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The Difficulties of Quantum Mechanics and its Investigations of Development

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THEDIFFICULTIESOFQUANTUMMECHANICSANDITSINVESTIGATIONSOFDEVELOPMENT

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The Difficulties of Quantum Mechanics and its Investigations of Development

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Abstract- When the states and properties of microscopic particles were described by linear SchrÖdinger equation the quantum mechanics had a lot of difficulties, which cause a centenary controversy in physics and have not a determinate conclusions till now. Thus we used a nonlinear SchrÖdinger equation to replace the linear SchrÖdinger equation and to depict and study in detail the states, properties and rules of motion of microscopic particles. Concretely speaking, we here investigated the properties and rules of wave-particle duality of microscopic particles and its stability, the invariances and conservation laws of motion of particles, the classical rule of motion, Hamiltonian principle of particle motion, corresponding Lagrangian and Hamilton equations for the microscopic particle, the mechanism and rules of particle collision in different nonlinear systems, the uncertainty relation of position and momentum of particles, the features of reflection and transmission of the particles at interfaces as well as the eigenvalue and eigenequations of the particles, and so on. The results obtained from these investigations show clearly that the microscopic particles described by the nonlinear SchrÖdinger equation have exactly a wave- particle duality, motions of its centre of mass meet not only classical equation of motion but also the Lagrangian and Hamilton equations, its mass, energy and momentum and angular momentum satisfy corresponding invariances and conservation laws, their collision has the feature of collision of classical particles, the uncertainty of position and momentum of the particles has a minimum, it is a bell-type solitary wave contained envelope and carrier wave, which but differs from not only KdV solitary wave but linear wave, its eigenvalues and eigenequations obey Lax principle and possess plenty of unusual peculiarities. The above natures and properties of the particles are different completely and in essence from those described the linear SchrÖdinger equation in the quantum mechanics. This shows that to use nonlinear SchrÖdinger equation changes the intrinsic natures of microscopic particles in quantum mechanics and makes the microscopic particles have the wave-particle duality. Thus the centenary controversy in physics could be solved by this idea and theory. At the same time, these results impel the quantum mechanics to develop toward the nonlinear theory, or speaking, nonlinear quantum theory is a necessary direction of development of quantum mechanics.

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I. The Nature and Properties of the Microscopic Particles Described by the Schrödinger Equation in the Quantum Mechanics

a) *Fundamental hypothesizes of quantum mechanics*

As known, quantum mechanics established by several great scientists, such as, Bohr, Born, SchrÖdinger and Heisenberg, etc., in the early 1900s [1-7] has served as the foundation of modern science in the history of physics and science and was extensively used to describe the states and properties of motion of microscopic particles. When the developmental history of quantum mechanics are remembered we find that its some fundamental hypothesizes and principles were disputed also about one century. What is this? This is due to plenty of difficulties and contradictions existed in quantum mechanics. An astonishing problem is that these disputations have not an unitized conclusion up to now[8-15]. This means that these difficulties are stern and crucial. After undergoing one centenary disputation we now ask what are the successes and shortcomings of quantum mechanics on earth? what are in turn the roots and reasons resulting in these difficulties and questions? Can or cannot these difficulties be solved ? How do we solve these difficulties? A series of problems need us to solve and are also worth to solve seriously, at present. In order to investigate and solve these problems we have to look back to these fundamental hypothesizes of quantum mechanics. These hypothesizes can be outlined as follows[1-12].

(1) The states of microscopic particles is represented by a vector of states $|\psi\rangle$ in Hilbert space, or a wave function $\psi(\vec{r},t)$ in coordinate representation. It reflects the properties of wave of motion of the microscopic particles and can be normalized (i.e. $\langle\psi|\psi\rangle=1$). If β is a constant number, then both $|\psi\rangle$ and $\beta|\psi\rangle$ describe a same state.

(2) A mechanical quantity of microscopic particle, such as, coordinate x, momentum p and energy E, etc., is represented by an operator in Hibert space. An observable mechanical quantity corresponds to a Hermitean operator, the eigenvectors of its state constructs a basic vector in the Hibert space. This shows that the values

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of the physical quantity are just eigenvalues of these operators. The eigenvalues of Hermitean operator are some real numbers. The eigenvectors corresponding to different eigenvalues are orthogonal with each other. A common eigenstates of commutable Hermitean operators are constituted as an orthogonal and complete set, $\{\psi_L\}$. Any vector of state, $\psi(\vec{r},t)$ may be expanded by it into a series as follows:

$$\psi(\vec{r},t)(=\sum_L C_L\psi_L(\vec{r},t), \quad \text{or} \quad |\psi(\vec{r},t)\rangle = \sum_L \langle\psi_L|\psi\rangle|\psi_L\rangle \tag{1}$$

where $C_L = \langle\psi_L|\psi\rangle$ is the wave function in representation L. If the spectrum of L is continuous, then the summation in Eq.(1) should be replaced by an integral: $\int dL\cdots$. Equation(1) can be regarded as a projection of wave function $\psi(\vec{r},t)$ of the microscopic particle system in the representation, The Eq.(1) is the foundation of transformation between different representations and of measurement of physical quantities in quantum mechanics. In the quantum state described by $\psi(\vec{r},t)$, the probability getting the L' in the measurement of L is $|C_L|^2$ = $|\psi_{L'}|^2 = |\langle\psi_{L'}|\psi\rangle|^2$ in the case of discrete spectrum, the probability is $|\langle\psi_{L'}|\psi\rangle|^2 dL$ in the case of continuous spectrum. In a single measurement of any a mechanical quantity, only one of the eigenvalues of corresponding operator can be obtained, the system is then said to be in the eigenstate belonging to this eigenvalue.

The two hypothesizes are the most important assumptions and stipulate how the states of the microscopic particles are represented in quantum mechanics

(3) A mechanical quantity in an arbitrary state $|\psi\rangle$ can only take an average value by

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle/\langle\psi\|\psi\rangle, \text{ or } \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle \tag{2}$$

when Ψ is normalized, i.e., possible values of the physical quantity, A, may be obtained by calculating this average. In order to find out these possible values, a wave function of states must be firstly known. Condition for determinate value the quantity, A, has in this state is $\overline{\langle\Delta A\rangle^2}$ =0. Thus we can obtain the eigenequation of the operator Â to be as follows

$$\hat{A}\psi_L = \hat{A}\psi_L \tag{3}$$

From this equation we can determine the spectrum of eigenvalues of the operator Â and its corresponding eigenfunction Ψ_L. The eigenvalues of Â are possible values observed from experiment for this physical quantity. All possible values of Â in any other states are nothing but its eigenvalues in its own eigenstates. This hypothesis reflects the statistical nature in the description of microscopic particle in quantum mechanics.

(4) Hilbert space is a linear one and the mechanical quantity corresponds to a linear operator. Then corresponding eigenvector of state, or wave function, satisfies the linear superposition principle, i.e., if two states, $|\psi_1\rangle$ and $|\Psi_2\rangle$ are ones of a particle, then their linear superposition:

$$|\psi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle, \tag{4}$$

describes also the state of the particle, where C_1 and C_2 are two arbitrary constants,. The linear superposition principle of quantum state is determined by the linear characteristics of the operators and this is why the quantum theory is referred to as linear quantum mechanics. However, it is noteworthy to point out that such a superposition is different from that of classical wave, it does not result in changes in probability and intensity of particle.

(5) Correspondence principle. If the classical mechanical quantities, A and B, satisfy the Poisson brackets:

$$\{A,B\} = \sum_n\left(\frac{\partial A}{\partial q_n}\frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial q_n}\right)$$

where q_n and p_n are generalized coordinate and momentum in classical system, respectively, then the corresponding operators Â and $\hat{B}$ in quantum mechanics satisfy the following commutation relations:

$$[A,B]=(AB-BA)=-ih\{A,B\} \tag{5}$$

where i= $\sqrt{-1}$ and h is the Planck constant. If A and B are substituted by q_n and p_n respectively, we have:

$$[\hat{p}_n,\hat{q}_m]=-ih\delta_{nm}, \quad [\hat{p}_n,\hat{p}_m]=0$$

This reflects the fact that the values taken for physical quantity are quantized. Based on this fundamental hypothesis, the Heisenberg uncertainty relation can be obtained as follows:

$$\overline{(\Delta A)^2}\,\overline{(\Delta B)^2} \geq \frac{|C|^2}{4} \tag{6}$$

where $[\hat{A},\hat{B}]=i\hat{C}$ and $\Delta A = <\hat{A} - <\hat{A}>>$. For the coordinate and momentum operators, the Heisenberg uncertainty relation takes the usual form:

$$\Delta x \Delta p \geq \hbar/2$$

(6) The time dependence of a quantum state $|\psi\rangle$ of a microscopic particle is determined by the following SchrÖdinger Equation:

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}|\psi\rangle=\hat{H}|\psi\rangle \tag{7a}$$

or

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+\hat{V}(\vec{r},t)\psi \tag{7b}$$

where $\hat{T}=\hbar^2\nabla^2/2m$ is the kinetic energy operator, $\hat{V}(\vec{r},t)$ is the externally applied potential operator, m is the mass of particles, $\psi(\vec{r},t)$ is a wave function describing the states of particles, $\vec{r}$ is the coordinate or position of the particle, and t is the time. This is a fundamental dynamic equation of the microscopic particle in time-space. Corresponding Hamiltonian operator of the systems, H, is assumed to give by

$$\hat{\mathrm{H}}=\hat{T}+\hat{V}=-\frac{\hbar^2}{2m}\nabla^2+\hat{V} \tag{8}$$

This fundamental hypothesis amounts to assume that the independence of Hamiltonian operator of the systems with wave function of states of particles, and the SchrÖdinger equation (7) is a linear one for the wave function $\psi(\vec{r},t)$ in quantum mechanics. This is an another of reasons to be referred to it as linear quantum mechanics. This hypothesis shows that the states and properties of the systems or microscopic particles at any time are determined by the Hamiltonian of the systems, or nonlinear SchrÖdinger equation (7).

If the state vector of the system at time t_0 is $|\psi(t_0)\rangle$ then the mechanical quantity and wave vector at time t are associated with those at time t_0 by a unitary motive operator $\hat{U}(t,t_0)$, namely

$$|\psi(t_0)\rangle=\hat{U}(t,t_0)\,|\psi(t_0)\rangle$$

here $\hat{U}(t,t_0)=1$, $\hat{U}^+\hat{U}=\hat{U}\hat{U}^+=I$. If let $\hat{U}(t,0)=\hat{U}(t)$ then the equation of motion becomes

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\hat{U}(t)=\hat{H}\hat{U}(t) \tag{9}$$

When H does not depend explicitly on an time t and $\hat{U}(t)=\exp[i\hat{H}t/\hbar]$, If H is an explicit function of time t, we then have

$$\hat{U}(t) = 1 + \frac{1}{i\hbar}\int_0^t dt_1 \hat{H}(t_1) + \frac{1}{(i\hbar)^2}\int_0^t dt_1 \hat{H}(t_1)\int_0^{t_1} dt_2 \hat{H}(t_2) + \cdots \quad (10)$$

This equation shows a causality relation of the microscopic law of motion. Obviously, there is an important assumption in quantum mechanics, i.e., the Hamiltonian operator of the system is independent on the wave function of state of the particle. This is a fundamental assumption in quantum mechanics.

(7) Principle of full-identity. No new physical state occurs when same two particles exchange mutually their positions in the systems, in other words, it satisfies $p_{kj}|\psi\rangle = \lambda|\psi\rangle$, where $\hat{p}_{kj}$ is a exchange operator. The wave function for an system consisting of identical particles must be either symmetric, ψ_s, $(\lambda = +1)$ or antisymmetric, ψ_a, $(\lambda = -1)$, and this property remains invariant with time and is determined only by the nature of the particle. The wave function of a boson particle is symmetric and that of fermion is antisymmetric.

(8) Assumption of measurement of physical quantities in quantum systems. There was no assumption made about measurement of physical quantities at the beginning of quantum mechanics. It was introduced later to make the quantum mechanics complete. The foundation of this hypothesis is the equations (1) and (3). However, this is a nontrivial and controversial topic which has been a focus of scientific debate. This problem will not be discussed here. Interested reader can refer to some literatures.

In one word, these hypothesizes stipulate the representation forms of states and mechanical quantities and Hamiltonian of microscopic particles and the relationships satisfied by them. Concretely speaking, the states of microscopic particle is represented by a wave function, which satisfies the linear SchrÖdinger equation (7) and linear superposition principle in Eq.(4) and normalization condition, the square of its absolute value represents the probability of the particle at certain point in space, and is used to indicate the corpuscle feature of microscopic particle. The mechanical quantities of microscopic particle are represented by the operators, which satisfy the commutation relation in Eq.(5) and uncertainty relation in Eq.(6), their values are denoted by some possible average values or eigenvalues of corresponding operators in any states or eigenstates, respectively. The Hamiltonian operator of the systems is independent on the wave function of state of the particles and denoted only by the sum of kinetic and potential energy operators in Eq.(8), which determine the states of the particles by virtue of Eq.(7). These are just the quintessence and creams of quantum mechanics.

b) The Successes and Difficulties as Well as Disputations in Quantum Mechanics

On the basis of several fundamental hypothesises mentioned above, Heisenberg, SchrÖdinger, Bohn, Dirac, etc[1-7]., have founded up the theory of quantum mechanics which describes the law and properties of motion of the microscopic particles. This theory states that once the externally applied potential fields and the conditions at the initial states for the particle are given, the states and features of the particles at any time later and any position can be easily determined by linear SchrÖdinger equations (7)-(8) in the case of nonrelativistic motion. The quantum states and their occupations of electronic systems, atom, molecule, and the band structure of solid state matter, and any given atomic configuration are completely determined by the above equations. Macroscopic behaviors of the systems, such as, mechanical, electrical and optical properties may be also determined by these equations. This theory can also describes the properties of microscopic particle systems in the presence of external electromagnetic-field, optical and acoustic waves, and thermal radiation. Therefore, to a certain degree, the linear SchrÖdinger equation describes the law of motion of microscopic particles of which all physical systems are composed. It is the foundation and pillar of modern science.

One of the great creative point of quantum mechanics is just to forsake completely traditional representations of particles in classical physics, in which the wave functions or vectors are used to describe the state of microscopic particles and the operators are introduced to represent the mechanical quantity of the particles. their applied results show that this is available. Thus quantum mechanics had great achievements in descriptions of motion of microscopic particles[1-12], such as, electron, phonon, photon, exciton, atom, molecule, atomic nucleus and elementary particles, and in predictions of properties of matter based on the motion of these quasi-particles. For example, energy spectra of atoms (such as hydrogen atom, helium atom) and molecules (such as hydrogen molecule) and compounds, the electric, magnetic and optical properties of atoms and condensed matters can be calculated based on the quantum mechanics, and these calculated results are in basic agreement with experimental measurements. Thus the establishment of the theory of quantum mechanics has revolutionized not only physics but also many other science branches, such as, chemistry, astronomy, biology, etc., and at the same time, created many new branches of science, for example, quantum statistics, quantum field theory, quantum electronics, quantum chemistry, quantum biology, quantum optics, etc.. One of the great successes of quantum mechanics is the explanation of the fine energy-spectra of hydrogen atom, helium atom and hydrogen molecule. The energy

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spectra predicted by quantum mechanics for these simple atoms and molecules are completely in agreement with experimental data. Furthermore, modern experiments have demonstrated that the results of Lamb shift and superfine structure of hydrogen atom and anomalous magnetic moment of electron predicted by the theory of quantum electrodynamics are coincident with experimental data within an order of magnitude of 10^{-5}. It is therefore believed that the quantum electrodynamics is one of successful theories in modern physics.

Despite the great successes of quantum mechanics, it nevertheless encountered some problems and difficulties[7-15]. In order to overcome these difficulties, Einstein had disputed with Bohr and others for the whole of their life, but very sorry that these difficulties remained still up to now. The difficulties of quantum mechanics are well known and have been reviewed by many scientists. When one of founders of the quantum mechanics, Dirac, visited to Australia in 1975, he give a speech on development of quantum mechanics in New South Wales University. During his talk, Dirac mentioned that at the time, great difficulties existed in the quantum mechanical theory. One of the difficulties referred to by Dirac was about an accurate theory for interaction between charged particles and an electromagnetic field. If the charge of a particle is considered as concentrated at one point, we shall find that the energy of a point charge is infinite. This problem had puzzled physicists for more than 40 years. Even after the establishment of renormalization theory, no actual progress had been made……..Therefore, Dirac concluded his talk by marking the following statements: It is because of these difficulties, I believe that the foundation for the quantum mechanics has not been correctly laid down. I cannot accept that the present foundation of the quantum mechanics is completely correct."

However, have what difficulties in the quantum mechanics on earth evoked these contentions and raised doubts about the theory among physicists in the world? It was generally accepted that the fundamentals of the quantum mechanics consist of Heisenberg matrix mechanics, SchrÖdinger wave mechanics, Born's explanation of probability for the wave function and Heisenberg uncertainty principle, etc.. These were also the focal points of debate and controversy[12-15]. The debate was about how to interpret the quantum mechanics. Some of the questions being debated concern the interpretation of the wave-corpuscle duality, probability explanation of wave function, Heisenberg uncertainty principle, Bohr complementary (corresponding) principle, the quantum mechanics which describes whether the law of motion for a single particle or for an assembly consisting of a great number of particles. The following is a brief summary of issues being debated and disputed in quantum mechanics. (1) First, the correctness and completeness of the quantum mechanics were challenged. Is quantum mechanics correct? Is it complete and self-consistent? Can the properties of microscopic particle systems be completely described by the quantum mechanics? or speaking, Whether the SchrÖdinger equation (7) can describe completely the states and properties of microscopic particles in a realistically physical systems. Meanwhile, the quantum mechanics in principle can describe the physical systems with many-body and many particles, but it is not easy to solve such a system and plenty of approximations must be used to obtain some approximate solutions. In doing this, a lot of true and important phenomena and effects of the systems are artificially neglected or thrown away. This is very sorry for physics. Do the fundamental hypothesizes contradict each other? , whether the hypothesis for the independence of Hamiltonian operator of the systems on wave function of states of particles in Eq.(8) is correct.

(1) Is the quantum mechanics a dynamic or a statistical theory? Does it describe the motion of a single particle or a system of particles? The dynamical equation (7) seems as equation of a single particle, but its mechanical quantities are determined based on the concepts of probability and statistical average. This caused confusion about the nature of the theory itself.

(2) How to describe the wave-corpuscle duality of microscopic particles? What is the nature of a particle defined on the hypotheses of the quantum mechanics? The wave-corpuscle duality is established by the de Broglie relations. Can the statistical interpretation of wave function correctly describe such a property? There are also difficulties in using wave package to represent the corpuscle nature of microscopic particles. Thus wave-corpuscle duality was a major challenge to the quantum mechanics.

(3) Was the uncertainty principle due to the intrinsic properties of microscopic particle or a result of uncontrollable interaction between the measuring instruments and the system being measured?

(4) A particle appears in space in the form of a wave, and it has certain probability to be at a certain location. However, it is always a whole particle, rather than a fraction of it, being detected in a measurement. How can this be interpreted? Is the explanation of this problem based on wave package contraction in the measurement correct?

Since these are important issues concerning the fundamental hypotheses of the quantum mechanics, many scientists were involved in the debate. Unfortunately, after being debated for almost a century, there are still no definite answers to most of these questions. While many enjoyed the successes of the quantum mechanics, other were wondering whether the quantum mechanics is the right theory of real microscopic physical world, the microscopic particle has or has not wave-corpuscle duality on earth, because of the problems and difficulties it encountered. Modern quantum mechanics was born in 1920s, but these problems were always the topics of heated disputes among different views and different schools till now. It was quite exceptional in the history of physics that so many prominent physicists from different institutions were involved and the scope of the debate was so wide. The group in Copenhagen School headed by Bohr represented the view of the main stream in these discussions. In as early as 1920s, heated disputes on statistical explanation and incompleteness of wave function arose ever between Bohr and other physicists, including Einstein, de Broglie, SchrÖdinger, Lorentz, etc, who has doubted and continuously criticized Bohr's interpretations. This results in a life-long disputations between Bohr and Einstein, which is unprecedented and went through three stages.

