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A Semi-Symmetric Metric S-Connection in a Generalised Co-Symplectic Manifold

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Abstract- In the present paper, we define a new type of connection called semi-symmetric S-connection in a generalised co-symplectic manifold and studied some of its properties. The relation between the curvature tensor with respect to this connection and the curvature tensor with respect to the Riemannian connection is establised.

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Ref

1. Mishra, R.S., Almost Contact Metric Manifold, Monograph (I), Tensor Society of India, Lucknow, (1991).

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Abstract- In the present paper, we define a new type of connection called semi-symmetric S-connection in a generalised co-symplectic manifold and studied some of its properties. The relation between the curvature tensor with respect to this connection and the curvature tensor with respect to the Riemannian connection is establised.

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I. Introduction

An n-dimensional differentiable manifold M_n is an almost Contact manifold, if it admidts a tensor field F of type $(1,1)$, a vector field ξ and a 1-form η satisfying for arbitrary vector field X , such that

$$\overline{\overline{X}} + X = \eta(X)\xi \tag{1.1}$$

$$\overline{\xi} = 0 \tag{1.2}$$

where

$$\overline{X} = FX$$

Again equations (1.1) and (1.2) gives

$$\eta(\overline{X}) = 0 \tag{1.3}$$

$$\eta(\xi) = 1 \tag{1.4}$$

An almost contact manifold M_n in which a Riemannian metric tensor g of type (0,2) satisfies

$$g(\overline{X}, \overline{Y}) = g(X, Y) - \eta(X)\eta(Y) \tag{1.5}$$

$$g(X, \xi) = \eta(X) \tag{1.6}$$

for arbitrary vector field X, Y, is called an almost Contact Metric Manifold [1].

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Let us put

$$F'(X,Y) = g(\overline{X},Y)$$

then we have

$$F'(\overline{X},\overline{Y}) = F'(X,Y) \tag{1.7}$$

$$F'(X,Y) = g(\overline{X},Y) = -g(X,\overline{Y}) = -F'(Y,X) \tag{1.8}$$

An almost contact metric manifold satisfying

$$(\nabla_X F')(Y,Z) = \eta(Y)(\nabla_X \eta)(\overline{Z}) - \eta(Z)(\nabla_X \eta)\overline{Y} \tag{1.9}$$

and

$$(\nabla_X F')(Y,Z) + (\nabla_Y F')(Z,X) + (\nabla_Z F')(X,Y) + \eta(X)[(\nabla_Y \eta)\overline{Z} - (\nabla_Z \eta)\overline{Y}]$$

$$+\eta(Y)[(\nabla_Z \eta)\overline{X} - (\nabla_X \eta)\overline{Z}] + \eta(Z)[(\nabla_X \eta)\overline{Y} - (\nabla_Y \eta)\overline{X}] = 0 \tag{1.10}$$

for arbitrary vector field X,Y, Z . Then M_n is called Generalised Co-Symplectic and Generalised Quasi-Ssasakian Manifold[2] .

If in M_n

$$(\nabla_X \eta)\overline{Y} = -(\nabla_{\overline{X}} \eta)Y = (\nabla_Y \eta)\overline{X} \tag{1.11}$$

$$(\nabla_X \eta)Y = (\nabla_{\overline{X}} \eta)\overline{Y} = -(\nabla_Y \eta)X \tag{1.12}$$

$$(\nabla_\xi F) = 0 \tag{1.13}$$

Then ξ is said to be of the first class and the manifold is said to be first class [2].

If in an almost Contact metric Manifold M_n , ξ satisfies

$$(\nabla_X \eta)\overline{Y} = (\nabla_{\overline{X}} \eta)Y = -(\nabla_Y \eta)\overline{X} \Leftrightarrow (\nabla_X \eta)Y = -(\nabla_{\overline{X}} \eta)\overline{Y} = (\nabla_Y \eta)X \tag{1.14}$$

$$(\nabla_\xi F) = 0. \tag{1.15}$$

Then ξ is said to be of the second class and the manifold M_n is said to be of the second class [2].

