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Super-Symmetry Quantum Factorization Of Radial Schrodinger Equation for the Doubly An-Harmonic Oscillator Via Transformation to Bi-Confluent Heun's Differential Operators

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I. Introduction

Heuns differential equation and its confluent forms have been subject of many investigations in last years due to a large number of their applications in mathematical physics and quantum mechanics [13, 9]. They indeed play a central role in a number of physical problems, like quasi-exactly solvable systems [10], higher dimensional correlated systems [11], Kerr-de Sitter black holes [12], Calogero-Moser-Sytherland system [14], finite lattice Bethe-ansatz systems [15], etc. Besides, this equation appears as a natural generalization of the hypergeometric equation and its special cases including the Gauss hypergeometric, confluent hypergeometric, Mathieu, Ince, Lame, Bessel, Legendre, Laguerre equation, etc. The general second order Heun's differential equation (GHE) can be written, in canonical form, as follows [6]

\begin{equation} $$\mathcal{D}^2\mathcal{Y} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\mathcal{D}\mathcal{Y} + \frac{(\alpha\beta x - q)}{x\,(x-1)(x-a)}\mathcal{Y} = 0, \qquad (1.1)$$ \end{equation}

Where $\mathcal{D} = \frac{d}{dx}, \{\alpha, \beta, \gamma, \delta, \epsilon, a, q\}\ (a \neq 0,1)$ are parameters, generally complex and arbitrary, linked by the Fuschian constraint $\{\alpha + \beta + 1 = \gamma + \delta + \epsilon.$ This equation has four regular singular points at $\{0,1,a,\infty\},$ with the exponents of these singularities being respectively, $\{0,1,-\gamma\},\{0,1-\delta\},\{0,1-\epsilon\},$ and $\{\alpha,\beta\}.$ The equation (1.1) can be transformed into the following other multi-parameters equations one of which is (14): Bi-confluent Heun's equation (BHE) (see page 131 of [13]).

\begin{equation} $$\mathcal{D}^2\mathcal{Y} + \left(\frac{\alpha+1}{x} - \beta - 2x\right)\mathcal{D}y + \left(\gamma - \alpha - z^2 - \frac{\beta+(\alpha+1)\beta}{2x}\right)\mathcal{Y} = 0, \qquad (1.2)$$ \end{equation}

This equation has many important features in obtaining solutions to Schrodinger equations.
Consider the Schrodinger equations.

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\begin{equation}

$$\mathcal{Y}''(x_1) + \{\mathrm{E} - \mu x_1^2 - \lambda_1^4 - \eta x_1^6\}\mathcal{Y}(x) = 0, \qquad \text{\end{equation}} \tag{1.3}$$

Is transformed into multi-parameters equation Bi-confluent Heun differential equation via the transformation

$$\text{\begin{equation}} \quad \mathcal{Y}(x_1) = x^{\frac{1}{2}} e - \left(\tfrac{1}{4}\alpha C \propto_1^4 + \tfrac{1}{2}\beta C \propto_1^2\right)\xi(x), x_1^2 = \left(\tfrac{2}{\alpha_C}\right)^{\frac{1}{2}} x \quad \text{\end{equation}}$$

The Bi-confluent equation obtained was
\begin{equation}

$$x\xi(x)'' + \{\tfrac{3}{2} + \beta_C(\tfrac{2}{\alpha_C})^{\frac{1}{2}} x - 2x^2\}\xi'(x) + \{\tfrac{1}{2\alpha_C}(\beta^2{}_c - 3\alpha_C - \mu)x + \tfrac{1}{4}(\tfrac{2}{\alpha_C})^{\frac{1}{2}}(\mathrm{E} + 3\beta_c)\}\xi(x) = 0$$

\end{equation} (1.4)

with

$$\alpha = \tfrac{1}{2}, \beta = \beta_c\,(\tfrac{2}{\alpha_C})^{\frac{1}{2}},\ \gamma = \tfrac{1}{2\alpha_C}\,(\beta_c^2 - 3\alpha_c - \mu) + \tfrac{3}{2}, \delta = -\frac{E}{(2\alpha_c)^{\frac{1}{2}}}$$

Comparing equation (1.2) and (1.4), the following parameter relations were deduced in page 201 of [13],

$$\alpha = 2(l+1)\alpha_F^{\frac{1}{2}} - 1;\ \beta = \frac{2\beta F}{\alpha_F^{\frac{1}{2}}};\ \gamma = \frac{\epsilon}{\alpha F} + 2(l+1)\,(\alpha_F^{\frac{1}{2}} - 1);\ \delta = \frac{2}{\alpha_F^{\frac{1}{2}}}\{-a + 2\beta_F(l+1)(1 - \alpha_F^{\frac{1}{2}})\}$$

At this point, it is necessary to emphasize that the super-symmetry method of factorization of the Bi-confluent Heun's differential equation can be use to obtain solutions of the named Schrodinger equation. To do this we briefly discuss the SUSY method of factorization.

