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Using Spin, Twist and Dial Homeomorphisms to Generate Homeotopy Groups

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Using Spin, Twist and Dial Homeomorphisms to Generate Homeotopy Groups

David Sprows

I. Introduction

In this paper we consider some problems concerned with the isotopy classification of homeomorphisms of multiply punctured compact 2-manifolds, i.e., manifolds of the form $X - \cup_{i=1}^{m} \mathring{D}_i - P$ where X is a closed 2-manifold, $\{D_i : 1 \leq i \leq m\}$ is a family of disjoint discs in X and $P = \{p_{m+1}, \ldots, p_n\}$ is a finite subset of X disjoint from each D_i. Inparticular we will show that various homeotopy groups for these manifolds are generated by the isotopy class of three types of homeomorphisms. In the case X is the two sphere we will give a complete presentation of these homeotopy groups. Parts of the material in this paper have been considered in [5], [6] and [7], but this is the first time that a full treatment of this topic, including detailed illustrations of the isotopies involved, has been submitted for publication. For an alternate approach to the material in this paper see (for example) [2], and [].

Definitions and Notation

Let x be a 2-manifold (connected, triangulated).
Let $A \subset X$ and $B \subset X$.
$G(X)$ = group of all homeomorphisms of X onto itself with the compact-open topology.
$G_0(X)$ = arc component of 1_X in $G(X)$. Clearly $G_0(X)$ is normal in $G(X)$.
$H(X) = G(X)/G_0(X)$ = the homeotopy group of X.
$G(X,A) = \{g \in G(X) : g/A \in G(A)\}$.
$G_0(X,A)$ = arc component of 1_X in $G(X, A)$.
$H(X,A) = G(X,A)/G_0(X,A)$ = the homeotopy group of (X, A).
$G`(X,A) = \{g \in G(X) : g / A = 1_A\}$.
$G`_0(X,A)$ = arc component of $1_X$ in $G`(X,A)$.
$H`(X,A) = G`(X,A)/G`_0(X,A)$
$H^*(X,A) = (G`(X,A) \cap G_0(X))/G`_0(X,A)$.
$G(X,A,B) = \{g \in G(X) : g / A \in G(A), g / B \in G(B)\}$.
$G_0(X,A,B)$ = arc component of 1_X in $G(X,A,B)$.
$H(X,A,B) = G(X,A,B)/G_0(X,A,B)$.
$\mathcal{S}_n$= symmetric group on n letters.

Remarks:

1) $f \in G_0(X)$ iff f is isotopic to 1_X (denoted $f \simeq 1_X$).
2) $f \in G_0(X, A)$ iff f is isotopic to 1_X by an isotopy which keeps A invariant.

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5. D. Sprows, "Homeotopy groups of compact manifolds" Fundamenta Mathemaiticae, 90 (1975) 99-103.

3) $f \in G`_0(X, A)$ iff f is isotopic to 1_X by an isotopy which is fixed on A (denoted $f = 1_X$ (rel A)).
4) If A is a finite set, then $G_0(X,A) = G`_0(X,A)$ and if also $A = A` \cup A``$, then $G_0(X, A`, A``) = G_0(X,A)$.

II. Homeotopy Groups

In the fpllowing we develop some of the basic tools used in studying homeotopy groups of multiply punctured compact manifolds. We also obtain a presentation for $H^*(S^2, F_n)$where F_n is a finite subset of S^2.

Lemma1: Let X be a compact 2-manifold, $F = \{p_i : 1<i<n\}$a finite subset of X.

Define k: $H(X, F) \rightarrow H(X) \times H(F)$ by $k(fG_0(X, F)) = (fG_0(X), (f/F)G_0(F)) = (fG_0(X), f/F)$.Then k is an epimorphism with kernel $H^*(X, F) = (G`(X, F)\cap G_0(X))/G`_0(X, F)$.Therefore, we have the exact sequence $0 \rightarrow H^*(X, F) \rightarrow H(X, F) \xrightarrow{k} H(X) \times \$_n \rightarrow 0$ where we have identified H(F) with $\$_n$.

Proof. The map k is clearly well-defined and a homomorphism. Let $[gG_0(X), \alpha] \in H(X) \times \$_n$. Suppose $g(p_i) = y_i$. By the homogeneity of X for finite subsets of X we can assume $g(p_i) = p_i$. By the remark following Lemma 1.2 (or homogeneity again), we can find a homeomorphism $h \in G_0(X)$ with $h/F = \alpha$. So k is an epimorphism.

Now $fG_0(X,F) \in \ker k$ if and only if $f \in G_0(X)$ and $f/F = 1_F$. That is $f \in G_0(X)\cap G`(X, F)$. But for finite sets, $G_0(X,F) = G`_0(X,F)$. Thus,
$\ker f = (G_0(X)\cap G`(X, F))/G`_0(X, F) = H^*(X, F)$.

Remark 1. By Lemma 1.1, $H(X,F) \cong H(X-F)$ so we also have the short exact sequence $0 \rightarrow H^*(X, F) \rightarrow H(X, F) \rightarrow H(X) \times \$_n \rightarrow 0$.

Remark 2. Similarly, if $\{D_i : 1\leq i\leq n\}$ is a family of disjoint discs in a closed 2-manifold X, then by Theorem 1.6, we have the short exact sequence

$$0 \rightarrow H^*(X, F) \rightarrow H(X - \cup_{i=1}^{n} \mathring{D}_i) \xrightarrow{k1} H(X) \times \$_n \rightarrow 0.$$

Lemma 2 Let X be a closed 2-manifold, $\{D_i : 1\leq i\leq m\}$ a family of disjoint discs in X with $p_i \in \mathring{D}_i$. Let $\{p_i: m+1\leq i\leq m+r\}$be a set of points in X disjoint from $\cup_{i=1}^{m} D_i$.Consider the composition $\phi : H(X- \cup_{i=1}^{m} \mathring{D}_i, \cup_{j=m+1}^{m+r} p_j) \xrightarrow{k1} H(X- \cup_{i=1}^{m} \mathring{D}_i) \times \$_n \xrightarrow{(\psi, 1)} H(X, \cup_{i=1}^{m} p_i) \times \$_n \xrightarrow{(k2,1)} H(X) \times \$_m \times \$_r$ where k_1 is as in Remark 2 after Lemma 3.1, ψ is as in Theorem 1.6 and k_2 is as in Lemma 3.1. Then ϕ is an epimorphism with kernal$\cong H^*(X,\cup_{i=1}^{m+r} p_i)$.

Proof.ϕ is an epimorphism, since k_1, k_2, and ψ are epimorphisms. We have $fG_0(X- \cup_{i=1}^{m} \mathring{D}_i, \cup_{m+1}^{m+r} p_j)$ in the kernal of ϕ provided 1) $f_C \in G_0(X)$, 2) $f(\partial D_i) = \partial D_i$, $1\leq i\leq m$, and 3) $f(p_i) = p_i$, $m+1\leq i\leq m+r$. Now by Corollary 1.8 $H(X- \cup_{i=1}^{n} \mathring{D}_i,\cup_{m+1}^{m+r} p_j) \cong H(X- \cup_{1}^{m} p_i, \cup_{m+1}^{m+r} p_j)$and the image of $\ker \phi$ under this isomorphism is $H^*(X, \cup_{i=1}^{m+r} p_i)$. This follows since $G_0(X, \cup_{1}^{m} p_i, \cup_{m+1}^{m+r} p_i) = G_0(X, \cup_{1}^{m+r} p_i)$.

Remarks.

1) Note that we have the short exact sequence $0 \rightarrow H^*(X, \cup_{1}^{m+r} p_i) \rightarrow H(X - \cup_{i=1}^{m} \mathring{D}_i - \cup_{m+1}^{m+r} p_i) \rightarrow H(X) \times \$_m \times \$_r \rightarrow 0$.

2) Since H(X) is known for X a closed 2-manifold (see Sections 2 and 6 of [1]), Lemmas 1 and 2 show that once we determine $H^*(X,F)$ for any finite set F and any closed 2-manifold X, the homeotopy group of any multiply punctured compact 2-manifold will be determined up to a group extension.
3) In the following we will determine a presentation for $H^*(S^2, F)$ for any finite subset F of S^2.

Lemma.3 Let X be a 2-manifold and let $x_0 \in \mathring{X}$. Then
1) G(X) is a fibre bundle over $\mathring{X}$with fibre $G(X, x_0)$
2) $i_* : \pi_i(\mathring{X}, x_0) \cong \pi_i(X, x_0)\ \forall i$.

Proof. 1) is given in Lemma 4.10 of [3].
2) is given in Lemma 4.11 of [3].

Remarks.
1) The homotopy sequence of this fibration together with the isomorphism in 2) yields $\to \pi_1(G(X, x_0),1_X) \xrightarrow{i^*} \pi_1(G(X),1_X) \xrightarrow{p^*} \pi_1(X, x_0) \xrightarrow{d^*} \pi_0(G(X, x_0), 1_X) \xrightarrow{i^*} \pi_0(G(X),1_X) \to 0$.

The map $i_* : \pi_0(G(X, x_0),1_X) \to \pi_0(G(X),1_X)$ is onto because any homeomorphism of X is isotopic to one that fixes x_0.

