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On Some Bicomplex Modules

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Abstract- A class of entire bicomplex sequences denoted by *B*, studied by Srivastava & Srivastava in 2007 is studied and is shown to be a bicomplex module. The subclasses of this class,studied by Wagh in 2008, are also shown to be bicomplex modules. Further they have been shown to form module structure over the class *B*.

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ONSOMEBICOMPLEXMODULES

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Ref

5. Srivastava, Rajiv K. : "Certain topological aspects of Bicomplex space", Bull. Pure & Appl. Math. Dec., 2(2), 2008, 222-234.

On Some Bicomplex Modules

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I. Section: C

Bicomplex Numbers were introduced by Corrado Segre(1860–1924) in 1892. In[3], he defined an infinite set of algebras and gave the concept of multicomplex numbers. For the sake of brevity, we confine ourselves to the bicomplex version of his theory. The space of bicomplex numbers is the first in an infinite sequence of multicomplex spaces. The set of bicomplex numbers is denoted by C_2 and is defined as follows:

$$C_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0\}$$

Or equivalently

$$C_2 = \{z_1 + i_2 z_2 : z_1, z_2 \in C_1\}$$

where $i_1^2 = i_2^2 = -1$, $i_1 i_2 = i_2 i_1$ and C_0, C_1 denote the space of real and complex numbers respectively.

The binary compositions of addition and scalar multiplication on C_2 are defined coordinate wise and the multiplication in C_2 is defined term by term. With these binary compositions, C_2 becomes a commutative algebra with identity. Algebraic structure of C_2 differs from that of C_1 in many respects [2]. Few of them, which pertain to our work, are mentioned below:

a) *Idempotent Elements*

Besides 0 and 1, there are exactly two nontrivial idempotent elements in C_2 defined as $e_1 = (1 + i_1i_2) / 2$, $e_2 = (1 - i_1i_2) / 2$.

Note that $e_1 + e_2 = 1$ and $e_1.e_2 = e_2.e_1 = 0$.

A bicomplex number $\xi = z_1 + i_2z_2$ has a unique idempotent representation, [5] as

$\xi = {}^1\xi\, e_1 + {}^2\xi\, e_2$ where ${}^1\xi = z_1 - i_1z_2$, ${}^2\xi = z_1 + i_1z_2$.

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b) Two Principal Ideals

The Principal Ideals in C_2 generated by e_1 and e_2 are denoted by I_1 and I_2 respectively; thus

$$I_1 = \{\xi e_1 : \xi \in C_2\},$$

$$I_2 = \{\xi e_2 : \xi \in C_2\}.$$

Since $\xi = {}^1\xi e_1 + {}^2\xi e_2$, where ${}^1\xi$ and ${}^2\xi$ are the idempotent components of ξ, therefore these ideals can also be represented as

$$I_1 = \{(z_1 - i_1 z_2)e_1 : z_1, z_2 \in C_1\} = \{z\, e_1 : z \in C_1\}$$

$$I_2 = \{(z_1 + i_1 z_2)e_2 : z_1, z_2 \in C_1\} = \{z\, e_2 : z \in C_1\}$$

Note that $I_1 \cap I_2 = \{0\}$ and $I_1 \cup I_2 = O_2$, the set of all singular elements of C_2.

c) Zero Divisors

As we have seen, $e_1 . e_2 = e_2 . e_1 = 0$. Thus zero divisors exist in C_2. In fact, two Bicomplex numbers are divisors of zero if and only if one of them is a complex multiple of e_1 and the other is a complex multiple of e_2. In other words, two Bicomplex numbers are divisors of zero if and only if one of them is a member of $I_1 \sim \{0\}$ and the other is a member of $I_2 \sim \{0\}$.

d) Conjugates of a Bicomplex number

The i_2- conjugate of a bicomplex number is defined in [2] as follows:

$$\xi^{\#} = z_1 - i_2 z_2 = {}^2\xi\, e_1 + {}^1\xi\, e_2 \text{ where } \xi = z_1 + i_2 z_2$$

e) Norm of a Bicomplex number

The norm in C_2 is defined as

$$\|\xi\| = \left\{ |z_1|^2 + |z_2|^2 \right\}^{1/2} = \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2} \right]^{1/2}$$

C_2 becomes a modified Banach algebra with respect to this norm in the sense that

$$\| \xi.\eta \| \le \sqrt{2} \| \xi \| . \| \eta \|$$

f) Entire functions

A function f of a bicomplex variable is said to be an entire function if it is holomorphic in the entire bicomplex space C_2.

g) Entire Bicomplex Sequence

If $f(\xi) = \sum_{k \ge 1} \alpha_k (\xi - \eta)^k$ represents an entire function, the series $\sum \alpha_k$ is called entire bicomplex series and the sequence $\{\alpha_k\}$ is called entire bicomplex sequence.

