Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 16 Issue 4 Version 1.0 Year 2016
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A Survey on Developments in Sequence Spaces and introduction to Bicomplex Duals

By Dr. Mamta Amol Wagh
University of Delhi

Abstract- A survey on recent developments in the duality theory of sequence spaces has been done and duality in some Bicomplex sequence spaces is introduced which gives rise to sixteen types of different duals.

Keywords: *köthe – toeplitz dual, generalized köthe – toeplitz dual, bicomplex köthe – toeplitz dual, difference sequence spaces, generalized difference sequence spaces.*

GJSFR-F Classification : *MSC 2010: 47B35* eJournal · Print Journal · epaper · open standard · 3d virtual journal

Ref

10. Köthe, G & Toeplitz, O: "Lineare Raume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen", Jour. Reine angew. Math. 171, 1934, 193 – 226.

A Survey on Developments in Sequence Spaces and introduction to Bicomplex Duals

Mamta Amol Wagh

Abstract- A survey on recent developments in the duality theory of sequence spaces has been done and duality in some Bicomplex sequence spaces is introduced which gives rise to sixteen types of different duals.

Keywords: köthe – toeplitz dual, generalized köthe – toeplitz dual, bicomplex köthe – toeplitz dual, difference sequence spaces, generalized difference sequence spaces.

I. Introduction

In several branches of analysis, for instance, the structural theory of topological vector spaces, Schauder basis theory, Summability theory, and the theory of functions, the study of sequence spaces occupies a very prominent position. The impact and importance of this study can be appreciated when one sees the construction of numerous examples of locally convex spaces obtained as a consequence of the dual structure displayed by several pairs of distinct sequence spaces. There is an ever increasing interest in the theory of sequence spaces that has made remarkable advances in enveloping summability theory via unified techniques effecting matrix transformations from one sequence space into another.

Thus we have several important applications of the theory of sequence spaces and therefore we attempt to present a survey on recent developments in sequence spaces and their different kinds of duals.

There are two types of dual of a sequence space, namely Algebraic dual and Topological dual. The set of all linear functionals, on a linear space V, with domain as V and range as K is denoted by $L(V, K) = V^{\#}$ and is called algebraic dual of V. If we consider the set of all continuous linear functional, then we get topological dual denoted by V^{*}.

From the point of view of the duality theory, the study of sequence spaces is much more profitable. Köthe & Toeplitz were the first to recogonize the problem that it is difficult to find the topological duals of sequence spaces equipped with linear topologies. To resolve it, they introduced a kind of dual, α - dual, in quite many familiar and useful sequence spaces. In the same paper [10], they also introduced another kind of dual namely β – dual which together with the given sequence space forms a nice dual system. A still more general notion of a dual, γ - dual was later

Author: Assistant Professr, Department of Mathematics, Deen Dayal Upadhyay College, University of Delhi, Karampura, Shivaji Marg, New Delhi – 110015, India. e-mail: nigam.mamta@gmail.com

introduced by Garling [6]. For symmetric sequence spaces there is another notion of a dual, called a δ – dual due to Garling [7] and Ruckle [28].

Let λ be a sequence space, and ω is the family of all sequences $\{x_n\}$ with $x_n \in K$, $n \geq 1$. Define α - , β - , γ- , and δ – dual, respectively as follows:

$$1.\ \alpha-dual \quad \lambda^{\alpha}=\left\{x : x\in\omega, \sum_{i\geq 1}|x_i y_i|<\infty, \forall y\in\lambda\right\}$$

$$2.\ \beta-dual \quad \lambda^{\beta}=\left\{x : x\in\omega, \left|\sum_{i\geq 1} x_i y_i\right|<\infty, \forall y\in\lambda\right\}$$

$$3.\ \gamma-dual \quad \lambda^{\gamma}=\left\{x : x\in\omega, \sup_n\left|\sum_{i=1}^{n} x_i y_i\right|<\infty, \forall y\in\lambda\right\}$$

$$4.\ \delta-dual \quad \lambda^{\delta}=\left\{x : x\in\omega, \sum_{i\geq 1}|x_i y_{\rho(i)}|<\infty, \forall y\in\lambda\, and\, \rho\in\pi\right\}$$

Here π is the set of all permutations of N.

$\lambda^{\alpha}, \lambda^{\beta}, \lambda^{\gamma}, and\ \lambda^{\delta}$ are sequence spaces, and $\phi\subset\lambda^{\delta}\subset\lambda^{\alpha}\subset\lambda^{\beta}\subset\lambda^{\gamma}$.