2) $\pi_0(G(X, x_0),1_X) \cong H(X, x_0)$ and $\pi_0(G(X),1_X) \cong H(X)$ and making these replacements above we get the homeotopy sequence of X

$$\to \pi_1(G(X, x_0),1_X) \xrightarrow{i^*} \pi_1(G(X),1_X) \xrightarrow{p^*} \pi_1(X, x_0) \xrightarrow{d^*} \pi_0(H(X, x_0), 1_X) \xrightarrow{i^*} H(X) \to 0.$$

3) Let X be a closed 2-manifold. Let $F_{n+1} = \{p_0, \dots, p_n\}$ be a set of n+1 points in X and let $F_n = \{p_0, \dots, p_{n-1}\} = F_{n+1} - p_n$. Consider the homeotopy sequence of $X\text{-}F_n$ where we use p_n for the base point of $X\text{-}F_{n,}$ the last few terms are $\dots \to \pi_1(X\text{-}F_n, p_n) \xrightarrow{d^*} H(X\text{-}F_n, p_n) \to H(X\text{-}F_n) \to 0$.

If, in this sequence, we replace $H(X\text{-}F_n, p_n)$ by a subgroup L which still contains im $d_* = \ker i_*$ and replace $H(X\text{-}F_n)$ by i_*L then the new sequence will also be exact. Now, since $G_0(X, F_{n+1}) = G'_0(X, F_{n+1})$, we have that $H^*(X, F_{n+1})$ is a subgroup of $H(X, F_{n+1})$. The restriction map $h \to h/X\text{-}F_n$ defines an isomorphism of $H^*(X, F_{n+1})$ with a subgroup L of $H(X\text{-}F_n, p_n)$.

Now, a homeomorphism g of $X\text{-}F_n$ represents an element of L if and only if its unique extension $\bar{g}$toXx sends p_i to itself, $0 \le i \le n$, and is isotopic to 1_X (all p_i being allowed to move). $g \in G(X\text{-}F_n)$ represents an element of $\ker i_*$ provided its extension $\bar{g}$ sends each p_i to itself and is isotopic to 1_X (all p_n being allowed to move). Hence, $\ker i_* \subset$ L.

$f \in G(X\text{-}F_n)$ represents an element of i_*L provided $\bar{f}$ takes each p_i to itself $0 \le i \le n\text{-}1$ and $\bar{f}$ is isotopic to 1_X (all p_i being allowed to move). Thus, i_*L can be identified with $H^*(X, F_n)$.

Now in the homeotopy sequence of $X\text{-}F_n$ with p_n as base point, we replace $H(X\text{-}F_n, p_n)$ by its subgroup L and $H(X\text{-}F_n)$ by i_*L. Next, we replace L by its isomorph $H^*(X, F_{n+1})$ and i_*L by its isomorph $H(X, F_n)$ getting the exact sequence

$$\dots \to \pi_1(X\text{-}F_n,p_n) \xrightarrow{d} H^*(X,F_{n+1}) \xrightarrow{e} H^*(X, F_n) \to 0.$$

4) Note if $[\tau]$ $\pi_1(X\text{-}F_n,p_n)$ where τ is a loop in $X\text{-}F_n$ based at p_n, then $d([\tau])$ $H^*(X, F_{n+1})$ is determined as follows:

Let H_t be an isotopy of X beginning at 1_X which drags the point p_n about the loop τ while leaving each p_i, $0 \le i \le n\text{-}1$ fixed; then $d([\tau])$ is represented by the

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1. D.B.A. Epstein, “Curves on 2-manifolds and isotopies” Acta. Math. 115 (1966) 83-107.

homeomorphism H_1. Thus, the isotopy class (rel F_{n+1}) of a homeomorphism h represents d([τ]), provided $h = 1_X$ (relF_n) and the isotopy H_t from 1_X to h is such that $H_t(p_n) = \tau$. Note: The map e, on the level of homeomorphisms, takes each h to itself.

Lemma 4 Let $F_n = \{p_0, \ldots, p_{n-1}\}$ denote a set of n points in S^2. It $n \geq 3$, then $\pi_1(G(S^2\text{-}F_n), 1) = 0$.

Proof. This is stated in Theorem 3.1 of [1]. See also page 303 of [3].

Remark. Let $p_0, p_1, \ldots, p_n, \ldots$ be a sequence of distinct points in S^2 and let $F_n = \{p_0, \ldots, p_{n-1}\}$. Then, in view of Lemma 3.4 we have $0 \to \pi_1(S^2\text{-}F_n, p_n) \to H^*(S^2\text{-}F_n) \to 0$ is exact for $n \geq 3$.

Lemma 3.5 Given an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. Let $\{a_i, \ldots, a_n\}$ generate A and $\{c_1, \ldots, c_m\}$ generate C. Suppose $b_1, \ldots, b_r$ are elements of B satisfying $f(a_i) = b_i$, 1 i r and $b`_1, \ldots, b`_m$ in B satisfy $g(b`_i) = c_i$, $1 \leq i \leq m$, then $\{b_1, \ldots, b_r, b`_1, \ldots, b`_m\}$ generates B.

Proof. Let $b \in B$. Then $g(b) \in C$, i.e., $g(b) = w_1(c_i)$ where $w_1(c_i)$ is a word in $c_1, \ldots, c_m$. Thus, $g(b) = w_1(g(b`_i)) = g(w_1(b`_i))$. So $g(bw_1(b`_i))^{-1}) = 1_C$. Now, $bw_1(b`_i)^{-1} \in \ker g = \text{lm } f \Rightarrow bw_1(b`_i)^{-1} = f(w_2(a_i))$ where $w_2(a_i)$ is a word in $a_1, \ldots, a_r$. Hence, $bw_1(b`_i)^{-1} = w_2(f(a_i)) = w_2(b_i)$. So $b = w_1(b`_i)w_2(b_i)$.

Next, we define "twist homeomorphisms". These will be used to obtain generators for $H^*(S^2, F_n)$.

Let $P = p_0, p_1, \ldots$ be a sequence of points in the interior of a disc D which converge to $p_\infty \in \mathring{D}$. Let α be an oriented simple closed curve in $\mathring{D}$-P and let D` denote the closure of that component of D-α which forms an open disc. We define the "twist homeomorphism of D supported by a``, denoted h_α, as follows:

Let A denote a collar neighborhood of α in the disc D` with $A \cap P = \emptyset$ (see Figure 1). Let e: $S`\times I \to A$ be a homeomorphism with $e/S`\times I = \alpha$ where $S`\times I$ is oriented as in Figure 2. Define g: $S`\times I \to S`\times I$ by $g(x,t) = (x\text{-}t,t)$ where $S` = R/Z$ (see Figure 2). Finally, we define h_α: $(D, P) \to (D, P)$ by $h_\alpha(x) = ege^{-1}(x)$ for $x \in A$ and $h_\alpha(x) = x$ for $x \in D\text{-}A$.

Now, suppose $D \subset \mathring{} X$ where X is a 2-manifold. Since h_α is the identity on ∂D we can extend h_α to X by the identity. This new homeomorphism will also be denoted h_α and called the twist homeomorphism of X supported by α.

Remark 1. Different choices for the annulus A, satisfying the above conditions, result in different choices for h_α. However, each of these possibilities for h_α are ambient isotopic (rel P) to one another by the regular neighborhood theorem (applied to ∂D` in D`-P). Since we are interested only in the isotopy classes (rel P) of homeomorphisms, we will abuse the notation slightly and use h_α to denote any homeomorphism which results from the above process.

Remark 2. In [6] twist homeomorphisms (or "c-homeomorphisms") are defined for any simple closed curve B in a closed 2-manifold X as follows: Let e`: $S`\times I \to A`$ be a regular neighborhood of B in X with $e`/S` \times æ = B$ and define g`: $A` \to A`$ as indicated in Figure 3. Define the "c-homeomorphism corresponding to B" to be the homeomorphism obtained by conjugating g` by e` and extending by the identity. In the case B bounds a disc D`, this definition yields our definition of $h_B$ provided we add the condition $e`(S` \times 0) \subset D`$. Without this condition it is possible for two different embeddingse`: $S`\times I \to A`$ and e``: $S`\times I \to A``$ to yield two non-isotopic "c-homeomorphisms", namely a homeomorphism and its inverse. Thus, the isotopy class of a "c-homeomorphism" is not uniquely determined by the curve B.

Notes

Remark 3. With the notational abuse mentioned in Remark 1 we can write $h^{-1}_{\alpha} = (h_\alpha)^{-1}$.

Remark 4. For any n, h_α is defined in X-F_n where $F_n = \{p_0, \ldots , p_{n-1}\}$ to be the restriction of h_α to X-F_n.