In recent work [4], the concept of factorization method, super-symmetry quantum mechanics (SUSY QM) and shape invariant techniques have been extended to Sturn-Liouville (SL) equations to solve Schrödinger equations. In the present work this concept shall be extended to the Bi-confluent Heun's differential equation.

II. Factorization of BH Operator

a) *General method of factorization of SL operators*

In this section, we extend to BH the general method of factorization of SL operators developed in the previous work [4] to construct new solvable potentials for Heun's operators. For a matter of convenience we first briefly recall the results of [4].

b) *Brief Review of general method of factorization of SL Operator*

Following [4], the concept of factorization method was extended to SL equation SUSY QM and shape invariance were widely developed to solve Schrödinger equations and reviewed. Consider the one-dimensional second order differential equation

$$\mathrm{H}\Phi = \xi\Phi',\ \ \Phi\Phi' \in AC_{loc}\,(]a\,b,[), \tag{2.1}$$

where

$$\mathrm{H} = -\sigma(x)\frac{d^2}{dx^2} - \tau(x)\frac{d}{dx} + V(x). \tag{2.2}$$

ξ is a constants, $\sigma(x), \tau(x)$ and $V(x)$ are real function defined on the open interval, $(]a\,b,[) \subseteq \mathrm{R}$ and $AC_{loc}(a,b)$ is the set of local absolute continuous functions given by

$$AC_{loc}(a,b) = \{f \in AC[\alpha_1\beta_1],\ \forall[\alpha_1\beta_1] \subset (a,b), [\alpha_1\beta_1]\ \text{compact}\}, \tag{2.3}$$

$$AC[\alpha_1\beta_1] = \left\{f \in C[\alpha_1\beta_1], f(x) = f(\alpha_1) + \int_{\alpha_1}^{x} g(t)dt,\ g \in L^1[\alpha_1\beta_1]\right\}. \tag{2.4}$$

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4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005)

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The suitable Hilbert space $\mathbb{H} = L^2\ ([a,b], \rho\ (x)dx)$ with the inner product defined by means of non-negative weight function $\rho(x)$ on $(]a\,b,[)$:

$$\langle u,v\rangle = \int_a^b \overline{u}\,(x)v(x)p(x)dx, \qquad u(x), v(x) \in \mathbb{H}, \tag{2.5}$$

Where $\overline{u}$ is the complex conjugate of u. The domain of H will be examined below. Choosing the weight function $\rho(x)$ such that the Pearson equation.

$$[\sigma(x)\rho(x)]' = \tau(x)p(x), \tag{2.6}$$

Is satisfied. The differential equation (2.1) can be reduced to the self-adjoint form [4]

$$[\sigma(x)p(x)\Psi'(x)]' - [V-\xi] \quad \Psi(x)p(x) = 0, \tag{2.7}$$

and the operator (2.2) can be written in the equivalent form of SL operators [4]

$$H = \frac{1}{\rho(x)}\left(-\frac{d}{dx}\rho(x)\frac{d}{dx} + q(x)\right), \tag{2.8}$$

Where $p(x) = \sigma(x)\rho(x)\ and\ q(x)V(x)p(x)$. Eq. (2.5), together with the following boundary condition:

$$\rho(x)\sigma(x)\left[\overline{u}(x)v'(x) - \overline{u}'(x)v(x)\right]\Big|_a^b = 0,\ \forall u,v \in \mathbb{H}, \tag{2.9}$$

Is called *Sturm-Liouville system* [4]. The boundary condition ensures the self-adjointness of the operator H. Since we want the operator to be self-adjoint, we take on the Hilbert space *H* as

$$\mathcal{D}(H) = \{u \in \mathbb{H}, u, pu' \ \in\ AC_{\text{loc}}(\text{a},\text{b}), Hu\ \in \mathbb{H}\},$$

$$p(x)\ \left[\overline{u}(x)v'(x) - \overline{u}'(x)v(x)\right]\Big|_a^b = 0,\ \forall u,v \in \mathbb{H}. \tag{2.10}$$

It is clear that $\mathcal{D}(H)$ is dense in $\mathbb{H}$ since $C_0^\infty(]a\,b,[) \subset \mathcal{D}(H)$. by requiring:

(i) $p \in AC_{\text{loc}}\ (]a\,b,[), \rho' \in L^2_{loc}(]a\,b,[), \rho^{-1} \in L^2_{loc}(]a\,b,[)$ positive and real-valued;

(ii) $q \in L^2_{loc}(]a\,b,[)$, be real-valued;

(iii) $\in L^1_{loc}(]a\,b,[), \text{p}^{-1} \in L^1_{loc}(]a\,b,[)$, positive and real valued. Then, the operator (H, $\mathcal{D}(H)$) is self adjoint [4].

