\eta\in H_{-q}^{\chi}$ and $\varphi,\psi\in H_q^{\chi}$, *we have*

(i) $\xi\diamond_{\chi}\eta\in H_{-q}^{\chi}$,

(ii) $\varphi\diamond_{\chi}\quad\in H_q^{\chi}$.

Proof. If $\xi=\sum\limits_{m=0}^{\infty}\xi_m q_m^{\chi}$, $\eta=\sum\limits_{n=0}^{\infty}\eta_n q_n^{\chi}\in H_{-q}^{\chi}$, then for some $q_1\in\mathbb{N}$ we have

$$\sum_{m=0}^{\infty}|\xi_m|^2K^{-q_1m}<\infty \text{ and } \sum_{n=0}^{\infty}|\eta_n|^2K^{-q_1n}<\infty. \tag{4.2}$$

Ref

39. G. C. Wick, The evaluation of the collinear matrix, *Phys. Rev.* **80** (1950), 268-272.

We note that

$$\xi \diamond_\chi \eta = \sum_{m,n=0}^{\infty} \xi_m \eta_n q^\chi_{m+n} = \sum_{l=0}^{\infty} \left(\sum_{m+n=l}^{\infty} \xi_m \eta_n \right) q^\chi_l = \sum_{l=0}^{\infty} \zeta_l q^\chi_l, \tag{4.3}$$

where $\zeta_l = \sum_{m+n=l}^{\infty} \xi_m \eta_n$. With $q = q_1 + p$ we have

$$\begin{aligned} \sum_{l=0}^{\infty} |\zeta_l|^2 K^{-ql} &= \sum_{l=0}^{\infty} \left| \sum_{m+n=l}^{\infty} \xi_m \eta_n \right|^2 K^{-q_1 l} K^{-pl} \\ &\le \sum_{l=0}^{\infty} \left(\sum_{m+n=l}^{\infty} |\xi_m|^2 K^{-q_1 m} \right) \left(\sum_{m+n=l}^{\infty} |\eta_n|^2 K^{-q_1 n} \right) K^{-pl} \\ &\le \left(\sum_{l=0}^{\infty} K^{-pl} \right) \left(\sum_{m=0}^{\infty} |\xi_m|^2 K^{-q_1 m} \right) \left(\sum_{n=0}^{\infty} |\eta_n|^2 K^{-q_1 n} \right) \\ &< \infty, \end{aligned} \tag{4.4}$$

which proves (i). The proof of (ii) is similar. ■

The following important algebraic properties of the χ-Wick product follow directly from Definition 4.1.

Lemma 4.2. For each $\xi, \eta, \zeta \in H^\chi_{-q}$, we get

(i) $\xi \diamond_\chi \eta = \eta \diamond_\chi \xi$ *(Commutative law),*

(ii) $\xi \diamond_\chi (\eta \diamond_\chi \zeta) = (\xi \diamond_\chi \eta) \diamond_\chi \zeta$ *(Associative law),*

(iii) $\xi \diamond_\chi (\eta + \zeta) = \xi \diamond_\chi \eta + \xi \diamond_\chi \zeta$ *(Distributive law).*

Remark 4.1. According to Lemmas 4.1 and 4.2, we can conclude that the spaces H^χ_{-q} and H^χ_q form topological algebras with respect to the χ-Wick product.

As shown in Lemmas 4.1 and 4.2, the χ-Wick product satisfies all the ordinary algebraic rules for multiplication. Therefore, one can carry out calculations in much the same way as with usual products. But, there are some problems when limit operations are involved. To treat these situations it is convenient to apply a transformation, called the χ-Hermite transform, which converts χ-Wick products into ordinary (complex) products and convergence in H^χ_{-q} into bounded, pointwise convergence in a certain neighborhood of 0 in $\mathbb{C}$. The original Hermite transform, which first appeared in Lindstrøm et al. [29], has been applied by the authors in many different connections. Now, we give the definition of the χ-Hermite transform and discuss its basic properties.

Definition 4.2. Let $\xi = \sum_{n=0}^{\infty} \xi_n q^\chi_n \in H^\chi_{-q}$ with $\xi_n \in \mathbb{C}$. Then, the χ-Hermite transform of ξ, denoted by $\mathcal{H}_\chi \xi$, is defined by

$$\mathcal{H}_\chi \xi(z) = \sum_{n=0}^{\infty} \xi_n z^n \in \mathbb{C} \quad \text{(when convergent)}. \tag{4.5}$$

In the following we define for $0 < M, q < \infty$ the neighborhoods $\mathbb{O}_q(M)$ of zero in $\mathbb{C}$ by

$$\mathbb{O}_q(M) = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} |z^n|^2 K^{qn} < M^2 \right\}. \tag{4.6$$

Ref

29. T. Lindstrom, B. Oksendal and J. Uboe, Stochastic differential equations involving positive noise. In M. Barlow and N. Bingham (editors): *Stochastic Analysis*. Cambridge University Press: Cambridge, MA, (1991), 261-303.

