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Mixed Quadrature Rule for Double Integrals of Simpsons $\frac{1}{3}rd$ and Gauss Legendre Two Point Rule in Two Variables

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Mixed Quadrature Rule for Double Integrals of Simpsons $\frac{1}{3}rd$ and Gauss Legendre Two Point Rule in Two Variables

Saumya Ranjan Jena α & Subash Chandra Mishra σ

Abstract- In this article we are concerned with mixed quadrature rule of higher degree of precision for double integrals with double variables. The rule is numerically tested taking some suitable texts and the error bound is determined.

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I. Introduction

A Mixed quadrature rule of higher degree of precision has been formed by [2], [3], [4], [5], [6], [7]. This rule is meant for single integral. In the same vein here we develop a mixed quadrature rule of degree of precision-7 for double integrals taking the convex combination of Simpson's $\frac{1}{3}$rd and Gauss-Legendre-2 point rule each of degree of precision 5. This paper is designed as follows. Section -2 contains formulation of quadrature of constituent rules and the corresponding errors in-2 variables. Section-3 is devoted to construction of mixed quadrature rule. The error analysis is done in section-4. In section-5 the rule is numerically verified by taking suitable examples. The conclusions are drawn in section-6.

II. Formulation of Quadrature Rule in Two Variables X and Y

For approximate evaluation of real definite integral

$$I(f)=\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \tag{2.1}$$

The Simpson's $\frac{1}{3}$ rd rule is

$$I(f)\cong R_{s_{\frac{1}{3}}}(f)=\frac{1}{9}\begin{bmatrix}\{f(-1,-1)+4f(0,-1)+f(1,-1)\}\\ +4\{f(-1,0)+4f(0,0)+f(1,0)\}\\ +\{f(-1,1)+4f(0,1)+f(1,1)\}\end{bmatrix} \tag{2.1}$$

And the Gauss-Legendre's two point rule is

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Ref

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$$I(f) \cong R_{GL2}(f) = f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)+f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \tag{2.3}$$

where each rule of equation (2.2) and equation (2.3) is of precision 3.

Hence

$$I(f) = R_{s\frac{1}{3}}(f) + E_{s\frac{1}{3}}(f) \tag{2.4}$$

$$I(f) = R_{GL2}(f) + E_{GL2}(f) \tag{2.5}$$

Where $E_{S\frac{1}{3}}(f)$ and $E_{GL2}(f)$ are error in approximating the integrals $I(f)$ by equation (2.2) and equation (2.3) respectively.

Now assuming $f(x,y)$ to be sufficiently differentiable in $-1 \le x \le 1, -1 \le y \le 1$

We can write equation (2.1) using Maclaurin's expansion

$$I(f) = \int_{-1}^{1}\int_{-1}^{1} \left[\begin{array}{l} f(0,0) + \{x\, f_{1,0}(0,0) + y\, f_{0,1}(0,0)\} \\ +\frac{1}{2!}\{x^2\, f_{2,0}(0,0) + 2xy\, f_{1,1}(0,0) + y^2\, f_{0,2}(0,0)\} \\ +\frac{1}{3!}\{x^3\, f_{3,0}(0,0) + 3x^2y\, f_{2,1}(0,0) + 3xy^2\, f_{1,2}(0,0) + y^3\, f_{0,3}(0,0)\} \\ +\frac{1}{4!}\left\{\begin{array}{l} x^4\, f_{4,0}(0,0) + 4x^3y\, f_{3,1}(0,0) + 6x^2y^2\, f_{2,2}(0,0) \\ +4xy^3\, f_{1,3}(0,0) + y^4\, f_{0,4}(0,0) \end{array}\right\} \\ +\frac{1}{5!}\left\{\begin{array}{l} x^5\, f_{5,0}(0,0) + 5x^4y\, f_{4,1}(0,0) + 10x^3y^2\, f_{3,2}(0,0) \\ +10x^2y^3\, f_{2,3}(0,0) + 5xy^4\, f_{1,4}(0,0) + y^5\, f_{0,5}(0,0) \end{array}\right\} \\ +\frac{1}{6!}\left\{\begin{array}{l} x^6\, f_{6,0}(0,0) + 6x^5y\, f_{5,1}(0,0) + 15x^4y^2\, f_{4,2}(0,0) \\ +20x^3y^3\, f_{3,3}(0,0) + 15x^2y^4\, f_{2,4}(0,0) \\ +6xy^5\, f_{1,5}(0,0) + y^6\, f_{0,6}(0,0) \end{array}\right\} \end{array}\right] dxdy \tag{2.6}$$