The Nijenhuis tensor in Generalised Co-Symplectic Manifold is given by

$$N(X,Y) = (\nabla_{\overline{X}} F)Y - (\nabla_{\overline{Y}} F)X - \overline{\nabla_X F)Y} + \overline{\nabla_Y F)X} \tag{1.16}$$

$$N'(X,Y,Z) = (\nabla_{\overline{X}} F')(Y,Z) - (\nabla_{\overline{Y}} F')(X,Z) + (\nabla_X F')(Y,\overline{Z}) - (\nabla_Y F')(X,\overline{Z}) \tag{1.17}$$

Ref

2. Ojha , R.H. and Prasad S. , On Semi-Symmetric Metric s-Connextion in a Sasakian Manifold , Indian Jour. Pure and Appl. Math. , (Vol. 16(4), 341-344(1985).

II. A Semi-Symmetric Metric S-Connection

Let E be an affine connection and E is said to be metric if

$$(E_X g) = 0 \tag{2.1}$$

The metric connection satisfying

$$(E_X F)Y = \eta(Y)X - g(X,Y)\xi \tag{2.2}$$

is called S-connection [3].
A metric S-Connection E is called semi -symmetric metric S-Connection if

$$E_X Y = \nabla_X Y - \eta(X)\overline{Y} \tag{2.3}$$

which implies

$$S(X,Y) = \eta(Y)\overline{X} - \eta(X)\overline{Y} \tag{2.4}$$

where S is the torsion tensor of connection E. We know that

$$E_X(g(Y,Z)) = (E_X g)(Y,Z) + g(E_X Y, Z) + g(Y, E_X Z).$$

Using equation (2.3)

$$(E_X g)(Y,Z) = 0, where X, Y, Z \epsilon M_n. \tag{2.5}$$

Therefore , linear connection E defined by equation (2.3) and satisfying equations (2.4) and (2.5) is semi-symmetric metric connection ,we have

$$S(X,Y) = -S(Y,X)$$

This implies S is semi-symmetric. Now let E be a linear connection defined on a generalised Co-Symplectic manifold M_n by

$$E_X Y = \nabla_X Y + P(X,Y) \tag{2.6}$$

where P is a tensor of type (1,2) defined on M_n. Now , from equations (2.5) and (2.6) , we have

$$E_X(g(Y,Z)) = (E_X g)(Y,Z) + g(E_X Y, Z) + g(Y, E_X Z)$$

$$\Leftrightarrow g(P(X,Y),Z) + g(Y, P(X,Z)) = 0$$

$$g(P(X,Y),Z) + g(P(X,Z),Y) = 0 \tag{2.7}$$

from equation (2.6) , we get

$$S(X,Y) = P(X,Y) - P(Y,X). \tag{2.8}$$

Using equation (2.8), we get

$$g(S(X,Y),Z) + g(S(Z,X),Y) + g(S(Z,Y),X) = 2g(P(X,Y),Z) \tag{2.9}$$

Ref

3. Sasaki, S., Almost Contact Manifold, I, II, III, A lecture Note, Tohoku University, (1967),(1967),(1968).

and

$$P(X,Y) = \frac{1}{2}[\eta(Y)\overline{X} - \eta(X)\overline{Y}] \tag{2.10}$$

Now from equations (2.6) and (2.10), we get

$$E_X Y = \nabla_X Y + \frac{1}{2}[\eta(Y)\overline{X} - \eta(X)\overline{Y}] \tag{2.11}$$

Further for a 1-form η on a generalised Co-Symplectic manifold M_n, we have,

Theorem 2.1. *A Generalised Co-Symplectic Manifold M_n admitting a connection E , is uniquely determined by the contact form η and tensor field F satisfies*

$$(E_X\eta)Y = (\nabla_X\eta)Y \tag{2.12}$$

$$(E_X\eta)(FY) = (\nabla_X\eta)(FY) \tag{2.13}$$

$$E_X(FY) = \eta(Y)X - g(X,Y)\xi + \overline{\nabla_X Y} - \frac{1}{2}[\eta(Y)X - \eta(X)Y] \tag{2.14}$$

$$E_X(FY) - E_Y(FX) = \overline{\nabla_X Y} - \overline{\nabla_Y X} \tag{2.15}$$

Proof: Using equations (2.6), (2.10) and (2.11) , we have the results (2.12),(2.13),(2.14) and (2.15).

