A Semi-Symmetric Metric S-Connection in a Generalised Co-Symplectic Manifold

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I. Introduction

An n-dimensional differentiable manifold $M_n$ is an almost Contact manifold, if it admits a tensor field $F$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying for arbitrary vector field $X$, such that

$$\overline{X} + X = \eta(X)\xi$$ (1.1)

$$\xi = 0$$ (1.2)

where

$$\overline{X} = FX$$

Again equations (1.1) and (1.2) gives

$$\eta(\overline{X}) = 0$$ (1.3)

$$\eta(\xi) = 1$$ (1.4)

An almost contact manifold $M_n$ in which a Riemannian metric tensor $g$ of type $(0,2)$ satisfies

$$g(\overline{X}, \overline{Y}) = g(X,Y) - \eta(X)\eta(Y)$$ (1.5)

$$g(X, \xi) = \eta(X)$$ (1.6)

for arbitrary vector field $X, Y$, is called an almost Contact Metric Manifold [1].
Let us put

\[ F'(X,Y) = g(\overline{X},Y) \]

then we have

\[ F'(\overline{X}, \overline{Y}) = F'(X,Y) \] (1.7)

\[ F'(X,Y) = g(\overline{X},Y) = -g(X,\overline{Y}) = -F'(Y,X) \] (1.8)

An almost contact metric manifold satisfying

\[ (\nabla_X F')(Y,Z) = \eta(Y)(\nabla_X \eta)(\overline{Z}) - \eta(Z)(\nabla_X \eta)\overline{Y} \] (1.9)

and

\[ (\nabla_X F')(Y,Z) + (\nabla_Y F')(Z,X) + (\nabla_Z F')(X,Y) + \eta(X)[(\nabla_Y \eta)\overline{Z} - (\nabla_Z \eta)\overline{Y}] \]

\[ + \eta(Y)[(\nabla_Z \eta)\overline{X} - (\nabla_X \eta)\overline{Z}] + \eta(Z)[(\nabla_X \eta)\overline{Y} - (\nabla_Y \eta)\overline{X}] = 0 \] (1.10)

for arbitrary vector field X, Y, Z. Then \( M_n \) is called Generalised Co-Symplectic and Generalised Quasi-Sasakian Manifold[2].

If in \( M_n \)

\[ (\nabla_X \eta)\overline{Y} = -(\nabla_X \overline{\eta})Y = (\nabla_Y \eta)\overline{X} \] (1.11)

\[ (\nabla_X \eta)Y = (\nabla_X \overline{\eta})\overline{Y} = -(\nabla_Y \eta)X \] (1.12)

\[ (\nabla_\xi F') = 0 \] (1.13)

Then \( \xi \) is said to be of the first class and the manifold is said to be first class [2].

If in an almost Contact metric Manifold \( M_n \), \( \xi \) satisfies

\[ (\nabla_X \eta)\overline{Y} = (\nabla_X \overline{\eta})Y = -(\nabla_Y \eta)\overline{X} \iff (\nabla_X \eta)Y = -(\nabla_X \overline{\eta})\overline{Y} = (\nabla_Y \eta)X \] (1.14)

\[ (\nabla_\xi F') = 0. \] (1.15)

Then \( \xi \) is said to be of the second class and the manifold \( M_n \) is said to be of the second class [2].

The Nijenhuis tensor in Generalised Co-Symplectic Manifold is given by

\[ N(X,Y) = (\nabla_X F')Y - (\nabla_Y F')X - (\nabla_X F')\overline{Y} + (\nabla_Y F')\overline{X} \] (1.16)

\[ N'(X,Y,Z) = (\nabla_X F')(Y,Z) - (\nabla_Y F')(X,Z) + (\nabla_X F')(Y,\overline{Z}) - (\nabla_Y F')(X,\overline{Z}) \] (1.17)
II. A SEMI-SYMMETRIC METRIC S-CONNECTION

Let $E$ be an affine connection and $E$ is said to be metric if

$$ (E_X g) = 0 $$

(2.1)

The metric connection satisfying

$$ (E_X F)Y = \eta(Y)X - g(X,Y)\xi $$

(2.2)

is called S-connection [3].