Constantine G. Lascarides [11] in his paper, "A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer", has examined the Köthe – Toeplitz reflexivity of certain sequence spaces and characterized some classes of matrix transformations defined on them.

Matrix transformation: let *X*, *Y* be two nonempty subsets of the spaces of all complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers a_{nk} $(n, k = 1,2,\ldots)$. For every $x=(x_k)\in X$ and every integer n we write

$$A_n(x) = \sum_k a_{nk}x_k ,$$

Where the sum without limits is always taken from $k = 1$ *to* $k = \infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. we say that $A \in (X, Y)$ if and only if $Ax\in Y$ whenever $x\in X$.

The following classes of sequences were defined by Maddox [17]:

$$l(p)=\left\{x : \sum_k |x_k|^{p_k}<\infty\right\},$$

$$l_\infty(p)=\left\{x : \sup_k |x_k|^{p_k}<\infty\right\},$$

$$c(p)=\left\{x : |x_k-l|^{p_k}\to 0\ for\ some\ l\right\},$$

$$c_0(p)=\left\{x : |x_k|^{p_k}\to 0\right\}.$$

When all the terms of (p_k) are constant and all equal to p>0 we have $l(p)=l_p, l_\infty(p)=l_\infty, c(p)=c, and\ c_0(p)=c_0$, where l_p, l_∞, c, c_0, are respectively the spaces of p – summable, bounded, converget, and null sequences.

Ref

6. Garling, D.G.H.: "The β - and γ - duality", *Proc. Cambridge Philos. Soc. (1967), 963 – 981.*

It was shown in [14],[15],[17], that the sets $l(p), l_\infty(p), c(p)$ and $c_0(p)$, are linear spaces under coordinatewise addition and scalar multiplication if and only if $p \in l_\infty$.

Let E be a nonempty subset of s. Then we denote by $E^\dagger$ the generalized Köthe – Toeplitz dual of E, i.e.,

$$E^\dagger = \left\{ a : \sum_k a_k x_k \text{ converges for every } x \in E \right\}$$

Some properties of dual spaces:

Lemma 1:

The Köthe – Toeplitz duality has the following properties:

(i) $E^\dagger$ is a linear subspace of s for every $E \subset s$.

(ii) $E \subset F$ implies $E^\dagger \supset F^\dagger, \forall E, F \subset s$.

(iii) $E^{\dagger\dagger} \equiv (E^\dagger)^\dagger \supset E, \forall E \subset s$.

(iv) $(\bigcup E_j)^\dagger = \bigcap E$, *for every family* $\{E_j\}$ *with* $E_j \subset s$.

A nonempty subset E of s is said to be perfect or Köthe – Toeplitz reflexive if and only if $E^{\dagger\dagger} = E$. $E^\dagger$ is perfect for every E. If E is perfect then it is a linear space. The converse is not always true, e.g., c is a linear space with Köthe – Toeplitz dual l_1 and not perfect.

Let E(p) be any of the sets $l(p), l_\infty(p), c(p), c_0(p)$, Let $E(p;1) = E^\dagger(p), E(p;2) = E^{\dagger\dagger}(p)$ etc. Then $E(p;1) = E(p;2n+1), \forall n \geq 0$.

Köthe – Toeplitz dual of the above classes of sequences:

Lemma 2 : (I) If $0 < p_k \leq 1, \forall k$, then $l(p;1) = l_\infty(p)$

(ii) If $p_k > 1, \forall k$, then $l(p;1) = M(p)$ where

$$M(p) = \bigcup_{N \geq 1} \left\{ a : \sum_k |a_k|^{q_k} N^{-q_k} < \infty \right\}$$

With $p_k^{-1} + q_k^{-1} = 1$. For convenience we write $r_k = p_k^{-1}, s_k = q_k^{-1}$

(iii) For every $p = (p_k)$ we have

$$l_\infty(p;1) = M_\infty(p), \text{ where}$$

$$M_\infty(p) = \bigcap_{N > 1} \left\{ a : \sum_k |a_k| N^{r_k} < \infty \right\}$$

(iv) Also for every $p = (p_k), c_0(p;1) = M_0(p)$ where

$$M_0(p) = \bigcup_{N > 1} \left\{ a : \sum_k |a_k| N^{-r_k} < \infty \right\}.$$

Theorem 1: For every $p = (p_k)$ we have $c(p;1) = c_0(p;1) \bigcap \gamma$ where γ is the space of all convergent series.