Ref

2. Price, G. Baley : "An Introduction to Multicomplex spaces and functions", Marcel Dekker, Inc., 1991.

II. Class B and the Bicomplex Modules

Let's see at these classes for the ready reference.

The classes B, B', B'' of entire Bicomplex sequences:

Srivastava & Srivastava [4] defined the class B as

$$B = \left\{ f : f = \{ \xi_k \} = \{ {}^1\xi_k e_1 + {}^2\xi_k e_2 \} : \sup_{k \ge 1} k^k \left| {}^1\xi_k \right| < \infty, \sup_{k \ge 1} k^k \left| {}^2\xi_k \right| < \infty \right\} \tag{2.1.1}$$

Every element of class B is the sequence of coefficients of an entire function and is, therefore, an entire bicomplex sequence.

Algebraic structure of B, given by [4]

Binary compositions on B are defined as follows:

Let $f = \{ \xi_k \}$ and $g = \{ \eta_k \}$ be arbitrary members of B and $a \in C_0$

1. Addition : $f + g = \{ \alpha_k \}$ where $\alpha_k = \xi_k + \eta_k, \forall k \ge 1.$
2. Scalar multiplication : $a.f = \{ \beta_k \}$ where $\beta_k = a.\xi_k$, $\forall\, k \ge 1$, $a \in C_0$.
3. Weighted Hadamard Multiplication : $f \times g = \{ \gamma_k \}$,

where $\gamma_k = k^k \ \xi_k \times \eta_k, \forall k \ge 1$

They have shown that B is a commutative algebra with identity, the element $u = \{ k^{-k} \}$ being the identity element of B.

Two subclasses of the class B have been defined in [6] as follows:

$$B' = \left\{ f : f = \{ {}^1\xi_k e_1 \} : \sup_{k \ge 1} k^k \ \left| {}^1\xi_k \right| < \infty \right\} \tag{2.1.2}$$

$$B'' = \left\{ f : f = \{ {}^2\xi_k e_2 \} : \sup_{k \ge 1} k^k \left| {}^2\xi_k \right| < \infty \right\} \tag{2.1.3}$$

The elements of B' and B'' are the sequences with members in A_1 and A_2, respectively where A_1 and A_2 are the auxiliary space.

Note first that B' is closed with respect to the binary compositions induced on B' as a subset of B, owing to the consistency of idempotent representation and the algebraic structure of bicomplex numbers.

Norm in B' is defined as follows:

$$\| f \| = \sup_{k \ge 1} \left| {}^1\xi_k \right| , \ f = \{ {}^1\xi_k . e_1 \} \in B'.$$

B' is Gel'fand subalgebra of B [7]. B' is an algebra ideal of B which is not a maximal ideal [7]. Zero divisors, Invertible and quasi invertible elements have also been characterised for this subclass [7].

Ref

4. Srivastava, Rajiv K. & Srivastava, Naveen K. : "On a class of Entire Bicomplex sequences", South East. Asian J. Math & Math. Sc. 5(3), 2007, pp. 47-68.

Definition 2.1.1: Bicomplex Modules ($BC-$ module or $T-$ module)

A $BC-$ module X over the ring BC of bicomplex numbers consists of an abelian group *(X, +)* and an operation $BC\times X\rightarrow X$ such that for all $\xi,\eta\in BC$ and $x,y\in X$, we have

1. $\xi(x+y)=\xi x+\xi y$
2. $(\xi+\eta)x=\xi x+\eta x$
3. $(\xi\eta)x=\xi(\eta x)$
4. $1_{BC}\, x=x$, 1_{BC} is the multiplicative identity of BC.

The members of X are known as vectors and members of C_2 are known as scalar and in most of the books scalar are always in left and vector in right side, for example, $(x\,y)\xi=x(y\,\xi)$. As the ring of bicomplex numbers is commutative, we don't need to define left or right BC - modules.

If X is a BC - module, then some structural peculiarities of the set BC are immediately manifested in any bicomplex module, in contrast to the case of real, complex or even quaternionic linear spaces where the structure of linear space does not imply anything immediate about the space itself.

Consider the sets

$$X_{e_1}=e_1.X \text{ and } X_{e_2}=e_2.X .$$

Since, we know that

$$X_{e_1}\cap X_{e_2}=\{0\}$$

and

$$X=X_{e_1}+X_{e_2}, \qquad (2.1.4)$$

We can define two mappings:

$$P:X\rightarrow X\ ,\ Q:X\rightarrow X$$

by

$$P(x)=e_1 x \qquad Q(x)=e_2 x.$$

Since

$$P+Q=Id_X\, ,P\circ Q=Q\circ P=0, P^2=P, Q^2=Q\ ,$$

the operators P and Q are mutually complementary projectors. (2.1.4) is called the idempotent decomposition of *X*, and it will play an important role in the development of the theory of bicomplex duals.