A metric S-Connection $E$ is called semi-symmetric metric S-Connection if

$$ E_X Y = \nabla_X Y - \eta(X)\nabla_Y $$

(2.3)

which implies

$$ S(X,Y) = \eta(Y)\nabla_X - \eta(X)\nabla_Y $$

(2.4)

where $S$ is the torsion tensor of connection $E$. We know that

$$ E_X (g(Y,Z)) = (E_X g)(Y,Z) + g(E_X Y,Z) + g(Y,E_X Z). $$

Using equation (2.3)

$$ (E_X g)(Y,Z) = 0, \text{where } X,Y,Z \in M_n. $$

(2.5)

Therefore, linear connection $E$ defined by equation (2.3) and satisfying equations (2.4) and (2.5) is semi-symmetric metric connection, we have

$$ S(X,Y) = -S(Y,X) $$

This implies $S$ is semi-symmetric. Now let $E$ be a linear connection defined on a generalised Co-Symplectic manifold $M_n$ by

$$ E_X Y = \nabla_X Y + P(X,Y) $$

(2.6)

where $P$ is a tensor of type (1,2) defined on $M_n$. Now, from equations (2.5) and (2.6), we have

$$ E_X (g(Y,Z)) = (E_X g)(Y,Z) + g(E_X Y,Z) + g(Y,E_X Z) $$

$$ \Rightarrow g(P(X,Y),Z) + g(Y,P(X,Z)) = 0 $$

$$ g(P(X,Y),Z) + g(P(X,Z),Y) = 0 $$

(2.7)

from equation (2.6), we get

$$ S(X,Y) = P(X,Y) - P(Y,X). $$

(2.8)

Using equation (2.8), we get

$$ g(S(X,Y),Z) + g(S(Z,X),Y) + g(S(Z,Y),X) = 2g(P(X,Y),Z) $$

(2.9)
and

\[ P(X, Y) = \frac{1}{2}[\eta(Y)X - \eta(X)Y] \]  \hspace{2cm} (2.10)

Now from equations (2.6) and (2.10), we get

\[ E_X Y = \nabla_X Y + \frac{1}{2}[\eta(Y)X - \eta(X)Y] \]  \hspace{2cm} (2.11)

Further for a 1-form \( \eta \) on a generalised Co-Symplectic manifold \( M_n \), we have,

**Theorem 2.1.** A Generalised Co-Symplectic Manifold \( M_n \) admitting a connection \( E \), is uniquely determined by the contact form \( \eta \) and tensor field \( F \) satisfies

\[ (E_X \eta)Y = (\nabla_X \eta)Y \]  \hspace{2cm} (2.12)

\[ (E_X \eta)(FY) = (\nabla_X \eta)(FY) \]  \hspace{2cm} (2.13)

\[ E_X(FY) = \eta(Y)X - g(X,Y)\xi + \nabla_X Y - \frac{1}{2}[\eta(Y)X - \eta(X)Y] \]  \hspace{2cm} (2.14)

\[ E_X(FY) - E_Y(FX) = \nabla_X Y - \nabla_Y X \]  \hspace{2cm} (2.15)

**Proof:** Using equations (2.6), (2.10) and (2.11), we have the results (2.12), (2.13), (2.14) and (2.15).

Again covariant differentiation of the torsion tensor \( S \) is given by

\[ E_X(S(Y, Z)) = (E_X S)(Y, Z) + S(E_X Y, Z) + S(Y, E_X Z). \]

Using equation (2.4), we have

\[ (E_X S)(Y, Z) = ((E_X \eta)Z)Y - ((E_X \eta)Y)Z \]  \hspace{2cm} (2.16)

Let us define

\[ \overline{S}(X, Y, Z) = g(S(X, Y), Z) \]  \hspace{2cm} (2.17)

Then from the equations (2.4) and (2.17), we get

\[ \overline{S}(X, Y, Z) + \overline{S}(Y, Z, X) + \overline{S}(Z, X, Y) = 0 \]  \hspace{2cm} (2.18)

and

\[ g(P(X, Y), Z) + g(P(X, Z), Y) = 0 \]  \hspace{2cm} (2.19)

The torsion tensor \( S \) of the connection \( E \) satisfies the following relations

(a) \( S(\overline{X}, Y) = 0 \)

(b) \( S(X, \xi) = \overline{X} \)

(c) \( S(\overline{X}, \xi) = \overline{X} \)

(d) \( S(X, \xi) - S(\overline{X}, \xi) = 2X - 2\eta(X)\xi \)

(e) \( S(\overline{X}, Y) = \eta(X)\eta(Y)\xi - \eta(Y)X \)

(f) \( S(X, Y) = \eta(Y)\overline{X} \)

(g) \( \eta(S(X, Y) = 0 \)