The purpose of this section is to introduce a factorization model with an annihilator operator of the form

$$A = \kappa\ \left[\frac{d}{dx} + W(x)\right], \tag{2.11}$$

with domain:

$$\mathcal{D}(A) = \{u \in \mathbb{H}, \kappa u' + \kappa W u \in \mathbb{H}\}, \tag{2.12}$$

where κ and W are continuous functions on $(]a\,b,[)$. We infer that $\mathcal{D}(A)$ dense in $\mathbb{H}$ since $H^{1,2}\ ((]a\,b,[), p(x)dx)$ is dence in $\mathbb{H}$since $H^{1,2}\ ((]a\,b,[), p(x)dx) \subset \mathcal{D}(A)$ where $H^{m,n}\ (\Omega)$ is a sobolev space of indices $\{m,n\}$. The operator *A* is closed in $\mathbb{H}$. The adjoint operator A^+ is given by [4].

$$\mathcal{D}(A^+) = \{v \in \mathbb{H}\}|\exists\overline{v} \in \mathbb{H}\}: \quad \langle Au,\ v\rangle = \langle Au,\ \overline{v}\rangle, \forall u, \in \mathcal{D}(A), A^+v = \overline{v}. \tag{2.13}$$

The explicit expression of (A^+) is given through the following theorem

Theorem 2.1 [4] Suppose the following boundary condition

$$\kappa(x)\rho(x)u(x)v(x)\Big|_a^b = 0, \forall u\ \in \mathcal{D}(A) \text{ and } v\ \mathcal{D}(A^+)\ , \tag{2.14}$$

is verified. then the operator A^+ *can be written as*

$$A^\dagger = \kappa(x)\left[-\frac{d}{dx} + W(x) + \mu(x)\right], \quad (2.15)$$

where $\mu(x)$ is a real continuous function defined by $\mu(x) = \frac{d}{dx}\ln[\kappa(x)\rho(x)]$.

Let H_1 and H_2 be the product operators A^+A and AA^+, respectively,

$$H_1 = A^+A, \qquad H_2 = AA^+, \quad (2.16)$$

with the corresponding domains

$$\mathcal{D}(H_1) = \{u \in \mathcal{D}(A), v = Au \in \mathcal{D}(A^\dagger) \text{ and } A^\dagger v \in \mathbb{H}\},$$

$$\mathcal{D}(H_2) = \{u \in \mathcal{D}(A^\dagger), v = A^\dagger u \in \mathcal{D}(A) \text{ and } Av \in \mathbb{H}\}. \quad (2.17)$$

Remark that

$$H^{1,2}(]a\,b[, p(x)dx) \subset \mathcal{D}(A) \subset \mathcal{D}(A^+),$$

$$\mathcal{D}(H_1), \mathcal{D}(H_2) \supset H^{2,2}(]a\,b[, \rho(x)dx).$$

We infer the $\mathcal{D}(H_1)$ and $\mathcal{D}(H_2)$ are dense in $\mathbb{H}$. Furthermore, the following theorem gives the additional conditions to subject to the function of κ and the potentials V so that the operator H factorizes in terms of A and A^+.

Theorem 2.2 *[4] Suppose that*

(i) κ *and* μ *are related to* σ *and* τ *as:*

$$\kappa^2 = \sigma; \quad \kappa(\kappa^1 - \kappa\mu) = \tau; \quad (2.18)$$

(ii) *the potential function* V *is related to the* W *by the Riccati type equation*

$$V - \xi_0 = \sigma(W^2 - W) - \tau W'. \quad (2.19)$$

Then the operators $H_{1,2}$ *are self-adjoint and*

$$H_1 = A^+A = H - \xi_0 = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + \sigma(W^2 - W') - \tau W',$$

$$H_2 = AA^+ = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + \sigma(W^2 - W') + (\tau - \sigma')W + \kappa(\kappa\mu)'. \quad (2.20)$$