Ref

14. Maddox, I. J., "Paranormed sequence spaces generated by infinite matrices", Proc. Cambridge Philos. Soc. 64 , 1968, 335 – 340.

Also he has characterized the second Köthe – Toeplitz duals and discussed the reflexivity of the sets $l(p), l_\infty(p), and\ c_0(p)$,

Theorem 2: For every $p=(p_k)$ we have $c_0(p;2)=\lambda_1$ where

$$\lambda_1 = \bigcap_{N>1}\left\{y : \sup_k |y_k| N^{r_k} < \infty\right\}.$$

Theorem 3: For every $p=(p_k)$ we have $l_\infty(p;2)=\lambda_2$ where

$$\lambda_2 = \bigcup_{N>1}\left\{a : \sup_k |a_k| N^{-r_k} < \infty\right\}.$$

I.J. Maddox [12] in his paper "Generalized Köthe – Toeplitz duals" has characterized the α and β – dual spaces of generalized l_p spaces where $0<p\le\infty$. The question of when the α and β – dual spaces coincide is also considered.

Let $A=(A_k)$ denote a sequence of linear, but not necessarily bounded, operators on X into Y. Following results are given there:

Theorem 4 : Let $0<p\le 1$. Then $A\in l_p^\beta(x)$ if and only if there exists $m\in \mathrm{N}$ such that A_k is bounded, for all $k\ge m$, and

$$H=\sup_{k\ge m}\|A_k\|<\infty .$$

Theorem 5: If $0<p\le 1$ then

$$l_p^\alpha(X)=l_p^\beta(X).$$

Theorem 6: Let $1<p<\infty$.

$$M=\sum_{k=m}^{\infty}\|A_k\|^q<\infty .$$

Theorem 7: Let $1<p<\infty$. Then $A\in l_p^\beta(X)$ if and only if there exists $m\in\mathrm{N}$ such that A_k is bounded for all $k\ge m$, and

$$\sup\sum_{k=m}^{\infty}\left\|A_k^* f\right\|^q<\infty$$

Where the supremum is over all $f\in Y^*$ with $\|f\|\le 1$.

Theorem 8: $A\in l_\infty^\alpha(X)$ if and only if there exists $m\in\mathrm{N}$ such that A_k is bounded for all $k\ge m$, and

$$\sum_{k=m}^{\infty}\|A_k\|<\infty .$$

Theorem 9: $A\in l_\infty^\beta(X)$ if and only if there exists $m\in\mathrm{N}$ such that A_k is bounded for all $k\ge m$, and

Ref

12. Maddox, I. J. : "Generalized Köthe – Toeplitz duals", International, J. Math. & Math. Sci. 3 (3), 1980, 423 – 432.

$$\sup \sum_{k=m}^{m+n} \|A_k x_k\| < \infty$$

$$\sup \left\| \sum_{k=m}^{m+n} A_k x_k \right\| \to 0 \ (m \to \infty)$$

where the suprema are over all $n \geq 0$ and all $x_k \in X$ with $\|x_k\| \leq 1$.

Theorem 10: If $1 < p < \infty$ then there are Banach spaces X and Y such that $l_p^\alpha(X) \subset l_p^\beta(X)$ with strict inclusion.

Theorem 11: If $1 < p < \infty$ and Y is finite dimensional then for any X we have

$$l_p^\alpha(X) = l_p^\beta(X).$$

For certain values of p, and any X, the next result is the converse of previous theorem.

Theorem 12: If $2 < p < \infty$ and $l_p^\alpha(X) = l_p^\beta(X)$ then Y must be finite dimensional.

Theorem 13: Let Y be a Hilbert space and suppose $l_2^\alpha(X) = l_2^\beta(X)$. Then Y must be finite dimensional.

In this paper Maddox [13] has established relations between several notions of solidity in vector valued sequence spaces, and has introduced a generalized Köthe – Toeplitz dual space in the setting of a Banach algebra.

Topologies on a sequence space, involving β and γ – duality have been examined by Garling [6], who noted that $E^\alpha = E^\beta = E^\gamma$ when E is solid (or normal), i.e., when $x \in E$ and $|y_k| \leq |x_k|$ for all $k \in N$ together imply that $y \in E$.

For example, the space c_0 of null sequences is solid, but the space c of convergent sequences is not.