The first stage was during the period from 1924 to 1927, when the theory of quantum mechanics had just been founded. Einstein proceeded from his own philosophical belief and his scientific goal for an exact description of causality in the physical world, and expressed his extreme unhappiness with the probability interpretation of the quantum mechanics. In a letter to Born on December 4, 1926, He said that " Quantum mechanics is certainly imposing. But an internal sound tells me that it is not the real thing (der Wahre Jakob). The theory says a lot, but it does not bring us any closer to the secret of the "Old one." I, at any rate, am convinced that He is not playing at dice."

The second stage was from 1927 to 1930. After Bohr had put forward his complementary principle and had established his interpretation as main stream interpretation, Einstein was extremely unhappy. His main criticism was directed at the Heisenberg uncertainty relation on which Bohr's complementary principle was based.

The third stage was from 1930 until the death of Einstein. The disputation during this period is reflected in the debate between Einstein and Bohr over the "EPR" paradox proposed by Einstein together with Podolsky and Rosen[13-15]. This paradox concerned the fundamental problem of the quantum mechanics, i.e. whether it satisfied the deterministic localized theory and the microscopic causality. The disputation to this problem maintains a longer period.

To summarize, the long-dated disputation between Bohr and Einstein schools was focused on three problems: (1)Einstein upheld to belief that the microscopic world is no different from the macroscopic one, particles in microscopic world are matters and they exist regardless of the methods of measurements, any theoretical description to it should in principle be determinant. (2) Einstein always considered that the theory of the quantum mechanics was not an ultimate and complete one. He believed that quantum mechanics is similar to the classical optics. Both of them are correct theories based on statistical laws, i.e., when the probability $|\psi(\vec{r},t)|^2$ of a particle at moment t and location r is known,, an average value of observable quantity can be obtained using statistical method and then compared with the experimental results. However, understanding to processes involving single particle was not satisfactory. Hence, $\psi(\vec{r},t)$ has not give everything about a microscopic particle system, and the statistical interpretation cannot be ultimate and complete. (3)The third issue concerns the physical interpretation of the quantum mechanics. Einstein was not impressed with the attempt to completely describing some single processes using quantum mechanics, which he made very clear in a speech at the fifth Selway International Meeting of physics. In an article, "Physics and Reality", published in 1936 in the Journal of the Franklin Institute, Einstein again mentioned that what the wave function $\psi(\vec{r},t)$ describes can only be a many-particle system, or an assemble in terms of statistical mechanics, and under no circumstances, the wave function can describe the state of a single particle. Einstein also believed that the uncertainty relation is a result of incompleteness of the description of a particle by $\psi(\vec{r},t)$, because a complete theory should give precise values for all observable quantities. Einstein also did not accept the statistical interpretation because he did not believe that an electron possess free will. Thus, Einstein's criticism against the quantum mechanics was not directed towards the mathematic formalism of the quantum mechanics, but to its fundamental hypotheses and its physical interpretation. He considered that this is due to the incomplete understanding of the microscopic objects. Moreover, the contradiction between the theory of relativity and the fundamental of the quantum mechanics was also a central point of disputation. Einstein made effort to unite the theory of relativity and quantum mechanics, and attempted to interpret the atomic structure using field theory. The disagreements on several fundamental issues of the quantum mechanics by Einstein and Bohr and their followers were deep rooted and worth further study. This brief review on the disputes between the two great physicists given above should be useful to our understanding on the nature and problems of the quantum mechanics.

c) The Roots Produced these Difficulties in Quantum Mechanics

From the above description we see that the difficulties and questions of quantum mechanics exist hugely and extensively, the disputations on it between Einstein and Bohr are drastic, their branches are also considerable. However, what are the reasons and roots generating these difficulties on earth? This is just a key problem on quantum mechanism and its development. Only if the roots are sought we can solve these difficulties and improve and develop quantum mechanics. For this purpose we have to look carefully at the essences and significances of the above hypothesizes of quantum mechanics.

As known, the linear SchrÖdinger equation (7) is a wave equation describing the properties and rules of motion of microscopic particles. In the light of this theory we can find the solutions of the equation[8-12] and know thus the states and properties of the particles, if only the externally applied potential is known, whether or no complicated systems and interactions among the particles. This is very simple process finding solutions. However, for all externally applied potentials, the solutions of the equation are always a linear or dispersive wave, for example, at $V(\vec{r},t)=0$, its solution is a plane wave as follows:

$$\psi(\vec{r},t)=A'\exp[i(\vec{k}\cdot\vec{r}-\omega t)] \tag{11}$$

where *k* is the wavevector of the wave, ω is its frequency, and *A'* is its amplitude. This solution denotes the state of a freely moving microscopic particle with an eigenenergy of

$$E=\frac{p^2}{2m}=\frac{1}{2m}(p_x^2+p_y^2+p_z^2),(-\infty<p_x,p_y,p_y<\infty) \tag{12}$$

This is a continuous spectrum. It states that the probability of the particle to appear at any point in the space is the same, thus the microscopic particle propagates and distributes freely in a wave in total space, this means it cannot be localized and have nothing about corpuscle feature.

If the free particle is artificially confined in a small finite space, such as, a rectangular box of dimension *a*, *b* and *c*, then the solution of Eq.(7) is a standing waves as follows

$$\psi(x,y,z,t)=A\sin\left(\frac{n_1\pi x}{a}\right)\sin\left(\frac{n_2\pi y}{b}\right)\sin\left(\frac{n_3\pi z}{c}\right)e^{-iEt/\hbar} \tag{13}$$

In such a case, this microscopic particle is still localized, it appears still at each point in the box with a determinant probability. Difference from Eq.(12) is that the eigenenergy of the particle in this case is quantized as follows

$$E=\frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{a^2}+\frac{n_2^2}{b^2}+\frac{n_3^2}{c^2}\right), \tag{14}$$

the corresponding momentum is also quantized. This means that the wave feature of microscopic particle has be not changed because of the variation of itself boundary condition.

If the potential field is further varied, for example, the microscopic particle is subject to a conservative time-independent field, $U(\vec{r},t)=U(\vec{r})\neq 0$, then the microscopic particle satisfies the time-independent linear SchrÖdinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi'+V(\vec{r})\psi'=E\psi' \tag{15}$$

where

$$\psi=\psi'(\vec{r})e^{-iEt/\hbar} \tag{16}$$

When $V=\vec{F}\cdot\vec{r}$, here $\vec{F}$ is a constant field, such as, a one dimensional uniform electric field $V(x)=-e\varepsilon x$, the solution of Eq.(15) is $\psi'=A\sqrt{\xi}H_{1/2}^{(1)}\left(\frac{2}{3}\xi^{3/2}\right),\left(\xi=\frac{x}{l}+\lambda\right)$, where $H^{(1)}(x)$ is the first kind Hankel function, *A* is a normalized constant, *l* is the characteristic length and λ is a dimensionless quantity. The solution is still a dispersed wave. When $\xi\to\infty$, it approaches $\psi'(\xi)=A'\xi^{-1/4}e^{-2\xi^{3/2}/3}$ to be a damped wave.

If $V(x)=Fx^2$, the eigenenergy and eigenwave function are $\psi'(x)=N_n e^{-a^2x^2/2}H_n(\alpha x)$, $E_n=(n+\frac{1}{2})\hbar\omega$, (n=0,1,2,…), respectively, here $H_n(\alpha x)$ is the Hermite polynomial. The solution obviously has a decaying feature, and so on.

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The above practical examples show clearly that the solutions of linear SchrÖdinger equation (7) is only a wave and always have not the corpuscle feature whether or no changes of externally applied potentials. In such a case Born had to introduce the probability of $|\psi(\vec{r},t)|^2$ to represent the particulate feature of the particle, but which together with the hypothesis of average values of mechanical quantity results in turn in a controversy on the quantum mechanics which describes whether the law of motion for a single particle or for an assembly consisting of a great number of particles. These results indicate clearly that the essence of microscopic particles described by the quantum mechanics based on the above hypothesizes is just a wave, The wave feature of microscopic particles is fully incompatible and contradictory with the traditional concept of particles[7-9] and cannot be changed by the externally applied potentials. Just so, a series of difficulties and questions of quantum mechanics occur subsequently, such as, the uncertainty relation between the position and momentum, the probability interpretation of wave function, the concept of statistical average of the mechanical quantities as mentioned above, and so on. This shows clearly that these difficulties and contradictions in quantum mechanics are the intrinsic and inherent, or speaking cannot overcome.

Very obviously, the roots or reason generating these difficulties and contradictions in quantum mechanics should focus on the fundamental hypothesizes of dynamic equation of microscopic particles in Eq. (7) and Hamiltonian of the systems in Eq.(8)[12-15]. They are too simple to useful to represent a realistic physical systems. As far as Hamiltonian operator in Eq.(8) and dynamic equation (7) are concerned, they consist only of externally applied potential term, $V(\vec{r})$, and kinetic energy term, $(h^2/2m)\nabla^2 = \hat{p}^2/2m$, of particles. The former is not related to the state or wave function of the particle, so it cannot change the natures, only can vary the shapes and outlines of the particle, such as amplitude and velocity. The nature of the particles are mainly determined by the kinetic energy term in Eqs.(7)-(8). However, the effect of the kinetic energy term is a dispersive effect and it makes the particle be permanently in motion. The dispersive effect cannot be balanced and suppressed by an external potential field $V(\vec{r},t)$ in Eq.(7). Thus the microscopic particle has only a wave feature in quantum mechanics. Therefore, the root generating these difficulties, contradictions and disputations are just the simplicity of quantum mechanics, or speaking, the dynamic equation (7) and Hamiltonian in Eq.(8) of the particles composed only of externally applied potential and kinetic energy terms are the basic reasons giving rise to these difficulties, contradictions and disputations in quantum mechanics. In fact, in quantum mechanical calculation including the systems of many-particles and many-bodies we separate always the studied particle from other particles, ignore further the real and complicated interactions including some nonlinear interactions among the particles or between the particle and background field, such as, lattices, and replace artificially these complicated interactions among these particles by an averagely external applied-potential unrelated to the wave function of the particle in virtue of various approximation methods in quantum mechanics. Or else, quantum mechanics cannot use to investigate these quantum systems at all. However, plenty of basic natures of the microscopic particles have been blotted out and denied in this calculation. In such a case the microscopic particles cannot be also localized. Therefore, all microscopic particles have only wave or dispersive feature, not corpuscle nature in quantum mechanics whether the particle is in what systems or accepted how much interactions. The character of quantum mechanics is intrinsic and inherent and cannot be permanently changed. This is just the essence of quantum mechanics. It is also the roots that the microscopic particles have only a wave feature, not the corpuscle feature at all.

II. The Considerable and Essential Changes of Nature and Properties of the Microscopic Particles Depicted by the Nonlinear Schrödinger Equation

a) *The establishments of nonlinear SchrÖdinger equation with nonlinear interaction*

i. *The nonlinear interaction between the particles and establishment of nonlinear SchrÖdinger equation*

From the above investigation we sought the roots that the microscopic particles have only a wave feature, not the corpuscle feature, or the reasons generating plenty of difficulties and centenary disputations in quantum mechanics. It is just due to simple dynamic equation (7) and Hamiltonian operator in Eg.(8). At the same time, we know that the wave feature of microscopic particle cannot be permanently changed, the difficulties and constructions of quantum mechanics are intrinsic and inherent, and cannot be overcome and solved in itself framework. If these difficulties and constructions want solve, then we must break through some basic hypothesizes and fundamental framework of quantum mechanics, such as, independence of Hamiltonian operator of the systems on the states of particles, and seek a interaction, which can inhibit and suppress the dispersive effect of kinetic energy term in Eqs.(7)-(8), to make the particles be localized in virtue of taking into account these interactions, especially the nonlinear interactions, among the particles or between the particle and background field[16-20]. The above roots and reasons arising from the difficulties in quantum mechanics awakens and evokes also us to rivet one's attention on the interaction related to the states of microscopic particles among the particles or between the particle and background field. I expect that the natures of the microscopic particles can be changed hugely, when these nonlinear interactions are considered in dynamic equation of the particles and Hamiltonian operator of the systems[18-22].

In accordance with this idea we study the dynamic features of microscopic particles in nonlinear interaction systems. As known, the interaction among the particles or between the particle and background field is, in general, described by virtue of a model of interaction of two bodies. When the interaction between the two bodies is considered, their dynamical equations of the two microscopic particles can be often represented, respectively, by

$$i\hbar\frac{\partial}{\partial t}\phi=-\frac{\hbar^2}{2m}\nabla^2\phi+V(x,t)+\chi\phi\frac{\partial F}{\partial x} \tag{17}$$

and

$$\frac{\partial^2 F}{\partial t^2}-v_0^2\frac{\partial^2 F}{\partial x^2}=-\chi\frac{\partial}{\partial x}|\phi|^2, \tag{18}$$

where ϕ denotes the state of a studied microscopic particle, equation (17) describes then the dynamics of the microscopic particle having a coupling interaction with other particle or field. F is the state of a background field, such as a lattice or another particle, such as a phonon, equation (18) expresses its dynamics, or , a forced vibration of the background field or another particle with velocity v_0 due to the interaction arising from the changes of state of the studied microscopic particle studied[18-22]. χ represents just the coupling interaction effect between them. This coupling can change the states and natures of both the studied particle and other particle. This implies that we replaced not the practical interaction between them by an averagely external-applied potential, but treated and described the dynamics of two particles in same way in virtue of the coupling interaction between them. This is a new investigated idea and method, which differs completely from of those of quantum mechanics. From Eq.(18) we can obtain

$$\frac{\partial F}{\partial x}=-\frac{\chi}{v^2-v_0^2}|\phi|^2 \tag{19}$$

Inserting Eq.(19) into Eq.(17) yields

$$i\hbar\frac{\partial}{\partial t}\phi=-\frac{\hbar^2}{2m}\nabla^2\phi+V(x,t)-b|\phi|^2\phi \tag{20}$$

where $b=\frac{\chi^2}{v^2-v_0^2}$ is a nonlinear interaction coefficient. This equation is just so-called nonlinear SchrÖdinger equation relative to the linear SchrÖdinger equation (7). It is just the new dynamic equation of the microscopic particle in the system with nonlinear interactions among the particles or between the particle and a background field. we see clearly from Eq.(20) that this coupling among the particles or between the particle and background field result in a nonlinear interaction, $b|\phi|^2\phi$, related to the wave function of the particles to occur. This shows clearly the nonlinear interaction comes from the interactions among the particles or between the particles and background field In such a case the states of the microscopic particles are no longer described by the linear SchrÖdinger equation (7), but by the nonlinear SchrÖdinger equation (20).

In general, the nonlinear interactions can be produced by the following three mechanisms by means of the interaction among particles and between the medium and the particles[18-22]. In the first mechanism, the attractive effect is due to interactions between the microscopic particle and other particles. This is called a self-interaction. A familiar example is the Bose-Einstein condensation mechanism of microscopic particles because of an attraction among the Bose particles. Here the mechanism is referred to as self-condensation. In the second mechanism, the medium has itself anomalous dispersion effect (*i.e.* $k''=\partial^2 k/\partial\omega^2\big|_{\omega_0<0}$) and nonlinear features resulting from its anisotropy and nonuniformity. Motion of the microscopic particles in the system are modulated by these nonlinear effects. This mechanism is called self-focusing. The third mechanism is called self-trapping. It is produced by interaction between the microscopic particles and background, such as, lattice or medium as mentioned above.

If other complicated interactions and damping effect of medium are considered, then equation (20) should be replaced by the following nonlinear SchrÖdinger equations[18-19]:

$$i\hbar\frac{\partial\phi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\phi\pm b|\phi|^2\phi+V(\vec{r},t)\phi+A(\phi), \tag{21}$$

or

$$\mu\frac{\partial\phi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\phi\pm b|\phi|^2\phi+V(\vec{r},t)\phi+A(\phi) \tag{22}$$

where A(ϕ) represents the complicated interactions among the particles related to wave function $\phi(\vec{r},t)$ of the particles, μ is a complex number, equation (22) is often used to depict the motion of microscopic particles in damping medium. The wave function, $\phi(\vec{r},t)$, can be written as

$$\phi(\vec{r},t)=\varphi(\vec{r},t)e^{i\theta(\vec{r},t)} \tag{23}$$

where both the amplitude $\varphi(\vec{r},t)$ and phase $\theta(\vec{r},t)$ of the wave function are functions of space and time.

We here should point out that physicists and mathematicians derived plenty of nonlinear SchrÖdinger equation with different forms from different systems and conditions[20-29], at present, but we obtain the nonlinear SchrÖdinger equations (20)-(22) from features of motion and interaction of microscopic particles. They are some special nonlinear SchrÖdinger equations, have the symmetry of time-space, thus are quite appropriate to the microscopic particles[18-22].

ii. *The effects of nonlinear interaction on the states of particles*

In Eqs.(21)-(22) there are also the nonlinear interaction term, $b|\phi|^2\phi$. Since it relates always to the states of particles, then the nature of the particles must be changed under its action. We expect that the nonlinear interaction can balance and suppress the dispersion effect of the kinetic term in Eq. (20) to make the particles be localized[18-19,21-22], eventually makes the particle becomes a soliton with wave-corpuscle duality. Why? This is due to the fact that the interaction can distort and collapse the dispersive wave, thus can obstruct and suppress the dispersive effect of kinetic energy and make the microscopic particles to be eventually localized.

To see clearly this, we now study carefully the motion of water wave in sea. When a water wave approaches the beach, its shape variants gradually from a sinusoidal cross section to triangular, and eventually a crest which moves faster than the rest. This is a result of the nonlinear nature of wave. As the water wave approaches the beach the wave will be broken up due to the fact that the nonlinear interaction is enhanced. Since the speed of wave propagation depends on the height of the wave in such a case, so, this is a nonlinear phenomenon. If the phase velocity of the wave, v_c, depends weakly on the height of the wave, *h*, then $v_c=\frac{\omega}{k}=v_{co}+\Theta_1 h$, where $\Theta_1=\frac{\partial v_c}{\partial h}\big|_{h=h_0}$, h_0 is the average height of the wave surface, v_{co} is the linear part of the phase velocity of the wave, Θ_1 is a coefficient denoting the nonlinear effect. Therefore, the nonlinear interaction results in changes in both form and velocity of waves. This is the same for the dispersion effect, but their mechanism and rules are different. When the dispersive effect is weak, the velocity of a wave is denoted by $v_c=\frac{\omega}{k}=v'_{co}+\Theta_2 k^2$, where v_{co} is a dispersionless phase velocity, $\Theta_2=\frac{\partial^2 v_c}{\partial k^2}\big|_{k=k_0}$ is the coefficient of the dispersion feature of the wave. Generally speaking, the lowest order dispersion occurring in the phase velocity is proportional to k^2, and the term proportional to *k* gives rise to the dissipation effect. If the two effects act simultaneously on a wave, then it is necessary to change the nature of the wave.

To further explore the effects of nonlinear interaction on the behaviors of microscopic particles, we consider a simple motion as follows

$$\phi_e+\phi\phi_x=0 \tag{24}$$

where $\phi\phi_x$ is a nonlinear interaction. There is no dispersive term in this equation. It is easy to verify that $\phi=\Phi'(x-\phi t)$ satisfies Eq.(24). This solution indicates that as time elapses, the front side of wave gets steeper and steeper, until it becomes triple-valued function of *x* due to the nonlinear interaction, which does not occur for a general wave equation. This is a deformation effect of wave resulting from the nonlinear interaction. If let $\phi=\Phi'=\cos\pi x$ at $t=0$, then at $x=0.5$ and $t=\pi^{-1},\phi=0$ and $\phi_x=\infty$. The time $t_B=\pi^{-1}$ at which the wave becomes very steep is called destroyed period of the wave. However, the collapsing phenomenon can be suppressed by adding a dispersion term ϕ_{xxx} as in the KdV equation[13-14]. Then, the system has a stable soliton, $\mathrm{sec}h^2(X)$, in such a case. Therefore, a stable soliton, or a localization of particle can occur only if the nonlinear interaction and dispersive effect exist simultaneously in the system, and can be balanced and canceled each other. Otherwise, the particle cannot be localized, and a stable soliton cannot be formed.

However, if ϕ_{xxx} is replaced by ϕ_{xx}, then Eq. (24) becomes

$$\phi_t + \phi\phi_x = v\phi_{xx}, (v > 0) \tag{25}$$

This is the Burgur's equation. In such a case, the term $v\phi_{xx}$ cannot suppress the collapse of the wave, arising from the nonlinear interaction $\phi\phi_{xx}$. Therefore, the wave is damped. In fact, utilizing the Cole-Hopf transformation $\phi = -2\gamma \frac{d}{dx}(\log\psi')$, equation (48) becomes $\frac{\partial\psi'}{\partial t} = v\frac{\partial^2\psi'}{\partial x^2}$. This is a linear equation of heat conduction or diffusion equation, which has a damping solution. Therefore, the Burgur's equation (25) is not a equation with soliton solution[13-17].