Consider the component – wise operations on *X*:

If $x=e_1 x+e_2 x\in X\, ,y=e_1 y+e_2 y\in Y$ and if $\lambda=\lambda_1 e_1+\lambda_2 e_2$, then

$$x+y=e_1 x+e_2 x+e_1 y+e_2 y=(e_1 x+e_1 y)+(e_2 x+e_2 y),$$

$$\lambda x=(\lambda_1 e_1+\lambda_2 e_2)(e_1 x+e_2 x)=\lambda_1 x e_1+\lambda_2 x e_2 .$$

Notes

Notes

Since, X_{e_1} and X_{e_2} are $\mathrm{R}-, C_1(i_1)-$ and $C_1(i_2)-$ linear spaces as well as BC - modules , we have that

$$X = X_{e_1} \oplus X_{e_2} \;,$$

where the direct sum $\oplus$ can be understood in the sense of $\mathrm{R}-, C_1(i_1)-$ and $C_1(i_2)-$ linear spaces, as well as BC - modules. By saying that X_{e_1} and X_{e_2} are $\mathrm{R}-, C_1(i_1)-$ and $C_1(i_2)-$ linear spaces as well as BC - modules, we mean that X_{e_1} and X_{e_2} are linear spaces over $\mathrm{R}, C_1(i_1), C_1(i_2)$ and are modules over the ring of bicomplex numbers.

If we consider X as a $C_1(i_1)-$ linear space, we denote it as $X_{C_1(i_1)}$ and if it is considered as $C_1(i_2)-$ linear space, then it is denoted as $X_{C_1(i_2)}$.

Theorem 2.1.1: Prove that the class B is a BC - module.

Proof: Let us take two bicomplex scalars $\alpha, \beta \in BC$ and $f = (\xi_k), g = (\eta_k) \in B$. *(B, +)* is an abelian group.

The operation $BC \times B \to B$ is well defined as

$\sup_k k^k \|\alpha f\| = \|\alpha\| \sup_k k^k \|f\| < \infty, \because f \in B$. This implies that $\alpha f \in B$

Now,

1. $\alpha(f+g) = \alpha(\xi_k + \eta_k) = \alpha\xi_k + \alpha\eta_k = \alpha f + \alpha g$,

   also $\sup_k k^k \|\alpha f + \beta g\| = \sup_k k^k \|\alpha\xi_k + \beta\eta_k\| \le \|\alpha\| \sup_k k^k \|\xi_k\| + \|\beta\| \sup_k k^k \|\eta_k\| < \infty$
2. $(\alpha+\beta) f = (\alpha+\beta)\xi_k = (\alpha\xi_k + \beta\xi_k) = \alpha f + \beta f$.
3. $(\alpha\beta) f = \alpha(\beta f)$
4. $1.f = f.$

Thus, we can say that B is a BC - module.

Theorem 2.1.2: Prove that the class B' is a BC - module.

Proof: Let $\alpha, \beta \in BC$ be two bicomplex scalars and let $f = ({}^1\xi_k\, e_1), g = ({}^1\eta_k\, e_1) \in B'$.
(B, +) is an abelian group.

The operation $BC \times B' \to B'$ is well defined as

$$\sup_k k^k \|\alpha f\| = \sup_k k^k \left\| ({}^1\alpha e_1 + {}^2\alpha e_2)({}^1 f\, e_1 + {}^2 f\, e_2) \right\|$$

$$= \sup_k k^k \left\| {}^1\alpha\, {}^1\xi_k\, e_1 + {}^2\alpha . 0\, e_2 \right\|, \because {}^2 f = 0$$

$$= \left|{}^1\alpha\right| \sup_k k^k \left|{}^1\xi_k\right| < \infty$$

This implies that $\alpha f \in B'$.

Now,

1. $\alpha(f+g) = \alpha({}^1\xi_k + {}^1\eta_k)e_1 + 0\, e_2$

$$= {}^1\alpha\,{}^1\xi_k\, e_1 + {}^1\alpha{}^1\eta_k\, e_1 + {}^2\alpha\, 0 e_2 + {}^2\alpha\, 0 e_2$$

$$= \left({}^1\alpha\,{}^1\xi_k\, e_1 + {}^2\alpha\, 0 e_2\right) + \left({}^1\alpha\,{}^1\eta_k\, e_1 + {}^2\alpha\, 0 e_2\right)$$

$$= \left({}^1\alpha\, e_1 + {}^2\alpha\, e_2\right)\left({}^1\xi_k\, e_1 + 0 e_2\right) + \left({}^1\alpha\, e_1 + {}^2\alpha\, e_2\right)\left({}^1\eta_k\, e_1 + 0 e_2\right)$$

$$= \alpha f + \alpha g$$

Also,

$$\sup_k k^k \left\|\alpha f + \alpha g\right\| = \sup_k k^k \left\|\alpha\,{}^1\xi_k\, e_1 + \alpha\,{}^1\eta_k e_1\right\| \le \left|{}^1\alpha\right| \sup_k k^k \left|{}^1\xi_k\right| + \left|{}^1\alpha\right| \sup_k k^k \left|{}^1\eta_k\right| < \infty$$

