Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 16 Issue 4 Version 1.0 Year 2016
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Yet Another New Proof of Feuerbach's Theorem

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GJSFR-F Classification : MSC 2010: 11J83, 11D41

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Yet Another New Proof of Feuerbach's Theorem

Dasari Naga Vijay Krishna

Abstract- In this article we give a new proof of the celebrated theorem of Feuerbach.

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I. Introduction

The Feuerbach's Theorem states that "*The nine-point circle of a triangle is tangent internally to the in circle and externally to each of the excircles*".

Feuerbach's fame is as a geometer who discovered the nine point circle of a triangle. This is sometimes called the Euler circle but this incorrectly attributes the result. Feuerbach also proved that the nine point circle touches the inscribed and three escribed circles of the triangle. These results appear in his 1822 paper, and it is on the strength of this one paper that Feuerbach's fame is based. He wrote in that paper:-

The circle which passes through the feet of the altitudes of a triangle touches all four of the circles which are tangent to the three sides of the triangle; it is internally tangent to the inscribed circle and externally tangent to each of the circles which touch the sides of the triangle externally.

The nine point circle which is described here had also been described in work of Brianchon and Poncelet the year before Feuerbach's paper appeared. However John Sturgeon Mackay notes in [4] that Feuerbach gave:-

... the first enunciation of that interesting property of the nine point circle

namely that "it is internally tangent to the inscribed circle and externally tangent to each of the circles which touch the sides of the triangle externally." The point where the incircle and the nine point circle touch is now called the Feuerbach point.

In this short paper we deal with an elementary concise proof for this celebrated theorem.

II. Notation and Background

Let ABC be a non equilateral triangle. We denote its side-lengths by a, b, c, its semi perimeter by $s=\frac{1}{2}(a+b+c)$, and its area by Δ. Its *classical centers* are the circum center S, the in center I, the centroid G, and the orthocenter O. The nine-point center

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N is the midpoint of SO and the center of the nine-point circle, which passes through the side-midpoints A', B', C' and the feet of the three altitudes. The Euler Line Theorem states that G lies on SO with OG : GS = 2 : 1 and ON : NG : GS = 3 : 1 : 2. We write I_1, I_2, I_3 for the excenters opposite A, B, C, respectively, these are points where one internal angle bisector meets two external angle bisectors. Like I, the points I_1, I_2, I_3 are equidistant from the lines AB, BC, and CA, and thus are centers of three circles each tangent to the three lines. These are the excircles. The *classical radii* are the circum radius R (= SA = SB = SC), the in radius r, and the exradii r_1, r_2, r_3.

The following formulas are well known

(a) $\Delta=\dfrac{abc}{4R}=rs=r_1(s-a)=r_2(s-b)=r_3(s-c)=\sqrt{s(s-a)(s-b)(s-c)}$

(b) $r=4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$, $r_1=4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$, $r_2=4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$,

$r_3=4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

(c) $r_1+r_2+r_3-r=4R$, $r+r_2+r_3-r_1=4R\cos A$, $r+r_1+r_3-r_2=4R\cos\mathrm{B}$,

$r+r_1+r_2-r_3=4R\cos\mathrm{C}$

II. Basic Lemma's

Lemma -1

If A, B and C are the angles of the triangle ABC then

(1) $\cos A+\cos B+\cos C=1+\dfrac{r}{R}$

(2) $\cos B+\cos C-\cos A=\dfrac{r_1}{R}-1$

(3) $\cos A+\cos C-\cos B=\dfrac{r_2}{R}-1$

(4) $\cos A+\cos B-\cos C=\dfrac{r_3}{R}-1$

Proof:

Using the formula (c) and by little algebra, we can arrive at the conclusions (1), (2), (3) and (4)

Lemma -2

If N is the center of Nine point circle of the triangle ABC then its radius is $\dfrac{R}{2}$

Proof:

Clearly Nine point circle is acts as the circum circle of the medial triangle, so the radius of nine point circle is the circum radius of medial triangle.

It is well known that by the midpoints theorem if a, b and c are the sides of the reference(given) triangle and Δ is its area then a/2, b/2 and c/2 are the sides of its medial triangle whose area is $\dfrac{\Delta}{4}$.

If R^1 is the radius of nine point circle then using (a) we have

Notes

Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

$$R' = \frac{\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)}{4\left(\frac{\Delta}{4}\right)} = \frac{\frac{4R\Delta}{8}}{\Delta} = \frac{R}{2}$$

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Hence proved

Lemma -3

If N is the nine point center of the triangle ABC then

$$AN = \frac{1}{2}\sqrt{R^2 + c^2 + b^2 - a^2} = \frac{1}{2}\sqrt{R^2 + 2bc\cos A}$$

$$BN = \frac{1}{2}\sqrt{R^2 + c^2 + a^2 - b^2} = \frac{1}{2}\sqrt{R^2 + 2ac\cos B}$$