This example tells us that the deformational effect of nonlinearity on the wave can suppress its dispersive effect, thus a soliton solution of dynamic equations can then occur in such a case [18-19]. The nonlinear term in nonlinear SchrÖdinger equation (20) sharpens the peak, while its dispersion term has the tendency to leave it off, thus Eq.(20) has a soliton solution, then the microscopic particle described by the nonlinear SchrÖdinger equation (20) can be localized in such a case. This example also verifies sufficiently that a stable soliton or localization of particle cannot occur in the absence of nonlinear interaction and dispersive effect or weak nonlinear interaction relative to the dispersive effect in the nonlinear SchrÖdinger equation in Eq.(20).

However, Whether can these phenomena occur for the above nonlinear SchrÖdinger equations in Eqs.(20)-(22)? Or speaking, what are the changes of states and properties of microscopic particles under action of the nonlinear interaction on earth? What are the influences on quantum mechanics when the states and properties of microscopic particles are described by the above nonlinear SchrÖdinger equations? These are both some very interesting and challenged problems and worth studying carefully and completely. In the following we study deeply these problems.

b) Display and exhibition of wave-corpuscle duality of microscopic particle described by the nonlinear SchrÖdinger equation

i. *The solutions of nonlinear SchrÖdinger equation and wave-corpuscle duality of particles*

In the one-dimensional case, the equation (20) at V(x,t)= 0 becomes as

$$i\phi_{t'} + \phi_{x'x'} + b|\phi|^2\phi = 0 \tag{26}$$

where $x' = x/\sqrt{\hbar^2/2m}$, $t' = t/\hbar$. We now assume the solution of Eq.(26) to have the form of Eq.(23). Inserting Eq.(23) into Eq.(26) we can obtain

$$\varphi_{x'x'} - \varphi\theta_{t'} - \varphi\theta_{x'}^2 - b\varphi^2\varphi = 0....(b > 0)......... \tag{27}$$

$$\varphi\theta_{x'x'} + 2\varphi_{x'}\theta_{x'} + \varphi_{t'} = 0............................. ... \tag{28}$$

If let $\theta = \theta(x' - v_c t'), \varphi = \varphi(x' - v_e t'),$ then Eqs.(27)-(28) become as

$$\varphi_{x'x'} - v_c\varphi\theta_{t'} - \varphi\theta_{x'}^2 - b\varphi^3 = 0.................................. \tag{29}$$

$$\varphi\theta_{x'x'} + 2\varphi_{x'}\theta_{x'} - v_e\varphi_{t'} = 0............................ \tag{30}$$

If fixing the time t' and further integrating Eq.(30) with respect to x' we can get

$$\varphi^2(2\theta_{x'} - v_e) = A(t') \tag{31}$$

Now let integral constant A(t')=0, then we can get $\theta_{x'} = v_e/2$. Again substituting it into Eq.(29), and further integrating this equation we then yield[23-24]

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{Q(\varphi)}} = x' - v_e t' \tag{32}$$

where $Q(\varphi) = -b\varphi^4/2 + (v_e^2 - 2v_c v_e)\varphi^2 + c'$.

When c'=0, $v_e^2 - 2v_c v_e > 0$, then $\varphi = \pm\varphi_0, \varphi_0 = [2(v_e^2 - 2v_c v_e)/b]^{1/2}$ is the root of $Q(\varphi) = 0$ except for $\varphi = 0$. Thus from Eq.(32) we obtain the solution of Eqs.(27)-(28) to be

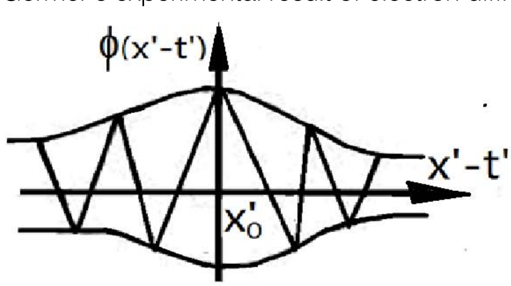
$$\varphi(x',t') = \varphi_0 \sec h[\sqrt{\frac{b}{2}}\varphi_0 (x' - v_e t')] \quad (33)$$

Then the solution of nonlinear SchrÖdinger equation (20) eventually is of the form

$$\phi(x,t) = A_0 \sec h\left\{\frac{A_0\sqrt{b}}{\sqrt{2}\hbar}\left[\sqrt{2m}(x - x_0) - v_e t\right]\right\} e^{iv_e\left[\sqrt{2m}(x-x_0)-v_c t\right]/2\hbar} \quad (34)$$

where $A_0 = \sqrt{\frac{v_e^2 - 2v_c v_e}{2b}}$. The solution of Eq.(34) can be also found by the inverse scattering method[25-26]. This solution is completely different from Eq.(11), and contains a envelop and carrier waves, the former is $\varphi(x,t) = A_0 \sec h\left\{A_0\left[\sqrt{2m}(x - x_0) - v_e t\right]/\sqrt{2}\hbar\right\}$ and a bell-type non-topological soliton with an amplitude A_0, the latter is the $\exp\left\{iv_e\left[\sqrt{2m}(x - x_0) - v_e t\right]/2\hbar\right\}$. v_e is the group velocity of the particle, v_c is the phase speed of the carrier wave. This solution is shown in Fig.1. Therefore, the microscopic particle described by nonlinear SchrÖdinger equation (20) is a soliton[18-19,23-24]. The envelop φ(x,t) is a slow varying function and the mass center of the particle, the position of the mass center is just at x_0, A_0 is its amplitude, and its width is given by $W = 2\pi\hbar/(\sqrt{mb}A_0)$.Thus, the size of the particles is $A_0 W = 2\pi\hbar/\sqrt{mb}$ and a constant. This shows that the particle has exactly a mass centre and determinant size, thus is localized at x_0. For a certain system, v_e, v_c and size of the particle are determinant and do not change with time. According to the soliton theory [18-19,23-24], the bell-type soliton in Eq.(34) can move freely over macroscopic distances in a uniform velocity v_e in space-time retaining its form, energy, momentum and other quasi-particle properties. Just so, the vector $\vec{r}$ or x denotes exactly the positions of the microscopic particles at time t. Then, the wavefunction $\phi(\vec{r},t)$or φ(x,t) can represent exactly the states of the particle at the position $\vec{r}$ or x and time t. These features are consistent with the concept of particles. Thus the feature of corpuscle of microscopic particles is displayed clearly and outright.

However, the envelope of the solution in Eq.(19) is a solitary wave. It has a certain wavevector and frequency as shown in Fig. 1(b), and can propagate in space-time, which is accompanied with the carrier wave. The feature of propagation depends only on the concrete nature of the particle. Figure 1(b) shows the width of the frequency spectrum of the envelope φ(x,t), the frequency spectrum has a localized structure around the carrier frequency ω_0. Therefore, the microscopic particle has exactly a wave-particulate duality[18,27-34]. This consists of Davisson and Germer's experimental result of electron diffraction on double seam in 1927[8-12].



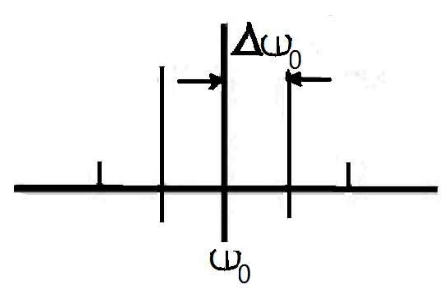


Fig. 1: The solution in Eq. (34) at V=0 in Eq. (20) and its features

We can verify that this nature of wave-corpuscle duality of microscopic particles is not changed with varying the externally applied potentials. As a matter of fact, for $V(x') = \alpha x' + c$ in Eq.(20) , where α and c are some constants. In this case equation(28) is replaced by

$$\varphi_{x'x'} - \varphi\theta_{t'} - \varphi\theta_{x'}^2 - b\varphi^2\varphi = \alpha x' + c. \quad (35)$$

Now let[23-24,27-32]

$$\varphi(x',t')=\varphi(\xi),\xi=x'-u(t'),u(t')=-\alpha(t')^2+vt'+d \quad (36)$$

where $u(t')$ describes the accelerated motion of $\varphi(x'.t')$. The boundary condition at $\xi\to\infty$ requires $\varphi(\xi)$ to approach zero rapidly. Equation (29) in such a case can be written as

$$-\dot{u}\frac{\partial\varphi}{\partial\xi}+2\frac{\partial\varphi}{\partial\xi}\frac{\partial\theta}{\partial\xi}+\varphi\frac{\partial^2\theta}{\partial\xi^2}=0 \quad (37)$$

where $\dot{u}=\frac{du}{dt'}$. If $2\partial\theta/\partial\xi-\dot{u}\neq 0$, Equation (37) may be written as

$$\varphi^2=\frac{g(t')}{(\partial\theta/\partial\xi-\dot{u}/2)} \text{ or } \frac{\partial\theta}{\partial x'}=\frac{g(t')}{\varphi^2}+\frac{\dot{u}}{2} \quad (38)$$

Integration of Eq.(38) yields

$$\theta(x',t')=g(t')\int_0^{x'}\frac{dx'}{\varphi^2}+\frac{\dot{u}}{2}x'+h(t') \quad (39)$$

where $h(t')$ is an undetermined constant of integration. From Eq.(39) we can get

$$\frac{\partial\theta}{\partial t'}=\dot{g}(t')\int_0^{x'}\frac{dx'}{\varphi^2}-\frac{g\dot{u}}{\varphi^2}+\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}+\frac{\ddot{u}}{2}x'+\dot{h}(t') \quad (40)$$

Substituting Eqs. (29) and (30) into Eq.(35) ,we have

$$\frac{\partial^2\varphi}{\partial(x')^2}=[(\alpha x'+c)+\frac{\ddot{u}}{2}x'+\dot{h}(t')+\frac{\dot{u}^2}{4}+\dot{g}\int_0^{x'}\frac{dx'}{\varphi^2}+\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}]\varphi-b\varphi^3+\frac{g^2}{\varphi^3} \quad (41)$$

Since $\frac{\partial^2\varphi}{\partial(x')^2}=\frac{d^2\varphi}{d\xi^2}$ is a function of ξ only, in order for the right-hand side of Eq. (41) to be also a function of ξ only, it is necessary that $g(t')=g_0=\text{const}$,

$$(\alpha x'+c)+\frac{\ddot{u}}{2}x'+\dot{h}(t')+\frac{\dot{u}^2}{4}+\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}=\overline{V}(\xi) \quad (42)$$

Next, we assume that $V_0(\xi)=\overline{V}(\xi)-\beta$ where β is real and arbitrary. Then

$$\alpha x'+c=V_0(\xi)-\frac{\ddot{u}}{2}x'+[\beta-\frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}-\dot{h}(t')-\frac{\dot{u}^2}{4}] \quad (43)$$

Clearly, in the case being discussed , $V_0(\xi)=0$, and the function in the brackets in Eq. (43) is a function of t'. Substituting Eqs.(42) and (43) into Eq.(41) ,we can get

$$\frac{\partial^2\varphi}{\partial\xi^2}=\beta\varphi-b\varphi^3+\frac{g_0^2}{\varphi^3} \quad (44)$$

This shows that $\varphi=\varphi(\xi)$ is the solution of Eq.(44) when β and g are constant. For large $|\xi|$, we may assume that $|\varphi|\leq\beta/|\xi|^{1+\Delta}$, when Δ is a small constant. To ensure that $d^2\varphi/d\xi^2$, and φ approach zero when $|\xi|\to\infty$, only the solution corresponding to $g_0=0$ in Eq.(44) is kept to be stable. Therefore we choose $g_0=0$ and obtain the following from Eq. (38)

$$\frac{\partial\theta}{\partial x'}=\frac{\dot{u}}{2} \tag{45}$$

Thus, we obtain from Eq. (43)

$$\alpha x'+c=-\frac{\ddot{u}}{2}x'+\beta-\dot{h}(t')-\frac{\dot{u}^2}{4}$$

$$h(t')=(\beta-c-\frac{1}{4}\upsilon^2)t'-\frac{1}{3}\alpha^2(t')^3+\upsilon\alpha(t')^2/2 \tag{46}$$

Substituting Eq.(46) into Eqs.(29) and (30), we obtain

$$\theta=(-\alpha t'+\frac{1}{2}\upsilon)x'+(\beta-c-\frac{1}{4}\upsilon^2)t'-\frac{1}{3}\alpha^2(t')^3+\upsilon\alpha(t')^2/2 \tag{47}$$

Finally, substituting the above into Eq.(44), we can get

$$\frac{\partial^2\varphi}{\partial\xi^2}-\beta\varphi+b\varphi^3=0 \tag{48}$$

When $\beta>0$, the solution of Eq.(48) is of the form[18-19,23-24,,33-34]

$$\varphi=\sqrt{\frac{2\beta}{b}}\sec h(\sqrt{\beta}\xi) \tag{49}$$

Thus

$$\phi=\sqrt{\frac{2\beta}{b}}\sec h[\sqrt{\beta}(\sqrt{\frac{2m}{\hbar^2}}(x-x_0)+\frac{\alpha t^2-\upsilon t-d}{\hbar})]\times\exp\{i[(\frac{-\alpha t}{\hbar}+\frac{\upsilon}{2})\sqrt{\frac{2m}{\hbar^2}}x+ (\beta-c-\frac{1}{4}\upsilon^2)\frac{t}{\hbar}-\frac{\alpha^2t^3}{3\hbar^3}+\frac{\upsilon\alpha t^2}{2\hbar}]\} \tag{50}$$

This is also soliton solution. If V(x')=c, the solution can represent as

$$\phi=\sqrt{\frac{2\beta}{b}}\sec h\left\{\sqrt{\beta}[(x'-x_0')-v_e(t-t_0)]\right\}\exp i\left\{\frac{v_e}{2\hbar}[(x'-x_0')-(\beta-\frac{v_e^2}{4}-C)]t)\right\} \tag{51}$$

If $V(x')=ax'$ and $b=2$, the solution can represent by

$$\phi=2\eta\,\mathrm{sech}\left[2\eta\left(x'-x_0'-4\xi t'+2\alpha t'^2\right)\right]\times \exp\left\{-i\left[2\left(\xi-\alpha t'\right)x'+\frac{4\alpha^2t'^3}{3}-4\alpha\xi t'^2+4\left(\xi^2-\eta^2\right)t'+\theta_0\right]\right\} \tag{52}$$

In this calculation we used the transformation [35-36] :

$$\phi\left(x',t'\right)=\phi'\left(\tilde{x}',\tilde{t}\right)e^{-i\alpha\tilde{x}'\tilde{t}-i\alpha^2\left(\tilde{t}'\right)^3/3},\ x'=\tilde{x}'-\alpha\tilde{t}'^2,\ t'=\tilde{t}', \tag{53}$$

Under this transformation and in this case thus Eq. (20) becomes

$$i\phi'_{\tilde{t}'} + \phi'_{\tilde{x}'\tilde{x}'} + 2|\phi'|^2 \phi' = 0,$$

Utilizing Eq.(44), its solution in Eq.(52) then can be obtained immediately.

For a more complicated potential $V(x)$ in Eq.(20), for example, $V(x) = kx^2 + A(t)x + B(t)$, utilizing the above method the soliton solution in Eq.(27) of Eq. (20) can be written as[18-19,23-24]

$$\phi = \varphi(x - u(t))e^{i\theta(x,t)} \tag{54}$$

where $\varphi(x-u(t)) = \sqrt{\frac{2B}{b}} \operatorname{sec}h(a[(x-x_0)-u(t)]), u(t) \equiv 2\cos(2\sqrt{kt}+\beta)+u_0(t)$

$$\theta(x,t) = \left[-2\sqrt{k}\sin(2\sqrt{kt}+\beta)+\frac{u_0}{2}\right]+\lambda_0 t+g_0$$

$$-\int_0^1\left\{\left[u_0(t^{'})-k(2\cos(2-\sqrt{kt^{'}}+\beta))\right]^2+B(t^{'})+\left[\frac{u_0}{2}-2\sqrt{k}\sin(2\sqrt{kt^{'}}+\beta)\right]\right\}dt^{'}$$

When $A(t) = B(t) = 0$, $u(t) = 2\cos(2\sqrt{kt}) + u_0(t)$,

$$\theta(x,t) = -2\sqrt{k}\sin(2\sqrt{kt}+\frac{u_0}{2}x)+g_0$$

$$-\int_0^1\left\{\left[-k(2\cos(2\sqrt{kt})+u_0(t)\right]^2+\left[\frac{u_0}{2}-2\sqrt{k}\sin(2\sqrt{kt^{'}})\right]\right\}$$

For the case of $V_0(x') = \alpha^2 x'^2$, which is a harmonic potential, where α is constant. In this condition in accordance with above way the solution can be denoted by[35-36]

$$\phi = 2\eta\, sec\,h\left\{2\eta(x'-x_0^{'}) - \frac{4\xi\eta}{\alpha} sin\left[2\alpha(t'-t_0')\right]\right\}\times$$

$$exp\left\{i\left[2\xi x' cos\, 2\alpha(t'-t_0') - \frac{\xi^2}{\alpha} sin\left[4\alpha(t'-t_0')\right] + 4\eta^2(t'-t_0') + \theta_0'\right]\right\} \tag{55}$$

where $2\sqrt{2/b}\eta = A_0$, and $2\sqrt{2}\xi = v_c$ are the amplitude and group velocity of the particles in Eqs.(53) and (55), respectively. From Eqs.(51)-(53) and (55) we see clearly that these solutions of the nonlinear SchrÖdinger equation in Eq.(20) have all same shape as shown in Fig.1 or Eq.(27) or (54) and similar natures, such as, they contain a envelop and carrier waves, and are also some bell-type soliton with certain amplitude A_0 and the group velocity v_e and phase speed v_c. Meanwhile, these microscopic particles have also a mass center and possess an amplitude, width and sizes, thus are localized at x_0. Thus we can conclude that these microscopic particles have all the wave-corpuscle duality in the light of previous explanation. However, the differences among these solutions are only distinctions of the amplitude, velocity and frequency, the velocity for some particles are related to time, some frequencies is oscillatory. The above features indicate that the localization feature or wave-corpuscle duality of a microscopic particle cannot be changed with varying the external potential $V(x)$, the latter alters only the sizes of amplitude, velocity and frequency of microscopic particle, therefore its influence is secondary. This shows also that the fundamental nature of microscopic particles is mainly determined by the combined effect of dispersion forces and nonlinear interaction in such a case. It is the nonlinear interaction that makes dispersive microscopic particle become a localized soliton.

From the above results we see clearly that the microscopic particles are a soliton which can denote all by $\phi(x',t') = \varphi(x',t')e^{i\theta(x',t')}$ in Eq.(23). According to the soliton theory [18-20], the bell-type soliton in Eq.(34) can move freely over macroscopic distances in a uniform velocity v_e in space-time retaining its shape, energy, momentum and other quasi-particle properties. This means that its mass, momentum and energy are constants, and can be represented by [18-19,23-24]

$$N_s = \int_{-\infty}^{\infty} |\phi|^2 dx' = 2\sqrt{2}A_0$$

$$p = -i\int_{-\infty}^{\infty} \left(\phi^* \phi_{x'} - \phi\phi_{x'}^*\right) dx' = 2\sqrt{2}A_0 v_e = N_s v_e = const$$

$$E = \int_{-\infty}^{\infty} \left[|\phi_{x'}|^2 - \frac{1}{2}|\phi|^4 \right] dx' = E_0 + \frac{1}{2} M_{sol} v_e^2 \quad (56)$$

where $x' = x/\sqrt{\hbar^2/2m}$, $t' = t/\hbar$, and $M_{sol} = N_s = 2\sqrt{2}A_0$ is just effective mass of the microscopic particle, which is a constant. Obviously, the energy, mass and momentum of the particle cannot be dispersed in its motion. Just so, the position vector $\vec{r}$ or position *x* in Eq.(20) or (26) has definitively physical significance, and denotes exactly the positions of the particles at time *t*. Thus, the wave function $\phi(\vec{r},t)$ or φ(x,t) can represent exactly the states of microscopic particles at the position $\vec{r}$ or *x* and time *t*. This is consistent with the concept of classical particles or corpuscles.