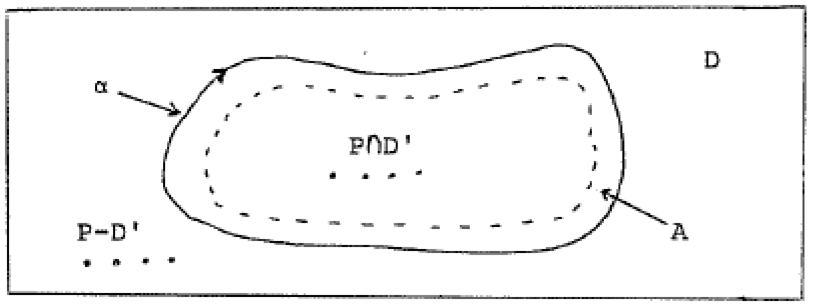

Figure 1

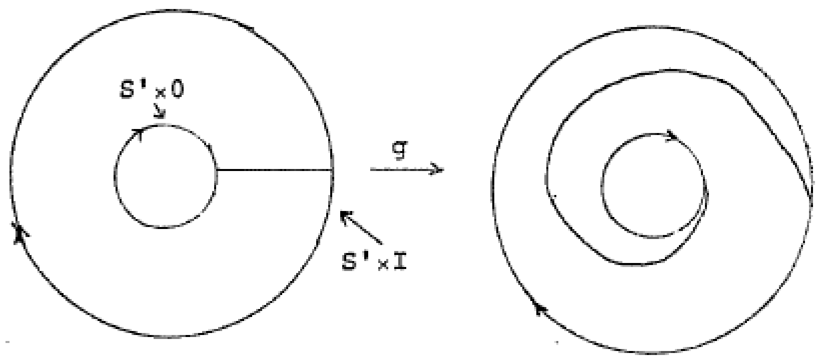

Figure 2

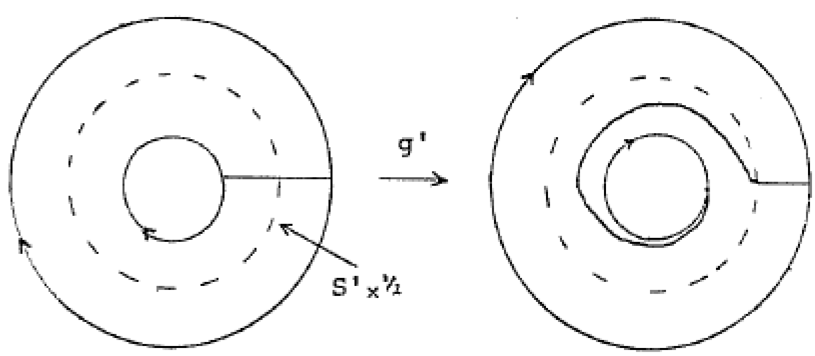

Figure 3

The following two lemmas are versions of lemmas given in [7]. The proofs follow almost immediately from the definitions.

Lemma 6 Let x be a 2-manifold and let F be a finite set of points in $\mathring{X}$. Let D be a disc in $\mathring{X}$ with F ⊂ $\mathring{D}$ and let α be a simple closed curve in $\mathring{D}$-F. Let f: (X, F) -> (X, F) be a homeomorphism; then $h_{f\circ\alpha} \simeq f h_\alpha f^{-1}$(rel F).

Proof. If e : S`×I → A is the embedding used to define $h_\alpha$, then $f\circ e$ : S`×I → f(A) can be used to define $h_{f\circ\alpha}$. Hence, $h_{f\circ\alpha}(x) = (f\circ e)g(f\circ e)^{-1}(x)$ on f(A) and $h_{f\circ\alpha}(x) = x$ elsewhere. That is, $h_{f\circ\alpha}(x) = f\circ h_\alpha \circ f^{-1}(x)$ on f(A) and $h_{f\circ\alpha}(x) = x$ elsewhere. But $f\circ h_\alpha\circ f^{-1}(x) = x$ for x ˜ϵ f(A). Thus, $h_{f\circ\alpha} = f\circ h_\alpha \circ f^{-1}$everywhere on X.

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7. D. Sprows, "Boundary fixed homeomorphisms of 2-manifolds with boundary" Global Journal of Science Frontier Research 11 (2011) 13-15.

Lemma 7 Let F, D, and X be as in Lemma 3.6. Let $\alpha, \beta \in \mathring{D}$-F be simple closed curves with $\alpha \simeq \beta$ (rel F), i.e., α is ambient isotopic to β (rel F) in X. Then $h_\alpha \simeq h_\beta$(rel F).

Proof. Let $H_t : X \to X$ be the ambient isotopy with $H_1\alpha = \beta$, then by Lemma 3.6 $h_\alpha \simeq H_1 h_\alpha H_1^{-1}$. Thus, $F_t = H_t h_\alpha H_t^{-1}$. is an isotopy between h_α and h_β.

We now concentrate on S^2. Let $P = p_0, \ldots, p_n, \ldots$ be a sequence of points in S^2 converging to a point p. Let i¡j and let α_{ij} be the simple closed curve in S^2 given in Figure 4, i.e., α_{ij} is oriented in a clockwise direction about p and encloses the points p_i and p_k for $k \geq j$ as indicated. Note that for pictorial purposes S^2 appears as a disc with q as its boundary, but on S^2, q is identified to a single point. We let a_{ij} denote the homeomorphism $h_{\alpha ij}$ where to define $h_{\alpha ij}$we take a disc about P not containing the point q.

Lemma 8 Let $n \geq 3$ and let $F_n = \{p_0, \ldots, p_{n-1}\} \subset P$. Consider the short exact sequence $0 \to \pi_1(S^2\text{-}F_n, p_n) \xrightarrow{d} H^*(S^2, F_{n+1}) \xrightarrow{e} H^*(S^2, F_n) \to 0$ given in the remark following Lemma 3.4. If a_{ij} is defined as above, then the image of d is generated by $\{\bar{a}_{in} : 1 \leq i \leq n\}$ where $\bar{a}_{in}$= isotopy class of a_{in} in $H^*(S^2, F_{n+1})$.

Proof. The loops α_{in} can be deformed slightly to yield loops B_{in} passing thru p_n such that the homotopy classes $[B_{in}^{-1}]$, $1 \leq i < n$, generate $\pi_1(S^2\text{-}F_n, p_n)$. Let b_{in}^{-1} be the homeomorphism which dials the point p_n once around B_{in}^{-1} as indicated in Figure 5. The identity is clearly isotopic to b_{in}^{-1} (relF_n) by an isotopyH_t with the property that $H_t(p_n) = B_{in}^{-1}$. Thus, as indicated in Remark 4 following Lemma 3, we have $d[B_{in}^{-1}] = \bar{b}_{in}^{-1}$. On the other hand,$\bar{b}_{in}^{-1} = \bar{a}_{in}$, i.e., $b_{in}^{-1} \simeq a_{in}^{-1}$ keeping F_{n+1} fixed. To see this just note that on the disc about p_i bounded by B_{in}^{-1} we have b_{in}^{-1} restricted to the boundary of this disc is the identity. Hence, b_{in}^{-1} restricted to this can be isotope to the identity on this disc by an isotopy which keeps the boundary of the disc fixed. Extending this isotopyby the identity to all of S^2 we see b_{in}^{-1} is isotopic to a_{in} (relF_{n+1}). Note if we denote the isotopy from b_{in}^{-1} to a_{in} by I_t, then $I_{1/2}$ is given in Figure 6.

Theorem 9 For $n \geq 3$, $H^*(S^2, F_{n+1})$ is generated by$\cup_{k=3}^{n} \{\bar{a}_{ik} : 1 \leq i < k\}$where $\bar{a}_{ik}$is the isotopy class of a_{ik} in $H^*(S^2, F_{n+1})$.

Proof. Recall the short exact sequence $0 \to H^*(S^2, F_{n+1}) \to H^*(S^2, F_n) \xrightarrow{k} H(S^2) \times \mathcal{S}_n \to 0$ of Lemma 3.1. By Theorem 3.1(c) of [1], $H^*(S^2, F_3) \simeq H(S^2) \times \mathcal{S}_3$. Hence, $H^*(S^2, F_3) = 0$. In particular, letting n=3 in the sequence $0 \to \pi_1(S^2 - F_n, p_n) \xrightarrow{d} H^*(S^2, F_{n+1}) \xrightarrow{e} H^*(S^2, F_n) \to 0$ we have $\pi_1(S^2 - F_3, p_3) \cong H^*(S^2, F_4)$. Thus, by Lemma 3.8, $H^*(S^2, F_4)$ is generated by $\{\bar{a}_{13}, \bar{a}_{23}\}$. Inductively, assume ${}^{n-1}\cup_{k=3} \{\bar{a}_{ik} G_0(S^2, F_n) : 1 \leq i < k\}$generates $H^*(S^2, F_n)$. Now, $e(a_{ik} G_0(S^2, F_{n+1})) = a_{ik} G_0(S^2, F_n)$. Hence, $\cup_{k=3}^{n-1} \{e(\bar{a}_{ik}) : 1 \leq i < k\}$ generates $H^*(S^2, F_n)$. By Lemma 3.8, $\{\bar{a}_{ln}, \ldots, \bar{a}_{n-ln}\}$ generates the image of d. The Theorem now follows by applying Lemma 3.5.

Notation: We let $G_n = \cup_{k=3}^{n} \{\bar{a}_{ik} : 1 \leq i < k\} \subset H^*(S^2, F_{n+1})$.

Remark. G_n is a generating subset of $H^*(S^2, F_{n+1})$ by Theorem 9. The remainder of this chapter will be devoted to finding a complete set of relations among these generators.