If X is a complex normed linear space, denote by B the closed unit ball of X and by *B(X)* the space of all bounded linear operators on *X*. X^* denotes the continuous dual space of *X*. Two subspaces of *s(X)* that has been considered are

$$l_\infty(X) = \left\{ x \in s(X) : \sup_k \|x_k\| < \infty \right\},$$

$$l_1(X) = \left\{ x \in s(X) : \sum_{k=1}^{\infty} \|x_k\| < \infty \right\}.$$

These spaces generalize the classical spaces l_∞ *and* l_1 which are subspaces of *s*. Consider the following statements, each of which expresses some notion of solidity for linear subspace E of *s(X)*.

(1) $x \in E$ *and* $\lambda \in l_\infty$ *imply* $\lambda x \in E$

(2) $x \in E$ *and* $|\lambda_n| \leq 1$ *imply* $\lambda x \in E$

(3) $x \in E$ *and* $|\lambda_n| = 1$ *imply* $\lambda x \in E$

(4) $x \in E$ *and* $\|y_n\| = \|x_n\|$ *imply* $y \in E$

Ref

6. Garling, D.G.H.: "The β - and γ - duality", *Proc. Cambridge Philos. Soc. (1967), 963 – 981.*

(5) $x \in E$ *and* $\|y_n\| \le \|x_n\|$ *imply* $y \in E$

(6) $x \in E$ *and* $\|A_n\| \in l_\infty$ *imply* $Ax \in E$

(7) $x \in E$ *and* $\|A_n\| \le 1$ *imply* $Ax \in E$

(8) $x \in E$ *and* $\|A_n\| = 1$ *imply* $Ax \in E$

Equivalences:

Theorem 14: In any complex linear space X the statements (1), (2) and (3) are equivalent.

Theorem 15: In any normed linear space X the statements (4), (5), (6), (7) and (8) are equivalent.

Theorem 16: In any normed linear space X any one of the statements (4) to (8) implies all of the statements (1) to (3). But (1) is equivalent to (4) if and only if X is one dimensional.

The notion of difference sequence spaces was first introduced by Kizmaz[9]. He defined the sequence spaces

$$l_\infty(\Delta) = \{x = (x_k) : \Delta x \in l_\infty\}, i.e., \left\{x = (x_k) : \sup_k \|x_k - x_{k+1}\| < \infty\right\}$$
$$c(\Delta) = \{x = (x_k) : \Delta x \in c\}, i.e. \ \{x = (x_k) : \lim \|x_k - x_{k+1}\| \ exists\}$$
$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\}, i.e. \ \{x = (x_k) : \lim \|x_k - x_{k+1}\| = 0\}$$

Where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaces with norm

$$\|x\|_1 = |x_1| + \|\Delta x\|_\infty .$$

The notion of generalized difference sequence spaces was further generalized by Et. and Colak.

Et and Colak [4] generalized the above sequence spaces to the following sequence spaces.

$$l_\infty(\Delta^m) = \{x = (x_k) : \Delta^m x \in l_\infty\},$$
$$c(\Delta^m) = \{x = (x_k) : \Delta^m x \in c\},$$
$$c_0(\Delta^m) = \{x = (x_k) : \Delta^m x \in c_0\}$$

Where, $m \in N, \Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1}), \Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v} .$$

These are Banach spaces with norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty .$$

We can see that $c_0(\Delta^m) \subset c_0(\Delta^{m+1}), c(\Delta^m) \subset c(\Delta^{m+1}), l_\infty(\Delta^m) \subset l_\infty(\Delta^{m+1})$ and

Ref

9. Kizmaz, H. : "On certain Sequence Spaces", Canad. Math. Bull. 24 (1981), 169 – 176.

$c_0(\Delta^m) \subset c(\Delta^m) \subset l_\infty(\Delta^m)$ are satisfied and strict [et and colak].

Colak [3] defined the sequence space $\Delta_v(X) = \{x = (x_k) : \Delta_v x_k \in X\}$ where $(\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and X is any sequence space, and investigated some of its topological properties.

Later Etand Esi [5] have defined the sequence spaces

$$\Delta_v^m(l_\infty), \Delta_v^m(c) \; and \; \Delta_v^m(c_0), (m \in N)$$

and some topological properties, inclusion relations of these sequence spaces have been given, their continuous and Köthe – Toeplitz duals have been computed.