Let us remark that the condition $\kappa(\kappa^1 - \kappa\mu) = \tau$ of (2.19) can be done from the Pearson equation (2.6) and the constraint $\kappa^2 = \sigma$. The quantity α can also be expressed as $\alpha = \frac{\kappa^1}{\kappa} - \frac{\tau}{\sigma}$. by means of the operation A and $A^\dagger$ we can form a superalgebra as follows;

$$\{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i = H_{ss}\delta_{ij}, \qquad [H_{ss}Q_i] = 0; \qquad i,j = 1,2,$$

Where $Q_1 = (Q^+ + Q^-)/\sqrt{2}$ and $Q_2 = (Q^+ + Q^-)/i\sqrt{2}$

$$\text{With } Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix},\ Q^- = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \qquad H_{ss} = \begin{pmatrix} A^+A & 0 \\ 0 & AA^+ \end{pmatrix}. \quad (2.21)$$

We can rewrite the operators $H_{1,2}$ as

$$H_1 = A^+A = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + V_1 \text{ and } H_2 = AA^+ = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + V_2 \quad (2.22)$$

Ref

4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005)

Where

$$V_1 = \sigma\,(W^2 - W') - \tau W, \quad V_2 = \sigma\,(W^2 - W') - (\tau - \sigma')W + \kappa(\kappa\mu)'. \tag{2.23}$$

It clearly appears the SUSY QM is extended to SL operators. We design here that the operators H_1, H_2 as SUSY partner. V_1, V_2 are SUSY partner potentials. The expression [4]

$$H_{ss} = [-D^2 + W^2(x)]I_2 + W'(x)\sigma_3, \tag{2.24}$$

which gives the superalgebra in terms of the superpotentials takes here the form

$$H_{ss} = \quad -[\sigma \frac{d^2}{dx^2} - \tau(x)\frac{d}{dx} - \sigma W^2 + \tau W - \frac{1}{2}(\sigma' W + \kappa(\kappa\mu)')]I_2$$

$$[\sigma W' + \frac{1}{2}(\sigma' W + (\kappa\mu)']\sigma_3,] \tag{2.25}$$

Where $\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the Pauli spin matrix and I_2 designs the 2×2 identity matrix. The equation (2.23) are the Riccati equation relating the partner potentials to the superpotentials. We denote, for $n \geq 0$, and $\xi_n^{(1)}$ the energy eigenfunctions and eigenvalues of H_1, respectively and$\Phi_n^{(2)}$ and $\xi_n^{(1)}$of H_2, respectively.

The pair of SL operators $H_{1,2}$ satisfies the interwining relation

$$H_2 A = AH, \quad H_1 A^+ = A^+ H_{2,} \tag{2.26}$$

and their states are related by SUSY transformations

$$H_2\left(A\Phi_n^{(1)}\right) = \xi_n^{(1)}\left(A\Phi_n^{(1)}\right), \qquad H_1\left(A\Phi_n^{(2)}\right) = \xi_n^{(2)}\left(A\Phi_n^{(2)}\right). \tag{2.27}$$

It is then straight forward to show that the eigenvalues of H_1 and H_2 are positive definite $\xi_n^{(1,2)} \geq 0$ and isospectra, i.e. they have almost the same energy eigenvalues, except for the ground state energy of H_1. Their energy spectra are related by the same equations.

$$\xi_n = \xi_n^{(1)} + \xi_{0,}\, \Psi_n = \Psi_n^{(1)}, \quad \xi_n^{(2)} = \xi_{n+1,}^{(1)} \qquad \Psi_n^{(2)} = [\xi_{n+1}^{(1)}]^{-1/2} A\Psi_{n+1}^{(1)},$$
$$\Psi_{n+1}^{(2)} = [\xi_n^{(1)}]^{-1/2} A^+ \Psi_n^{(2)}, \qquad n = 0,1,2,\ldots \tag{2.28}$$

We remark that the SL operator (2.2) is related to the Schrodinger-type operator. If we make the chance of variable such that $x = x(t)$ such that $\frac{d}{dx} = \kappa(x(t))$ and define the new function

$$\Psi_n(t) = \sqrt{(\kappa\big(x(t)\big)\rho x(t)\phi_n(x(t))} \tag{2.29}$$

Then equation (2.1) turns to an equation of the Schrodinger type

$$\frac{d^2}{dt^2}\Psi_n(t) + V(t)\,\Psi_n = \xi_n \Psi_n(t), \tag{2.30}$$

Where

$$V(t) = \left[V(x) + \frac{\tau(x)\,((\kappa(x)p(x))' + \kappa^2(x)(\kappa(x)\rho(x))''}{2\kappa(x)\rho(x)} - \frac{3\kappa^2(x)((\kappa(x)\rho(x))')^2}{4(\kappa(x)\rho(x))^2}\right]_{x=x(t)} \tag{2.31}$$

We shall denote various forms of corresponding superpotentials and partner-potentials of BHE by $BHEW_j$ and $BHEV_j$ and $j = 1, \ldots k$. The integer k depends on the number of the corresponding solutions of the Bi-confluent Heun's equation.