In the light of this method and formulae we can find out the effective masses, momentums and energies of the microscopic particles described by Eq.(36) at A(ϕ)=0, V(x)=c, $V(x') = \alpha x'$, $V(x') = \alpha x' + c$, $V_0(x') = \alpha^2 x'^2$ and $V(x) = kx^2 + A(t)x + B(t)$, respectively.

ii. *The linear SchrÖdinger equation is a special case at the nonlinear interaction to equal to zero*

However, we also demonstrate that the solution of Eq.(20) is not the solution Eq.(11) of linear SchrÖdinger equation in Eq.(7), even though the nonlinear interaction approaches to zero. To see this clearly, we first examine the velocity of the skirt of the soliton given in Eq.(34). For weak nonlinear interaction (b<<1) and small skirt $\phi(x',t')$, it may be approximated by (for $x > \upsilon_e t$)

$$\phi = 2\sqrt{2/b}ke^{-\sqrt{2}k(x'-\upsilon_e t')}e^{i\upsilon_e(x'-\upsilon_c t')/2} \quad (57)$$

where $2^{3/2}k/b^{1/2} = A_0$. Thanks to the small term b$|\phi|^2\phi$, then Eq. (11) can be approximated by

$$i\phi_{t'} + \phi_{x'x''} \approx 0 \quad (58)$$

Substituting Eq. (57) into Eq. (58), we get $\upsilon_e \approx 2\sqrt{2}k$, which is the group speed of the particle. (Near the top of the peak, we must take both the nonlinear and dispersion terms into account because their contributions are of the same order. The result is the group speed.). Here, we have only checked the formula for the region where $\phi(x,t)$ is small; that is, when a particle is approximated by Eq.(49), it satisfies the approximate wave equation (50) with $\upsilon_e \approx 2\sqrt{2}k$.

However, if Eq.(50) is treated as a linear SchrÖdinger equation, its solution is of the form:

$$\phi'(x,t) = Ae^{i(kx-\omega t)} \quad (59)$$

We now have $\omega = k^2$, which gives the phase velocity ω/k as $\upsilon_c = k$ and the group speed $\partial\omega/\partial k = \upsilon_{gr} = k$. Apparently, this is different from $\upsilon_e = 2\sqrt{2}k$. This is because the solution Eq. (57) is essentially different from Eq. (59). Therefore, the solution Eq. (59) is not the solution of nonlinear SchrÖdinger equation (20) with V(x,t)= 0 in the case of weak nonlinear interactions. Solution Eq.(57) is a "divergent solution" ($\phi(x,t) \to \infty$ at $x \to -\infty$), which is not an "ordinary plane wave". The concept of group speed does not apply to a divergent wave. Thus, we can say that the soliton is made from a divergent solution, which is abandoned in the linear waves. The divergence develops by the nonlinear term to yield waves of finite amplitude. When the nonlinear term is very weak, the soliton will diverge; and suppression of divergence will result in no soliton. These circumstances are clearly seen from the soliton solution in Eq.(34) in the case of nonlinear coefficient $b \neq 1$. If the nonlinear term approaches zero ($b \to 0$), the solitary wave diverges $(\phi(x,t) \to \infty)$. If we want to suppress the divergence, then we have to set $k = 0$. In such a case, we get Eq. (59) from Eq.(34). This illustrates that the nonlinear SchrÖdinger equation can reduce to the linear

SchrÖdinger equation if and only if the nonlinear interaction and the group speed of the particle are zero. Therefore, we can conclude that the microscopic particles described by the nonlinear SchrÖdinger equation in the weak nonlinear interaction limit is also not the same as that in linear SchrÖdinger equation in quantum mechanics. Only if the nonlinear interaction is zero, the nonlinear SchrÖdinger equation can reduce to the linear SchrÖdinger equation. However, real physical systems or materials are made up of a great number of microscopic particles, and nonlinear interactions always exist in the systems. The nonlinear interactions arise from the interactions among the microscopic particles or between the microscopic particles and the environment as mentioned above. The nonlinear SchrÖdinger equation should be the correct and more appropriate theory for real systems. It should be used often and extensively, even in weak nonlinear interaction cases. However, the linear SchrÖdinger equation in quantum mechanics is an approximation to the nonlinear SchrÖdinger equation and can be used to study motions of microscopic particles in systems in which there exists only very weak and negligible nonlinear interactions.

iii. *The reason for localization of the microscopic particles*

However, how could a microscopic particle be localized in such a case? In order to shed light on the conditions for localization of microscopic particle in the nonlinear SchrÖdinger equation, we return to discuss the property of nonlinear SchrÖdinger equation (20). The time-independent solution of Eq.(20) is assumed to have the form of[18-19,23-24,33-34]

$$\phi(x,t) = \varphi'(x,t)e^{-iEt/\hbar} \tag{60}$$

then Equation(20) becomes as

$$-\frac{\hbar^2}{2m}\nabla^2\varphi' + \left[V(\vec{r}) - E\right]\varphi' - b|\varphi'|^2\varphi' = 0. \tag{61}$$

For the purpose of showing clearly the properties of this system, we here assume that $V(\vec{r})$ and *b* are independent on $\vec{r}$. Then in one-dimensional case, equation (61) may be written as

$$\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2} = -\frac{d}{d\varphi'}V_{eff}(\varphi') \tag{62}$$

with

$$V_{eff}(\varphi') = \frac{1}{4}b|\varphi'|^4 - \frac{1}{2}(V-E)|\varphi'|^2. \tag{63}$$

When $V > E$ and $V < E$, the relationship between $V_{eff}(\varphi')$ and φ' is shown in Fig.2. From this figure we see that there are two minimum values of the potential, corresponding to two ground states of the microscopic particle in the system, *i.e.* $\varphi_0' = \pm\sqrt{\frac{V-E}{b}}$. This is a double-well potential, and the energies of the two ground states are $-(V-E)^2/4b \le 0$. This shows that the microscopic particle can be localized due to the fact that the microscopic particle has negative binding energy. This localization is achieved through repeated reflection of the microscopic particle in the double-well potential field. The two ground states limit the energy diffusion, thus the energy of the particle is gathered, soliton is formed, the particle is eventually localized.. Obviously, this is s result of the nonlinear interaction because the particle is in an expanded state if *b*=0. In the latter, there is only one ground state of the particle which is $\varphi' = 0$. Therefore, only if $b \ne 0$, the system can have two ground states, and the microscopic particle can be localized. Its binding energy, which makes the particle to be localized, is provided by the attractive nonlinear interaction , -b$(\varphi'\varphi'^*)^2$, in the systems.

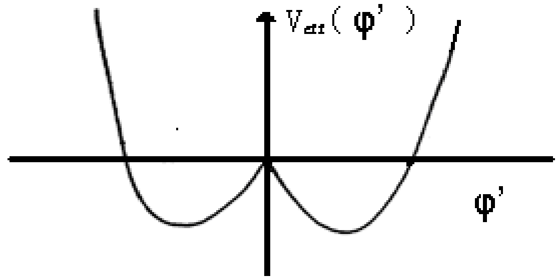


Fig. 2: The effective potential of nonlinear SchrÖdinger equation

From Eq. (63), we know that when $V>0$, $E>0$ and $V<E$, or $|V|>E$, $E>0$ and $V<0$, for $b>0$, the microscopic particle may not be localized by the mechanisms mentioned above. On the other hand, we see from (61)-(63) that if the nonlinear self-interaction is of repelling type (*i.e,* $b<0$.), then, equation (20) becomes

$$i\hbar\frac{\partial}{\partial t}\phi+\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2}-|b||\phi|^2\phi=V(x,t)\phi. \tag{64}$$

It is impossible to obtain a bell-type soliton solution, with full matter features, of this equation. However, if $V(x,t)=V(x)$ or a constant, solution of kink soliton type exists. In this case inserting (60) into (64), we can get

$$\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2}-|b|\varphi'^3+\left[E-V(x)\right]\varphi'=0. \tag{65}$$

If V is independent of *x* and $0<V<E$, equation (64) has the following solution

$$\varphi'=\frac{\sqrt{2(E-V)}}{|b|}\tanh\left[\sqrt{\frac{2(E-V)}{h^2}}(x-x_0)\right]. \tag{66}$$

This is the kink soliton solution when $|V|>E$ and $V<0$. In the case of $V(x)=0$, Zakhorov and Shabat et al[25-26] obtained dark soliton solution which was experimentally observed in optical fiber and was discussed in the Bose-Einstein condensation model .

c) The stability of wave-corpuscle duality of microscopic particles

i. *The instability of microscopic particles in quantum mechanics*

As mentioned above, the microscopic particles depicted by linear SchrÖdinger equation (7) are always dispersive, thus also unstable. What is so-called dispersion effect? The concept of dispersion comes from optics. We know from optics that so-called dispersion of light is just a beam white light to split into several beams of lights with different velocities, when the beam passes through a prism, in which the matter the light wave is propagated is referred to as a dispersive medium. The relationship between the wave length and frequency of the light (wave) in this phenomenon is called a dispersed relation, which can be expressed as $\omega=\omega(\vec{k})$ or $G(\omega,\vec{k})=0$, where $\det\frac{\partial^2\omega}{\partial k_i\partial k_j}\neq 0$ or $\frac{\partial^2\omega}{\partial^2 k^2}\neq 0$ in one-dimensional case. It specifies how the velocity of frequency of the wave (light) depends on its wavelength or wavevector. The equation depicts a wave propagation in a dispersive median and is called as dispersion equation. The linear SchrÖdinger equation (1) in quantum mechanics is a dispersion equation[8-12]. If Eq.(11) is inserted into Eq.(7), we can get $\omega=\hbar k^2/2m$, here $E=\hbar\omega, \vec{p}=\hbar\vec{k}$. The quantity $v_e=\omega/k$ is called the phase velocity of the microscopic particle (wave), but the wave vector $\vec{k}$ is a vector designating the direction of the wave propagation. Thus the phase velocity is given by $\vec{v_e}=(\omega/k^2)\vec{k}$. This is a standard dispersion relation. Thus, the solutions of the linear SchrÖdinger equation (1) are a dispersive wave[8-12].

But how does the dispersive effect influence the state of a microscopic particle? To this end, we consider the dispersive effect of a wave-packet which is often used to explain the corpuscle feature of microscopic particles in quantum mechanics. The wave-packet is formed from a linear superposition of several plane waves in Eq.(11) with wavevector *k* distributed in a range of $2\Delta k$. In the one dimensional case, a wave-packet is can be expressed as[8-12]

$$\Psi(x,t)=\frac{1}{2\pi}\int_{k_0-\Delta k}^{k_0+\Delta k}\psi(k,t)e^{i(kx-\omega t)}dk\ . \tag{67}$$

We now expand the angular frequency at ω_0 by

$$\omega=\omega_0+(\frac{d\omega}{dk})_{k_0}\Delta k+\frac{1}{2!}(\frac{d^2\omega}{dk^2})_{k_0}(\Delta k)^2+\cdots \tag{68}$$

If we consider only the first two terms in the dispersive relation, *i.e.* $\omega=\omega_0+(\frac{d\omega}{dk})_{k_0}\xi$, here $\xi=\Delta k=k-k_0$, then

$$\Psi(x,t)=\psi(k_0)e^{i(k_0x-\omega_0t)}\int_{-\Delta k}^{\Delta k}d\xi e^{i\left[x-(d\omega/dk)_{k_0}t\right]\xi}$$

$$=2\psi(k_0)\frac{\sin\left\{\left[x-(\frac{d\omega}{dk})_{k_0}t\right]\Delta k\right\}}{x-(\frac{d\omega}{dk})_{k_0}t}e^{i(k_0x-\omega_0t)}, \tag{69}$$

where the coefficient of $e^{i(k_0x-\omega_0t)}$ is the amplitude of the wave-packet. Its maximum is $2\psi(k_0)\Delta k$ which occurs at $x=0$, but it is zero at $x=x_n=n\pi/\Delta k\,(n=\pm1,\pm2\cdots)$. Obviously, the amplitude of the wave-packet decreases with increasing distance of propagation due to the dispersion effect. Hence, the dispersion effect results directly in damping of the microscopic particle (wave). This means that a wave-packet could eventually collapse with increasing transported time. Thus a wave-packet is unstable and cannot express the corpuscle feature of particles Therefore, the microscopic particles are instable in quantum mechanics[9-12]. Obviously, this is due to the dispersion effect of the kinetic energy term $(h^2/2m)\nabla^2=\vec{p}^{\,2}/2m$ in Eq.(7) or Eq.(8), which cannot always be balanced and suppressed by an external potential field $V(\vec{r},t)$.

ii. *The stability of microscopic particles depicted by nonlinear SchrÖdinger equation*

As known, stability of particle designates its corpuscle feature, in classical physics the particles are stable. However, whether is the above wave-corpuscle duality of microscopic particles depicted by nonlinear SchrÖdinger equation stable? This need to prove further. In the absence of an externally applied field, the stability of the microscopic particles can be demonstrated by means of the initial and structural stabilities[18-20]. However, how does the stability of macroscopic particles exposed in an externally applied field be proved ? If the motion of a macroscopic particles is located in a finite range where the potential is lowest, we can say that the particle is stable according to the minimum theorem of energy. As a matter of fact, when there are a lot of particles with complicated interactions in the system, then we are very difficult to define the individual behavior of each particles in this case. Thus we cannot use again same strategies as those used in the discussions of initial stability and collision of particles to determine their stability[18-19,23-24,37]. Instead, we apply the fundamental work- energy theorem in classical physics to determine their stability. The theorem of minimum energy can be desribed as follows. If a mechanical system is in a state of minimal energy, then we can say it is stable because in order to change this state, external energy must be supplied. We apply this fundamental concept to demonstrate the stability of the microscopic particles described by the nonlinear SchrÖdinger equation (20), which is outlined in the following.

Let $\phi(x,t)$ represent the field of the particle, and assume that it has derivatives of all orders, and all integrations, and is convergent and finite. The Lagrange density function corresponding to the nonlinear SchrÖdinger equation (20) is as follows:

$$\mathcal{L}=\frac{i\hbar}{2}(\phi^*\phi_t-\phi\phi_t^*)-\frac{\hbar^2}{2m}(\nabla\phi.\nabla\phi^*)-V(x)\phi^*\phi+\frac{b}{2}(\phi^*\phi)^2 \tag{70}$$

The momentum density of this field is defined as p= $\partial L/\partial\phi$. Thus, the Hamiltonian density of the field is as follows

$$\mathcal{H}=\frac{i\hbar}{2}\left(\phi^*\partial_t\phi-\phi\partial_t\phi^*\right)-\mathcal{L}=\frac{\hbar^2}{2m}(\nabla\phi.\nabla\phi^*)+V(x)\phi^*\phi-\frac{b}{2}(\phi^*\phi)^2 \tag{71}$$

From Eqs.(70)-(71), we see clearly that the Lagrange and Hamiltonian operators of the systems corresponding to Eq. (20) are all related to the state wave function of particles and involve the nonlinear interactional energy, $b(\phi\phi^*)^2$ related to the states of microscopic particles. This is in essence different from the Hamiltonian operator in Eq.(8) in quantum mechanics. Then the natures and features of microscopic particles should be together determined by the kinetic and nonlinear interaction terms in this case. Just so, there is a force or energy to obstruct and suppress the dispersing effect of kinetic energy in the system, thus the microscopic particles cannot disperse and propagate again in total space, and eventually is localized all the time. This is just the essential reason that the microscopic particles have a particulate nature or corpuscle-wave duality as mentioned above. Therefore, we can say that the systems described by the nonlinear SchrÖdinger equation (20) and correspond- ding Lagrange and Hamiltonian in Eqs.(70)-(71) breaks through the fundamental hypothesis for the independence of Hamiltonian operator with the wave function of the particles in the quantum mechanics[18-19]. This is a new development of quantum mechanics.

In the general case, the total energy of the particles is a function of t' and is represented by

$$E(t') = \int_{-\infty}^{\infty}\left[\left|\frac{\partial\phi}{\partial x'}\right|^2 - \frac{b}{2}\left|\phi\phi^*\right|^2 + V(x')\left|\phi\right|^2\right]dx' \tag{72}$$

However, in this case, b and $V(x')$ are not functions of t'. So, the total energy of the systems is a conservative quantity, i.e., $E(t')$=E=const.. We can demonstrate that when $x' \to \pm\infty$, the solutions of Eq.(20) and $\phi(x',t')$should tend to zero rapidly, i.e.[18-19,37-41],

$$\underset{|x'|\to\infty}{Lim}\,\phi(x',t') = \underset{|x|\to\infty}{Lim}\frac{\partial\phi}{\partial x'} = 0$$

Then

$$\int_{-\infty}^{\infty}\phi^*\phi dx' = const.\text{ or a function of } t'$$

The position of mass centre of microscopic particle can be represented by

$$\langle x'\rangle = x'_g = x_0 = \int_{-\infty}^{\infty}\phi'^{*}x'\phi dx' \Big/ \int_{-\infty}^{\infty}\phi^*\phi dx' \tag{73}$$

Thus, the velocity of mass centre of microscopic particle can be denoted by

$$v_g = \frac{d\langle x'\rangle}{dt'} = \frac{d}{dt'}\left\{\frac{\int\phi * x'\phi dx'}{\int\phi * \phi dx'}\right\} = -2i\frac{\int_{-\infty}^{\infty}\phi * \frac{\partial\phi'}{\partial x'}dx'}{\int\phi * \phi dx'} \tag{74}$$

However, for different solutions of the same nonlinear SchrÖdinger equation (20), $\int_{-\infty}^{\infty}\phi^*\phi dx'$, $\langle x'\rangle$ and dx'/dt' can have different values. Therefore, it is unreasonable to compare the energy between a definite solution and other solutions. We should compare the energy of one particular solution to that of another solution. The comparison is only meaningful for many microscopic particle systems that have the same values of $\int_{-\infty}^{\infty}\phi^*\phi dx' = k.$, $\langle x'\rangle = u$ and $d<x'>/dt' = \dot{u}$ at the same time t'_0. Based on these, we can determine the stability of the solutions of Eq.(20), for example, Eq.(34). Thus, we assume that the different solutions of the nonlinear SchrÖdinger equation (20) satisfy the following boundary conditions at definite time t'_0:

$$\int_{-\infty}^{\infty}\phi^*\phi dx' = k,\ \langle x'\rangle\Big|_{t'=t'_0} = u(t'_0),\ \left.\frac{d<x'>}{dt'}\right|_{t'=t'_0} = \dot{u}(t'_0), \tag{75}$$

Now we assume the solution of nonlinear SchrÖdinger equation (20) to have the form of Eq.(23).
Substituting Eq.(23) into Eq.(72), we obtain the energy formula:

$$E=\int_{-\infty}^{\infty}\left[\left(\frac{\partial\varphi}{\partial x'}\right)^2+\varphi^2\left(\frac{\partial\theta}{\partial x'}\right)^2-b\varphi^4+V(x')\varphi^2\right]dx' \tag{76}$$

At the same time, equation (24) becomes

$$\int_{-\infty}^{\infty}\varphi^2 dx'=k\,,\quad \frac{\int_{-\infty}^{\infty}x'\varphi^2 dx'}{\int_{-\infty}^{\infty}\varphi^2 dx'}=u\left(t_0{}'\right),\quad \frac{2\int_{-\infty}^{\infty}\varphi^2\frac{\partial\theta}{\partial x'}dx'}{\int_{-\infty}^{\infty}\varphi^2 dx'}=\dot{u}(t_0') \tag{77}$$

Finding the extreme value of the functional Eq.(76) under the boundary conditions Eq. (77) by means of the Lagrange uncertain factor method, we obtain the following Euler equations:

$$\frac{\partial^2\varphi}{\partial(x')^2}=\left\{\begin{matrix}V(x')+C_1(t_0')C_2(t_0')[x'-u(t_0')]+\\ C_3(t_0')\left[2\frac{\partial\theta}{\partial t'}-\dot{u}(t_0')\right]+\left(\frac{\partial\theta}{\partial t'}\right)^2\end{matrix}\right\}\varphi-b\varphi^3=0 \tag{78}$$

$$\frac{\partial^2\varphi}{\partial(x')^2}\varphi^2+2\frac{\partial\theta}{\partial t'}\varphi\frac{\partial\varphi}{\partial t'}+2C_3(t_0')\varphi\frac{\partial\varphi}{\partial t'}=0 \tag{79}$$

where the Lagrange factors C_1, C_2 and C_3 are all functions of t'. Now, let $C_3(t_0')=-\frac{1}{2}\dot{u}(t_0')$