**Theorem 2.2.** A generalised Co-symplectic Manifold \( M_n \) satisfies the following relations i.e. (a), (b), (c), (d), (e), (f) and (g) defined above.
Theorem 2.3. In a Generalised Co-Symplectic manifold \( M_n \) with connection \( E \), we have
\[
\begin{align*}
(a) & \quad \tilde{P}(X, Y, Z) = \frac{1}{2}[\eta(Y) F'(X, Y) - \eta(X) F'(Y, Z)] \\
(b) & \quad \tilde{P}(X, Y, Z) = 1/2[\eta(X) g(Y, Z) - \eta(X) \eta(Y) \eta(Z)] \\
(c) & \quad \tilde{P}(\bar{X}, \bar{Y}, \bar{Z}) = 0 \\
(d) & \quad \tilde{P}(\bar{X}, Y, Z) = 0 = \tilde{S}(\bar{X}, Y, Z) = 0 \\
(e) & \quad \tilde{P}(X, Y, Z) = \frac{1}{2}[\eta(Y) F'(X, \bar{Z}) - \eta(X) F'(Y, \bar{Z})] \\
\end{align*}
\]
where \( \tilde{P}(X, Y, Z) = g(P(X, Y), Z) \)

Theorem 2.4. In a Generalised Co-Symplectic Manifold \( M_n \) admitting connection \( E \) satisfied the following properties:
\[
\begin{align*}
(a) & \quad (E_X F')(Y, Z) = (\nabla_X F')(Y, Z) - \frac{1}{2}[\eta(Y) F'(X, Z) + \eta(Z) F'(Y, X)] \\
(b) & \quad E_F F'(Y, Z) = (\nabla_{E X} F')(Y, Z) \\
\end{align*}
\]
Proof: (a) We have
\[
\begin{align*}
X(F'(Y, Z)) & = (E_X F')(Y, Z) + F'(E_X Y, Z) + F'(Y, E_X Z) \\
X(F'(Y, Z)) & = (\nabla_X F')(Y, Z) + F'(\nabla_X Y, Z) + F'(Y, \nabla_X Z)
\end{align*}
\]
which implies
\[
(E_X F')(Y, Z) = (\nabla_X F')(Y, Z) + F'(\nabla_X Y, Z) + F'(Y, \nabla_X Z) - F'(E_X Y, Z) - F'(Y, E_X Z) \\
= (\nabla_X F')(Y, Z) - 1/2[\eta(Y) F'(X, Z) + \eta(Z) F'(Y, X)]
\]
Using equation (2.10), we get (a). Again baring (a), we get (b) and (c).

### III. Curvature Tensor of \( M_n \) with Respect to Connection \( E \)

Let \( \tilde{R} \) be the curvature tensor with respect to the semi-symmetric metric connection \( E \) on a generalised co-symplectic manifold \( M_n \). Then
\[
\tilde{R}(X, Y, Z) = E_X E_Y Z - E_Y E_X Z - E_{[X,Y]} Z \tag{3.1}
\]
We have the following results:

Theorem 3.1. In a Generalised Co-Symplectic Manifold \( M_n \) curvature tensor \( \tilde{R} \) is given by
\[
\begin{align*}
\tilde{R}(X, Y, Z) & = R(X, Y, Z) + \frac{1}{2}[S(X, \nabla_Y Z) + S(\nabla_X Z, Y)] + \frac{1}{2}\eta(Z) S'(X, Y) \\
& \quad + \frac{1}{2}\eta(Z) [\nabla_X Y - \nabla_Y X] + \frac{1}{2}[\eta(Y) \nabla_X Z - \eta(X) \nabla_Y Z] \\
& \quad - \frac{1}{2}(\nabla_Y \eta(Z)) X - \frac{1}{2}\eta(Z) [X, Y] + \frac{1}{2}[\eta([X,Y]) Z - ([\nabla_X \eta] Y) Z] \tag{3.2}
\end{align*}
\]
where
\[
S'(X, Y) = \eta(X) Y - \eta(Y) X \\
R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z
\]
Proof: Let \( \tilde{R}(X,Y,Z) \) be the curvature tensor for generalised co-symplectic manifold with respect to the semi-symmetric metric S-connection \( E \), then

\[
\tilde{R}(X,Y,Z) = E_X E_Y Z - E_Y E_X Z - E_{[X,Y]} Z
\]

By using equations (1.1),(1.2),(1.11),(1.12),(1.13) ,(2.9) and (2.12), we get equation (3.1), where \( R(X,Y,Z) \) is curvature tensor of \( M_n \) with respect to the Riemannian Curvature \( \nabla \).