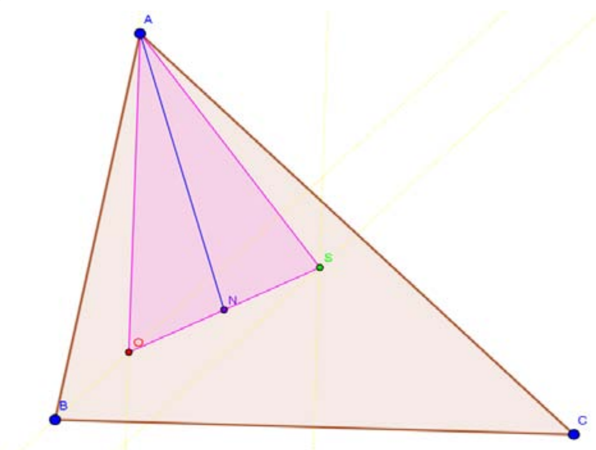
$$CN = \frac{1}{2}\sqrt{R^2 + a^2 + b^2 - c^2} = \frac{1}{2}\sqrt{R^2 + 2ab\cos C}$$

Proof:

We are familiar with the following results

If G is the centroid and S is the circum center then $GS^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$

Nine point center(N) lies on the Euler's Line and N acts as the mid point of the line segment formed by joining the Orthocenter(O) and Circum center(S), So AN is the median of the triangle AOS



Hence by Apollonius theorem, we have $AN = \frac{1}{2}\sqrt{2AO^2 + 2AS^2 - OS^2}$ ....(A)

Since ON : NG : GS = 3 : 1 : 2, we have $OS^2 = 9GS^2$

And also we know that $AO = 2R\cos A$, $AS = R$

By replacing AO, AS and OS in (A) and by some computation, we get

$AN = \frac{1}{2}\sqrt{R^2 + c^2 + b^2 - a^2} = \frac{1}{2}\sqrt{R^2 + 2bc\cos A}$ (by cosine and sine rule)

Similarly we can find BN and CN

Lemma -4

If I is the In center of the triangle ABC whose sides are a, b and c and M be any point in the plane of the triangle then

$$IM^2 = \frac{a\,AM^2 + b\,BM^2 + c\,CM^2 - abc}{a+b+c}$$

Proof:
The proof of above lemma can be found in [1]
Similarly we can prove that

If I_1, I_2 and I_3 are excenters of the triangle ABC whose sides are a, b and c and M be any point in the plane of the triangle then

Notes

$$I_1M^2 = \frac{-a\,AM^2 + b\,BM^2 + c\,CM^2 + abc}{b+c-a}$$

$$I_2M^2 = \frac{a\,AM^2 - b\,BM^2 + c\,CM^2 + abc}{a+c-b}$$

$$I_3M^2 = \frac{a\,AM^2 + b\,BM^2 - c\,CM^2 + abc}{a+b-c}$$

Lemma -5

If r_1 and r_2 are the radii of two non concentric circles whose centers are at a distance of d then the circles touch each other

(i) internally only when $d = |r_1 - r_2|$

(ii) externally only when $d = r_1 + r_2$

Proof:

We know that If r_1 and r_2 are the radii of two circles whose centers are at a distance of d units then the length of their direct common tangent $= \sqrt{d^2 - (r_1 - r_2)^2}$

And the length of their transverse common tangent $= \sqrt{d^2 - (r_1 + r_2)^2}$

Now if two circles touch each other internally then their length of direct common tangent is zero

So $\sqrt{d^2 - (r_1 - r_2)^2} = 0$, it implies that $d = |r_1 - r_2|$

In the similar manner, if two circles touch each other externally then their length of transverse common tangent is zero. So $\sqrt{d^2 - (r_1 + r_2)^2} = 0$, it implies that $d = r_1 + r_2$

III. Main Results

Theorem -1

Nine point circle of triangle ABC touches its in circle internally, that is if N and I are the centers of nine point circle and in circle respectively whose radii are $\frac{R}{2}$ and r then $NI = \left|\frac{R}{2} - r\right|$

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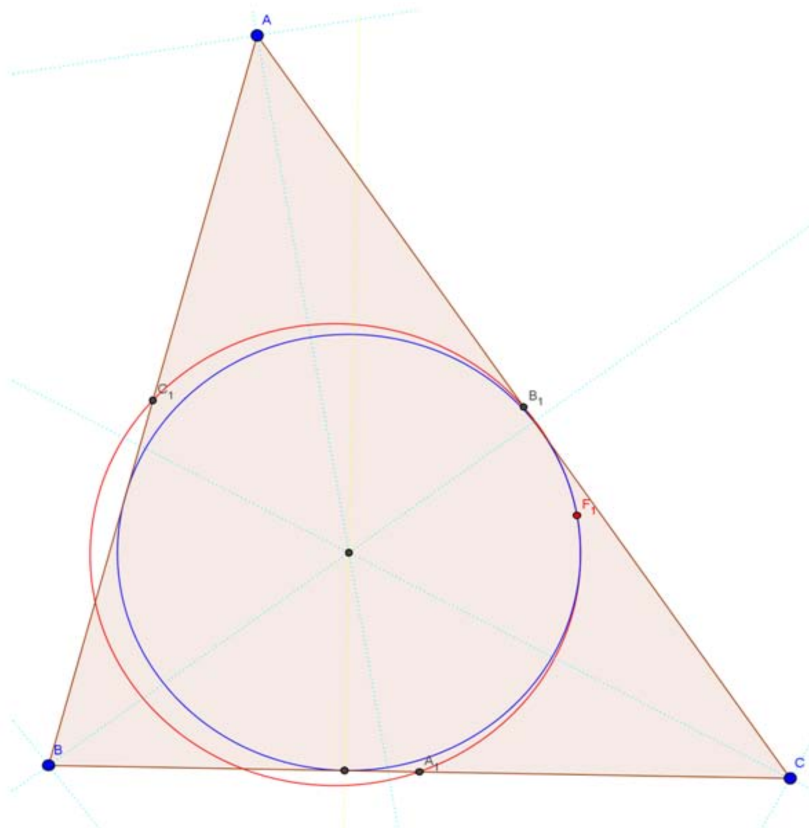
Proof:



Clearly by lemma -4, for any M we have $IM^2 = \dfrac{a\,AM^2 + b\,BM^2 + c\,CM^2 - abc}{a+b+c}$

Now fix M as Nine point center(N),

So $$IN^2 = \frac{a\,AN^2 + b\,BN^2 + c\,CN^2 - abc}{a+b+c}$$

It can be rewritten as $$IN^2 = \frac{\sum_{a,b,c} a\left(\frac{1}{4}(R^2 + 2bc\cos A)\right) - abc}{a+b+c}$$

Using lemma -1, (a) and by some computation, we can arrive at the conclusion

$$IN^2 = (\frac{R}{2} - r)^2 \quad (\text{since } R \ge 2r)$$

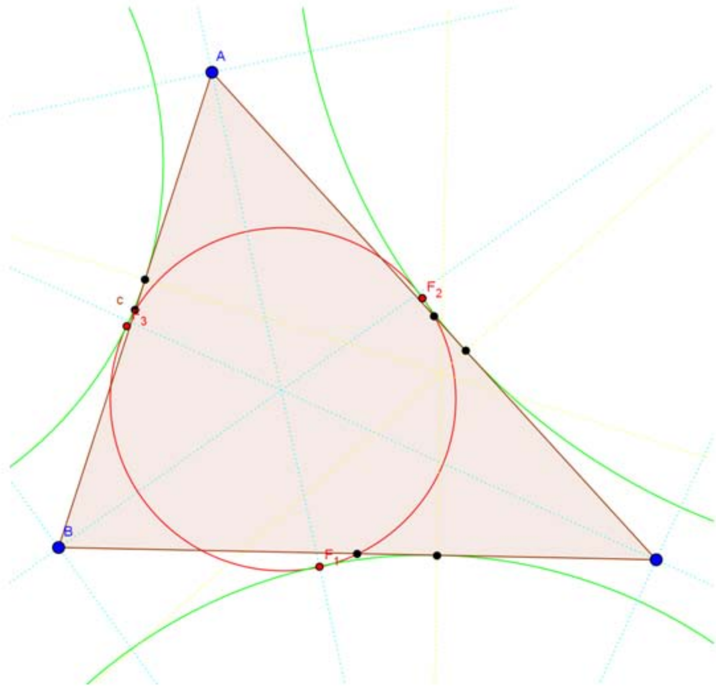
It further gives $NI = \left|\dfrac{R}{2} - r\right|$

Hence proved

Theorem -2

Nine point circle of triangle ABC touches their excircles externally, that is if N and I_1, I_2, I_3 are the centers of nine point circle and excircles respectively whose radii are $\dfrac{R}{2}$ and r_1, r_2, r_3 then $NI_1 = \dfrac{R}{2} + r_1$, $NI_2 = \dfrac{R}{2} + r_2$ and $NI_3 = \dfrac{R}{2} + r_3$

Proof:



Clearly by lemma -4, for any M we have $I_1M^2 = \dfrac{-a\,AM^2 + b\,BM^2 + c\,CM^2 + abc}{b+c-a}$

Now fix M as Nine point center(N),

So $$I_1N^2 = \frac{-a\,AN^2 + b\,BN^2 + c\,CN^2 + abc}{-a+b+c}$$

It can rewritten as $$I_1N^2 = \frac{\frac{R^2}{4}(-a+b+c) + \frac{abc}{2}\left(\cos B + \cos C - \cos A\right) + abc}{-a+b+c}$$

Using lemma -1, (a) and by some computation,

we can arrive at the conclusion $I_1N^2 = (\frac{R}{2} + r_1)^2$

It further gives $NI_1 = \dfrac{R}{2} + r_1$

Similarly we can prove $NI_2 = \dfrac{R}{2} + r_2$ and $NI_3 = \dfrac{R}{2} + r_3$

Hence proved

Now we are in a position to deal with the concise proof of celebrated Feuerbach's Theorem.

Theorem - 3(Feuerbach, 1822)

In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and externally tangent to the three excircles.

Proof:

Theorem -1 and Theorem-2 completes the proof of Feuerbach's Theorem.
For historical details see [4] , [5], [6] and [9]

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Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

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