If $$2\frac{\partial\theta}{\partial x'}-\dot{u}(t_0')\neq 0$$

we can get from Eq.(79)

$$\frac{2}{\varphi}\frac{\partial\varphi}{\partial x'}=\frac{-\frac{\partial^2\theta}{\partial x'^2}}{-\frac{\partial\theta}{\partial x'}-\frac{1}{2}\dot{u}(t_0')}$$

Integration of the above equation yields

$$\varphi^2=\frac{g(t')}{\frac{\partial\theta}{\partial x'}-\frac{1}{2}\dot{u}(t_0')}\quad\text{or}\quad\left.\frac{\partial\theta}{\partial x'}\right|_{t'=t_0'}=\frac{g(t_0')}{\varphi^2}+\frac{\dot{u}(t_0')}{2} \tag{80}$$

where $g(t_0')$ is an integral constant. Thus,

$$\theta(x',t')=g(t_0')\int_0^x\frac{dx'}{\varphi^2}+\frac{\dot{u}(t_0)}{2}x'+M(t_0') \tag{81}$$

Here, $M(t_0')$ is also an integral constant. Again let

$$C_2(t_0')=\frac{1}{2}\ddot{u}(t_0') \tag{82}$$

Substituting Eqs.(80)-(82) into Eq.(79), we obtain

$$\frac{\partial^2\varphi}{\partial(x')^2}=\left\{V(x')+\frac{\ddot{u}(t_0')}{2}x'+\left[C_1(t_0')-\frac{\ddot{u}(t_0')}{2}u(t_0')+\frac{u^2(t_0')}{4}\right]\right\}\varphi-b\varphi^3+\frac{g^2(t_0')}{\varphi^3} \tag{83}$$

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Letting

$$C_1(t_0') = \frac{u(t_0')\ddot{u}(t_0')}{2} - \frac{\dot{u}^2(t_0')}{2} + M(t_0') + \beta' \tag{84}$$

where β' is an undetermined constant, which is a function of t'-independent, and assuming $Z = x' - u(t_0')$, then $\frac{\partial^2\varphi}{\partial(x')^2} = \frac{\partial^2\varphi}{\partial Z^2}$ is only a function of Z. To make the right-hand side of Eq.(34) be also a function of Z, the coefficients of φ, φ^3 and $1/\varphi^3$ must also be functions of Z, thus, $g(t_0') = g_0 = const$, and

$$V(x') + \frac{\ddot{u}(t_0')}{2}x' + M(t_0') - \frac{u^2(t)}{4} = \tilde{V}_0(Z)$$

Then, equation (83) becomes

$$\frac{\partial^2\varphi}{\partial(x')^2} = \{\tilde{V}[x' - u(t_0')] + \beta'\}\varphi - b\varphi^3 + \frac{g^2(t_0')}{\varphi^3} \tag{85}$$

Since $\tilde{V}(Z) = \tilde{V}_0[x' - u(t_0')] = 0$ in the present case. Hence, equation(85) becomes

$$\frac{\partial^2\varphi}{\partial(x')^2} = \beta'\varphi - b\varphi^3 + \frac{g^2(t_0')}{\varphi^3} \tag{86}$$

Therefore, φ is the solution of Eq.(86) for the parameters β' =constant and $g(t_0')$=constant. For sufficiently large $|Z|$ we may assume[18,19,37-41] that $|\varphi| \le \tilde{\beta}/|Z|^{1+\Delta}$, where Δ is a small constant. However, in Eq.(86) we can only retain the solution $\varphi(Z)$ corresponding to $g(t_0')$ to ensure that $Lim_{|\varepsilon|\to\infty} d^2\varphi/dZ^2 = 0$, thus, Eq. (86) becomes

$$\frac{\partial^2\varphi}{\partial(x')^2} = \beta'\varphi - b\varphi^3 \tag{87}$$

As a matter of fact, if $\partial\theta/\partial t' = \dot{u}/2,$ and considering Eqs.(84)-(85) we can verify that the solution in Eq.(34) can satisfy Eq.(87). In such a case, it is not difficult to show that the energy corresponding to the solution Eq.(34) of Eq.(87) has a minimal value under the boundary conditions of Eq.(87). Thus, we can conclude that the solution of Eq. (20), or the wave-corpuscle duality of microscopic particles depicted by nonlinear SchrÖdinger equation (20) is stable in such a case.. This indicates the microscopic particles have a feature of classical particles.

d) The classical features of motion of microscopic particles

i. *The feature of Newton's motion of microscopic particles*

Since the microscopic particle described by the nonlinear SchrÖdinger equation (20) has a corpuscle feature and is also quite stable as mentioned above. Thus its motion in action of a potential field in space-time should have itself rules of motion. We now study this rule of motion.
Now utilizing Eq. (20) and its conjugate equation as follows:

$$-i\hbar\frac{\partial\phi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi^* \pm b|\phi|^2\phi^* + V(\vec{r},t)\phi^*, \tag{88}$$

we can obtain[18,37-41]

$$\frac{d}{dt'}\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx' = \int_{-\infty}^{\infty}\phi_{t'}^*\phi_{x'}dx' + \int_{-\infty}^{\infty}\phi^*(\phi_{t'})_{x'}dx' = i\int_{-\infty}^{\infty}\{\phi^*\frac{\partial}{\partial x'}[\phi_{x'x'} + b\phi^*\phi^2$$

$$-V\phi] - [\phi_{x'x'}^* - b\phi(\phi^*)^2 - V\phi^*]\phi_{x'}\}dx' = i\int_{-\infty}^{\infty}\phi^*\frac{\partial V}{\partial x'}\phi dx' \tag{89}$$

where $x'=x/\sqrt{\hbar^2/2m}$, $t'=t/\hbar$. We here utilize the following relations and the boundary conditions:

$$\int_{-\infty}^{\infty}(\phi^*\phi_{x'x'x'}-\phi^*_{x'x'}\phi_{x'})dx'=0,\int_{-\infty}^{\infty}b(\phi^{*2}\phi\phi_{x'}+\phi^*\phi^2\phi^*_{x'})dx'=0$$

$$\underset{|x'|\to\infty}{Lim}\phi(x',t')=\lim_{|x'|\to\infty}\phi_{x'}(x',t')=0 \text{ and } \int_{-\infty}^{\infty}\phi^*\phi dx'=const.\underset{|x'|\to\infty}{Lim}\phi^*x'\phi_{x'}=\underset{|x'|\to\infty}{Lim}\phi^*_{x'}x'\phi=0$$

where $\phi_{x'}=\frac{\partial\phi}{\partial x'},\phi_{x'x'x'}=\frac{\partial^3\phi}{\partial x'^3}$. Thus, we can get

$$\frac{d}{dt'}\int_{-\infty}^{\infty}\phi^*x'\phi dx'=\int_{-\infty}^{\infty}(\frac{\partial\phi^*}{\partial t'}x'\phi+\phi^*x'(\frac{\partial\phi}{\partial t'})dx'=-2i\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx' \tag{90}$$

In the systems, the position of mass centre of microscopic particle can be represented by Eq.(73), thus the velocity of mass centre of microscopic particle is represented by Eq.(74). Then, the acceleration of mass centre of microscopic particle can also be denoted by

$$\frac{d^2}{dt'^2}<x'>=-2i\frac{d}{dt'}\{\int_{-\infty}^{\infty}\phi^*\phi_{x'}dx'/\int_{-\infty}^{\infty}\phi^*\phi dx'\}=-2\int_{-\infty}^{\infty}\phi^*V_{x'}\phi^*dx'=-2<\frac{\partial V}{\partial x'}> \tag{91}$$

If ϕ is normalized, i.e., $\int_{-\infty}^{\infty}\phi^*\phi dx'=1$, then the above conclusions also are not changed.

We expand $\frac{\partial V}{\partial x'}$ at the mass centre $x'=<x'>=x_0'$ as[18,37-41]

$$\frac{\partial V(x')}{\partial x'}=\frac{\partial V(<x'>)}{\partial<x'>}+(x'-<x'>)\frac{\partial^2V(<x'>)}{\partial<x'>^2}+\frac{1}{2!}(x'-<x'>)^2\frac{\partial^3V(<x'>)}{\partial<x'>^3}+\cdots$$

Finding the expectation value to the above formula, thus we get

$$\left\langle\frac{\partial V(x')}{\partial x'}\right\rangle=\frac{\partial V(<x'>)}{\partial<x'>}+\frac{1}{2!}<(x'-<x'>)^2>\frac{\partial^3V(<x'>)}{\partial<x'>^3}$$

For the microscopic particle described by Eq.(20) or Eq.(26), the position of the mass center of the particle is known and determinant, which is just $<x'>=x_0'$ =constant, or 0. Since we here study only the rule of motion of the mass centre x_0, which means that the terms containing x_0' in $<x'^2>$ are effective, thus $<x'^2>$ =, $<x'><x'>$, then $<(x'-<x'>)^2>$ =0 can yield. Thus

$$\left\langle\frac{\partial V(x')}{\partial x'}\right\rangle=\frac{\partial V(<x'>)}{\partial<x'>} \tag{92}$$

Finally, we can get the acceleration of the mass center of the particle to be of the form

$$\frac{d^2}{dt'^2}<x'>=-2\frac{\partial V(<x'>)}{\partial<x'>} \text{ or } m\frac{d^2x_0}{dt^2}=-\frac{\partial V}{\partial x_0} \tag{93}$$

where x_0' =<x'> is the position of the mass centre of microscopic particle. Equation (93) is a Newton-type classical equation of motion. This shows clearly that the motion of the mass centre of microscopic particles satisfies the Newton law, when the microscopic particles are described by the nonlinear SchrÖdinger equation in Eq.(20). Therefore, we can say that the microscopic particle has some properties of the classical particle.

The above equation of motion of microscopic particles can also be derived from the nonlinear *SchrÖdinger* equation (20) by another method. As is known, the momentum of microscopic particle depicted by Eq.(20) is denoted by $P=\frac{\partial L}{\partial\phi}=-i\int_{-\infty}^{\infty}\left(\phi^*\phi_{x'}-\phi^*_{x'}\phi\right)dx'$. The energy $E(t')=\int_{-\infty}^{\infty}\left[\left|\frac{\partial\phi}{\partial x'}\right|^2-\frac{b}{2}|\phi\phi^*|^2+V(x')|\phi|^2\right]dx'$ and quantum number

$N_s = \int_{-\infty}^{\infty} |\phi|^2 dx'$ in this system are integral invariant. However, the momentum P is not conserved. From Eq.(20) we has

$$\frac{dP}{dt'} = \int_{-\infty}^{\infty} 2V(x')\frac{\partial}{\partial x'}|\phi|^2 dx' = -2\int_{-\infty}^{\infty} \frac{\partial V}{\partial x'}|\phi|^2 dx' = -2\left\langle \frac{\partial V(x')}{\partial x'} \right\rangle \tag{94}$$

where the boundary condition is $\phi(x') \to 0$ as $|x'| \to \infty$. Utilizing again Eqs.(88) and (92) we can get thet the acceleration of the mass center of the particle to be

$$\frac{dP}{dt'} = -2\frac{\partial V(x_0')}{\partial x_0'} \quad \text{or} \quad m\frac{d^2 x_0}{dt^2} = -\frac{\partial V}{\partial x_0} \tag{95}$$

where x_0' is the position of the center of the mass of the macroscopic particle. This is the same as Eq. (93). It resembles the Newton's equation for a classical particle.

ii. *Lagrangian and Hamilton Equations of microscopic particle*

Using the above variables ϕ and ϕ^* one can determine the Poisson bracket and write further the equations of motion of microscopic particles in the form of Hamilton's equations. For Eq. (20) with $V(\vec{r},t) = 0$, the variables ϕ and ϕ^* satisfy the Poisson bracket [42] :

$$\left\{\phi^{(a)}(x), \phi^{(b)}(y)\right\} = i\delta^{ab}\delta(x-y) \tag{96}$$

where

$$\{A, B\} = i\int_{-\infty}^{\infty}\left(\frac{\delta A}{\delta\phi}\frac{\delta B}{\delta\phi^*} - \frac{\delta B}{\delta\phi}\frac{\delta A}{\delta\phi^*}\right)$$

The corresponding Lagrangian density $\mathcal{L}$ in Eq. (70) associated with Eq. (20) can be written in terms of $\phi(x,t)$ and its conjugate ϕ^* viewed as independent variables. The action of the system can be written as

$$S(\phi, \phi^*) = \int_{t_b}^{t_1}\int_D L' dx dt \tag{97}$$

and its variation for infinitesimal $\delta\phi$ and $\delta\phi^*$ is of the form[42]

$$\delta S = \int_{t_B}^{t}\int_D \left[\frac{\partial L'}{\partial \phi}\delta\phi + \frac{\partial L'}{\partial \nabla\phi}\delta\nabla\phi + \frac{\partial L'}{\partial \phi_t}\delta\phi_t\right] dx\, dt + \text{c.c.} \tag{98}$$

where $L' = \mathcal{L}$, $\partial L'/\partial(\nabla\varphi)$ denotes the vector with components $\partial L'/\partial(\partial_i\phi)(i=1,2,3)$. After integrating by parts, we get

$$\delta S = \int_{t_b}^{t}\int_D \left[\frac{\partial L'}{\partial\phi} - \nabla\cdot\left(\frac{\partial L'}{\partial\nabla\phi}\right) - \partial_t\left(\frac{\partial L'}{\partial\phi_t}\right)\right]\delta\phi dx dt + \left[\frac{\partial L'}{\partial\phi_t}\delta\phi\right]_{t_0}^{t_1} + \text{c.c.} \tag{99}$$

A necessary and sufficient condition for a function $\phi(x,t)$ with known values $\phi(x,t_0)$ and $\phi(x,t_1)$ to yield an extremum of the action S is that it must satisfy the Euler-Lagrange equation:

$$\frac{\partial L'}{\partial\phi} = \nabla\cdot\left(\frac{\partial L'}{\partial\nabla\phi}\right) + \partial_t\left(\frac{\partial L'}{\partial\phi_t}\right) \tag{100}$$

Equation(100) can give the nonlinear SchrÖdinger equation (20) if the Lagrangian density Eq. (70) is used. Therefore, the dynamic equation, or the nonlinear *SchrÖdinger* equation can be derived from the Euler-Lagrange equation, if the Lagrangian function of the system is known. This is different from quantum mechanics, in which a dynamic equation, or the linear SchrÖdinger equation, cannot be obtained from the Euler-Lagrange equation.

The above derivation of the nonlinear *SchrÖdinger* equation based on the variational principle is a foundation for other methods, such as, the "the collective coordinates", the "variational approach", and the "Rayleigh-Ritz optimization principle", where a solution is assumed to maintain a prescribed approximate profile (often bell-type)[10-12]. Such methods greatly simplify the problem, reducing it to a system of ordinary differential equations for the evolution of a few characteristics of the systems.

The Hamiltonian density H corresponding to Eq.(20) is Eq.(71)[42]. Introducing the canonical variables,

$$q_1=\frac{1}{2}\left(\phi+\phi^*\right),\quad p_1=\frac{\partial L'}{\partial\left(\partial_t q_1\right)};q_2=\frac{1}{2i}\left(\phi-\phi^*\right),\quad p_2=\frac{\partial L'}{\partial\left(\partial_t q_2\right)}$$

where L' $=\mathcal{L}$, the Hamiltonian density takes the form

$$\mathcal{H}=\sum_i p_i\partial_t q_i - \mathcal{L}$$

and the corresponding variation of the Lagrangian density $\mathcal{L}$ = L' can be written as

$$\delta L' =\sum_i \frac{\delta L'}{\delta q_i}\delta q_i+\frac{\delta L'}{\delta\left(\nabla q_i\right)}\delta\left(\nabla q_i\right)+\frac{\delta L'}{\delta\left(\partial_t q_i\right)}\delta\left(\partial_t q_i\right) \tag{101}$$

From Eq.(101), the definition of p_i, and the Euler-Lagrange equation,

$$\frac{\partial L'}{\partial q_i}=\nabla.\frac{\partial L'}{\partial \nabla q_i}+\frac{\partial p_i}{\partial t}$$

one obtains the variation of the Hamiltonian in the form of

$$\delta H=\sum_i\int\left(\partial_t q_i\delta p_i-\partial_t p_i\delta q_i\right)dx$$

Thus, the Hamilton equation can be derived:

$$\frac{\partial q_i}{\partial t}=\delta\mathcal{H}/\delta p_i,\quad \frac{\partial p_i}{\partial t}=-\delta\mathcal{H}/\delta q_i \tag{102}$$

or in complex form:

$$i\hbar\frac{\partial\phi}{\partial t}=\frac{\delta H'}{\delta\phi^*},\text{ or } i\hbar\frac{\partial\phi^*}{\partial t}=-\frac{\delta H'}{\delta\phi}$$

This is interesting. It shows that the nonlinear *SchrÖdinger* equation describing the dynamics of microscopic particle can be also obtained from the classical Hamilton equation in the case, if the Hamiltonian of the system is known. Obviously, such methods of finding dynamic equations are impossible in the quantum mechanics. As is known, the Euler-Lagrange equation and Hamilton equation are important equations in classical theoretical (analytic) mechanics, and were used to describe laws of motions of classical particles. These equations are now used to depict properties of motions of microscopic particles. This shows sufficiently the classical features of microscopic particles described by nonlinear SchrÖdinger equation. On the other hand, from this study, we seek new ways of finding the equation of motion of the microscopic particles in nonlinear systems, i.e., if the Lagrangian or Hamiltonian of the system is known in the coordinate representation, then we can obtain the equation of motion of microscopic particles from the Euler-Lagrange or Hamilton equations.

On the other hand, from de Broglie relation[8-12] $E=h\upsilon=\hbar\omega$ and $\vec{p}=\hbar\vec{k}$ for microscopic particles which represent the wave-corpuscle duality in quantum theory, the frequency ω retains its role as the Hamiltonian of the system even in this complicated and nonlinear systems and

$$\frac{d\omega}{dt'}=\left.\frac{\partial\omega}{\partial k}\right|_{x'}\frac{dk}{dt}+\left.\frac{\partial\omega}{\partial x'}\right|_k\frac{\partial x'}{\partial t'}=0$$

as in the usual stationary media[18-19]. From the above result we also know that the usual Hamilton equation in Eq. (102) for the nonlinear systems remain valid for the microscopic particles. Thus, the Hamilton equation in Eq. (102) can be now represented by another form:

$$\frac{dk}{dt'} = -\frac{\partial \omega}{\partial x'}\Big|_k \text{ and } \frac{dx'}{dt'} = \frac{\partial \omega}{\partial k}\Big|_{x'} \tag{103}$$

in the energy picture, where $k = \partial\theta/\partial x'$ is the time-dependent wave number of the microscopic particle, $\omega = -\partial\theta/\partial t'$ is its frequency, θ is the phase of the wave function of the microscopic particles.

iii. *Confirmation of correctness of the above conclusions*

We now use some concrete examples to verify the correctness of the above laws of motion of microscopic particles.

(1). For $V=0$ or constants as shown in Eq.(34) we can get from Eq.(93) that $m\frac{d^2}{dt'^2}<x> = -\frac{\partial V(<x>)}{\partial <x>} = 0$. This shows that the acceleration of the mass centre of microscopic particle is zero, the velocity of the particle is a constant. In fact, if inserting Eq. (34) into Eq.(93) we can obtain $v_g = d<x'>/dt' = v_e$=constant, i.e., the microscopic particle moves exactly in uniform velocity in space-time in this case, the velocity is just the group velocity of the soliton. This shows that the energy and momentum of microscopic particle can be retained in the motion process.

On the other hand, from Eqs .(103) and (34) we can also get

$$\frac{dk}{dt} = 0 \text{ and } \frac{dx}{dt} = v_e$$

where

$$\omega = -\partial\theta/\partial t' = v_e v_c/2, \quad k = \partial\theta/\partial x' = v_c/2, \quad \theta = v_c\left[\sqrt{2m}\left(x - x_0\right) - v_e t\right]\Big/2\hbar$$

This result indicates that the acceleration of microscopic particle is zero, its velocity is a constant. This is same with those obtained from Eqs.(90) and (93). Thus the correctness of Eqs.(90), (102)- (103) and (93), or (95) are affirmed.