Integrating equation (2.6) we have

$$\begin{aligned} I(f) &= 4 f_{0,0}(0,0) + \frac{2}{3}\left[f_{2,0}(0,0) + f_{0,2}(0,0)\right] + \frac{1}{30}\left[f_{4,0}(0,0) + f_{0,4}(0,0)\right] \\ &+\frac{1}{9} f_{2,2}(0,0) + \frac{1}{180}\left[f_{4,2}(0,0) + f_{2,4}(0,0)\right] + \frac{4}{7!}\left[f_{6,0}(0,0) + f_{0,6}(0,0)\right] + \ldots. \end{aligned} \tag{2.7}$$

Again using Maclaurin's expansion

$$\begin{aligned} f(-1,-1) = f_{0,0} &- \left[f_{1,0}(0,0) + f_{0,1}(0,0)\right] + \frac{1}{2!}\left[f_{2,0}(0,0) + 2 f_{1,1}(0,0) + f_{0,2}(0,0)\right] \\ &-\frac{1}{3!}\left[f_{3,0}(0,0) + 3 f_{2,1}(0,0) + 3 f_{1,2}(0,0) + f_{0,3}(0,0)\right] \\ &+\frac{1}{4!}\left[f_{4,0}(0,0) + 4 f_{3,1}(0,0) + 6 f_{2,2}(0,0) + 4 f_{1,3}(0,0) + f_{0,4}(0,0)\right] \end{aligned}$$

$$-\frac{1}{5!}\left[f_{5,0}(0,0)+5f_{4,1}(0,0)+10f_{3,2}(0,0)+10f_{2,3}(0,0)+5f_{1,4}(0,0)+f_{0,5}(0,0)\right]$$

$$+\frac{1}{6!}\left[f_{6,0}(0,0)+6f_{5,1}(0,0)+15f_{4,2}(0,0)+20f_{3,3}(0,0)+15f_{2,4}(0,0)+6f_{1,5}(0,0)+f_{0,6}(0,0)\right]$$

$$+\ldots\ldots\ldots\ldots \tag{2.8}$$

Notes

Using equation (2.8) and $f(0,-1), f(1,-1), f(-1,0), f(1,0), f(0,1), f(-1,1)$ *and* $f(1,1)$ in equation (2.2)

$$R_{S\frac{1}{3}}(f)=4f_{0,0}(0,0)+\frac{2}{3}\{f_{2,0}(0,0)+f_{0,2}(0,0)\}+\frac{1}{18}\{f_{4,0}(0,0)+f_{0,4}(0,0)\}$$

$$+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{108}\{f_{4,2}(0,0)+f_{2,4}(0,0)\}+\frac{11}{9\times 6!}\{f_{6,0}(0,0)+f_{0,6}(0,0)\} \tag{2.9}$$

a) Error in Simpson's $\frac{1}{3}$ rd rule:

Using equation (2.6), equation (2.9) in equation (2.4) error associated with Simpson's $\frac{1}{3}rd$ rule is

$$E_{S\frac{1}{3}}(f)=I(f)-R_{S\frac{1}{3}}(f)$$

$$E_{S\frac{1}{3}}(f)=-\frac{1}{45}\{f_{4,0}(0,0)+f_{0,4}(0,0)\}-\frac{1}{270}\{f_{4,2}(0,0)+f_{2,4}(0,0)\}-$$

$$\frac{41}{63\times 6!}\{f_{6,0}(0,0)+f_{0,6}(0,0)\} \tag{2.10}$$

As the error contains at least fourth derivative of the integral function, it vanishes for all polynomials of degree ≤ 3 in x and y. Thus the formula becomes exact for all polynomials of degree up to 3 i.e. degree of precision of the formula is three.