2. $(\alpha+\beta)\, f = (\alpha+\beta)\,{}^1\xi_k\, e_1 = ({}^1\alpha\,{}^1\xi_k + {}^1\beta\,{}^1\xi_k)e_1 + ({}^2\alpha.0 + {}^2\beta.0)e_2$.

$$= ({}^1\alpha\, e_1 + {}^2\alpha\, e_2)\,{}^1\xi_k\, e_1 + ({}^1\beta\, e_1 + {}^2\beta\, e_2)\,{}^1\xi_k\, e_1$$

$$= \alpha f + \beta f$$

3. $(\alpha\beta)\, f = \left({}^1\alpha\,{}^1\beta\, e_1 + {}^2\alpha\,{}^2\beta\, e_2\right){}^1\xi_k\, e_1 = ({}^1\alpha\, e_1 + {}^2\alpha\, e_2).({}^1\beta\, e_1 + {}^2\beta\, e_2)\,{}^1\xi_k\, e_1$

$= \alpha(\beta f)$.

4. *1.f* $= f$, where *1* is the identity element of BC.

Thus, we can say that B' is a BC - module.

Theorem 2.1.3: Prove that the class B' is a BC - module.

Proof: Let $\alpha, \beta \in BC$ be two bicomplex scalars and $f = ({}^2\xi_k\, e_2), g = ({}^2\eta_k\, e_2) \in B''$. *(B′, +)* is an abelian group.

The operation $BC \times B'' \to B''$ is well defined as

$$\sup_k k^k \left\|\alpha f\right\| = \sup_k k^k \left\|({}^1\alpha e_1 + {}^2\alpha e_2)({}^1 f\, e_1 + {}^2 f\, e_2)\right\|$$

$$= \sup_k k^k \left\|{}^1\alpha.0 e_1 + {}^2\alpha.\,{}^2\xi_k\, e_2\right\|, \because {}^1 f = 0$$

$$= \left|{}^2\alpha\right| \sup_k k^k \left|{}^2\xi_k\right| < \infty$$

This implies that $\alpha f \in B''$.

Now,

1. $\alpha(f+g) = ({}^1\alpha\, e_1 + {}^2\alpha\, e_2)\left[\left({}^1 f + {}^1 g\right)e_1 + \left({}^2 f + {}^2 g\right)e_2\right]$

$$= {}^1\alpha\left({}^1 f + {}^1 g\right)e_1 + {}^2\alpha\left({}^2 f + {}^2 g\right)e_2$$

$$= {}^1\alpha\left(0+0\right)e_1 + {}^2\alpha\left({}^2\xi_k + {}^2\eta_k\right)e_2$$

$$= \left({}^1\alpha\, e_1 + {}^2\alpha\, e_2\right)\left(0 e_1 + {}^2\xi_k\, e_2\right) + \left({}^1\alpha\, e_1 + {}^2\alpha\, e_2\right)\left(0 e_1 + {}^2\eta_k\, e_2\right)$$

$$= \alpha f + \alpha g$$

Also

$$\sup_k k^k \left\|\alpha f + \alpha g\right\| = \sup_k k^k \left\|\alpha\,{}^2\xi_k\, e_2 + \alpha\,{}^2\eta_k e_2\right\|$$

Notes

$$\leq \left|{}^2\alpha\right|\sup_k k^k \left|{}^2\xi_k\right| + \left|{}^2\alpha\right|\sup_k k^k \left|{}^2\eta_k\right| < \infty$$

2. $(\alpha+\beta) f = (\alpha+\beta)\,{}^2\xi_k\, e_2 = [({}^1\alpha + {}^1\beta)e_1 + ({}^2\alpha + {}^2\beta)e_2]\left({}^2\xi_k\, e_2\right).$

$$= ({}^1\alpha + {}^1\beta).0.e_1 + ({}^2\alpha + {}^2\beta)\,{}^2\xi_k\, e_2$$
$$= \left({}^1\alpha\, e_1 + {}^2\alpha\, e_2\right)\left(0e_1 + {}^2\xi_k\, e_2\right) + \left({}^1\beta\, e_1 + {}^2\beta\, e_2\right)\left(0e_1 + {}^2\xi_k\, e_2\right)$$
$$= \alpha\, f + \beta\, f$$

3. $(\alpha\beta) f = \left({}^1\alpha\,{}^1\beta\, e_1 + {}^2\alpha\,{}^2\beta\, e_2\right){}^2\xi_k\, e_2 = ({}^1\alpha\, e_1 + {}^2\alpha\, e_2).({}^1\beta\, e_1 + {}^2\beta\, e_2)\,{}^2\xi_k\, e_2$

$$= \alpha(\beta f).$$

4. *1. f = f*, where *1* is the identity element of BC.

Thus, we can say that B'' is a BC - module.