(2). For $V(x') = \alpha x'$ in Eq.(20), its solution is Eq.(52).This solution has also a envelop, carrier wave and mass centre x'_0, which is the localized position of the particle. The characteristics of motion of the microscopic particle can be determined according to Eq. (93). Its accelerations of the center of mass is given by

$$\frac{d^2 x'_0}{dt'^2} = -2\frac{\partial V\left(\langle x'\rangle\right)}{\partial\langle x'\rangle} = -2\alpha = \text{constant} \tag{104}$$

From Eq. (52) we know that

$$\theta = 2(\xi - \alpha t')x' + \frac{4\alpha^2 t'^3}{3} - 4\alpha\xi t'^2 + 4\left(\xi^2 - \eta^2\right)t' + \theta_0, \tag{105}$$

From Eq.(103) we can find

$$k = 2(\xi - at'),$$

$$\omega = 2ax' - 4\left(2\xi - at'\right)^2 + \left(2\eta\right)^2 = 2ax' - k^2 + \left(2\eta\right)^2.$$

Thus, the group velocity of the microscopic particle is

$$\upsilon_g = \frac{d\tilde{x}'}{dt'} = \frac{\partial \omega}{\partial k}\Big|_{x'} = 4\left(2\xi - at'\right), \tag{106}$$

and its acceleration is given by

$$\frac{d^2\tilde{x}'}{dt'^2} = \frac{dk}{dt'} = -2a = \text{cons t an t}, \;\; \text{here}(x_0' = \tilde{x}') \tag{107}$$

Comparing Eq.(104) with Eq.(107) we find that they are same, which indicates that Eqs.(90), (102)- (103) and (93), or (95) are correct. In such a case the microscopic particle moves in uniform acceleration. This is similar with that of classical particle.

(3). For the case of $V_0(x') = \alpha^2 x'^2$, which is a harmonic potential, where α is constant, the solution of Eq.(20) is Eq.(55). This solution has also a envelop, carrier wave and mass centre x'_0, which is the localized position of the particle in such a condition. The properties of motion of the microscopic particle can be determined by Eq.(93). Then its accelerations of the center of mass is given by[35-36]

$$\frac{d^2 x_0'}{dt'^2} = -4\alpha^2 x_0' \tag{108}$$

From Eq. (55), we gain that

$$\theta = 2\zeta x' \, cos\left[2a\left(t'-t_0'\right) + \left(\frac{\zeta^2}{a}\right) sin\, 4a\left(t'-t_0'\right) + \right] 4\eta^2\left(t'-t_0'\right) + \theta_0', \tag{109}$$

From Eqs.(103) and (110) we can find

$$k{=}2\xi \cos 2\alpha\left(t'-t_0'\right),$$

$$\omega{=}4a\xi x' \sin 2\alpha\left(t'-t_0'\right) - 4\xi^2 \text{co} \;\; 4\alpha\left(t'-t_0'\right) - 4\eta^2$$

$$= 2\alpha^{x}\left(4\xi^2 - k^2\right)^{1/2} - 2^{k2} + 4\left(\xi^2 - \eta^2\right),$$

Thus, the group velocity of the microscopic particle is

$$v_g = \left.\frac{\partial\omega}{\partial k}\right|_{x'} = \frac{\alpha x'}{\xi}\frac{k}{\sqrt{1-k^2/4\xi^2}} - 2k = 2\alpha x' \,\text{ctg}\left[2\alpha\left(t'-t_0'\right)\right] - 4\xi \cos\left[2\alpha\left(t'-t_0'\right)\right],$$

while its acceleration is

$$\frac{dk}{dt'} = -\left.\frac{\partial\omega}{\partial x'}\right|_k = -2\alpha\sqrt{4\xi^2 - k^2} = -4\xi\alpha \sin\left[2\alpha\left(t'-t_0'\right)\right]. \tag{110}$$

Since $\frac{d^2\tilde{x}'}{dt'^2} = \frac{dk}{dt'}$, here $\left(\tilde{x}' = x_0'\right)$, we have

$$\frac{dp}{dt'} = \frac{d^2\tilde{x}'}{dt'^2} = -4\zeta\alpha \sin\left[2\alpha\left(t'-t_0'\right)\right],$$

and

$$\tilde{x}' = \frac{2\xi}{\alpha}\sin\left[2\alpha\left(t'-t_0'\right)\right]. \tag{111}$$

Finally, the acceleration of the microscopic particle is

$$\frac{d^2\overline{x}'}{dt'} = \frac{dp^k}{dt'} = -4\alpha^2\overline{x}'. \tag{112}$$

We see clearly that Eq. (112) are exactly the same as Eq.(108). Thus we confirm the validity of Eqs.(90), (102)- (103) and (93), or (95) . In such a case the microscopic particle moves in harmonic form. This resembles also with the result of motion of classical particle.

Since quantum mechanics has a lot of difficult and troubles, when the linear SchrÖdinger equation is used to describe the microscopic particles. However, when a nonlinear SchrÖdinger equation is used to describe the microscopic particles we find their law of motion and properties are greatly changed relative to that of quantum mechanics. In such a case we find that the motion of microscopic particle satisfies classical rule and obeys the Hamiltonian principle, Lagrangian and Hamilton equations. We verify further the correctness of these conclusions by the results of nonlinear SchrÖdinger equation under actions of different externally applied potential. At the same time we discover that a macroscopic object moves with a uniform velocity at V(x')=0 or constant, moves in an uniform acceleration, when $V\left(x'\right)=ax'$, which corresponds to the motion of a charge particle in a uniform electric field, but when $V\left(x'\right)=\alpha^2 x'^2$ the macroscopic object performs localized vibration with a frequency of 2α and an amplitude of $2\xi/\alpha$, the corresponding classical vibrational equation is $x'=x_0' \sin\omega t'$, with $\omega=2a$ *and* $x_0'=\xi/\alpha$. The equations of motion of the macroscopic particles are consistent with Eq. (93) and Eqs. (102) – (103) for the center of mass of microscopic particles in nonlinear systems. These correspondence between a microscopic particle and a macroscopic object shows that microscopic particles described by the nonlinear SchrÖdinger equation have exactly the same properties as classical particles, and their motion satisfy the classical laws of motion. We have thus demonstrated clearly from the dynamic equations (nonlinear SchrÖdinger equation), the Hamiltonian or Lagrangian of the systems, and the solutions of equations of motion, systems, that microscopic particles described by the nonlinear SchrÖdinger equation in nonlinear systems really have the corpuscle property in both uniform and inhomogeneous. Therefore, we should use the nonlinear SchrÖdinger equation to describe microscopic particles and develop further a nonlinear quantum theory[18,28-34,].

e) The general conservation laws of motion of particles described by nonlinear SchrÖdinger equation

i. *Conservation laws of mass, energy and momentum of particles in Eq.(26)*

It is known from classical physics that the invariance and conservation laws of mass, energy and momentum and angular momentum are some elementary and universal laws of matter including classical particles in nature. We demonstrate here also that the microscopic particles described by the nonlinear SchrÖdinger equation also have such properties. They satisfy the conventional conservation laws of mass, momentum and energy. This shows that the microscopic particles in the nonlinear quantum mechanics have a corpuscle feature.

For the quantum systems described by nonlinear SchrÖdinger equation(20) we can define the number density, number current, densities of momentum and energy for the particle as[19,37-41]

$$\left.\begin{aligned} &\rho=|\phi|^2,\ p=-i\hbar(\phi^*\phi_x-\phi\phi_x^*) \\ &J=i\hbar(\phi^*\phi_x-\phi\phi_x^*), \in=\frac{\hbar^2}{2m}|\phi_x|^2-\frac{b}{2}|\phi\phi^*|^2+V(x)|\phi|^2 \end{aligned}\right\} \quad (113)$$

where. $\phi_x=\frac{\partial}{\partial x}\phi(x,t), \phi_t=\frac{\partial}{\partial t}\phi(x,t)$. From Eq.(20) and its conjugate equation (88) as well as Eqs.(70)- (72) and (113) we can obtain

$$\frac{\partial p}{\partial t'}=\frac{\partial}{\partial x'}[2(\frac{\partial\phi}{\partial x'})^2+(b|\phi\phi^*|^2-2V|\phi|^2-(\phi^*\frac{\partial^2}{\partial x'^2}\phi+\phi\frac{\partial^2}{\partial x'^2}\phi^*)+2iV(\phi^*\frac{\partial\phi}{\partial x'})],$$

$$\frac{\partial\rho}{\partial t'}=\frac{\partial J}{\partial x'},\frac{\partial\in}{\partial x'}=\frac{\partial}{\partial x'}[\rho p+i(\frac{\partial\phi^*}{\partial x'}\frac{\partial^2\phi}{\partial x'^2}-\frac{\partial\phi}{\partial x'}\frac{\partial^2\phi^*}{\partial x'^2})-iV(\phi^*\frac{\partial\phi}{\partial x'}-\phi\frac{\partial\phi^*}{\partial x'})]$$

Thus, we get the following forms for the integral of motion

$$\frac{\partial}{\partial t'}M=\frac{\partial}{\partial t'}\int\rho dx'=0,\frac{\partial}{\partial t'}P=\frac{\partial}{\partial t'}\int p dx'=0,\frac{\partial E}{\partial t'}=\frac{\partial}{\partial t'}\int\in dx'=0, \quad (114)$$

These formulae represent just the conservation of mass, momentum and energy in such a case. This shows that the mass, momentum and energy of the microscopic particles described by the nonlinear SchrÖdinger equation in Eq.(20) in the quantum systems still satisfy conventional rules of conservation of matter including the classical particles in physics. Therefore, the microscopic particles described by the nonlinear SchrÖdinger equation in Eq.(20) reflects the common rules of motions of matter in nature. In the case of V(x,t)=0 or constant, we can find out easily the values of mass, momentum and energy of the particles of Eq.(26)or (34) [13-17] , as are shown in Eq.(56). These results show also that the microscopic particles in such a case have a corpuscle feature.

We understand clearly from the above investigations the really physical significance of wave function $\phi(\vec{r},t)$ in this case. It can represent in truth the states and properties of microscopic particles, the $|\phi(x,t)|^2$ represents the number or mass density of particles, instead of the probability occurred at a point in place-time in quantum mechanics. Although the representation in Eq.(23) can also seek in quantum mechanics, its physical significances are completely different from that described by the nonlinear SchrÖdinger equation in Eq.(20). The φ and θ are two independent physical quantities and denote the amplitude and phase of wave in quantum mechanics, respectively, but φ(x,t) and $\theta(x,t)$ in Eq.(23) represent the two different states of motion for envelope and carrier waves in the systems described by the nonlinear SchrÖdinger equation. From Eqs.(27)-(28) we see that the envelope and carrier waves are correlated with each other. Just the correlation the microscopic particles move in a soliton in the systems, thus wave-corpuscle duality of microscopic particles can occur. Therefore the wave functions of the particles have different physical significances in the two cases.

ii. *The invariance and conservation laws of particles in Eq.(20)*

We have learned from Eqs.(113)−(114) that some conservation laws for microscopic particles described by the nonlinear *SchrÖdinger* equation (20) are always related to the invariance of the action relative to several groups of transformations through the Noether theorem in light of Gelfand and Fomin's (1963) and Bulman and its Kermel's (1989) ideas (see C. Sulem and P. L. Sulem *et al*.'s book and references therein[42]). Therefore, we first give the Noether theorem for nonlinear *SchrÖdinger* equation.

To simplify the equation, we introduce the following notations:

$$\bar{\xi}=(t,x)=(\xi_0,\xi_1,\cdots,\xi_a)\ \partial_0=\partial_t,\partial=(\partial_0,\partial_1,\cdots,\partial_d)\ \text{and}\ \Phi=(\Phi_1,\Phi_2)=(\phi,\phi^*).$$

According to the Lagrangian Eq. (70) corresponding the nonlinear *SchrÖdinger* equation(20), then the action of the system[42]

$$S\{\phi\}=\int_{t_0}^{t_1}\int L'\left(\phi,\nabla\phi,\phi_t,\phi^*,\nabla\phi^*,\phi_t^*\right)dxdt$$

where L' = $\mathcal{L}$ is the Lagrange density function, now becomes

$$S\{\phi\}=\int_D\int_{x'}^{\infty}L'(\Phi,\partial\Phi)\,d\,\bar{\xi} \tag{115}$$

Under the action of a transformation T^{ε} which depends on the parameter ε, we have $\bar{\zeta}\to\tilde{\zeta}(\bar{\zeta},\Phi,\varepsilon),\Phi\to\tilde{\Phi}(\bar{\zeta},\Phi,\varepsilon)$, where $\tilde{\zeta}\, and\,\tilde{\Phi}$ are assumed to be differentiable with respect to ε. When $\varepsilon=0$, the transformation reduces to the identity. For infinitesimally small ε, we have $\tilde{\zeta}=\bar{\zeta}+\delta\varepsilon,\tilde{\Phi}=\Phi+\delta\Phi$. At the same time, $T^{\varepsilon},\Phi(\bar{\zeta})\to\tilde{\Phi}(\tilde{\zeta})$ by the transformation T^{ε}, and the domain of integration D is transformed into $\tilde{D}$,

$$S\{\phi\}\to\tilde{S}\{\tilde{\phi}\}=\int_D\int_{x^1}^{\infty}L'\left(\tilde{\Phi},\tilde{\partial}\tilde{\Phi}\right)d\,\tilde{\zeta}$$

where $\tilde{\partial}$ denotes differentiation with respect to $\tilde{\zeta}$. The change $\delta S=\tilde{S}\{\tilde{\phi}\}-S\{\phi\}$ in the limit of ε under the above transformation can be expressed as

$$\delta S=\int_D\int_{x^1}^{\infty}\left[L'\left(\tilde{\Phi},\tilde{\partial}\tilde{\Phi}\right)-L'\left(\tilde{\Phi},\partial\tilde{\Phi}\right)\right]d\,\bar{\xi}+\int_D\int_{x^1}^{\infty}L'(\Phi,\partial\Phi)\sum_{\upsilon=0}^{d}\frac{\partial\delta\xi_{\upsilon}}{\partial\xi_{\upsilon}}d\,\bar{\xi} \tag{116}$$

where we used the Jacobian expansion $\frac{\partial\left(\bar{\xi}_0,\ldots,\bar{\xi}_d\right)}{\partial\left(\xi_0,\ldots,\xi_d\right)}=1+\sum_{v=0}^{d}\frac{\partial\delta\xi_\upsilon}{\partial\xi_\upsilon}$, and $L'\left(\tilde{\Phi},\tilde{\partial}\tilde{\Phi}\right)$, in the second term on the right-hand side has been replaced by the leading term $L'(\Phi,\partial\Phi)$ in the expansion. Now define

$$\delta\tilde{\Phi}_i=\tilde{\Phi}_i\left(\bar{\xi}\right)-\tilde{\Phi}_i\left(\xi\right)=\partial_\upsilon\Phi_i\delta\xi_\upsilon+\delta\Phi_i\left(\xi\right)$$

$$\tilde{\partial}_\upsilon\tilde{\Phi}_i\left(\bar{\xi}\right)-\partial_\upsilon\Phi_i\left(\xi\right)=\left(\tilde{\partial}_\upsilon-\partial_\upsilon\right)\tilde{\Phi}_i\left(\bar{\xi}\right)+\partial_\upsilon\left[\tilde{\Phi}_i\left(\bar{\xi}\right)-\Phi_i\left(\xi\right)\right] \tag{117}$$

with.

$$\partial_\upsilon=\frac{\partial\bar{\xi}_\mu}{\partial\xi_\upsilon}\tilde{\partial}_\mu=\left(\delta_{\upsilon\mu}+\frac{\partial\delta\xi_\mu}{\partial\xi_\upsilon}\right)\tilde{\partial}_\mu=\tilde{\partial}_\upsilon+\frac{\partial\delta\xi_\mu}{\partial\xi_\upsilon}\tilde{\partial}_\mu$$

We then have

$$L'\left(\tilde{\Phi},\tilde{\partial}\tilde{\Phi}\right)-L'(\Phi,\partial\Phi)=\frac{\partial L'}{\partial\Phi_i}\left[\tilde{\Phi}_i\left(\bar{\xi}\right)-\Phi_i\left(\xi\right)\right]+\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\left[\tilde{\partial}_\upsilon\tilde{\Phi}_i\left(\bar{\xi}\right)-\partial_\mu\Phi_i\left(\xi\right)\right]$$

$$=\frac{\partial L'}{\partial\Phi_i}\partial\Phi_i+\partial_\mu\left(L'\delta\xi_\upsilon\right)-L'\frac{\partial\delta\xi_\upsilon}{\partial\xi_\upsilon}+\partial_\upsilon\left[\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\right]\delta\Phi_i-\partial_\mu\left[\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\right]\delta\Phi_i$$

Eq. (116) can now be replaced by

$$\delta S=\int_D\int_{x'}^{\infty}\left\{\frac{\partial L'}{\partial\Phi_i}-\frac{\partial}{\partial\xi_\upsilon}\left[\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\right]\right\}\delta\Phi_i\,\mathrm{d}\bar{\xi}+\int_D\int_{x'}^{\infty}\frac{\partial}{\partial\xi_\upsilon}\left[L'\delta\xi_\upsilon+\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\delta\Phi_i\right]\mathrm{d}\bar{\xi}$$

where we have used

$$\frac{\partial}{\partial\xi_\upsilon}\left(L'\delta\xi_\upsilon\right)=L'\frac{\partial\delta\xi_\upsilon}{\partial\xi_\upsilon}+\frac{\partial L'}{\partial\Phi_i}\partial_\upsilon\Phi_i\delta\xi_\upsilon+\frac{\partial^2L'}{\partial\left(\partial_\mu\Phi_i\right)}\partial^2{}_{\upsilon\mu}\Phi_i\delta\xi_\upsilon,$$

$$\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\partial_\upsilon\int_{x'}^{\infty}\delta\Phi_i\frac{\partial}{\partial\xi_\upsilon}\left[\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\delta\Phi_i\right]-\frac{\partial}{\partial\xi_\upsilon}\left[\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\delta\Phi_i\right]\delta\Phi_i$$

Using the Euler-Lagrange equation, the first term on the right-hand side in the equation of δS vanishes. We can get the Noether theorem[42], ***i.e.,*** (A) if the action Eq. (115) is invariant under the infinitesimal transformation of the dependent and independent variables $\phi\to\phi+\delta\phi,\bar{\xi}\to\bar{\xi}+\delta\bar{\xi}$ where $\bar{\xi}=\left(t,x_1\ldots x_d\right)$, the following conservation law holds[28-29]

$$\frac{\partial}{\partial\xi_\upsilon}\left[L'\delta\xi_\upsilon-\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\delta\Phi_i\right]=0,\text{or,}\frac{\partial}{\partial\xi_\upsilon}\left[L'\delta\xi_\upsilon+\frac{\partial L'}{\partial\left(\partial_\upsilon\Phi_i\right)}\left(\delta\Phi_i-\frac{\partial\Phi_i}{\partial\zeta_\mu}\delta\xi_\mu\right)\right]=0 \tag{118}$$

in terms of $\delta\hat{\Phi}_i$ defined above, where L’= *L*.

If the action is invariant under the infinitesimal transformation

$$t\to\bar{t}=t+\delta t\left(x,t,\phi\right),x\to\bar{x}=x+\delta x\left(x,t,\phi\right),$$

$$\phi(x,t) \to \overline{\phi}(\overline{t},\overline{x}) = \phi(t,x) + \delta\phi(t,x),$$

then

$$\int\left[\frac{\partial L'}{\partial \phi_t}\left(\partial_t \phi \partial_t + \nabla\phi\cdot\delta\vec{x} - \delta\phi\right) + \frac{\partial L'}{\partial \phi_t^*}\left(\partial_t \phi^* \partial_t + \nabla\phi^*\cdot\delta\vec{x} - \delta\phi^*\right) - L\delta t\right]dx$$

is a conserved quantity.

For the nonlinear *SchrÖdinger* equation (20) we have

$$\frac{\partial L'}{\partial \phi_t} = \frac{i}{2}\phi^*, \text{ and } \frac{\partial L'}{\partial \phi_t^*} = -\frac{i}{2}\phi$$

where L' = $\mathcal{L}$ is given in Eq.(70). Several conservation laws and invariance can be obtained from the Noether theorem.