Similarly using Maclaurin's expansion

We have

$$f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

$$=f(0,0)-\frac{1}{\sqrt{3}}\left[f_{1,0}(0,0)+f_{0,1}(0,0)\right]$$

$$+\frac{1}{2!}\left[\left(-\frac{1}{\sqrt{3}}\right)^2 f_{2,0}(0,0)+2\left(-\frac{1}{\sqrt{3}}\right)^2 f_{1,1}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^2 f_{0,2}(0,0)\right]$$

$$+\frac{1}{3!}\left[\left(-\frac{1}{\sqrt{3}}\right)^3 f_{3,0}(0,0)+3\left(-\frac{1}{\sqrt{3}}\right)^3 f_{2,1}(0,0)+3\left(-\frac{1}{\sqrt{3}}\right)^3 f_{1,2}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^3 f_{0,3}(0,0)\right]$$

$$+\frac{1}{4!}\left[\begin{array}{l}\left(-\frac{1}{\sqrt{3}}\right)^4 f_{4,0}(0,0)+4\left(-\frac{1}{\sqrt{3}}\right)^4 f_{3,1}(0,0)+6\left(-\frac{1}{\sqrt{3}}\right)^4 f_{2,2}(0,0)\\ +4\left(-\frac{1}{\sqrt{3}}\right)^4 f_{1,3}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^4 f_{0,4}(0,0)\end{array}\right]$$

$$+\frac{1}{5!}\left[\begin{array}{l}\left(-\frac{1}{\sqrt{3}}\right)^5 f_{5,0}(0,0)+5\left(-\frac{1}{\sqrt{3}}\right)^5 f_{4,1}(0,0)+10\left(-\frac{1}{\sqrt{3}}\right)^5 f_{3,2}(0,0)\\ +10\left(-\frac{1}{\sqrt{3}}\right)^5 f_{2,3}(0,0)+5\left(-\frac{1}{\sqrt{3}}\right)^5 f_{1,4}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^5 f_{0,5}(0,0)\end{array}\right]$$

$$+\frac{1}{6!}\left[\begin{array}{l}\left(-\frac{1}{\sqrt{3}}\right)^6 f_{6,0}(0,0)+6\left(-\frac{1}{\sqrt{3}}\right)^6 f_{5,1}(0,0)+15\left(-\frac{1}{\sqrt{3}}\right)^6 f_{4,2}(0,0)\\ +20\left(-\frac{1}{\sqrt{3}}\right)^6 f_{3,3}(0,0)+15\left(-\frac{1}{\sqrt{3}}\right)^6 f_{2,4}(0,0)\\ +6\left(-\frac{1}{\sqrt{3}}\right)^6 f_{1,5}(0,0)++\left(-\frac{1}{\sqrt{3}}\right)^6 f_{0,6}(0,0)\end{array}\right]\ldots\ldots \qquad (2.11)$$

Now *$f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, can be found out same ways as* (2.10)

Now substituting the value of

$$f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

in equation (2.3)

$$R_{GL2}(f)=4f(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{9\times 3!}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]$$

$$+\frac{20}{9\times 6!}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{1}{9}f_{2,2}(0,0)+\frac{4}{27\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] \qquad (2.12)$$

b) Error in Gauss-Legendre's two point rule

Now the error associated with the Gauss- Legendre 2-point rule is obtained substituting equation (2.6) and (2.11) in equation (2.5)

$$E_{GL2}(f)=I(f)-R_{GL2}(f)$$

$$=\frac{2}{135}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]+\frac{1}{405}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{1}{1701}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] \qquad (2.13)$$

Notes

In this case also, the error contains at least fourth derivative of the integral function. Thus the degree of the precision is 3.

III. Mixed Quadrature Rule

Now multiplying (2) by equation (2.10) and (3) by equation (2.13) and adding them, we get

Notes

$$I(f)=\frac{1}{5}\left[2R_{S\frac{1}{3}}(f)+3R_{GL2}(f)\right]+\frac{1}{5}\left[2E_{S\frac{1}{3}}(f)+3E_{GL2}(f)\right] \tag{3.1}$$

Where

$$I_{mix}(f)=R_{S\frac{1}{3}GL2}(f)+E_{S\frac{1}{3}GL2}(f) \tag{3.2}$$

Where

$$R_{S\frac{1}{3}GL2}(f)=\frac{1}{5}\left[2R_{S\frac{1}{3}}(f)+3R_{GL2}(f)\right] \tag{3.3}$$

Where $R_{S\frac{1}{3}GL2}(f)$ and $E_{S\frac{1}{3}GL2}(f)$ are mixed quadrature rule and its error obtained by Simpson's $\frac{1}{3}rd$ and Gauss-Legendre's 2 point rule respectively.

The truncation error generated by this approximation is given by

$$E_{S\frac{1}{3}GL2}=\frac{1}{5}\left[2E_{S\frac{1}{3}}(f)+3R_{GL2}(f)\right]=-\frac{1}{189\times 5!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] \tag{3.4}$$

The rule (3.3) may be called as mixed quadrature rule integrate exactly all polynomial of degree ≤ 5 in x and y.