Again covariant differentiation of the torsion tensor S is given by

$$E_X(S(Y,Z)) = (E_X S)(Y,Z) + S(E_X Y, Z) + S(Y, E_X Z).$$

Using equation (2.4) , we have

$$(E_X S)(Y,Z) = ((E_X\eta)Z)Y - ((E_X\eta)Y)Z \tag{2.16}$$

Let us define

$$\overline{S}(X,Y,Z) = g(S(X,Y),Z) \tag{2.17}$$

Then from the equations (2.4) and (2.17), we get

$$\overline{S}(X,Y,Z) + \overline{S}(Y,Z,X) + \overline{S}(Z,X,Y) = 0 \tag{2.18}$$

and

$$g(P(X,Y),Z) + g(P(X,Z),Y) = 0 \tag{2.19}$$

The torsion tensor S of the connection E satisfies the following relations

(a) $S(\overline{X},\overline{Y}) = 0$

(b) $S(X,\xi) = \overline{\overline{X}}$

(c) $S(\overline{X},\xi) = \overline{X}$,

(d) $S(X,\xi) - S(\overline{\overline{X}},\xi) = 2X - 2\eta(X)\xi$

(e) $S(\overline{\overline{X}},Y) = \eta(X)\eta(Y)\xi - \eta(Y)X$

(f) $S(X,Y) = \eta(Y)\overline{X}$

(g) $\eta(S(X,Y) = 0$

Theorem 2.2. *A generalised Co-symplectic Manifold M_n satisfies the following relations i.e. (a),(b),(c),(d),(e),(f) and (g)defined above.*

Notes

Theorem 2.3. *In a Generalised Co-Symplectic manifold M_n with connection E ,we have*

(a) $\tilde{P}(X,Y,Z) = \frac{1}{2}[\eta(Y)F^{'}(X,Y) - \eta(X)F^{'}(Y,Z)]$

(b) $\tilde{P}(X,\overline{Y},\overline{Z}) = 1/2[\eta(X)g(Y,Z) - \eta(X)\eta(Y)\eta(Z)]$

(c) $\tilde{P}(\overline{X},\overline{Y},\overline{Z}) = 0$

(d) $\tilde{P}(\overline{X},\overline{Y},Z) = 0 = \tilde{S}(\overline{X},\overline{Y},\overline{Z}) = 0$

(e) $\tilde{P}(X,Y,\overline{Z}) = \frac{1}{2}[\eta(Y)F^{'}(X,\overline{Z}) - \eta(X)F^{'}(Y,\overline{Z})]$

where $\tilde{P}(X,Y,Z) = g(P(X,Y),Z)$

Notes

Theorem 2.4. *In a Generalised Co-Symplectic Manifold M_n admitting connection E satisfied the following properties:*

(a) $(E_XF^{'})(Y,Z) = (\bigtriangledown_XF^{'})(Y,Z) - \frac{1}{2}[\eta(Y)F^{'}(\overline{X},Z) + \eta(Z)F^{'}(Y,\overline{X})]$

(b) $E_{FX}F^{'}(\overline{Y},\overline{Z}) = (\bigtriangledown_{FX}F^{'})(\overline{Y},\overline{Z})$