It is easy to see that

$$q \le p,\ N \le M \Rightarrow \mathbb{O}_p(N) \subseteq \mathbb{O}_q(M). \tag{4.7}$$

Note that if $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}, z \in \mathbb{O}_q(M)$ for some $0 < M, q < \infty$, we have the estimate

$$\begin{aligned} \sum_{n=0}^{\infty} |\xi_n||z^n| &= \sum_{n=0}^{\infty} |\xi_n||z^n| K^{-\frac{qn}{2}} K^{\frac{qn}{2}} \\ &\le \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn}\right)^{\frac{1}{2}} \\ &< M\left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn}\right)^{\frac{1}{2}} \\ &< \infty. \end{aligned} \tag{4.8}$$

The conclusion above can be stated as follows:

Proposition 4.1. If $\xi \in H_{-q}^{\chi}$, then $\mathcal{H}_{\chi}\xi$ converges for all $z \in \mathbb{O}_q(M)$ for all $q, M < \infty$.

A useful property of the χ-Hermite transform is that it converts the χ-Wick product into ordinary (complex) product.

Proposition 4.2. If $\xi, \eta \in H_{-q}^{\chi}$, then

$$\mathcal{H}_{\chi}(\xi \diamond_{\chi} \eta)(z) = \mathcal{H}_{\chi}\xi(z).\mathcal{H}_{\chi}\eta(z). \tag{4.9}$$

for all z such that $\mathcal{H}_{\chi}\xi$ and $\mathcal{H}_{\chi}\eta$ exist.

Proof. The proof is an immediate consequence of Definitions 4.1 and 4.2. ■

Let $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}$, with $\xi_n \in \mathbb{R}$. Then, the number $\xi_0 = \mathcal{H}_{\chi}\xi(0) \in \mathbb{R}$ is called the generalized expectation of ξ and is denoted by $\mathbb{E}(\xi)$. Suppose that $V \ni z \mapsto f(z) \in \mathbb{C}$ is an analytic function, where V is a neighborhood of $\mathbb{E}(\xi)$. Assume that the Taylor series of f around $\mathbb{E}(\xi)$ has coefficients in $\mathbb{R}$. Then, the χ-Wick version $f^{\diamond_{\chi}}$ of f is defined by

$$H_{-q}^{\chi} \ni \xi \mapsto f^{\diamond_{\chi}}(\xi) = \mathcal{H}^{-1}\big(f \circ \mathcal{H}_{\chi}(\xi)\big) \in H_{-q}^{\chi}. \tag{4.10}$$

Example 4.1. If the function $f : \mathbb{C} \to \mathbb{C}$ is entire, then $f^{\diamond_{\chi}}$ is defined for all $\xi \in H_{-q}^{\chi}$. For example, the χ-Wick exponential is defined by

$$\exp^{\diamond_{\chi}}(\xi) = \sum_{j=0}^{\infty} \frac{1}{j!} \xi^{\diamond_{\chi} n}. \tag{4.11}$$

Using χ-Hermite transform we see that χ-Wick exponential has the same algebraic properties as the usual exponential. For instance,

$$\exp^{\diamond_{\chi}}(\xi + \eta) = \exp^{\diamond_{\chi}}(\xi) \diamond_{\chi} \exp^{\diamond_{\chi}}(\eta),\ \xi, \eta \in H_{-q}^{\chi}. \tag{4.12}$$

From Proposition 4.1, we can deduce that χ-Hermite transform of any $\xi \in H_{-q}^{\chi}$ is a complex-valued analytic function on $\mathbb{O}_q(M)$ for all $q, M < \infty$. Moreover, the converse of this deduction is true, i.e., every complex-valued analytic function on $\mathbb{O}_q(M)$ (for some $q, M < \infty$) is the χ-Hermite transform of some element in H_{-q}^{χ}. To prove this, we need the following two auxiliary results.

Lemma 4.3. Let $f(z) = \sum_{n=0}^{\infty} \eta_n z^n$ be an analytic function in $z \in \mathbb{C}$ such that there exists $M < \infty, C > 0$ and $\delta > 0$ such that $|f(z)| \leq M$ when $z \in \mathbb{O} := \{z \in \mathbb{C} : C|z| \leq \delta^2\}$. Then $|\eta_n z^n| \leq M$ for all $z \in \mathbb{O}$ and $n \in \mathbb{N}$.