Let $l_\infty, c, and\ c_0$ be the linear spaces of bounded, convergent and null sequences. $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in N = \{1, 2, 3, \ldots\}$, the set of positive integers.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Define

$$\Delta_v^m(l_\infty) = \left\{x = (x_k) : \Delta_v^m x \in l_\infty\right\}$$

$$\Delta_v^m(c) = \left\{x = (x_k) : \Delta_v^m x \in c\right\}$$

$$\Delta_v^m(c_0) = \left\{x = (x_k) : \Delta_v^m x \in c_0\right\}$$

where

$$m \in N, \Delta_v^0 x = (v_k x_k), \Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1}), \; \Delta_v^m x_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$$

and so that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$$

It is trivial that $\Delta_v^m(l_\infty)$, $\Delta_v^m(c)$, $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_v = \sum_{i=1}^{m} |x_i v_i| + \|\Delta_v^m x\|_\infty$$

Theorem 17: Let

$$U_1 = \left\{a = (a_k) : \sum_k |a_k x_k| < \infty, \forall\, x \in X\right\} and$$

$$U_2 = \left\{a = (a_k) : \sup_k k^{-m} |a_k v_k| < \infty\right\}$$

Then

Ref

5. Et, D. D. & Esi, A.: "ON Köthe – Toeplitz Duals Of Generalized Difference Sequence Spaces", Bull. Malaysian Math. Sc. Soc.(second series) 23, 2000, 25 – 32.

$(i)\ \left(\Delta_v^m(l_\infty)\right)^\alpha = \left(\Delta_v^m(c)\right)^\alpha = \left(\Delta_v^m(c_0)\right)^\alpha = U_1$

$(ii)\ \left(\Delta_v^m(l_\infty)\right)^{\alpha\alpha} = \left(\Delta_v^m(c)\right)^{\alpha\alpha} = \left(\Delta_v^m(c_0)\right)^{\alpha\alpha} = U_2$

Corollary 1: $\left(\Delta_v^m(l_\infty)\right), \left(\Delta_v^m(c)\right)$ *and* $\left(\Delta_v^m(c_0)\right)$ are not perfect.

Corollary 2: If we take $(v_k) = (1,1,1....)$ and *m=1* in theorem 17, then we obtain for $X = l_\infty$ *or* c

$$(i)\ (\Delta^m(X))^\alpha = \left\{a = (a_k) : \sum_k k^{-m}|a_k| < \infty\right\},$$

$$(ii)\ (\Delta^m(X))^{\alpha\alpha} = \left\{a = (a_k) : \sup_k k^{-m}|a_k| < \infty\right\},$$

$$(iii)\ (\Delta^m(X))^\alpha = \left\{a = (a_k) : \sum_k k^{-m}\left|a_k v_k^{-1}\right| < \infty\right\}.$$

Corollary 3: If we take $v = (k^m)$ in theorem 17, then we obtain

$(i)\ \left(\Delta_v^m(l_\infty)\right)^\alpha = \left(\Delta_v^m(c)\right)^\alpha = \left(\Delta_v^m(c_0)\right)^\alpha = l_1$

$(ii)\ \left(\Delta_v^m(l_\infty)\right)^{\alpha\alpha} = \left(\Delta_v^m(c)\right)^{\alpha\alpha} = \left(\Delta_v^m(c_0)\right)^{\alpha\alpha} = l_\infty$

Prabhat Chandra and Binod Chandra Tripathy [2] introduced the concept of η - dual of sequence spaces. Further they established some results involving the perfectness of different sequence spaces relative to η - dual.

Denote by σ the space of all eventually alternating sequences i.e., if $(x_k) \in \sigma$, then there exists $k_0 \in N$ such that $x_k = -x_{k+1}, \forall k > k_0$. It is well known that $bv_0 = bv \cap c_0$.

Definition: Let E be a non – empty subset of ω and $r \geq 1$, then the η - dual of E is defined as

$$E^\eta = \left\{(y_k) \in \omega : (x_k y_k) \in l_r \ \textit{for all}\ (x_k) \in E\right\}.$$

A non – empty subset E of ω is said to be perfect or η – reflexive if $E^{\eta\eta} = E$. Taking r = 1 in the above definition we get the α – dual of E.

Theorem 18: $l_r^\eta = l_\infty, l_\infty^\eta = l_r$ and the spaces l_r *and* l_∞ are perfect spaces.

Theorem 19: $\sigma^\eta = l_r$ and σ is not perfect.

Theorem 20: Let $p > r \geq 1$, then $l_p^\eta = l_q$ where $p^{-1} + q^{-1} = r^{-1}$. The space l_p is perfect.

Theorem 21: $c_0^\eta = c^\eta = l_r$ and the sequence spaces c_0 *and* c are not perfect.

Theorem 22: $(bv)^\eta = l_r = (bv_0)^\eta$ and the spaces bv and bv_0 are not perfect.

T. Balasubramanian and A. Pandiarani [1] have given an account of some of the main developments which has occurred since Robinson's [19] paper of 1950.