(a) Invariance under time translation and energy conservation law

The action, Eq.(115), is invariant under the infinitesimal time translation $t \to t + \delta t$ with $\delta x = \delta\phi = \delta\phi^* = 0$, then equation (118) becomes

$$\partial_t\left[\nabla\phi\cdot\nabla\phi^* - \frac{b}{2}\left(\phi\phi^*\right)^2 + V(x,t)\phi^*\phi\right] - \nabla\cdot\left(\phi_t\nabla\phi^* + \phi_t^*\nabla\phi\right) = 0$$

This results in the conservation of energy

$$E = \int\left(\nabla\phi\cdot\nabla\phi^* - \frac{b}{2}\left(\phi^*\phi\right)^2 + V(x,t)\phi^*\phi\right)dx = constant \quad (119)$$

(b) Invariance of the phase shift or gauge invariance and mass conservation law

It is very clear that the action related to the nonlinear *Schrödinger* equation is invariant under the phase shift $\overline{\phi} = e^{i\theta}\phi$, which for infinitesimal θ gives $\delta\phi = i\theta\phi$, with $\delta t = \delta x = 0$. In this case, equation (118) becomes

$$\partial_t|\phi|^2 + \nabla\left\{i\left(\phi\nabla\phi^* - \phi^*\nabla\phi\right)\right\} = 0 \quad (120)$$

This results in the conservation of mass or number of particles.

$$N = \int|\phi|^2 dx = \text{constant}$$

and the continuum equation

$$\frac{\partial N}{\partial t} = \nabla\cdot\vec{j},$$

where $\vec{j}$ is the mass current density

$$\vec{j} = -i\left(\phi\nabla\phi^* - \phi^*\nabla\phi\right)$$

(c) Invariance of space translation and momentum conservation law

If the action is invariant under an infinitesimal space translation $x \to x + \delta x$ with $\delta t = \delta\phi = \delta\phi^* = 0$, then Eq.(118) becomes

$$\partial_t\left[i\left(\phi\nabla\phi^* - \phi^*\nabla\phi\right) + \nabla\cdot\left\{2\left(\nabla\phi^*\times\nabla\phi + \nabla\phi\times\nabla\phi^* + \mathcal{L}\right)\right\}\right] = 0$$

This leads to the conservation of momentum

$$\vec{P} = i\int \left(\phi\nabla\phi^* - \phi^*\nabla\phi\right)dx = \text{constant.} \tag{121}$$

Note that the center of mass of the microscopic particles is defined by

$$\langle x\rangle = \frac{1}{N}\int x|\phi|^2\, dx,$$

We then have

$$N\frac{d\langle x\rangle}{dt} = \int x\partial_t|\phi|^2\, dx = -\int x\nabla\left[i\left(\phi\nabla\phi^* - \phi^*\nabla\phi\right)\right]dx$$

$$= \int i\left(\phi\nabla\phi^* - \phi^*\nabla\phi\right)dx = \vec{P} = -\vec{J} = -\int \vec{j}dx \tag{122}$$

This is the definition of momentum in classical mechanics. It shows clearly that the microscopic particles described by the nonlinear *Schrödinger* equation have the feature of classical particles.

(d) Invariance under space rotation and angular momentum conservation law

If the action, Eq. (115), is invariant under a rotation of angle $\delta\theta$ around an axis $\vec{I}$ such that $\delta t = \delta\phi = \delta\phi^* = 0$ and $\delta\vec{x} = \delta\theta\vec{I}\times\vec{x}$, this leads to the conservation of the angular momentum

$$\vec{M} = i\int \vec{x}\times\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right)dx$$

Besides the above, Sulem also derived another invariance of the nonlinear *Schrödinger* equation the from the Noether theorem for nonlinear SchrÖdinger equation.

(e) Galilean Invariance

If the action is invariant under the Galilean transformation:

$$x \to x'' = x - \upsilon t, t \to t'' = t,$$

$$\phi(x,t) \to \phi''(x'',t'') = exp\{-i\left[\frac{1}{2}\upsilon x + \frac{1}{2}\vec{\upsilon}\cdot\vec{\upsilon}t\right]\}\phi(x,t),$$

which can also retain the nonlinear *Schrödinger* equation invariance. For an infinitesimal velocity $\upsilon, \delta\vec{x} = -\upsilon t, \delta t = 0$ and $\delta\hat{\phi} = \phi''(x'',t'') - \phi(x,t) = -(i/2)\upsilon x\phi(x,t)$. After integration over the space variables, equation (118) leads to the conservation law Eq. (122) which implies that the velocity of the center of mass of the microscopic particles is a constant. It is also the same, even though the particle is in motion. This exhibits clearly that the microscopic particles have the particulate nature.

f) Classical natures of collision of microscopic particles

i. The features of collision of the microscopic particle at b>0 in Eq.(20)

As is known, the most obvious feature of macroscopic particles is meeting the collision law or conservation law of momentum. Therefore, we often also use the law to determine the particulate feature of macroscopic particles. As a matter of fact, Zakharov *et al.*[25-26] used the inverse scattering method to find out the following solution of Eq.(26) at b=1>0. It now is denoted by

$$\phi_s(x',t') = 2\sqrt{2}\eta\,\mathrm{sec}\,h\{2\eta(x'-x_0') - 8\eta\xi t'\}\exp\{2i\xi x' - i4(\xi^2-\eta^2)t' + i\theta\} \tag{123}$$

where, $2\sqrt{2}\eta$ is the amplitude, $2\sqrt{2}\xi$ denotes the velocity, θ is its phase. At the same time, they studied further the collision feature of two microscopic particles based on the solution (123). From this study they obtain that the translations of mass centre $x_0^{'+}$ and phase θ^+ of each particles after collision can, respectively, represent by

$$x_{0m}^{'+} - x_{0m}' = \frac{1}{\eta_m}\prod_{p=m+1}^{N}\left|\frac{\varsigma_m - \varsigma_p}{\varsigma_m - \varsigma_p^*}\right| < 0, \text{ and } \theta_m^+ - \theta_m = -2\prod_{p=m+1}^{N}\arg\left(\frac{\varsigma_m - \varsigma_p}{\varsigma_m - \varsigma_p^*}\right)$$

where η_m and ς_m are some constants related to the amplitude and velocity of m[th] particles. The equations show that shift of position of mass centre of the particles and their variation of phase are a constants after collision. The collision process of two particles with different velocities and amplitudes can be described as follows. In the case of $t'\to-\infty$ the slowest soliton is in the front while the fastest at the rear, they collide with each other at t'=0, after the collision and $t'\to\infty$, they are just reversed. Thus Zakharov *et al.*[25-26] obtained that as the time *t* varies from $-\infty$ to ∞, the relative change of mass centre of two particles, $\Delta x'_{0m}$, and the relative change of their phases can, respectively, denoted by

$$\Delta x'_{0m}=x'^{+}_{0m}-x'^{-}_{0m}=\frac{1}{\eta_m}\left(\sum_{k=m+1}^{N}\ln\left|\frac{\varsigma_m-\varsigma_p}{\varsigma_m-\varsigma_p^*}\right|-\sum_{k=1}^{N-1}\ln\left|\frac{\varsigma_m-\varsigma_p}{\varsigma_m-\varsigma_p^*}\right|\right) \tag{124}$$

and

$$\Delta\theta_m=\theta_m^{+}-\theta_m=2\prod_{k=1}^{m-1}\arg\left(\frac{\varsigma_m-\varsigma_p}{\varsigma_m-\varsigma_p^*}\right)-2\prod_{k=m+1}^{N}\arg\left(\frac{\varsigma_m-\varsigma_p}{\varsigma_m-\varsigma_p^*}\right) \tag{125}$$

Equation (124) can be interpreted by assuming that the microscopic particles collide pairwise and every microscopic particle collides with others. In each paired collision, the faster microscopic particle moves forward by an amount of $\eta_m^{-1}\ln\left|\left(\varsigma_m-\varsigma_k^*\right)/\varsigma_m-\varsigma_k\right|$, $\varsigma_m>\varsigma_k$, and the slower one shifts backwards by an amount of $\eta_k^{-1}\ln\left|\left(\varsigma_m-\varsigma_k^*\right)/\varsigma_m-\varsigma_k\right|$. The total shift is equal to the algebraic sum of their shifts during the paired collisions. So that there is no effect of multi-particle collisions at all. In other word, in the collision processin each time the faster particle moves forward by an amount of phase shift, and the slower one shifts backwards by an amount of phase. The total shift of the particles is equal to the algebraic sum of those of the pair during the paired collisions. The situation is the same with the phases. This rule of collision of the microscopic particles described by the nonlinear SchrÖdinger equation is the same as that of classical particles, or speaking, meet also the collision law of macroscopic particles, i.e., during the collision these microscopic particles interact and exchange their positions in the space-time trajectory as if they had passed through each other. After the collision, the two microscopic particles may appear to be instantly translated in space and/or time but otherwise unaffected by their interaction. The translation is called a phase shift as mentioned above. In one dimension, this process results from two microscopic particles colliding head-on from opposite directions, or in one direction between two particles with different amplitudes. This is possible because the velocity of a particle depends on the amplitude. The two microscopic particles surviving a collision completely unscathed demonstrates clearly the corpuscle feature of the microscopic particles. This property separates the microscopic particles (solitons) described by nonlinear SchrÖdinger equation from the particles in the quantum mechanical regime. Thus this demonstrates the classical feature of the microscopic particles.

Using the above features of collision of two particles and following the approach of Zakharov and Shabat[25-26], Desem and Chu[43-44] obtained a solution corresponding to two discrete eigenvalues $\varsigma_{1,2}$ for the interacting two microscopic particles in the process of collision, which is represented by

$$\phi(x',t')=\frac{|\alpha_1|\cosh(a_1+i\theta_1)e^{i\theta_2'}+|\alpha_2|\cosh(a_2+i\theta_2)e^{i\theta_1'}}{\alpha_3\cosh(a_1)\cosh(a_2)-\alpha_4\left[\cosh(a_1+a_2)-\cos(A')\right]} \tag{126}$$

where $\theta'_{1,2}=2\left[2\left(\eta_{1,2}^2-\xi_{,2}^2\right)t'-x'\xi_{1,2}\right]+\left(\theta'_0\right)_{1,2}$, $A'=\theta'_2-\theta'_1+\left(\theta_2-\theta_1\right)$,

$$\alpha_{1,2}=2\eta_{1,2}\left(x'+4t'\xi_{1,2}\right)+\left(a_0\right)_{1,2},\ |\alpha_{1,2}|e^{i\theta_{1,2}}=\pm\left\{\left[\frac{1}{2\eta_{1,2}}-\frac{\eta}{\left(\Delta\xi^2+\eta^2\right)}\right]\pm i\frac{\Delta\xi}{\left(\Delta\xi^2+\eta^2\right)}\right\}$$

$$\alpha_3=\frac{1}{4\eta_1\eta_2},\alpha_4=\frac{1}{2\left(\eta^2+\Delta\xi^2\right)},\ \varsigma_{1,2}=\xi_{1,2}+i\eta_{1,2},\Delta\xi=\xi_2-\xi_1,\eta=\eta_1+\eta_2,$$

here η and ξ are the same as those in Eq. (123), and represent the velocities and amplitudes of the microscopic particle, $(a_0)_{1,2}$ the position, and $(\theta_0')_{1,2}$ the phase. They are all determined by the initial conditions.

Of particular interest here is an initial pulse waveform,

$$\phi(0,x') = \sec h(x' - x_0') + \sec h(x' + x_0')e^{i\theta} \tag{127}$$

which represents the motion of two microscopic particles into the system. Equation (127) will evolve into two particles described by Eq.(126) The interaction between the two microscopic particles given in Eq.(127) can therefore be analyzed through the two-particle function in Eq.(126). Given the initial separation x_0', phase difference θ between the two microscopic particles, the eigenvalues $\varsigma_{1,2}$,a_0 and θ'_0 can be evaluated by solving the Zakharov and Shabat equation (194), using Eq.(128) as the initial condition. Substituting the eigenvalues obtained into Eq.(126), we then obtain the description of the interaction between the two microscopic particles.

The two microscopic particles described by Eq.(127) interact through a periodic potential in t', through the $\cos A'$ term. The period is given by $\pi/(\eta_2^2 - \eta_1^2)$. The propagation of two microscopic particles with the initial conditions $\theta = 0$, $\xi_1 = \xi_2 = 0$, $(\theta_0')_1 = (\theta_0')_2 = 0$, obtained by Desem and Chu[43-44] is shown in Fig.3. The two microscopic particles with initially separated by x_0' coalesce into one microscopic particle at $A' = \pi$. Then they separate and revert to the initial state with separation x_0' at $A' = \pi$, and so on. An approximate expression for the separation between the microscopic particles as a function of the distance along the system can be obtained provided the two microscopic particles are well resolved. Assuming that the separation between the particles is sufficiently large, one can obtain the separation Δx as $\Delta x = \ln\left[\frac{2}{a}\left|\cos(at')\right|\right], a = 2e^{-x_0'}$. Thus the period of oscillations is approximately given by $t_p' = (\pi/2)e^{x_0'}$.

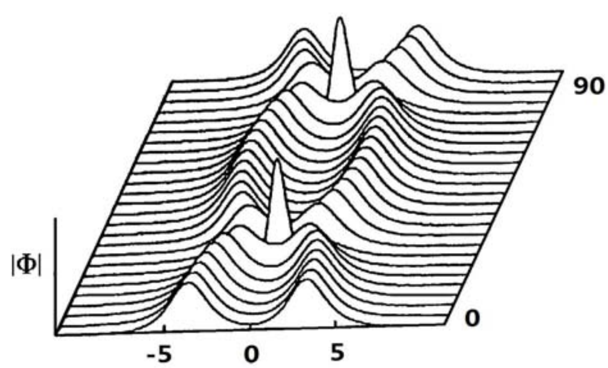

Fig. 3: Interaction with two equal amplitude microscopic particles. Initial microscopic particles separation=3.5 pulse width (pw)

The above collision features of the microscopic particle are obtained by using the inverse scattering method. However, the properties of collisions of microscopic particles can obtained by numerically solving Eq.(20). Numerical simulation can reveal more detailed feature of collision between two microscopic particles. For this purpose we begin by dividing Eq.(20) with V=constant into the following two-equations[45]

$$i\frac{\partial \phi'}{\partial t'} + \frac{\partial^2 \phi'}{\partial x'^2} = \phi' u, \quad \frac{\partial^2 u}{\partial t'^2} - \frac{\partial^2 u}{\partial x'^2} = \frac{\partial^2}{\partial x'^2}\left(|\phi'|^2\right). \tag{128}$$

Obviously, if $\xi = x' - vt'$ is assumed, we can get the nonlinear constant $b = 1/(1-v^2)$ in Eq.(20) at $\lim_{|x'|\to\infty} \phi_{x'}' = 0$,where ϕ' represents the state of a microscopic particle, then u denotes a background field or other particles, b is related to the velocity of particle v. The soliton solution of (128) can now be written as

$$\phi' = \sqrt{2(1-v^2)}\eta \sec h\left[\eta(x' - x_0' - vt')\right]\exp\left[\frac{i}{2}vx' - i\left(\frac{v^2}{4} - \eta^2\right)t' + i\theta\right]$$

$$u = -2\eta^2 \sec h^2\left[\eta(x' - x_0' - vt')\right].$$

The properties of the soliton depend on three parameters: η, v and θ, where η and v determine the amplitude and width of the particle, θ is the phase of the sinusoidal factor of ϕ' at $t'=0$. Tan et al.[45] carried out numerical simulation for the collision process between two particles using the Fourier pseudo-spectral method with 256 basis functions for the spatial discretization together with the fourth-order Runge-Kutta method for time-evolution. The system given in Eq.(128) has two exact integrals of motion, $N_s = \int_{-\infty}^{\infty} |\phi'|^2 dx'$ and $E_1 = \int_{-\infty}^{\infty} u dx'$, which can be used to check the accuracy of the numerical solutions.

For the collision experiments, the initial state is two solitary waves separated by distance x_0',

$$\phi' = \sqrt{2(1-v_1^2)}\eta_1 \sec h[\eta_1 x'] \exp\left[\frac{i}{2} v_1 x' + i\theta_1\right] +$$

$$\sqrt{2(1-v_2^2)}\eta_2 \sec h\left[\eta_2 (x' + x_0')\right] \exp\left[-\frac{v^2}{2}(x_2' + x_0') + i\theta_2\right],$$

$$u = -2\eta_1^2 \sec h^2(\eta_1 x') - 2\eta_2^2 \sec h^2\left[\eta_2 (x' + x_0')\right].$$

where the first term in each expression represents one particle (1) while the second term represents the other particle(2). It can be shown that the post-collision state of the particles is strongly dependent on both the initial phases and the initial velocities of the particles. Since ϕ' can be multiplied by an arbitrary phase factor, $\exp(i\theta\%)$, where $\theta\%$ is an arbitrary constant, and still remains a solution (which u unchanged), one of the phases is arbitrary, and only the difference of the two initial phases is significant. Thus we can set $\theta_1 = 0$ for the convenience of discussion.

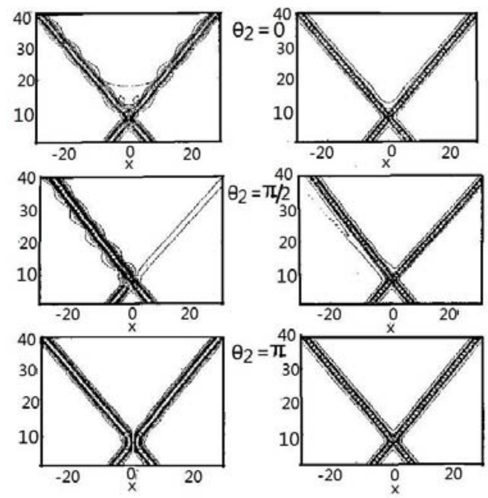

Fig. 4: Fast collisions of two microscopic particles. The initial ratio of velocities of the fast and slow particles to be1.8

Fig.4 shows the fast collisions obtained by Tan et al[45]., in which the initial ratio of velocities of the fast and slow particles is fixed to be 1.8. The absolute value of the ϕ' are shown using contours on the left in each pair of plots, with x' being the horizontal coordinate and t' increasing upward. The right panel shows the absolute value of u. The relative phase increases from 0 (top) to $\pi/2$ (middle) to π (bottom). All cases are identical except that θ_2 is increased by $\pi/2$ in each case, beginning with $\theta_2 = 0$ at the top. Before the collision, each of the initial particle contributes 0.7600 to N_s in all three cases ($\theta_2 = 0$, $\pi/2$ and π). When the relative phase is zero (top), the particles penetrate each other freely and then emerge with their shapes and velocity unchanged. When $\theta_2 = \pi/2$ (middle graphs), ϕ emerges from the collision asymmetrically, and a large particle which contributes 1.4272, moving to the left, at the same velocity as the initial speed of particle 2. Another small pulse, contributing 0.0928, travels to the right at the speed which is the same as the initial speed of particle 1. The post-collision energies are the same as those of pre-collision for ϕ' when $\theta_1 = 0$ and $\theta_2 = \pi$. For all values of θ_2, there is little change in the contributions of the particles in their u –field to energy E_1, and they are not shown here. When $\theta_2 = \pi$, as shown in the bottom panel of Fig.4, the u–components penetrate freely, but the ϕ'–components bounce off each other and change their directions, without interpenetration. The fourth case, $\theta_2 = 3\pi/2$, is not shown here because it is just the mirror image of the middle figure. That is

$$\phi'\left(x',t',\theta_2=\frac{3\pi}{2}\right)=\phi'\left(-x',t',\theta_2=\frac{\pi}{2}\right),\text{ and } u\left(x',t',\theta_2=\frac{3\pi}{2}\right)=u\left(-x',t',\theta_2=\frac{\pi}{2}\right).$$

The same is true for intermediate and slow collisions processes. However, the reflection principle cannot be generalized to all solutions which are different in their initial phases by π because the cases of $\theta_2 = 0$ and $\theta_2 = \pi$ are quite different in the general case.

Pang et al[46-48]. further simulated numerically the collision behaviors of two particles described nonlinear Schrodiger equation (20) at V(x)=constant using the fourth-order Runge_Kutta method. This result is shown in Fig.5. From figures 4-5, we see clearly that the two particles can go through each other while retaining their form after the collision, which is the same with that of the classical particles. Therefore, the microscopic particles depicted by the nonlinear SchrÖdinger equation (20) have an obvious corpuscle feature, their collision show the featuresof collision of classical particles. Thus we can conform that microscopic particles described by nonlinear SchrÖdinger equation.(20) has the corpuscle property.