Theorem 2.1.4: Prove that the subclasses B' and B'' are modules over the class B that is, they are bicomplex modules.

Proof: First of all observe that B given by (2.1.1) is a ring.
For all $f = (\xi_k), g = (\eta_k), h = (\zeta_k) \in B$

1. $f + g = (\xi_k + \eta_k) = (\eta_k + \xi_k) = g + f$
2. $(f+g)+h = (\xi_k + \eta_k) + \zeta_k = \xi_k + (\eta_k + \zeta_k) = f + (g+h)$
3. There exists $0 \in B$ such that $f + 0 = f$
4. There exists $-f = (-\xi_k) \in B$ such that $f + (-f) = 0$
5. $f(gh) = \xi_k\,(\eta_k\,\zeta_k) = (\xi_k\,\eta_k)\,\zeta_k = (fg)h$
6. $f\,(g+h) = \xi_k\,(\eta_k + \zeta_k) = \xi_k\,\eta_k + \xi_k\,\zeta_k = f\,g + f\,h$

$(f+g)h = (\xi_k + \eta_k)\,\zeta_k = \xi_k\,\zeta_k + \eta_k\,\zeta_k = f\,h + g\,h$

All the above properties can be easily verified with the help of idempotent representation.
Now the classes defined by Wagh [7], B' and B'' are shown to be modules over B.
First consider

$$B' = \left\{ f : f = \left\{{}^1\xi_k e_1\right\} : \sup_{k\geq 1} k^k \left|{}^1\xi_k\right| < \infty \right\}$$

In this direction, first it is needed to prove that B' is an additive abelian group;
For all $x = \left({}^1\xi_k\, e_1\right), y = \left({}^1\eta_k\, e_1\right), z = \left({}^1\zeta_k\, e_1\right) \in B'$

1. $x + y = \left({}^1\xi_k\, e_1\right) + \left({}^1\eta_k\, e_1\right) = \left({}^1\xi_k + {}^1\eta_k\right)e_1 = \left({}^1\eta_k + {}^1\xi_k\right)e_1 = y + x$, since ${}^1\xi_k, {}^1\eta_k$ are complex numbers and commutativity holds for them under addition.
2. By applying the same logic,

Ref

7. Wagh, Mamta A. : "On a Class of Bicomplex Sequences", Inter. J. Trends in Math. & Stats., 3(5), 2014, 158 – 171.

$$(x+y)+z=({}^1\xi_k\, e_1+{}^1\eta_k\, e_1)+\left({}^1\zeta_k\, e_1\right)=\{\left({}^1\xi_k+{}^1\eta_k\right)+\left({}^1\zeta_k\right)\}e_1$$

$$=\left\{\left({}^1\xi_k\right)+\left({}^1\eta_k+{}^1\zeta_k\right)\right\}e_1=x+(y+z)\,,$$

since associativity under addition holds for complex numbers.

3. Zero element, $0=0e_1$, exists in B' such that $x+0=x$.
4. Additive inverse of every element is present in B'.

Let $f=(\xi_k)\in B$ and $x=({}^1\zeta_k\, e_1)\in B'$

Now define an operation $B\times B'\to B'$. This map can be defined since

$$f\times x=x\times f=\left(k^k\,\xi_k\,{}^1\zeta_k\, e_1\right)=k^k\,({}^1\xi_k\, e_1+{}^2\xi_k\, e_2)\left({}^1\zeta_k\, e_1+0\,e_2\right)=k^k\,{}^1\xi_k\,{}^1\zeta_k\, e_1$$

and $$\sup_{k\geq1} k^k\left|k^k\,{}^1\xi_k\,{}^1\zeta_k\right|\leq\sup_{k\geq1} k^k\left|{}^1\xi_k\right|.\sup_{k\geq1} k^k\left|{}^1\zeta_k\right|<\infty,\because\left({}^1\xi_k\, e_1\right),\left({}^1\zeta_k\, e_1\right)\in B'\,.$$

Therefore, $f\times x=x\times f\in B'$.

Thus, the above map is well defined such that for all

$$f=(\xi_k),g=(\eta_k)\in B\ \ \&\ x=\left({}^1\zeta_k\, e_1\right),y=\left({}^1\mu_k\, e_1\right)\in B'$$