A sequence $x = (x_k)$ is said to be an entire sequence if $\lim_{k\to\infty} |x_k|^{1/k} = 0$.

2. Chandra, P. & Tripathy, B.C. : "On Generalized Köthe – Toeplitz Duals Of Some Sequence Spaces", Indian J. pure appl. Math., 33 (8), 2002, 1301- 1306.

A sequence $x=(x_k)$ is said to be an analytic sequence if $-|x_k|^{1/k}$ " = 0.

Let Γ *and* $\wedge$ respectively denote the linear space of all entire and analytic sequences. G_λ denote the linear space of all sequences $x=(x_k)$ such that

$$\sum \lambda_k^2 |x_k|^2 < \infty, \lambda_k \text{ is fixed}, \lambda_k > 0, \text{and } \frac{\lambda_{k+1}}{\lambda_k} \to 1 \, as\, k \to \infty .$$

Notes

If $(X, \| \, \|)$ is any Banach space over C then we define

$$\Gamma(X) = \left\{ x = (x_k) : \lim_{k\to\infty} \|x_k\|^{1/k} < \infty \right\}$$

$$\wedge(X) = \left\{ x = (x_k) : \sup_k \|x_k\|^{1/k} < \infty \right\}$$

$$G_\lambda(X) = \left\{ x = (x_k) : \sum \lambda_k^2 \|x_k\|^2 < \infty \right\}$$

Where (λ_k) *is fixed sequence of real numbers and* $\frac{\lambda_{k+1}}{\lambda_k} \to 1 \, as\, k \to \infty$

Theorem 23: $G_\lambda(X) \subset \Gamma(X) \subset \wedge(X)$.

Suppose in general that *(A_k)* is a sequence of linear but not necessarily bounded operators A_k mapping a Banach space *X* into a Banach space *Y*.

Theorem 24: $(A_k) \in \Gamma^{\beta}(X)$ if and only if there exist $m \in N$ such that

(i) $(A_k) \in B(X,Y)$

(ii) $\sup_{k \ge m} \|A_k\|^{1/k} < \infty$

Theorem 25: $(A_k) \in \wedge^{\beta}(X)$ if and only if

(i) $(A_k) \in \Gamma^{\beta}(X)$ and

(ii) $\|R_n\|^{1/n} \to 0 \, as\, n \to \infty$.

Theorem 26: $(A_k) \in G_\lambda^{\beta}(X)$ if and only if there exists $m \in N$ such that

(i) $(A_k) \in B(X,Y)$ *for all* $k \ge m$ and

(ii) $\sum_{k=m}^{\infty} \frac{1}{\lambda^2} \|A_k\|^2 < \infty$.

Theorem 27: $(A_k) \in \Gamma^{\alpha}(X)$ if and only if there exists $m \in N$ such that

(i) $(A_k) \in B(X,Y)$

(ii) $\left(\sum_{k=m}^{\infty} \|A_k\| \right)^{1/k} < \infty$.

It is clear from theorem 27 that these conditions are also necessary and sufficient for $(A_k) \in \wedge^{\beta}(X)$

Hence $\Gamma^{\alpha}(X)=\wedge^{\alpha}(X)$.

Zeren and Bektas [29] introduced and studied the new sequence spaces

$[V,\lambda,F,p,q,u]_0(\Delta_v^m)$, $[V,\lambda,F,p,q,u]_1(\Delta_v^m)$ and $[V,\lambda,F,p,q,u]_\infty(\Delta_v^m)$

which are generalized difference sequence spaces defined by a sequence of moduli in locally convex Hausdorff topological linear space.

II. Recent Developments Done in Bicomplex Kothe – Toeplitz Duals of Some Bicomplex Sequence Spaces

These are some of the recent developments in the Köthe – Toeplitz duality theory of sequence spaces. We in our research work are studying the spaces of bicomplex sequences and, which is the most recent generalization of complex sequences. Köthe – Toeplitz duals of these spaces are also studied and some important results are found

About C_2:

Bicomplex Numbers were introduced by Corrado Segre (1860 – 1924) in 1892. He published a paper [20] in which he defined an infinite set of algebras and gave the concept of multicomplex numbers. For the sake of brevity, we have confined ourselves to the bicomplex version of his theory. The spaceof bicomplex numbers is the first in an infinite sequence of multicomplex spaces. Price [18] may be referred to study more about bicomplex space.