Notes

Global Journal of Science Frontier Research (F) Volume XVI Issue VI Version I Year 2016

Lemma 10 As in Lemma 3.5, let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of groups. Let $\{a_1,\ldots,a_r\}$ generate A and $\{c_1,\ldots, c_r\}$ generate C. Let $b_i = f(a_i)$, $1\leq i\leq r$ and $g(b_1') = c_i$, $1\leq i\leq m$; then (i) B has a presentation with generators $\{b_1,\ldots, b_r, b_1',\ldots, b_m'\}$ in which ever relation has the form (1) $w(b_i) = w'(b_i')$, (2) $b_j' b_i b_j'^{-1}$, (3) $b_j'^{-1} b_i b_j' = w(b_i)$where $w(b_i)$ denotes a word in $b_1,\ldots, b_r$and $w'(b_i')$ denotes a word in $b_1',\ldots, b_m'$. (ii) Moreover, if the exact sequence splits, i.e., if there exists a homomorphism k: $C \to B$ with $g\circ k= 1_C$ and if we suppose that $b_i' = k(c_i)$ for $l\leq i\leq m$, then every relation of form (1) can be expressed as (1.1) $w(b_i) = l$ and (1.2) $w'(b_i') = 1$.

Proof. Since f (A) is normal in B we have that relations of the form (2) $b_j' b_i' b_j'^{-1} = w(b_i)$and (3) $b_j'^{-1} b_j' b_i' = w(b_i)$ exist for $l\leq i\leq r$ and $l\leq j\leq m$. Using relations of form (2) and (3) any other relator can be rewritten in the form $w(b_i)w'(b_i')$ and then transposed into form (1). Thus, part (i) of the lemma holds.

Part (ii) of Lemma 10 follows immediately from part (i), since if the sequence splits, then $f(A)\cap k(C) = \{l_B\}$, so any relation $w(b_i) = w'(b_i')$ reduces to $w(b_i) = l$ and $w'(b_i') = l$.

Remark. Suppose that for a given k and j we have that a relation of the form $b_k = b_j' w(b_i) b_j'^{-1}$ is a consequence of the relations of type (1) and (2) of Lemma 3.10. Then it follows that $b_j'^{-1} b_k b_j' = w(b_i)$. Hence, the corresponding relation of type (3) can be dropped from the given presentation.

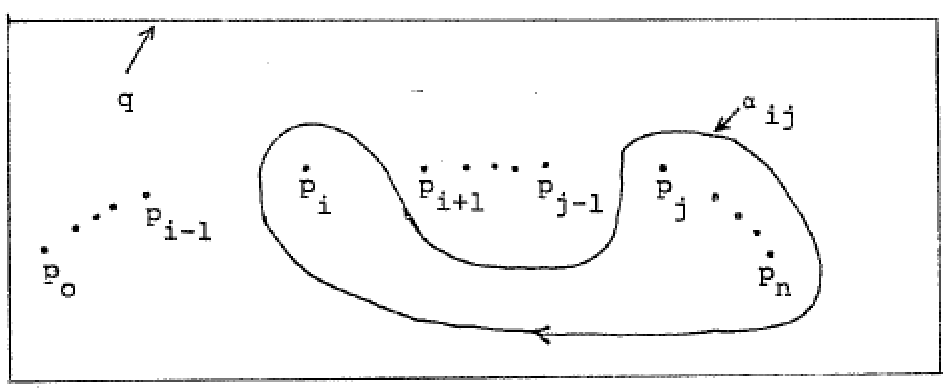

Figure 4

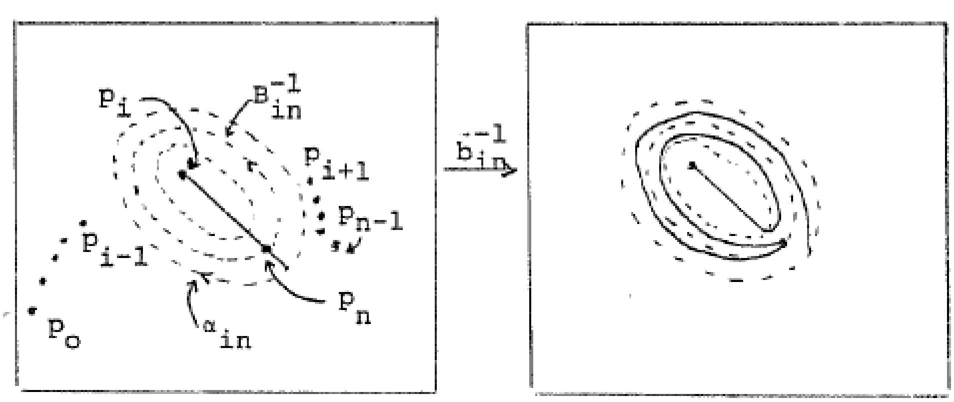

Figure 5

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Global Journal of Science Frontier Research (F) Volume XVI Issue VI Version I Year 2016

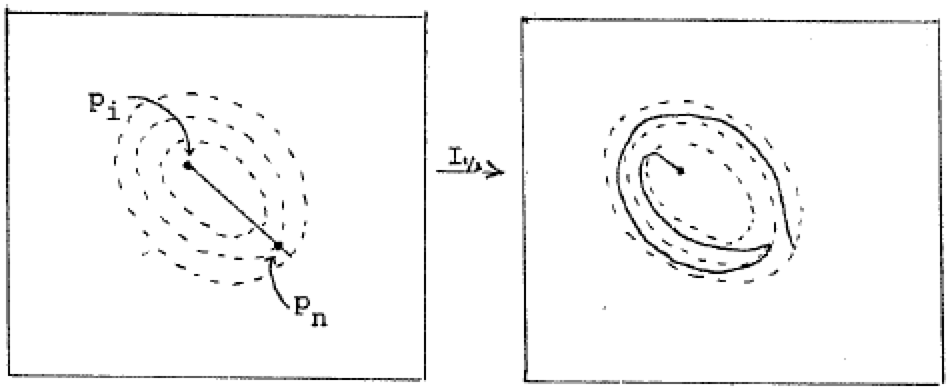

Figure 6

Remark 1. We will apply Lemma 10 to the short exact sequence $0 \to \pi_1(S^2\text{-}F_n, P_n) \xrightarrow{d} H^*(S^2, F_{n+1}) \xrightarrow{e} H^*(S^2, F_n) \to 0$. In the proof of Theorem 9 it is shown that the set $G_{n-1} = \cup_{k=3}^{n-1} \{\bar{a}_{ik} : 1 \leq i \leq k\}$ maps onto a generating set for $H^*(S^2, F_n)$ and the set $G_n - G_{n-1} = \{\bar{a}_{in}: 1 \leq i < n\}$ generates the image of d. Hence by part (i) of Lemma 3.10, $H^*(S^2, F_{n+1})$ has a presentation using these generators in which every relation has the form (1) $w(\bar{a}_{ik}) = w`(\bar{a}_{in})$, (2) $\bar{a}_{ik}\bar{a}_{rn}\bar{a}_{ik}^{-1} = w`(\bar{a}_{jn})$, (3) $\bar{a}_{ik}^{-1}\, \bar{a}_{rn}\bar{a}_{ik} = w`(\bar{a}_{jn})$ where in (1), (2), (3) we assume $k<n$.

Remark 2. The Technique for determining relations of types 2 and 3 is based on Lemmas 3.6 and 3.7. We will describe the approach for type 2 relations. Suppose $a_{ik}(\alpha_{rn}) = \gamma$. Then by Lemma 3.6, and the fact that a_{rn} is the twist homeomorphism corresponding to α_{rn}, we have $h_\gamma \simeq a_{ik}a_{rn}a_{ik}^{-1}$ $(\text{rel}F_{n+1})$. Suppose also we can find a product of homeomorphisms $w`(a_{jn})$ such that $w`(a_{jn})(\alpha_{rn}) = \gamma` \simeq \gamma$ $(\text{rel}F_{n+1})$, then by Lemma 3.6 we have $h_\gamma \simeq w`(a_{jn})a_{rn}w`(a_{ik})^{-1}$ $(\text{rel } F_{n+1})$. Now $h_\gamma \simeq h_\gamma$, $(\text{rel}F_{n+1})$ by Lemma 3.7. Hence $a_{ik}a_{rn}a_{ik}^{-1} \simeq w`(a_{jn})a_{rn}w`(a_{ik})^{-1}$and therefore $\bar{a}_{ik}\bar{a}_{rn}\bar{a}_{ik}^{-1} \simeq w`(\bar{a}_{jn})\bar{a}_{rn}w`(\bar{a}_{ik})^{-1} = w``(\bar{a}_{jn})$ is the desired relation of type 2. To adapt this technique to yield type 3 relations, just replace a_{ik} with a_{ik}^{-1}.

Remark 3. The next lemma is needed in the proof of Lemma 3.12, which in turn supplies the basis for applying the technique in Remark 2.

Lemma 11 Let D_n be a disc and $D_1,\ldots,D_{n-1}$ a family of disjoint discs in $\mathring{D}_n$. Let $B = \cup_{i=1}^{n-1} D_i$. Let r_i be an arc from ∂D_i to ∂D_{i+1}, $1 \leq i \leq n-1$, as indicated in Figure 3.7, i.e. the r_i are disjoint arcs which have only their end points in common with $B \cup \partial D_m$. Let $g: D_n \to D_n$ be a homeomorphism which is fixed on $B \cup \partial D_n$ and is such that $g\,(\cup_{i=1}^{n} r_i) = \cup_{i=1}^{n} r_i$, then $g \simeq 1$ $(\text{rel } B \cup \partial D_n)$.