1. $f\,(x+y)=\left\{\xi_k\left({}^1\eta_k\, e_1+{}^1\mu_k\, e_1\right)\right\}=\left(k^k\,{}^1\xi_k\,{}^1\eta_k\, e_1+k^k\,{}^1\xi_k\,{}^1\mu_k\, e_1\right)=f\,x+f\,y,$
   $\because e_1^{\,2}=e_1\,,e_2^{\,2}=e_2\,,e_1e_2=0$
2. $\left(f+g\right)x=\left(\xi_k+\eta_k\right)\left({}^1\zeta_k\, e_1\right)=k^k\left({}^1\xi_k\,{}^1\zeta_k\, e_1+{}^1\eta_k\,{}^1\zeta_k\, e_1\right)$
   $=k^k\,{}^1\xi_k\,{}^1\zeta_k\, e_1+k^k\,{}^1\eta_k\,{}^1\zeta_k\, e_1$
   $=f\,x+g\,x$
3. $(f\,g)x=\left(k^k\,\xi_k\,\eta_k\right)\left({}^1\zeta_k\, e_1\right)=k^k\left(k^k\,{}^1\xi_k\,{}^1\eta_k\,.{}^1\zeta_k\, e_1\right)=k^{2k}\,{}^1\xi_k\,{}^1\eta_k\,{}^1\zeta_k\, e_1$
   and $f\,(g\,x)=\left(\xi_k\right)\left(k^k\,{}^1\eta_k\,{}^1\zeta_k\, e_1\right)=k^k\,{}^1\xi_k\,.k^k\,{}^1\eta_k\,{}^1\zeta_k\, e_1=k^{2k}\,{}^1\xi_k\,{}^1\eta_k\,{}^1\zeta_k\, e_1$
   therefore, $(f\,g)x=f\,(g\,x)$.
4. $1_B=\left(k^{-k}\right)$ is the identity element of *B*, $\left(1_B\right)x=k^k.k^{-k}\,{}^1\zeta_k\, e_1=\left({}^1\zeta_k\, e_1\right)=x$

Hence, B' is a module over *B*.
Now consider other subclass:

$$B''=\left\{f:f=\left\{{}^2\xi_k e_2\right\}:\sup_{k\geq1} k^k\ \left|{}^2\xi_k\right|<\infty\right\}$$

In this also, first we prove that B'' is an additive abelian group;

For all $x=\left({}^2\xi_k\, e_2\right),y=\left({}^2\eta_k\, e_2\right),z=\left({}^2\zeta_k\, e_2\right)\in B''$

1. $x+y=\left({}^2\xi_k\, e_2\right)+\left({}^2\eta_k\, e_2\right)=\left({}^2\xi_k+{}^2\eta_k\right)e_2=\left({}^2\eta_k+{}^2\xi_k\right)e_2=y+x\,,$

Notes

since ${}^2\xi_k, {}^2\eta_k$ are complex numbers and commutativity holds for them under addition.

2. By applying the same logic

$$(x+y)+z=({}^2\xi_k\, e_2+{}^2\eta_k\, e_2)+\left({}^2\zeta_k\, e_2\right)=\{\left({}^2\xi_k+{}^2\eta_k\right)+\left({}^2\zeta_k\right)\}e_2$$

$$=\left\{\left({}^2\xi_k\right)+\left({}^2\eta_k+{}^2\zeta_k\right)\right\}e_2=x+\left(y+z\right),$$

since associativity under addition holds for complex numbers.

3. Zero element, $0=0\,e_2$, exists in B'' such that $x+0=x$.
4. Additive inverse of every element is present in B''.

Let $$f=\left(\xi_k\right)\in B \ \&\ x=\left({}^2\zeta_k\, e_2\right)\in B''$$

Now we define an operation $B\times B''\to B''$.

$$f\times x=x\times f=\left(k^k\,\xi_k\,{}^2\zeta_k\, e_2\right)=k^k\,({}^1\xi_k\, e_1+{}^2\xi_k\, e_2)\left(0e_1+{}^2\zeta_k e_2\right)=k^k\,{}^2\xi_k\,{}^2\zeta_k\, e_2$$

and $$\sup_{k\geq 1} k^k\left|k^k\,{}^2\xi_k\,{}^2\zeta_k\right|\leq\sup_{k\geq 1} k^k\left|{}^2\xi_k\right|.\sup_{k\geq 1} k^k\left|{}^2\zeta_k\right|<\infty,\because\left({}^2\xi_k\, e_2\right),\left({}^2\zeta_k\, e_2\right)\in B''.$$

Therefore, $f\times x=x\times f\in B''$.
Thus, the above map is well defined such that for all

$$f=\left(\xi_k\right),g=\left(\eta_k\right)\in B \ \&\ x=\left({}^2\zeta_k\, e_2\right),y=\left({}^2\mu_k\, e_2\right)\in B''$$

1. $f\,(x+y)=\left\{\xi_k\left({}^2\eta_k\, e_2+{}^2\mu_k\, e_2\right)\right\}=\left(k^k\,{}^2\xi_k\,{}^2\eta_k\, e_2+k^k\,{}^2\xi_k\,{}^2\mu_k\, e_2\right)=f\,x+f\,y$