Ref

4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005)

c) Factorization of Bi-confluent Heun's Differential Operator (BH)

The second order differential operator s corresponding to BHE reads as

$$H^{BHE} = -xD^2 - (1+\alpha-\beta x-2x^2)D - \left((\gamma-\alpha-2)x - {}^1\!/_2\,[\delta+\beta(1+\alpha)]\right), \quad (2.32)$$

Having the following factorization characteristics.

(I) $\sigma = x, \kappa^2 = x$ which impies $\kappa = \pm\sqrt{x}$,

(2) $\tau = (1+\alpha-\beta x-2x^2)$,

(3) $V = -[(\gamma-\alpha-2)x - {}^{[\delta+\beta(1+\alpha)]}\!/_2]$,

(4) $\mu = \frac{\beta x+2x^2-\alpha-{}^1\!/_2}{x}$

The operator H factorizes into two first order differentials operators $A = \kappa(x)\,(x)(D+W(x))$and $A^+ = \kappa(x)(D+W(x)+\mu)$. The operator H, also could be expressed I terms of $H_{1,2}$ as

$$H_1 = -\sigma(x)D^2 - \tau(x)D + V_1(x), \quad H_2 = -\sigma(x)D^2 - \tau(x)D + V_2(x),$$

$$V_1(x) = \sigma(W^2-W') - \tau W, \quad V_2(x) = \sigma(W^2-W') - (\tau-\sigma')W + \kappa(\kappa\alpha)'. \quad (2.33)$$

As accustomed, we set $V_1 = V - E_0$, with $E_0 = 0$ and $W = -z'(x)/z(x)$ into the Riccati equation of V_1 we obtain the original BHE equation given below

$$xD^2z + (1+\alpha-\beta x-2x^2)Dz + ((\gamma-\alpha-2)x - {}^1\!/_2\,[\delta+\beta(1+\alpha)]z = 0. \quad (2.34)$$

Also for BHE, from results in [13](pages 203-206), Eq. (2.34) has 16 forms of solutions corresponding to the super-potential $BHEW_j;\ j = 1,2\dots,16,$ where

$$BHEW_j(x) = -(ln_{zj}(x))'. \quad (3.35)$$

The $zj\,(x)$ are the solutions of (2.34). one of which is

$$z_1(x) = N(\alpha,\beta,\gamma,\delta;x), \quad z_2(x) = x^{-\alpha}N(-\alpha,\beta,\gamma,\delta;x),$$

when α is not a relative integer. Here $N(\alpha,\beta,\gamma,\delta;x) = \sum_{n\geq 0}\frac{A_n}{(1+\alpha)n}\frac{x^n}{n!}$, where $A_0 = 1;\ A_1 = \frac{1}{2}\left(\delta+\beta(1+\alpha)\right)$ and

$$A_{n+2} = \left\{(n+1)\beta + \tfrac{1}{2}\left(\delta+\beta(1+\alpha)\right)\right\}A_{n+1} - (n-1)(n+1+\alpha)(\gamma-2-\alpha-2n)A_n$$

The associated partner potentials reads as, for $j = 1,2\dots,16,$

$$BHEV_{2j} = x(BHEW_j^2 + BHEW_j^1 + (\alpha-\beta x-2x^2)BHEW_j + \frac{{}^1\!/_2+2x^2+\alpha}{\sqrt{x}}\,. \quad (2.36)$$

Thus, the BH operator factorizes as, for $j = 1,2\dots16, \varepsilon = \pm 1,$

$$A = \varepsilon\sqrt{x}\left(D + BHEW_j(x)\right),$$

$$A = \sqrt{x}\left(-D + BHEW_j(x) + \frac{{}^1\!/_2+2x^2+\alpha}{\sqrt{x}}\right)$$

Ref

13. A. Ronveaux, (ed.) Heun's Differential equation (Oxford university press, Oxford, 1995)

Haven derive the solution of (2.34), these solutions are related to equation (1.4) with parameter relation as stated above. These relation gives the solution of the equation (1.3). a super-potentials and partner potentials are obtained. It should be emphasizes at this stage that the partner-potentials and Super-potentials are not shape invariant.

III. Remarks

Apart from the confinement potentials of the Schrödinger equation, this method generates other potentials such as the partner potentials and super-potentials which are not shape invariant.

Notes

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