Proof. Let $S_n = D_n - \mathring{B}$ and let $g` = g/S_n$. We will show that $g` \simeq 1$ $(\text{rel}\partial S_n)$ and then conclude $g \simeq 1$ $(\text{rel}\partial D_n \cup B)$ by extending the isotopy for $g`$ over each disc $D_i$ by the identity. For each i, $g`/r_i$ is a homeomorphism which fixes the end points of r_i, thus it will be isotopic to the identity on $r_i$by an isotopy of $r_i$which keeps the end points fixed. Such an isotopy can be covered by an isotopy of S_n which is the identity off a small relative neighborhood of r_i (mod r_i) and is the identity on ∂S_n. Combining these isotopies we get $g`$ is isotopic $(\text{rel}\partial S_n)$ to a homeomorphism which fixes r_i for each i.

Notes

Global Journal of Science Frontier Research (F) Volume XVI Issue VI Version I Year 2016

Notes

Thus without loss of generality we can assume $g`/\cup r_i$ is the identity. Let D` be the disc formed by cutting S_n along $\cup_{i=1}^{n} r_i$. Let $p : D` \to S_n$ be the identification map. Define $h : D` \to D`$ by $h/\partial D` = 1$ and $h/D` - \partial D` = p^{-1}g`p$. By Lemma 1.4 $h \simeq l$ (rel$\partial$D`). Let $G`_t$: D` → D` be an isotopy with $G`o = h$ and $G`, =1$ where $G`_t$ is a homeomorphism which fixes ∂D` for each t. Let $G_t : S_n \to S_n$ be given by $G_t(x) = x$ if x is in $\cup r_i \cup \partial S_n$ and $G_t(x) = pG_t`p^{-1}(x)$ otherwise. Then $G_o = g`$ and $G_l = 1$ and also $G_t$ fixes $\partial S_n$ for all t. This $G_t$ is the desired isotopy between g` and the identity on S_n.

Lemma 12 Let D be a disc, $I = \{i_1,\ldots,i_p\}$, $J = \{j_1,\ldots, j_q\}$, $K = \{k_1,\ldots, k_r\}$ finite sets in $\mathring{D}$. Let γ_i, γ_j, γ_k,γ_{ij}, γ_{ik}, γ_{jk} and γ_{ijk} be the simple closed curves given in Figure 3.8 and let h_i, h_j, h_k, h_{ij}, h_{ik}, h_{jk}, and h_{ijk} be the corresponding twist homeomorphisms, then $h_{ik} \simeq h_i h_j h_k h_{ijk} h_{ik}^{-1} h_{jk}^{-1}$ (rel$\partial D \cup I \cup J \cup K$).

Moreover this isotopy can be taken to be fixed on suitably small discs D_1, D_2, D_3 in $\mathring{D}$about I, J, K respectively.

Proof. Let D_1, D_2, D_3 be discs about I, J, K as in Figure 3.9 and let r_1, r_2, $r_3$be arcs as given in Figure 3.9. Let $B = \cup_{i=1}^{3} D_i \cup \partial D$. By Lemma 3.11 it suffices to show we can isotope $h_{ik}(\cup_{i=1}^{3} r_i)$ to $h_i h_j h_k h_{ijk} h_{ik}^{-1} h_{jk}^{-1} (\cup_{i=1}^{3} r_i)$ (rel B). Figure 10 gives $h_{ik}(r_1)$ and Figure 12 gives $h_i h_j h_k h_{ijk} h_{ik}^{-1} h_{jk}^{-1} (r_1) = h_i h_j h_{jk}^{-1} (r_1)$. Clearly the curves in Figures 10 and 12 are isotopic (rel B). Figure 13 gives $h_{ik}(r_2)$. The curve γ given in Figure 16 is isotopic (rel B) to $h_{jk}^{-1} h_{ij}^{-1} (r_2)$ and hence the curve given in Figure 17 is isotopic (rel B) to $h_i h_j h_k h_{ijk} h_{ik}^{-1} h_{jk}^{-1}$. But clearly the curves in Figures 13 and.17 are isotopic.

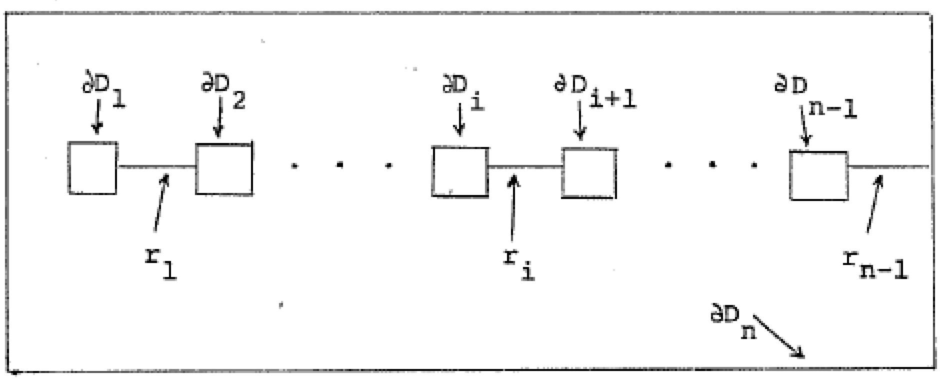

Figure 7

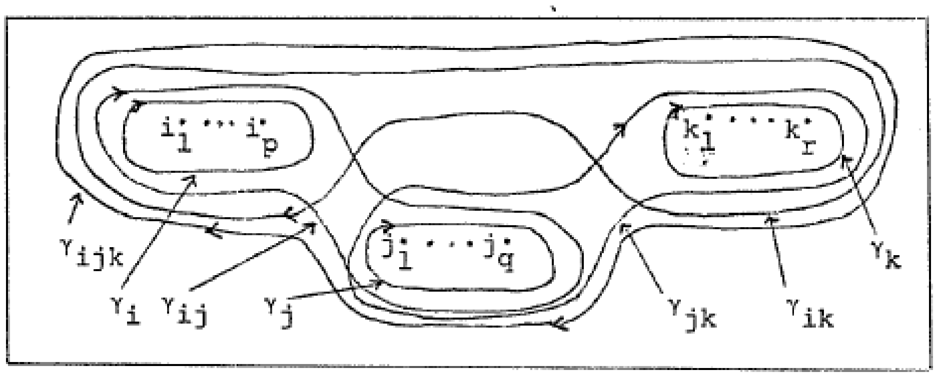

Figure 8

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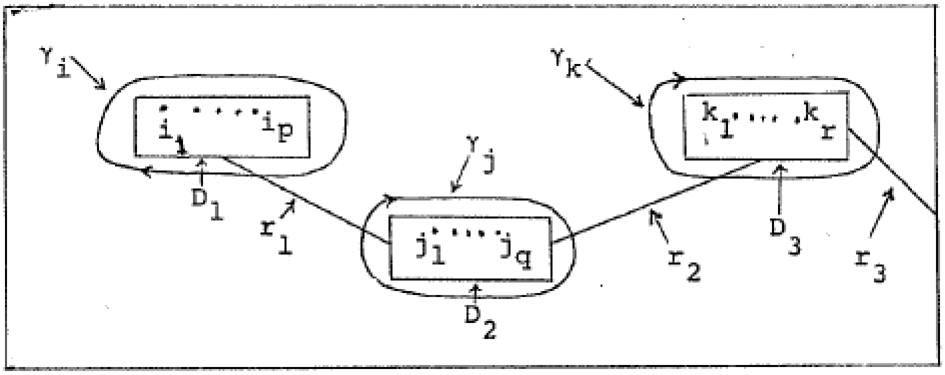