   Since, $e_1^2=e_1, e_2^2=e_2, e_1e_2=0$

2. $(f+g)\,x=(\xi_k+\eta_k)({}^2\zeta_k\, e_2)=k^k\,({}^2\xi_k\,{}^2\zeta_k\, e_2+{}^2\eta_k\,{}^2\zeta_k\, e_2)$

$$=k^k\,{}^2\xi_k\,{}^2\zeta_k\, e_2+k^k\,{}^2\eta_k\,{}^2\zeta_k\, e_2$$

$$=f\,x+g\,x$$

3. $\left(f\,g\right)x=\left(k^k\,\xi_k\,\eta_k\right)\left({}^2\zeta_k\, e_2\right)=k^k\left(k^k\,{}^2\xi_k\,{}^2\eta_k\,.{}^2\zeta_k\, e_2\right)=k^{2k}\,{}^2\xi_k\,{}^2\eta_k\,{}^2\zeta_k\, e_2$

   and $f\left(g\,x\right)=\left(\xi_k\right)\left(k^k\,{}^2\eta_k\,{}^2\zeta_k\, e_2\right)=k^k\,{}^2\xi_k\,.k^k\,{}^2\eta_k\,{}^2\zeta_k\, e_2=k^{2k}\,{}^2\xi_k\,{}^2\eta_k\,{}^2\zeta_k\, e_2$

   Therefore, $\left(f\,g\right)x=f\left(g\,x\right)$.

4. $1_B=\left(k^{-k}\right)$ is the identity element of B, $\left(1_B\right)x=k^k.k^{-k}\,{}^2\zeta_k\, e_2=\left({}^2\zeta_k\, e_2\right)=x$

Hence, B'' is also a module over B.

a) Construction of a BC - module, considering B and B' as complex linear spaces:

$$B'=\left\{f:f=\left\{{}^1\xi_k e_1\right\}:\sup_{k\geq 1} k^k\ \left|{}^1\xi_k\right|<\infty\right\}$$

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Let α be a complex number and $f \in B'$, then

$$\alpha . f = (\alpha\, {}^1\xi_k) e_1 \in B'$$

Therefore, both B' and B'' are complex linear spaces.
We know that B is a ring and

$$B' = A_1 e_1 = (B)_{e_1}$$
$$B'' = A_2 e_2 = (B)_{e_2}$$

We have also shown that B' and B'' are ideals in the ring B and they have the properties

$$(B)_{e_1} \cap (B)_{e_2} = \{0\} \text{ and}$$

$B = B_{e_1} + B_{e_2}$, idempotent decomposition of B.

Both the ideals are uniquely determined but their elements admit different representations.
$(\xi_k) \in B$ can be written as

$$\xi_k = z_{1k} + i_2 z_{2k} = \beta_{1k} e_1 + \beta_{2k} e_2 \qquad (3.1.1)$$

where,

$$\beta_{1k} = z_{1k} - i_1 z_{2k}$$
$$\beta_{2k} = z_{1k} + i_1 z_{2k}$$

(β_{1k}) and (β_{2k}) are complex sequences i.e., $\beta_{1k}, \beta_{2k} \in C_1(i_1)$.
(ξ_k) can also be written as

$$\xi_k = \eta_{1k} + i_1 \eta_{2k} = \gamma_{1k} e_1 + \gamma_{2k} e_2 \qquad (3.1.2)$$

where,

$$\eta_{1k} = x_{1k} + i_2 x_{2k},$$

$$\eta_{2k} = y_{1k} + i_2 y_{2k},$$

$$\gamma_{1k} = \eta_{1k} - i_2 \eta_{2k},$$

$$\gamma_{2k} = \eta_{1k} + i_2 \eta_{2k}$$

All these sequences are complex sequences in $C_1(i_2)$.
(3.1.1) and (3.1.2) can be equally called the idempotent representations of a bicomplex sequence in B.

We can say that (3.1.1) is the idempotent representation for B when we consider elements are from

$$C^2(i_1) = C_1(i_1) \times C_1(i_1)$$

And (3.1.2) is the idempotent representation for B when we consider elements are from

$$C^2(i_1) = C_1(i_2) \times C_1(i_2) .$$

Notes

We get similar results for both the representations. We can say that the consequences are similar but different.

Note 3.1.1: One point should be noted here that $(\beta_{1k})e_1=(\gamma_{1k})e_1$ and $(\beta_{1k})e_2=(\gamma_{1k})e_2$ although $\gamma_{1k},\gamma_{2k}\in C_1(i_2),\forall k\geq 1$ and $\beta_{1k},\beta_{2k}\in C_1(i_1),\forall k\geq 1$.
More specifically, given

$$\beta_{1k}=\mathrm{Re}(\beta_{1k})+i_1\,\mathrm{Im}(\beta_{1k}) \ ,$$

the equality

$$\beta_{1k}\,e_1=\gamma_{1k}\,e_1$$

is true if and only if

$$\gamma_{1k}=\mathrm{Re}\ \beta_{1k}-i_2\,\mathrm{Im}\,\beta_{1k}\,.$$

Similarly, given

$$\beta_{2k}=\mathrm{Re}(\beta_{2k})+i_1\,\mathrm{Im}(\beta_{2k})$$

the equality

$$\beta_{1k}\,e_1=\gamma_{1k}\,e_1$$

is true if and only if

$$\gamma_{2k}=\mathrm{Re}\ \beta_{2k}-i_2\,\mathrm{Im}\,\beta_{2k}\,.\ \text{[c.f. [1]]}$$