Proof: (a) We have

$$X(F^{'}(Y,Z)) = (E_XF^{'})(Y,Z) + F^{'}(E_XY,Z) + F^{'}(Y,E_XZ)$$

$$X(F^{'}(Y,Z)) = (\bigtriangledown_XF^{'})(Y,Z) + F^{'}(\bigtriangledown_XY,Z) + F^{'}(Y,\bigtriangledown_XZ)$$

which implies

$$(E_XF^{'})(Y,Z) = (\bigtriangledown_XF^{'})(Y,Z) + F^{'}(\bigtriangledown_XY,Z) + F^{'}(Y,\bigtriangledown_XZ) - F^{'}(E_XY,Z) - F(Y,E_XZ)$$

$$= (\bigtriangledown_XF^{'})(Y,Z) - 1/2[\eta(Y)F^{'}(\overline{X},Z) + \eta(Z)F^{'}(Y,\overline{X})]$$

Using equation (2.10) , we get (a). Again baring (a), we get (b) and (c).

III. Curvature Tensor of M_n with Respect to Connection E

Let $\tilde{R}$ be the curvature tensor with respect to the semi -symmetric metric connection E on a generalised co-symplectic manifold M_n. Then

$$\tilde{R}(X,Y,Z) = E_XE_YZ - E_YE_XZ - E_{[X,Y]}Z \tag{3.1}$$

We have the following results:

Theorem 3.1. *In a Generalised Co-Symplectic Manifold M_n curvature tensor $\tilde{R}$ is given by*

$$\tilde{R}(X,Y,Z) = R(X,Y,Z) + \frac{1}{2}[S(X,\bigtriangledown_YZ) + S(\bigtriangledown_XZ,Y)] + \frac{1}{2}\eta(Z)S^{'}(X,Y)$$
$$+\frac{1}{2}\eta(Z)[\bigtriangledown_X\overline{Y} - \bigtriangledown_Y\overline{X}] + \frac{1}{2}[\eta(Y)\bigtriangledown_X\overline{Z} - \eta(X)\bigtriangledown_Y\overline{Z}]$$
$$-\frac{1}{2}(\bigtriangledown_Y\eta(Z))\overline{X} - \frac{1}{2}\eta(Z)\overline{[X,Y]} + \frac{1}{2}\eta([X,Y])\overline{Z} - ((\bigtriangledown_X\eta)Y)\overline{Z} \tag{3.2}$$

where

$$S^{'}(X,Y) = \eta(X)Y - \eta(Y)X$$

$$R(X,Y,Z) = \bigtriangledown_X\bigtriangledown_YZ - \bigtriangledown_Y\bigtriangledown_XZ - \bigtriangledown_{[X,Y]}Z$$

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Proof: Let $\tilde{R}(X,Y,Z)$ be the curvature tensor for generalised co-symplectic manifold with respect the semi-symmetric metric S-connection E , then

$$\tilde{R}(X,Y,Z) = E_X E_Y Z - E_Y E_X Z - E_{[X,Y]} Z$$

By using equations (1.1),(1.2),(1.11),(1.12),(1.13) ,(2.9) and (2.12), we get equation (3.1), where R(X,Y,Z) is curvature tensor of M_n with respect to the Riemannian Curvature $\bigtriangledown$.

Let K and $\tilde{K}$ be curvature tensor of type (0,4) given by

$$K(X,Y,Z) = g(R(X,Y,Z),U)$$

$$\tilde{K}(X,Y,Z,U) = g(\tilde{K}(X,Y,Z),U)$$

Theorem 3.2. In a Generalised Co-Symplectic Manifold M_n,we have

$$\tilde{R}(X,Y,Z) + \tilde{R}(Y,Z,X) + \tilde{R}(Z,X,Y) = 0 \tag{3.3}$$

If

$$2[((\bigtriangledown_X \eta)Z)\overline{Y} + ((\bigtriangledown_Y \eta)X)\overline{Z} + ((\bigtriangledown_Z \eta)Y)\overline{X}]$$

$$+\eta([X,Z])\overline{Y} + \eta([Y,X])\overline{Z} + \eta([Z,Y])\overline{X} = 0 \tag{3.4}$$

and

$$\tilde{K}(X,Y,Z,U) + \tilde{K}(Y,X,Z,U) = 0 \tag{3.5}$$

If and only if

$$g((\bigtriangledown_Y \eta(Z))\overline{X},U) + g((\bigtriangledown_X \eta(Z))\overline{Y},U) = 0 \tag{3.6}$$

and

$$(\bigtriangledown_Y \eta)X + (\bigtriangledown_X \eta)Y = 0 \tag{3.7}$$

Proof: Using equation (3.2) and the first Bianchi identity

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0$$

with respect to Riemannian Connection $\bigtriangledown$, we get eq.(3.3) and (3.4).