Figure 9

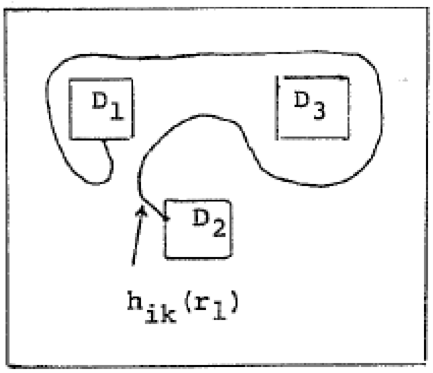

Figure 10

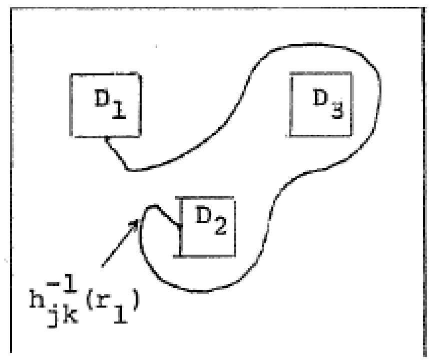

Figure11

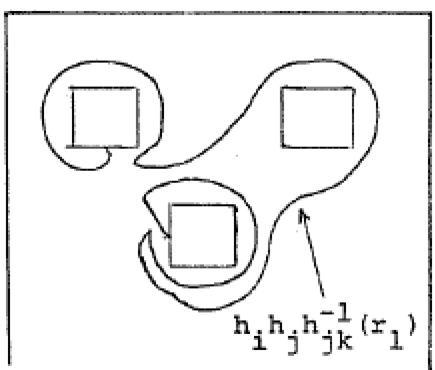

Figure 12

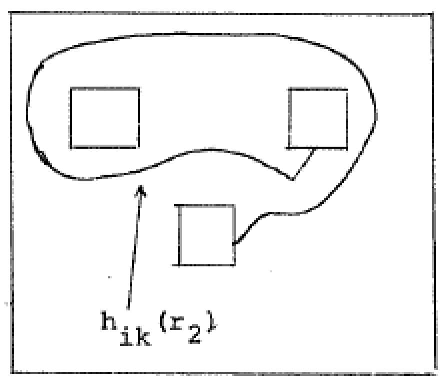

Figure 13

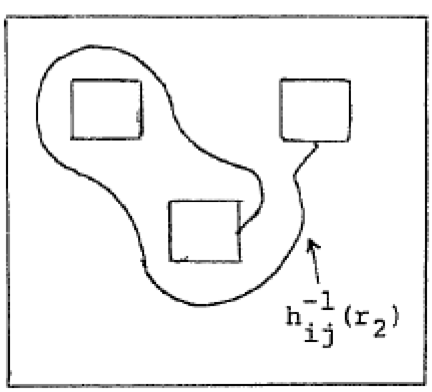

Figure14

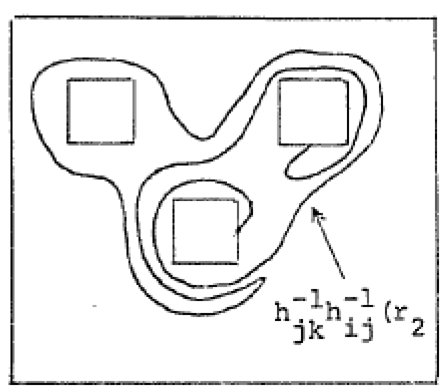

Figure 15

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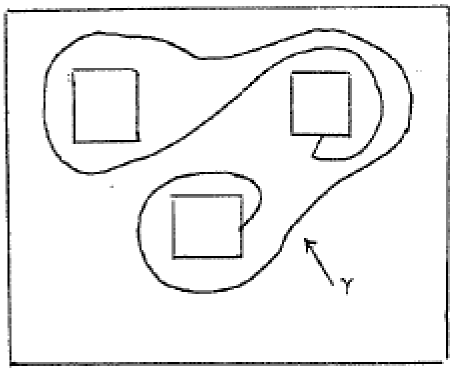

Figure 16

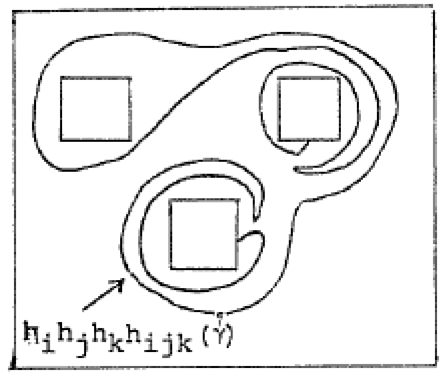

Figure 17

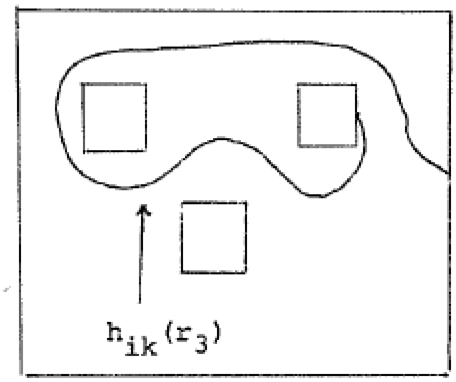

Figure 18

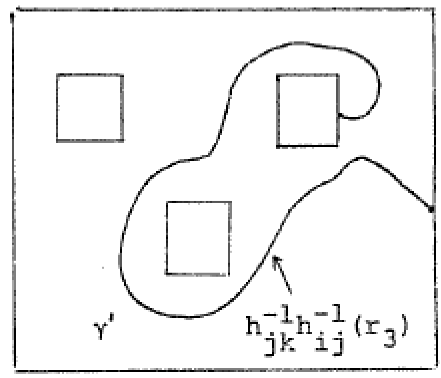

Figure 19

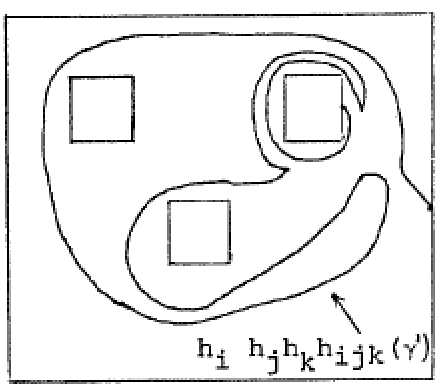

Figure 20

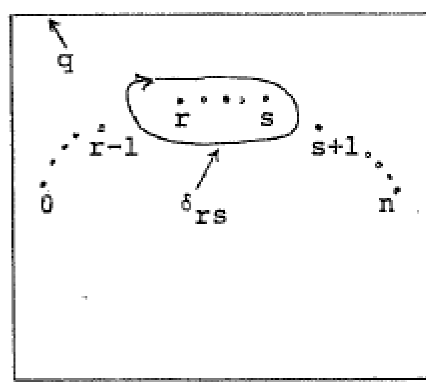

Figure 21

Finally the fact that the curves in Figures 18 and 20 are isotopic (rel B) implies that $h_{ik}(r_3)$ is isotopic (rel B) to $h_i h_j h_k h_{ijk} h_{ik}^{-1} h_{jk}^{-1}(r_3)$. Combining these isotopies we have the desired result.

Remark. In the above, if I, J, or K is singleton, then the corresponding twist homeomorphism h_i, h_j or h_k is isotopic to the identity and hence can be dropped from the statement of the lemma.

Remark 2. In Lemma 12, the homeomorphisms h_i, h_j or h_k commute with every homeomorphism in the lemma.

Remark 3. If the curves given in Lemma 12 are in the interior of a 2-manifold X, then we can consider the corresponding twist homeomorphisms as defined on X. In particular if F is a finite subset of X° and $L = I \cup J \cup K$ is a subset of F with F-L outside the disc bounded by γ_{ijk}, then we have $h_{ik} \simeq h_i h_j h_k h_{ijk} h_{ik}^{-1} h_{jk}^{-1}$ (rel F). Moreover this isotopy can be taken to be fixed on a family of disjoint discs in $\mathring{X}$ where each disc contains a single point of F in its interior. In the following we will sometimes refer to the statement in this remark as Lemma 12 (a).

Lemma 13 Let $r<s$ and let δ_{rs} be the simple closed curve in S^2 given in Figure 3.21 and let C_{rs} be the corresponding twist homeomorphism, then $C_{rs} \simeq a_{rr+1} a_{s+1s+2}^{s-r} a_{ss+1}^{-1} a_{s-1s+1}^{-1} \dots a_{rs+1}^{-1}$ (relF_{n+1}). Moreover this isotopy can be taken to be fixed a suitably small discs about each p_i in F_{n+1}.

Proof. Induct on p = s-r. If p=1, then $c_{s-1s} \simeq a_{s-1s} a_{s+1s+2} a_{ss+1}^{-1} a_{s-1s+1}^{-1}$ (relF_{n+1}) follows from Lemma 12 (a) by letting $\{i_1,\dots,i_p\} = \{s-1\}$, $\{j_1,\dots,j_q\} = \{s+1,\dots, n\}$ and $\{k_1,\dots, k_r\} = \{s\}$. See Figure 3.22. Now by induction assume (1) $c_{r+1s} \simeq a_{r+1r+2} a_{s+1s+2}^{s-r-1} a_{ss+1}^{-1} a_{s-1s+1}^{-1} \dots a_{r-1s+1}^{-1}$ (relF_{n+1}) claim (2) $c_{rs} \simeq a_{rr+1} a_{s+1s+2} a_{r-1r+2}^{-1} c_{r+1s} a_{rs+1}^{-1}$. Note that once (2) is established, substituting (1) into (2) gives the desired result. Letting $\{i_1,\dots,i_p\} = \{r\}$, $\{j_1,\dots,j_q\} = \{s+1,\dots, n\}$, $\{k_1,\dots, k_r\} = \{r+1,\dots, s\}$ in Lemma 3.12(a), we have (3) $c_{rs} \simeq a_{s+1s+2} c_{r+1s} a_{rr+1} a^{-1}_{r+1r+2} a^{-1}_{rs+1}$. See Figure 3.23. Keeping in mind which of these homeomorphisms commute, (2) now follows.

Lemma 14 Let $n \geq m \geq 3$ and $1 \leq r < m$ and $i<k$.

(1) If $i=r$ or $i<r \geq k$, then $a_{ik} a_{rm} a_{ik}^{-1} \simeq a_{rm}$ (relF_{n+1}).

(2) If $i>r$, then $a_{ik} a_{rm} a_{ik}^{-1} \simeq (a_{im}(a_{km} \dots a_{m-1m})) a_{rm} (a_{im}(a_{km} \dots a_{m-1m}))^{-1}$ (relF_{n+1}).

(3) If $i<r<k$, then $a_{ik} a_{rm} a_{ik}^{-1} \simeq ((a_{km} \dots a_{m-1m}) a_{im}) a_{rm} ((a_{km} \dots a_{m-1m}) a_{im})^{-1}$ (relF_{n+1}).