Note 3.1.2: The decomposition $B=B_{e_1}+B_{e_1}$ can be written in any of the two equivalent forms

$$\mathrm{B}=\mathrm{A}_1\,\mathrm{e}_1+\mathrm{A}_2\,\mathrm{e}_2\,;B=(A_1)'\,e_1+(A_2)'\,e_2\,,$$

where $A_1,A_2\subseteq C_1(i_1)$ and $(A_1)',(A_2)'\subseteq C_1(i_2)$.
If B is seen as $C_1(i_1)$ - linear space, then the first decomposition becomes a direct sum.

And if seen as $C_1(i_2)$ - linear space, then the second decomposition becomes a direct sum.

Note 3.1.3: We know that idempotent representation is unique but (3.1.1) and (3.1.2) contradicts this fact. But each of them is unique in the following sense:
It is easy to show that

$$z=\beta_{1k}\,e_1+\beta_{2k}\,e_2=\eta_{1k}\,e_1+\eta_{2k}\,e_2\Rightarrow\beta_{1k}=\eta_{1k}\,,\beta_{2k}=\eta_{2k}\ ,$$

where $\beta_{1k},\beta_{2k},\eta_{1k},\eta_{2k}\in C_1(i_1)$.

Similarly $z=\gamma_{1k}\,e_1+\gamma_{2k}\,e_2=\xi_{1k}\,e_1+\xi_{2k}\,e_2\Rightarrow\gamma_{1k}=\xi_{1k}\,,\gamma_{2k}=\xi_{2k}$,

where $\gamma_{1k},\gamma_{2k},\xi_{1k},\xi_{2k}\in C_1(i_2)$.

Hence, idempotent representation for B is unique.

Ref

1. Alpay D., Luna – Elizarrarás M. E., Shapiro M., Struppa D.C. : "Basics of Functional Analysis" with Bicomplex Scalars, and Bicomplex Shur Analysis", arXiv 2013.

Note 3.1.4: The idempotent decomposition of *B*, $B = B_{e_1} + B_{e_1}$, plays an important role, as it allows to realize component wise the operations on *B*:

$$(\xi_k) = ({}^1\xi_k e_1 + {}^2\xi_k e_2) \ , (\eta_k) = ({}^1\eta_k e_1 + {}^2\eta_k e_2)$$

$\lambda = {}^1\lambda e_1 + {}^2\lambda e_2$, then

$$\xi_k + \eta_k = ({}^1\xi_k e_1 + {}^2\xi_k e_2) + ({}^1\eta_k e_1 + {}^2\eta_k e_2) = ({}^1\xi_k + {}^1\eta_k) e_1 + ({}^2\xi_k + {}^2\eta_k) e_2$$

$$\lambda(\xi_k) = ({}^1\lambda\, {}^1\xi_k) e_1 + ({}^2\lambda\, {}^2\xi_k) e_2 .$$

When B is considered as $C_1(i_1)$ - linear space we write $B_{C_1(i_1)}$ and when as $C_1(i_2)$ - linear space, it is written as $B_{C_1(i_2)}$.

B_{e_1} and B_{e_2} are R - linear, $C_1(i_1)$ - linear, $C_1(i_2)$ - linear and BC - modules.

We can write $B = B_{e_1} \oplus B_{e_2}$, where the direct sum $\oplus$ can be understood in the sense of R - , $C_1(i_1)$ - , $C_1(i_2)$ - linear spaces, as well as BC - modules.

References Références Referencias

1. Alpay D., Luna – Elizarrarás M. E., Shapiro M., Struppa D.C. : "Basics of Functional Analysis" with Bicomplex Scalars, and Bicomplex Shur Analysis", arXiv 2013.
2. Price, G. Baley : "An Introduction to Multicomplex spaces and functions",Marcel Dekker, Inc., 1991.
3. Segre C. : "Le Rappresentazioni Reali Delle Forme complesse e Gli Enti Iperalgebrici" Math. Ann., 40, 1892, 413-467.
4. Srivastava, Rajiv K. & Srivastava, Naveen K. : "On a class of Entire Bicomplex sequences", South East. Asian J. Math & Math. Sc. 5(3), 2007, pp. 47-68.
5. Srivastava, Rajiv K. : "Certain topological aspects of Bicomplex space", Bull. Pure & Appl. Math. Dec., 2(2), 2008, 222-234.
6. Wagh, Mamta A. : "On certain spaces of Bicomplex sequences ", Inter. J. Phy. Chem. and Math. fund.(IJPCMF), 7(1), 2014, 1 – 6.
7. Wagh, Mamta A. : "On a Class of Bicomplex Sequences", Inter. J. Trends in Math. & Stats., 3(5), 2014, 158 – 171.

Notes