Proof. See [27], Lemma 2.6.10.

Proposition 4.3. Let $f(z) = \sum_{n=0}^{\infty} \eta_n z^n$, $\eta_n \in \mathbb{C}$ be a formal power series in $z \in \mathbb{C}$. Suppose there exist $q, M < \infty$ and $\delta > 0$ such that $f(z)$ is convergent for $z \in \mathbb{O}_q(\delta)$ and $|f(z)| \leq M$ for all $z \in \mathbb{O}_q(\delta)$. Then

$$\sum_{n=0}^{\infty} |\eta_n z^n| \leq M A(q) \quad \text{for all } z \in \mathbb{O}_{3q}(\delta) \tag{4.13}$$

where

$$A(q) := \sum_{n=0}^{\infty} K^{-qn} < \infty \quad (\text{Note that } K > 1). \tag{4.14}$$

Proof. It is evident that $z \in \mathbb{O}_{3q}$ $\delta)$ implies $K^q z \in \mathbb{O}_q$ $\delta)$. According to Lemma 4.3, we get

$$\begin{aligned} \sum_{n=0}^{\infty} |\eta_n z^n| &\leq \left(\sum_{n=0}^{\infty} |\eta_n|^2 |z^n|^2 K^{qn} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} K^{-qn} \right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{\infty} |\eta_n|^2 |(K^q z)^n|^2 K^{-qn} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} K^{-qn} \right)^{\frac{1}{2}} \\ &\leq M \sum_{n=0}^{\infty} K^{-qn}. \end{aligned} \tag{4.15}$$

Theorem 4.1. **(Characterization Theorem for H_{-q}^{χ})** *If $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}$, where $\xi_n \in \mathbb{C}$, then there exist $q < \infty$ and $R_q < \infty$ such that*

$$|\mathcal{H}_{\chi}\xi(z)| \leq R_q \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn} \right)^{\frac{1}{2}} \quad \forall z \in \mathbb{C}. \tag{4.16}$$

In particular, $\mathcal{H}_{\chi}\xi$ is a bounded analytic function on $\mathbb{O}_q(M)$ for all $M < \infty$. Conversely, suppose $f(z) = \sum_{n=0}^{\infty} \eta_n z^n$ is a given analytic power series of $z \in \mathbb{C}$ with $\eta_n \in \mathbb{C}$ such that there exist $q < \infty$ and $\delta > 0$, such that $f(z)$ is absolutely convergent when $z \in \mathbb{O}_q(\delta)$ and

$$\sup_{z \in \mathbb{O}_q(\delta)} |f(z)| < \infty. \tag{4.17}$$

Then, there exists a unique $\eta \in H_{-q}^{\chi}$ such that $\mathcal{H}_{\chi}\eta = f$, namely

$$\eta = \sum_{n=0}^{\infty} \eta_n q_n^{\chi}. \tag{4.18}$$

Proof. For each $z \in \mathbb{C}$, we have

$$|\mathcal{H}_{\chi}\xi(z)| \leq \sum_{n=0}^{\infty} |\xi_n||z^n| \leq \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn} \right)^{\frac{1}{2}}. \tag{4.19}$$

Since $\xi \in H_{-q}^{\chi}$, we see that $R_q^2 := \sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn} < \infty$ for all $q < \infty$.

Ref

27. H. Holden, B. Osendal, J. Uboe and T. Zhang, *Stochastic partial differential equations*, Springer Science+Business Media, LLC, (2010).

Notes

Conversely, Since $K > 1$, then $K^{-r} \in \mathbb{O}_r(\delta)$ for all $r < \infty$ and for some $\delta < \infty$. By virtue of Proposition 4.3, we have

$$\sum_{n=0}^{\infty} |\eta_n||z^n| \leq MA(q) \quad \text{for all } z \in \mathbb{O}_{3q}(\delta). \tag{4.20}$$

Hence, for $r \geq 3q$ and $z \in \mathbb{O}_{3q}(\delta)$, we get

$$\sum_{n=0}^{\infty} |\eta_n|^2 K^{-rn} \leq C \sum_{n=0}^{\infty} |\eta_n| K^{-rn} \leq C \sum_{n=0}^{\infty} |\eta_n||z^n| \leq CMA(q) < \infty. \tag{4.21}$$

where $C := \sup\{|\eta_n| : n \in \mathbb{N}\}$, and hence $\eta := \sum_{n=0}^{\infty} \eta_n q_n^{\chi} \in H_{-q}^{\chi}$, as claimed. ■

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