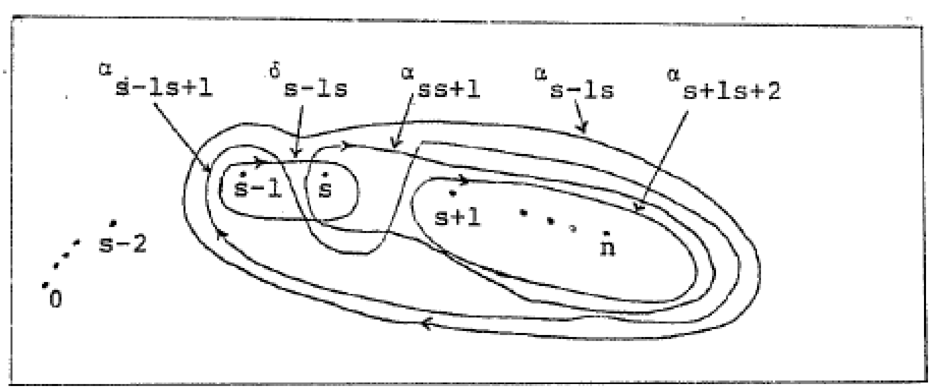

Figure 22

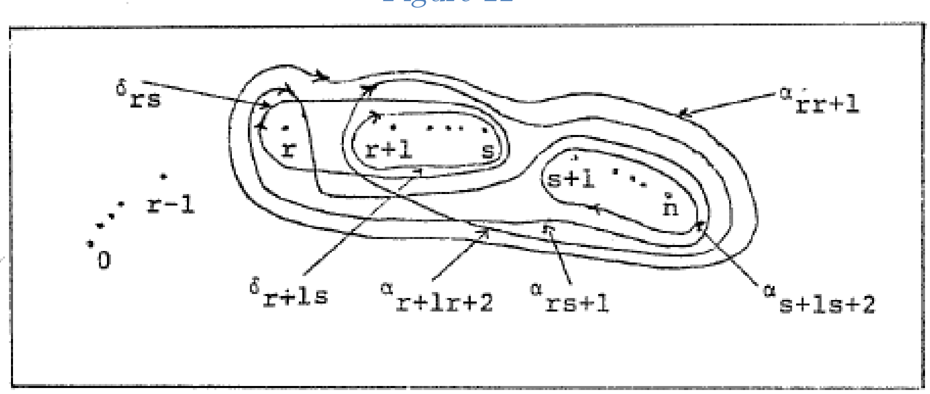

Figure 23

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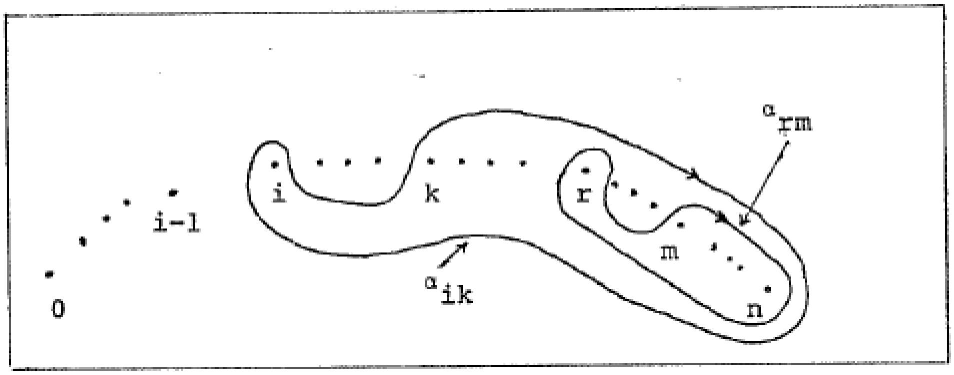

Figure 24

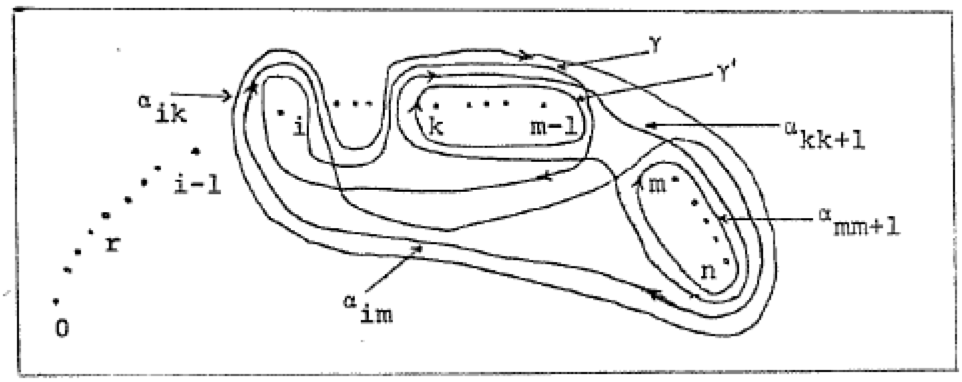

Figure 25

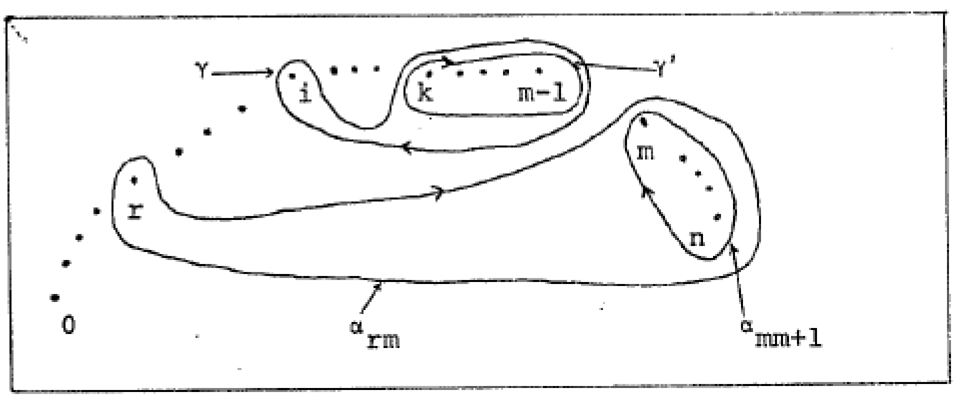

Figure 26

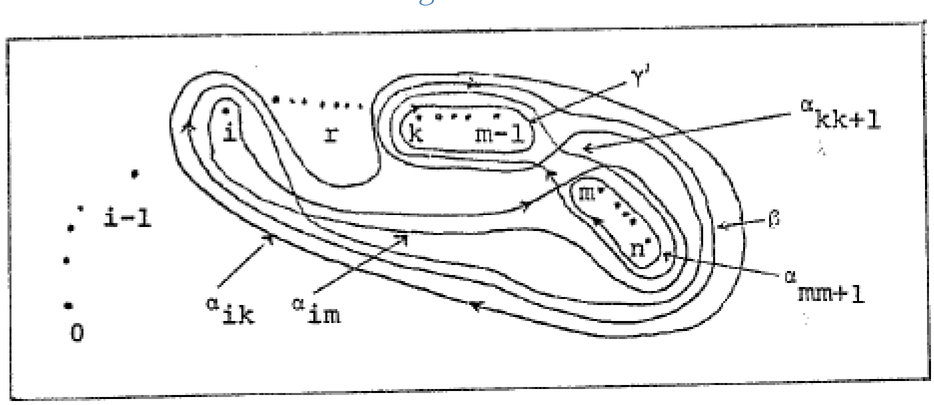

Figure 27

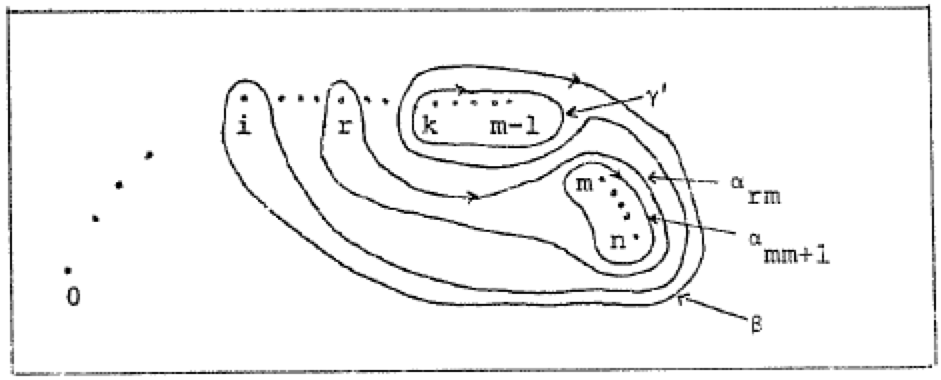

Figure 28

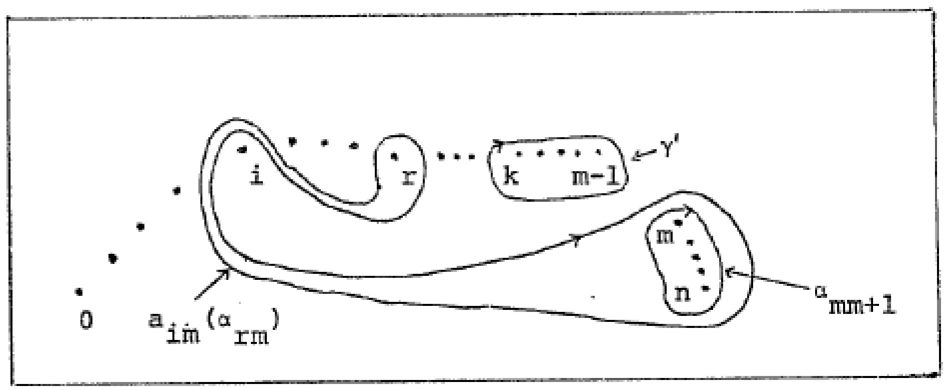

Figure 29

Moreover all the isotopes in Lemma 14 can be taken to be fixed on a family of disjoint discs $D_o,\dots,D_n$ with p_i in $\mathring{D}_I$ and to be equal to the identity outside a disc about F_n.