Let \( K \) and \( \tilde{K} \) be curvature tensor of type (0,4) given by

\[
K(X,Y,Z) = g(R(X,Y,Z), U)
\]

\[
\tilde{K}(X,Y,Z,U) = g(\tilde{R}(X,Y,Z), U)
\]

**Theorem 3.2.** In a Generalised Co-Symplectic Manifold \( M_n \), we have

\[
\tilde{R}(X,Y,Z) + \tilde{R}(Y,Z,X) + \tilde{R}(Z,X,Y) = 0 \tag{3.3}
\]

If

\[
2[ ((\nabla X \eta) Z) \nabla Y + ((\nabla Y \eta) X) \nabla Z + ((\nabla Z \eta) Y) \nabla X ] + \eta([X, Z]) \nabla Y + \eta([Y, X]) \nabla Z + \eta([Z, Y]) \nabla X = 0 \tag{3.4}
\]

and

\[
\tilde{K}(X,Y,Z,U) + \tilde{K}(Y,X,Z,U) = 0 \tag{3.5}
\]

If and only if

\[
g( (\nabla Y \eta(Z)) \nabla U, U) + g( (\nabla X \eta(Z)) \nabla U, U) = 0 \tag{3.6}
\]

and

\[
(\nabla Y \eta) X + (\nabla X \eta) Y = 0 \tag{3.7}
\]

Proof: Using equation (3.2) and the first Bianchi identity

\[
R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0
\]

with respect to Riemannian Connection \( \nabla \), we get eq.(3.3) and (3.4). We have

\[
\tilde{K}(X,Y,Z) = g(\tilde{R}(X,Y,Z), U)
\]

\[
= g(R(X,Y,Z), U) + \frac{1}{2} g(S(X, \nabla Y Z), U) + \frac{1}{2} g(S(\nabla X Z, Y), U)
\]

\[
+ \frac{1}{2} g(\eta(Z)[\nabla X \nabla Y - \nabla Y \nabla X], U) + \frac{1}{4} g(\eta(Z)[\eta(X)Y - \eta(Y)X]
\]

\[- g(((\nabla X \eta) Y) \nabla Z, U) + \frac{1}{2} g((\eta(Y) \nabla X Z - \eta(X) \nabla Y Z), U)
\]
We get

\[ \tilde{K}(X, Y, Z, U) = -\tilde{K}(Y, X, Z, U) \]  

(3.8)

If

\[ g(\nabla_Y \eta(Z), X, U) = -g(\nabla_X \eta(Z), Y, U) \]  

(3.9)

and

\[ (\nabla_Y \eta)X + (\nabla_X \eta)Y = 0 \]  

(3.10)

### IV. Nijenhuis Tensor of \(M_n\) with Respect to New Connection \(E\)

The Nijenhuis tensor with respect to \(E\) of \(F\) in a generalised Co-symplectic manifold \(M_n\) is a vector valued bilinear scalar function \(N_E\), is given by

\[ N_E(X, Y) = (\nabla_X F)Y + S(X, Y) - (\nabla_Y F)X + \overline{\nabla_X F}Y + \overline{\nabla_Y F}X \]  

(4.1)

Using equation (1.16), we get

\[ N_E(X, Y) = N(X, Y) + S(X, Y) + 2\overline{\nabla_X F}Y \]  

(4.2)

where \(N\) is Nijenhuis tensor with respect to Riemannian connection and \(S\) is the Torsion tensor of connection \(E\).

Again by using equation (1.17), we get

\[ N_E(X, Y, Z) = (E_X F')(Y, Z) - (E_Y F')(X, Z) - (E_Y F')(X, \overline{Z}) + \frac{1}{2} \eta(Y)[F'(X, \overline{Z}) + F'(X, Z)] \]  

(4.3)

and

\[ N_E(X, Y, Z) = N(X, Y, Z) \]  

(4.4)

If and only if

\[ (\nabla_X F')(Y, \overline{Z}) + (\nabla_Y F')(X, Z) - (\nabla_Y F')(X, \overline{Z}) = \frac{1}{2} \eta(X)[F'(Y, Z) + F'(Y, \overline{Z})] + \frac{1}{2} \eta(Y)[F'(X, \overline{Z}) + F'(X, Z)] \]  

(4.5)

### References

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