The set of bicomplex numbers is denoted by C_2 and defined as follows:

$$\mathrm{C}_2 = \{ x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0 \}$$

or equivalently as

$$\mathrm{C}_2 = \{ z_1 + i_2z_2 : z_1, z_2 \in C_1 \} ;$$

where $i_1^2 = i_2^2 = -1$; $i_1i_2 = i_2i_1$, and C_0 and C_1 denote the sets of real and complex numbers, respectively.

The binary compositions of addition and scalar multiplication on C_2 are defined coordinate wise and the multiplication in C_2 is defined term by term. With these binary compositions, C_2 becomes a commutative algebra with identity.

Algebraic structure of C_2 differs from that of C_1 in many respects. Few of them are mentioned below:

1. Non-invertible elements exist in C_2.
2. Non-trivial idempotent elements exist in C_2.
3. Non-trivial zero divisors exist in C_2.

If ω is the family of all sequences $\xi=(\xi_k)$ with $\xi_k \in C_2$, k ≥ 1 . Where C_2 is the space of bicomplex numbers.

$$1.\ \alpha-dual \quad \lambda^{\alpha} = \left\{ \xi : \xi \in \omega', \sum_{i\geq1} \|\xi_i.\eta_i\| < \infty, \forall y \in \lambda \right\}$$

$$2.\ \beta-dual \quad \lambda^{\beta} = \left\{ \xi : \xi \in \omega', \left\| \sum_{i\geq1} \xi_i.\eta_i \right\| < \infty, \forall y \in \lambda \right\}$$

Ref

18. Price, G. B., "An Introduction to Multicomplex Spaces And functions", Marcel Dekker, Inc., 1991.

3. $\gamma - dual \quad \lambda^{\gamma} = \left\{ \xi : \xi \in \omega', \sup_{n} \left\| \sum_{i \geq 1} \xi_i . \eta_i \right\| < \infty, \forall y \in \lambda \right\}$

4. $\delta - dual \quad \lambda^{\delta} = \left\{ \xi : \xi \in \omega', \sum_{i \geq 1} \left\| \xi_i . \eta_{\rho(i)} \right\| < \infty, \forall y \in \lambda \; and \; \rho \in \pi \right\}$

Various types of duals such as α - , β - , γ- , and δ – dual, can be defined on the so constructed bicomplex sequence spaces. In this way we have 16 types of duals for bicomplex sequence spaces viz.

α *α* – dual, *α* β – dual, *α* γ – dual, *α* *δ* – dual

β *α* – dual, ββ – dual, βγ – dual, β *δ* – dual

γ *α* – dual, γβ – dual, γγ – dual,γ *δ* – dual

δ *δ* – dual, *δ* β – dual , *δ* γ – dual, *δ* *δ* – dual

In [23] and [24] a functional analytic study of some bicomplex sequence spaces is done. In 2007, Srivastava and Srivastava [22] defined and studied a class B of bicomplex sequences associated with the functions which are holomorphic in the bicomplex space C_2, i.e., bicomplex entire functions. In [23], the study of Srivastava and Srivastava is furthered. The structure of the spectrum of an element in B has been formulated. Two subclasses B′ and B′′ have been defined and studied with a functional analytic viewpoint. Another class B* is also defined which has been shown to be a subalgebra of B which is not a normed subalgebra of B. However B* has been furnished with a Hilbert Space structure. In [24], certain properties of a particular class B′ of bicomplex sequences associated with the bicomplex functions which are holomorphic in the bicomplex space C_2, is investigated with a functional analytic view point. B′ has been provided with a modified Gelfand algebraic structure and it has been proved that B′ is an algebra ideal which is not a maximal ideal of B. Invertible and quasi invertible elements in B′ have been studied. A characterization of zero divisors is given and a sufficient condition for an element to be topological zero divisor has been derived. Algebra homomorphism between B and B′ has been investigated.In [25], [26] bicomplex duals of sequence spaces studied in [23], [24] are defined and analysed. [27] gives a study on some bicomplex modules. We are still working on duals and modules of these bicomplex sequence spaces and some interesting new results are expected in further study.

I shall always be thankful to Prof. Rajiv K. Srivastava for work done in this paper.

References Références Referencias

1. Balasubramanian T. & Pandiarani A. : "Duals of vector sequence spaces", Int. Journal of Math. Analysis, 3 (16), 2009, 779 – 785.
2. Chandra, P. & Tripathy, B.C. : "On Generalized Köthe – Toeplitz Duals Of Some Sequence Spaces", Indian J. pure appl. Math., 33 (8), 2002, 1301- 1306.
3. Colak R. "On some Generalized Sequence Spaces", Commun. Fac. Sci. Univ. Ank. Series A_1, 38, 1689, 35 – 46.
4. Colak R. & Et. M, "On some Generalized Difference sequence spaces", Soochow Journal of Math., 21, 1995, 377 – 386.
5. Et, D. & Esi, A.: "ON Köthe – Toeplitz Duals Of Generalized Difference Sequence Spaces", Bull. Malaysian Math. Sc. Soc.(second series) 23, 2000, 25 – 32.