We have

$$\tilde{K}(X,Y,Z) = g(\tilde{R}(X,Y,Z),U)$$

$$= g(R(X,Y,Z),U) + \frac{1}{2}g(S(X,\bigtriangledown_Y Z),U) + \frac{1}{2}g(S(\bigtriangledown_X Z,Y),U)$$

$$+\frac{1}{2}g(\eta(Z)[\bigtriangledown_X \overline{Y} - \bigtriangledown_Y \overline{X}],U) + \frac{1}{4}g(\eta(Z)[\eta(X)Y - \eta(Y)X]$$

$$-g(((\bigtriangledown_X \eta)Y)\overline{Z},U) + \frac{1}{2}g((\eta(Y)\bigtriangledown_X \overline{Z} - \eta(X)\bigtriangledown_Y \overline{Z}),U)$$

Notes

Notes

$$-\frac{1}{2}g((\nabla_Y\eta(Z)\overline{X},U)-\frac{1}{2}g(\eta(Z)\overline{[X,Y]},U)+\frac{1}{2}g(\eta([X,Y])\overline{Z},U)$$

We get

$$\tilde{K}(X,Y,Z,U)=-\tilde{K}(Y,X,Z,U) \tag{3.8}$$

If

$$g(\nabla_Y\eta(Z)\overline{X},U)=-g(\nabla_X\eta(Z)\overline{Y},U) \tag{3.9}$$

and

$$(\nabla_Y\eta)X+(\nabla_X\eta)Y=0 \tag{3.10}$$

IV. Nijenhuis Tensor of M_n with Respect to New Connection E

The Nijenhuis tensor with respect to E of F in a generalised Co-symplectic manifold M_n is a vector valued bilinear scalar function N_E , is given by

$$N_E(X,Y)=(\nabla_{\overline{X}}F)Y+S(X,Y)-(\nabla_{\overline{Y}}F)X+\overline{\nabla_X F)Y}+\overline{\nabla_Y F)X} \tag{4.1}$$

Using equation (1.16) , we get

$$N_E(X,Y)=N(X,Y)+S(X,Y)+2\overline{\nabla_X F)Y} \tag{4.2}$$

where N is Nijenhuis tensor with respect to Riemannian connection and S is the Torsion tensor of connection E.

Again by using equation (1.17) ,we get

$$N_E(X,Y,Z)=(E_{\overline{X}}F^{'})(Y,Z)-(E_{\overline{Y}}F^{'})(X,Z)-(E_YF^{'})(X,\overline{Z})+$$
$$\frac{1}{2}\eta(Y)[F^{'}(X,\overline{Z})+F^{'}(X,Z)] \tag{4.3}$$

and

$$N_E(X,Y,Z)=N(X,Y,Z) \tag{4.4}$$

If and only if

$$(\nabla_XF^{'})(Y,\overline{Z})+(\nabla_YF^{'})(X,Z)-(\nabla_YF^{'})(X,\overline{Z})=\frac{1}{2}\eta(X)[F^{'}(Y,Z)+F^{'}(\overline{Y},Z)]+$$
$$\frac{1}{2}\eta(Y)F^{'}(X,\overline{Z}) \tag{4.5}$$

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2. Ojha , R.H. and Prasad S. , On Semi-Symmetric Metric s-Connextion in a Sasakian Manifold , Indian Jour. Pure and Appl. Math. , (Vol. 16(4), 341-344(1985).
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