IV. Error Analysis

a) Theorem 4.1

Let $f(x,y)$ be sufficiently differentiable function in the closed interval $[-1,1]$. The bounds of truncatinal error $E_{S\frac{1}{3}GL2}(f)$ associated with the rule $R_{S\frac{1}{3}GL2}(f)$ is given by

$$\left|E_{S\frac{1}{3}GL2}(f)\right|\cong\frac{1}{189\times 5!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]$$

Proof:

The proof obviously follows from the equation (3.4)

Theorem 4.2

The bounds for the truncational error $\left|E_{S\frac{1}{3}GL2}(f)\right|\leq\frac{2M}{225}\left|\eta_2-\eta_1\right|$

Where $\eta_1,\eta_2\in[-1,1]$

Where $M=Max\left|f_{6,0}(0,0)+f_{0,6}(0,0)\right|,\ -1\leq x\leq 1,\ -1\leq y\leq 1$

Proof:
We have

$$E_{S\frac{1}{3}}(f) = -\frac{1}{45}\left|f_{4,0}(n_1) + f_{0,4}(n_1)\right|$$

$$E_{GL2}(f) = \frac{2}{135}\left|f_{4,0}(n_2) + f_{0,4}(n_2)\right|$$

Where

$$n_1, n_2 \in [-1,1]$$

Hence

$$E_{S\frac{1}{3}GL2}(f) = \frac{1}{5}\left|2E_{s\frac{1}{3}}(f) + 3E_{GL2}(f)\right|$$

$$= \frac{2}{225}\left[f_{4,0}(n_2,0) + f_{0,4}(0,n_2) - f_{4,0}(n_1,0) - f_{0,4}(0,n_1)\right]$$

$$= \frac{2}{225}\int_{n_1}^{n_2}\int_{n_1}^{n_2}\left[f_{5,0}(x,*) + f_{0,5}(*,y)\right]$$

Hence

$$\left|E_{S\frac{1}{3}GL2}(f)\right| \le \frac{2M\left|n_1 - n_2\right|}{225}$$

Where

$$M = \max\left|f_{5,0}(x,*) + f_{0,5}(*,y)\right| \qquad -1 \le x \le 1, \quad -1 \le y \le 1$$

Which, gives only the truncation error bound on n_1, n_2 are closed to each other.

b) Corollary

The error bound for the truncated error $\left|E_{S\frac{1}{3}GL2}(f)\right| \le \frac{4M}{225}$

When $\left|n_1 - n_2\right| \le 2$ [1]

V. Numerical Verification

The exact value of the integrals

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} e^{x+y}dxdy = 5.524391382167262$$

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} e^{-\left(x^2+y^2\right)}dxdy = 2.230985141404134$$

$$I_3 = \int_{0}^{1}\int_{0}^{1} \frac{\sin^2(x+y)}{(x+y)}dxdy = 0.613260369981918$$

Notes

Table-1 : (Comparison of exact result with approximate result)

Exact	$R_{S\frac{1}{3}}(f)$	$R_{GL2}(f)$	$R_{S\frac{1}{3}GL2}(f)$	$E_{S\frac{1}{3}GL2}(f)$
I_1 =5.524391382167262	5.577469135802469	5.488065843621398	5.523827160493826	0.00056421673436
I_3 =2.230985141404134	2.259259259259260	2.024691358024692	2.238518518518519	0.007533377114385
I_3 =0.613260369981918	0.612749599379052	0.609491314229328	0.610769318605380	0.002491051376538

VI. Conclusion

From the Table -1, we find that

$$\left|E_{S\frac{1}{3}}(f)\right| \leq \left|E_{GL2}(f)\right| \qquad (i)$$

$$\left|E_{S\frac{1}{3}GL2}(f)\right| \leq \left|E_{S\frac{1}{3}}(f)\right| \qquad (ii)$$

From (i) and (ii)

$$\left|E_{S\frac{1}{3}GL2}(f)\right| \leq \left|E_{S\frac{1}{3}}(f)\right| \leq \left|E_{GL2}(f)\right|$$

It is evident that the mixed quadrature rule $R_{S\frac{1}{3}GL2}(f)$ of degree of precision 5 provides us better result than constituent rule $R_{S\frac{1}{3}}(f), R_{GL2}(f)$ each of degree of precision three. Hence, the mixed quadrature rule is more efficient and numerically better convergent then exact result.

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