Ref

22. Srivastava, Rajiv K., Srivastava, Naveen K., "On a class of Entire Bicomple sequences", South East Asian J. Math. & Math. Sc. 5(3), 2007, 47 – 68.

6. Garling, D.G.H.: "The β - and γ - duality", *Proc. Cambridge Philos. Soc. (1967), 963 – 981.*
7. Garling, D.G.H.: "On symmetric sequence spaces", *Proc. London Math. Soc. 16 (3), 1966, 85 – 106.*
8. Kamthan, P.K & Gupta, M: " Sequence spaces and Series", *Marcel Dekker, Inc. New York and Bassel.*
9. Kizmaz, H. : "On certain Sequence Spaces", Canad. Math. Bull. 24 (1981), 169 – 176.
10. Köthe, G & Toeplitz, O: "Lineare Raume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen", Jour. Reine angew. Math. 171, 1934 , 193 – 226.
11. Lascarides, G. Constantine: "A Study of Certain sequence spaces of Maddox and A generalization of a theorem of Iyer",Pacific Journal of Mathematics, 38 (2), 1971.
12. Maddox, I. J. : "Generalized Köthe – Toeplitz duals", International, J. Math. & Math. Sci. 3 (3), 1980, 423 – 432.
13. Maddox, I. J. : "Solidity in Sequence Spaces", Revista Mathematica de la, Universidad Complutense de Madrid, 4 (2), 1991, 185 – 192.
14. Maddox, I. J., "Paranormed sequence spaces generated by infinite matrices", Proc. Cambridge Philos. Soc. 64 , 1968, 335 – 340.
15. Maddox, I. J., "Continuous and Köthe – Toeplitz duals of certain sequence spaces.", Proc. Cambridge Philos. Soc. 65 , 1969, 431 – 435,
16. Maddox, I. J., "Some properties of paranormed sequence spaces", J. London Math. Soc., (2), 1, 1969, 316 – 322.
17. Maddox, I. J., "Spaces of strongly summable sequences", Quaterly J. Math. Oxford, 2(18),1967, 345 – 355.
18. Price, G. B., "An Introduction to Multicomplex Spaces And functions", Marcel Dekker, Inc., 1991.
19. Robinson, A. R., "On functional transformations and summability", Proc. London Math. Soc., 52, 1995, 132 – 160.
20. Segre C., "Le Rappresentazioni Reali Delle forme complesse e Gli Enti Iperalgebrici", Math. Ann., 40, 1892, 413-467.
21. Srivastava, Rajiv K., "Certain topological aspects of Bicomplex space", Bull. Pure & Appl. Math. 2(2), 2008, 222-234.
22. Srivastava, Rajiv K., Srivastava, Naveen K., "On a class of Entire Bicomple sequences", South East Asian J. Math. & Math. Sc. 5(3), 2007, 47 – 68.
23. Wagh Mamta A., "On certain spaces of Bicomplex sequences ", Inter. J. Phy. Chem. and Math. fund.(IJPCMF), 7(1), 2014, 1 – 6.
24. Wagh Mamta A., "On a Class of Bicomplex Sequences", Inter. J. Trends in Math. & Stats., 3(5), 2014, 158 – 171.
25. Wagh, Mamta A. & Kumar, S., "Köthe – Toeplitz Duals of Certain Bicomplex Sequence Spaces", Inter. J. Math. & Comp. App. Research(IJMCAR), 4(3), 2014, 87–98.
26. Wagh, Mamta A. & Kumar, S., "On certain Bicomplex Duals", Global J. Sc. & Frontier Research (GJSFR), 14(6), 2014, 17 – 23.

Notes

27. Wagh, Mamta A., "On some Bicomplex Modules", to appear in Global J. Sc. & Frontier Research GJSFR, 16(3), 2016.
28. W. Ruckle, "Topologies on sequence spaces", Pacific Jour. Math. 42(1), 1972, 235-249.
29. Zeren S. & Bektas A. Cigdema. "Generalized difference sequence spaces defined by a sequence of moduli", Kragujevac Journal of Mathematics, 36(1), 2012, 83 – 91.

Notes