Proof. Let $B = \cup_{i=o}^{n} D_i\ (S - \mathring{D}`)$ where D` is a disc containing F_n. In the proof we use "h≃g" to denote "h≃g (rel B)".

(1) If i=r or i<r≥k, then α_{ik}can be takento be disjoint from α_{rm}. This means that the corresponding twist homeomorphisms commute. See Figure 3.24.

(2) Let i>r. By Lemma 12 (a) we have $h_\gamma \simeq h_{\gamma'} a_{mm+1} a_{ik} a_{kk+1}^{-1} a_{im}^{-1}$ where γ and γ` are are given Figure 25. Keeping in mind the commuting properties of the above homeomorphisms we can solve for $a_{ik}$ to yield $a_{ik} \simeq a_{im} a_{kk+1} a_{ik} h_\gamma h_{\gamma`}^{-1} a_{mm+1}^{-1}$. In particular, $a_{ik}(\alpha_{rm}) \simeq a_{im} a_{kk+1} a_{ik} h_\gamma h_{\gamma`}^{-1} a_{mm+1}^{-1} (\alpha_{rm})$. But $h_\gamma, h_\gamma$and a_{mm+1} are all the identity when restricted to α_{rm}, since the corresponding curves are disjoint from $\alpha_{rm}$when i>r. See Figure 3.26. Thus $a_{ik}(\alpha_{rm}) \simeq a_{im} a_{kk+1}(\alpha_{rm})$.

Now by Lemma 3.13, letting r=k and s=m-1 we have $h_{\gamma'} \simeq c_{km-1} \simeq a_{kk+1} a_{mm+1}^{k-m+1} a_{m-lm}^{-1} \dots a_{km}^{-1}$. Again keeping in mind which of the homeomorphisms commute we can solve for a_{kk+1} to yield a_{kk+1} $(a_{km}\dots a_{m-lm})$ $a_{mm+1}^{m-k+1} c_{km-1}$. Now c_{km-1} and a_{mm+1} are both the identity when restricted to α_{rm}, so $a_{kk+1}(\alpha_{rm}) \simeq (a_{km}\dots a_{m-lm})\ (\alpha_{rm})$. Thus $a_{ik}(\alpha_{rm}) \simeq a_{im} a_{kk+1}(\alpha_{rm}) \simeq a_{im}(a_{km}\dots a_{m-lm})(\alpha_{rm})$.

Notes

Part (2) now follows from Lemmas 3.6 and 3.7 as indicated in Remark 2 following Lemma 3.10.

(3) Let i<r<k. By Lemma 12(a) we have $a_{im} \simeq a_{ik} h_{\gamma`} a_{mm+1} a_{kk+1}^{-1} h_{\beta}^{-1}$ where γ` and β are given in Figure 3.27. As before we can solve for a_{ik} to obtain $a_{ik} a_{kk+1} a_{im} a_{mm+1}^{-1} h_{\gamma`}^{-1} h_{\beta}$. Now $a_{mm+1}$, $h_{\gamma`}$, h_{β} are all the identity on a_{rm} when i<r<k, see Figure 3.28. Thus $a_{ik}(\alpha_{rm}) \simeq a_{kk+1} a_{im}(\alpha_{rm})$. Now as in the proof of part (2), $a_{kk+1} \simeq (a_{km} \ldots a_{m-lm}) a_{mm+1}^{m-k+1} c_{rm-l}$.

Since c_{km-l} and a_{mm+1} are the identity on $a_{im}(\alpha_{rm})$, see Figure 3.29, we have $a_{ik}(\alpha_{rm}) \simeq (a_{km} \ldots a_{m-lm})\, a_{im}(\alpha_{rm})$. (3) now follows from Lemmas 3.6 and 3.7 as indicated above.

Theorem 15 Let $n \geq 3$, so that $H^*(S^2, F_{n+1})$ is generated by $G_n = {}^{n}\cup_{k=3} \{\bar{a}_{ik} : l \leq i \leq k\}$. In terms of these generators, $H^*(S^2, F_{n+1})$ has a presentation in which a complete set of relations is given as follows:

1. If $p \leq q$ and i=r or if p<q and $i<r \geq p$, then $\bar{a}_{ip}\bar{a}_{rq}\bar{a}^{-1}_{ip} = \bar{a}_{rq}$.
2. If $p<q$ and $i>r$, then $\bar{a}_{ip}\bar{a}_{rq}\bar{a}_{ip}^{-1} = (\bar{a}_{iq}(\bar{a}_{pq} \ldots \bar{a}_{q-lq}))\bar{a}_{rq}(\bar{a}_{iq}(\bar{a}_{pq} \ldots \bar{a}_{q-lq}))^{-1}$.
3. If p<q and i<r<p, then $\bar{a}_{ip}\bar{a}_{rq}\bar{a}_{ip}^{-1} = ((\bar{a}_{pq} \ldots \bar{a}_{q-lq})\bar{a}_{iq})\bar{a}_{rq}((\bar{a}_{pq} \ldots \bar{a}_{q-lq})\bar{a}_{iq})^{-1}$.

Proof. Theorem 3.9 shows G_n generates $H^*(S^2, F_{n+l})$. We prove the present theorem by induction on n, beginning with n=3. For n = 3, $H^*(S^2, F_4) = \pi_1(S^2\text{-}F_3, P_4)$ = the free group on 2 generators, as seen in the proof of Theorem 3.9. Moreover, $H^*(S^2, F_4)$ is generated by $\bar{a}_{13}$and $\bar{a}_{23}$, hence the present theorem is true in this case.

Assume Theorem 15 for n-1.

Let $\bar{\bar{a}}_{ip}$denote the equivalence class of a_{ip} in $H^*(S^2, F_n)$, to distinguish it from $\bar{a}_{ip}$, the equivalence class of a_{ip} in $H^*(S^2, F_{n+l})$. By the induction assumption $G`_{n-l} = \cup_{k=3}^{n-1} \{\bar{\bar{a}}_{ik} : l \leq i < k\}$ generates $H^*(S^2, F_n)$ with relations as in 1, 2, and 3 with ā replaced by $\bar{\bar{a}}$. If we then replace $\bar{\bar{a}}$ by ā, these relations hold in $H^*(S^2, F_{n+l})$ by Lemma 3.14. Hence the map k: $H^*(S^2, F_n) \rightarrow H^*(S^2, F_{n+l})$ defined by $k(\bar{\bar{a}}_{ik}) = \bar{a}_{ik}$ defines a homomorphism which splits the short exact sequence

$$o \rightarrow \pi_1(S^2\text{-}F_n, p_n) \rightarrow H^*(S^2, F_{n+l}) \rightarrow H^*(S^2, F_n) \rightarrow o.$$

Therefore by part (ii) of Lemma 3.10 $H^*(S^2, F_{n+l})$ has a presentation in terms of the generators G˙ in which every relation has the form (1.1) $w(\bar{a}_{rn}) = 1$, (1.2) $w`(\bar{a}_{ip}) = 1$, (2) $\bar{a}_{ip}\bar{a}_{rn}\bar{a}_{ip}^{-1} = w(\bar{a}_{rn})$ and (3) $\bar{a}_{ip}^{-1}\bar{a}_{rn}\bar{a}_{ip} = w(\bar{a}_{rn})$ where $\bar{a}_{rn}$is in $G_n - G_{n-l}$ and $\bar{a}_{ip}$ is in G_{n-l}. There can be no nontrivial relations of the form (1.1) since the subgroup generated by $G_n - G_{n-l}$, i.e. by $\{\bar{a}_{rn} : l \leq r < n\}$, is $d(\pi_1(S^2\text{-}F_n, p_n))$ which is free.

Since k is a monomorphism, $w`(\bar{\bar{a}}_{ip}) = 1$ if and only if $w`(\bar{a}_{ip}) = 1$. Hence by induction all relations of the form (1.2) are consequences of those of types 1, 2, and 3 given in the statement of the theorem.

Letting q = n, another application of Lemma 3.14 shows that the relations of type 1, 2, and 3 given in the statement of the present theorem supply the necessary relations of type (2) from Lemma 3.10. Finally we note that by 1 in Theorem 3.15, $\bar{a}_{ip}$ commutes with the elements $\bar{a}_{iq}$, $\bar{a}_{pq}, \ldots, \bar{a}_{q-lq}$ when p<q. Hence conjugating 2 and 3 by $\bar{a}_{ip}^{-1}$ - $(\bar{a}_{iq}(\bar{a}_{pq} \ldots \bar{a}_{q-lq}))^{-1}$ and using the above commuting properties we see that relations of type (3) from Lemma 10 are a consequence of 1, 2, and 3 as given in the theorem.

Notes

Corollary 16 $H^*(S^2, F_{n+1})$ made abelian is the free abelian group on 2+3+…+(n-1) = (n(n-1)/2) -1 generators.

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Notes