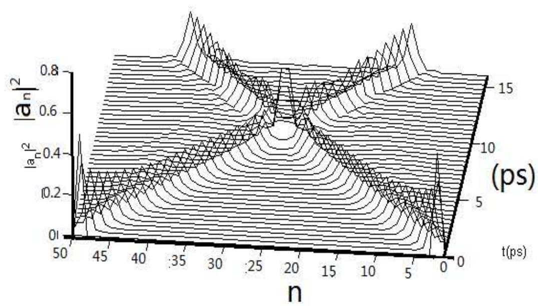

Fig. 5: The features of collision of microscopic particles

ii. *The rules of collision of microscopic particle at b<0 in Eq.(20)*

As known, in the case of $b<0$ the equation (20) has still soliton solution, when $\lim_{|x'|\to\infty}|\phi(x',t')|^2 \to$ constant, and $\lim_{|x'|\to\infty}\phi_{x'}=0$. The solutions are dark (hole) solitons, in contrast to the bright soliton when $b>0$. The bright soliton was observed experimentally in focusing fibers with negative dispersion, and the hole soliton solution was observed in defocusing fibers with normal dispersion effect by Emplit *et al*. and Krokel. In practice, it is an empty state without matter in microscopic world. Therefore $b>0$ corresponds to attractive between Bose particles, and $b<0$ corresponds to repulsive interaction between them. Thus, reversing the sign of *b* not only leads to changes in the physical picture of the phenomenon described by the nonlinear SchrÖdinger equation (20), but also requires considerable restructuring of the mathematical formalism for its solution. Solution of the nonlinear SchrÖdinger equation must be again analyzed and the collision rules of the microscopic particles in the case of $b<0$ must be studied separately using the inverse scattering method. Zakharov and Shabat[25-26] studied these problems. The soliton solution in such a case can be represented by

$$\frac{b}{2}|\phi(x',t')|^2 = 1-\frac{\nu^2}{\cosh^2[\nu(x'-x_0'-2\lambda t')]}$$

The parameter λ characterizes the amplitude ans velocity of the microscopic particles, and x_0' the position of its centre at t'=0, where $d(\ln\mu)/dt'=2\lambda\nu$ or $\ln\mu = 2\nu(x_0'+2\lambda t')$.

They studied further the features of collision described by the above formula. The displacements of the two microscopic particles after the collision were found to be

$$\delta x_1' = \frac{1}{2v_1}\ln\left(\frac{b_1^+}{b_1^-}\right)=\frac{1}{2v_1}\ln\left(\frac{1}{|a_2(\lambda_1,iv_1)|^2}\right)=\frac{Y}{2v_1},$$

$$\delta x_2' = \frac{1}{2v_1}\ln\left(\frac{b_2^+}{b_2^-}\right)=\frac{1}{2v_1}\ln\left(|a_1(\lambda_2,iv_2)|^2\right)=-\frac{Y}{2v_2}, \qquad (129)$$

where $Y = \ln\left[\frac{(\lambda_1-\lambda_2)^2+(v_1+v_2)^2}{(\lambda_1-\lambda_2)^2+(v_1-v_2)^2}\right]$.

Thus, the microscopic particle which has the greater velocity acquires a positive shift, and the other has only a negative shift. The microscopic particles behave like repelling each other of classical particles. From (129) we get[25-26]

$$v_1\delta x_1^{'}+v_2\delta x_2^{'}=0.$$

This relation was also obtained by Tsuzuki directly from Eq.(20) for $b<0$ by analyzing the motion of the center of mass of a Bose gas. It can be interpreted as the conservation of mass centre of the microscopic particles during the collision. This shows sufficiently the classical feature of microscopic particles described by Eq.(20).

The collision of many particles can be studied similarly. The result obtained shows that in the case of $t'\to-\infty$ the slowest soliton is in the front while the fastest at the rear, faster microscopic particle tracks the slower microscopic particle, they collide with each other at t'=0, after the collision and $t'\to\infty$, they are just reversed. So that each particle collides with each other particle. Going through the same analysis given above, we can verify that the total displacement of a particle, regardless of the details of the collisions, is equal to the sum of the displacements in individual collisions[25-26], i.e., $\delta_j = x_j^{'+}-x_j^{'-}=\sum_{i=1}\delta_{ij}$

where $\delta_{ij}=sign(\lambda_i-\lambda_j)\frac{1}{2v_i}\ln\left[\frac{(\lambda_i-\lambda_j)^2+(v_i+v_j)^2}{(\lambda_i-\lambda_j)^2+(v_i-v_j)^2}\right]$.

From the above studies we see that collisions of many microscopic particles described by the nonlinear SchrÖdinger equation (20), with both $b>0$ or $b<0$, satisfy rules of classical physics. This shows sufficiently the corpuscle feature of microscopic particles described by nonlinear SchrÖdinger equation.

iii. *The mechanism and properties of collision of microscopic particles at b<0*

In the following, we describe a series of laboratory and numerical experiments dedicated to investigate the detailed structure, mechanism and rules of collision between the microscopic particles described by the nonlinear Schrödinger equation (20) at b<0 . The properties and rules of such collision between two microscopic particles have been first studied by Aossey *et al*.[49]. Both the phase shift of the microscopic particles after their interaction and the range of the interaction are functions of the relative amplitude of the two colliding microscopic particles. The microscopic particles preserve the shape after the collision.

In accordance with Asossey et al's[49] representation the hole-particle or dark spatial soliton of Eq.(26) in the case of $b<0$ [26-27] is now given by

$$\phi(x',t')=\phi_0\sqrt{1-B^2\operatorname{sech}^2(\xi')}e^{\pm i\Theta\xi'} \tag{130}$$

where

$$\Theta(\xi')=\sin^{-1}\left[\frac{B\tanh(\xi')}{\sqrt{1-B^2\operatorname{sech}^2(\xi')}}\right],\xi'=\mu(x'-\upsilon_t t')$$

Here, B is a measure of the amplitude ("blackness") of the solitary wave (hole or dark soliton) and can take a value between -1 and 1, υ_t is the dimensionless transverse velocity of the particle center, and μ is the shape factor of the particle. The intensity (I_d) of the solitary wave (or the depth of the irradiance minimum of the dark soliton) is given by $B^2\phi_0^2$. Aossey *et al.* showed that the shape factor μ and the transverse velocity υ_t are related to the amplitude of the particles, which can be obtained from the nonlinear *Schrödinger* equation in the optical fiber to be

$$\mu^2=n_0|n_2|\mu_0^2B^2\phi_0^2,\upsilon_t\approx\pm\sqrt{(1-B^2)\frac{|n_2|\phi_0^2}{n_0}}$$

where n_0 and n_2 are the linear and nonlinear indices of refraction for the optical fiber material. We have assumed $|n_2|\phi_0^2 \ll n_0$. When two microscopic particles described by Eq.(26) collide, their individual phase shifts are given by

$$\delta x_j = \sqrt{\frac{n_0}{|n_2|\phi_0^2}}\frac{1}{2\mu_0 n_0 B_j}\ln\left[\frac{\left(\sqrt{1-B_1^2}+\sqrt{1-B_2^2}\right)^2+\left(B_1+B_2\right)^2}{\left(\sqrt{1-B_1^2}+\sqrt{1-B_2^2}\right)^2+\left(B_1-B_2\right)^2}\right] \tag{131}$$

The interaction of microscopic particles can be easily investigated numerically by using a split- step propagation algorithm, which was found, by Thusrston *et al.* [50], to closely predict experimental results. The results of a simulated collision between two equi-amplitude microscopic particles are shown in Fig.6 (a), which are similar to that of general microscopic particles (bright solitons) as shown in Fig.5. We note that the two particles interpenetrate each other, retain their shape, energy and momentum, but experience a phase shift at the point of collision. In addition, there is also a well- defined interaction length in z along the axis of time t that depends on the relative amplitude of two colliding microscopic particles. This case occurs also in the collision of two KdV solitons[51-52]. Cooney *et al*.[51] studied the overtaking collision, to verify the KdV soliton nature of an observed signal in the plasma experiment. In the following, we discuss a fairly simple model which was used to simulate and to interpret the experimental results on the microscopic particles described by nonlinear SchrÖdinger equation (26) at b<0 and KdV solitons.

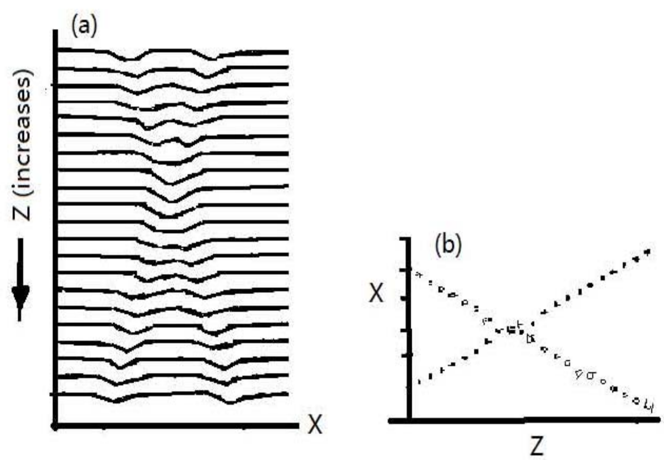

Fig. 6: Numerical simulation of an overtaking collision of equi-amplitude dark particles. (a) Sequence of the waves at equal intervals in the longitudinal position z . (b) Time-of-flight diagrams of the signal

The model is based on the fundamental property of solitons that two microscopic particles can interact and collide, but survive the collision and remain unchanged. Rather than using the exact functional form of $\mathrm{sech}\,\xi$ for microscopic particles described by Eq.(26), the microscopic particles are represented by rectangular pulses with an amplitude A_j and a width W_j where the subscript j denotes the jth microscopic particles. An evolution of the collision of two microscopic particles is shown in Fig.7(a). In this case, Aossey *et al.* [49] considered two microscopic particles with different amplitudes. The details of what occurs during the collision need not concern us here other than to note that the microscopic particles with the larger-amplitude has completely passed through the one with the smaller amplitude. In regions which can be considered external to the collision, the microscopic particles do not overlap as there is no longer an interaction between them. The microscopic particles are separated by a distance, $D = D_1 + D_2$, after the interaction. This manifests itself in a phase shift in the trajectories depicted in Fig. 7(b). This was noted in the experimental and numerical results. The minimum distance is given by the half-widths of the two.

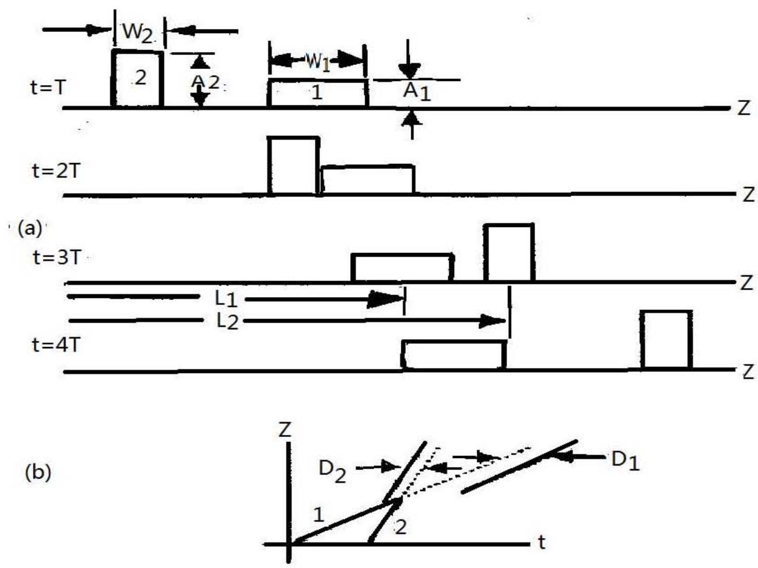

Fig. 7: Overtaking collision of two microscopic particles. (a) Model of the interaction just prior to the collision and just after the collision. After the collision, the two microscopic particles are shifted in phase. (b) Time-of-light diagram of the signals. The phase shifts are indicated

microscopic particles, $D \geq W_1/2 + W_2/2$. Therefore,

$$D_1 \geq \frac{W_1}{2} \text{ and } D_2 \geq \frac{W_2}{2} \tag{132}$$

Another property of the microscopic particles is that their amplitude and width are related. For the microscopic particles described by the nonlinear *Schrödinger* equation with $b < 0$ in Eq. (26) ($W \approx 1/\mu$), we have

$$B_j W_j = \text{constant} = K_1 \tag{133}$$

Using the minimum values in Eq.(132), we find that the ratio of the repulsive shifts for the microscopic particles described by the nonlinear *Schrödinger* equation (26) is given by

$$\frac{D_1}{D_2} = \frac{B_2}{B_1} \tag{134}$$

Results obtained from simulation of the kind of microscopic particle are presented in Fig. 6(a). The solid line in the figure corresponds to Eq.(134).

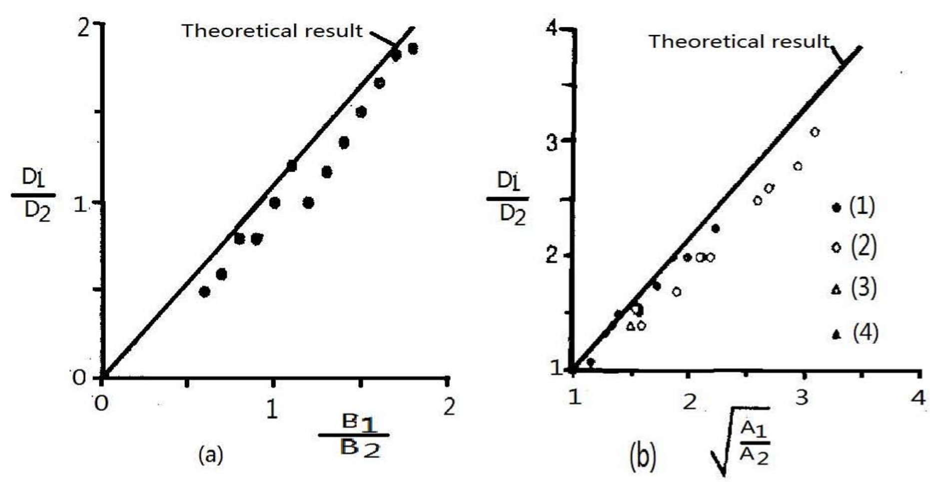

Fig. 8: Summary of the ratio of the measured phase shifts as a function of the ratio of amplitudes. (a) The particle in Eq.(26) at b<0, the solid line corresponds to Eq.(131). (b) KdV solitons, the data are from (1) this experiment, (2) Zabusky et al. [52], (3) Lamb's[41] and (4) Ikezi et al.'s [54] results. The solid line corresponds to Eq.(138)

In addition to predicting the phase shift that results from the collision of two microscopic particles, the model also allows us to estimate the size of the collision region or duration of the collision. Each microscopic particles depicted in Fig.6 travels with its own amplitude-dependent velocity υ_j. For the two microscopic particles to interchange their positions during a time ΔT, they must travel a distance L_1 and L_2,

$$L_1 = \upsilon_1 \Delta T \text{ and } L_2 = \upsilon_2 \Delta T \tag{135}$$

The interaction length must then satisfy the relation

$$L = L_2 - L_1 = (\upsilon_2 - \upsilon_1)\Delta T \approx W_1 + W_2 \tag{136}$$

Equation (135) can be written in terms of the amplitudes of the two microscopic particles. For the particle in Eq.(26) at b<0, combining Eqs. (132) and (136), Aossay *et al.* obtained

$$L \geq K_1 \left[\frac{1}{B_1} + \frac{1}{B_2} \right] \tag{137}$$

In Fig. 8(a), the results for the microscopic particles in Eq.(26) at b<0 are presented. The dashed line corresponds to Eq. (137) with $B_2 = 1$ and $K_1 = 6$. The interaction time (solid line) is the sum of the widths of the two microscopic particles, minus their repulsive phase shifts, and multiplied by the transverse velocity of microscopic particles 1. Since the longitudinal velocity is a constant, this scales as the interaction length. From the figure, we see that the theoretical result obtained using the simple collision model is in good agreement with that of the numerical simulation.

The discussion presented above and the corresponding formulae reveal the mechanism and rule of the collision between microscopic particles depicted by nonlinear SchrÖdinger equation (26) at b<0.

To verify the validity of this simple collision model, Aossey *et al.* studied the collision of the solitons using the exact form of $sech^2\xi$ for the KdV equation, $u_t + uu_x + d'u_{xxx} = 0$, and the collision model shown in Fig.6. For the KdV soliton they found that

$$A_j (W_j)^2 = \text{cons}\tan t = K_2 \text{ and } \frac{D_1}{D_2} = \frac{W_1/2}{W_2/2} = \sqrt{\frac{A_2}{A_1}} \tag{138}$$

where A_j and W_j are the amplitude and width of the jth KdV soliton, respectively. Corresponding to the above, Aossey *et al.* obtained

$$L \geq K_2 \left(\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} \right) = \frac{K_2}{\sqrt{A_1}} \left(1 + \sqrt{\frac{A_1}{A_2}} \right) \tag{139}$$

for the interaction length.

Aossey *et al.*[49]compared their results for the ratio of the phase shifts as a function of the ratio of the amplitudes for the KdV solitons, with those obtained in the experiments of Ikezi, Taylor, and Baker[53], and those obtained from numerical work of Zabusky and Kruskal[52] and Lamb[54], as shown in Fig.7(b). The solid line in Fig. 7(b) corresponds to Eq. (138). Results obtained by Aossey *et al.* for the interaction length are shown in Fig. 8(b) as a function of amplitudes of the colliding KdV solitons. Numerical results (which were scaled) from Zabusky and Kruskal are also shown for comparison. The dashed line in Fig. 8(b) corresponds to Eq. (139), with $A_1 = l$ and $K_2 = l$.

Since the theoretical results obtained by the collision model based on macroscopic bodies in Fig.6 are consistent with experimental data for the KdV soliton, shown in Figs. 7(b) and 8(b), it is reasonable to believe the validity of the above theoretic results of model of collision presented above , and results shown in Figs.7(a) and 8(a) for the microscopic particles described in the nonlinear Schrödinger equation (26) which are obtained using the same model as that shown in Fig 6. Thus, the above colliding mechanism for the microscopic particles shows clearly the corpuscle feature of the microscopic particles is described by nonlinear SchrÖdinger equation.

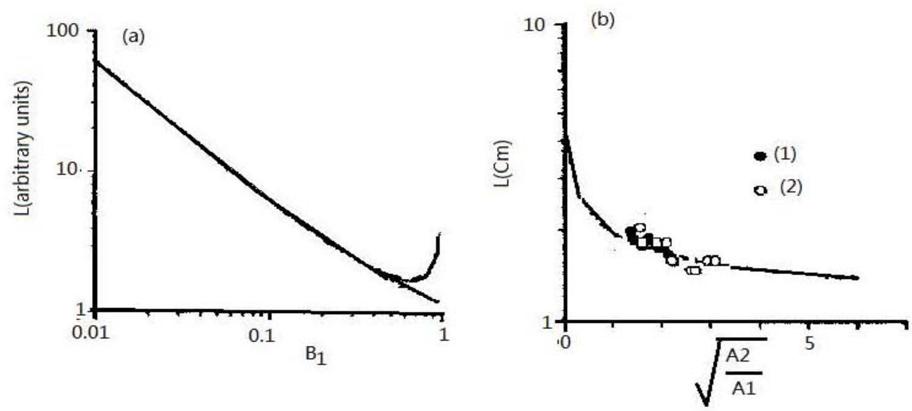

Fig. 9: Summary of the measured interaction length as a function of the amplitudes. **(a)** The particles described by Eq.(26) at b<0, the dashed line corresponds to Eq. (134) with $B_2=1$ and $K_1=6$. **(b)** KdV solitons, the symbols represent (1) experiment results of Ikezi et al., and (2) numerical results of Zabusky *et al.* [31]. The dashed line corresponds to Eq. (136) with $K_2=A_1=1$

g) The uncertainty relationship obeyed for microscopic particles

i. *Correct form of uncertainty relation in the quantum mechanics*

As known, in quantum mechanics the microscopic particle has not a determinant position, disperses always in total space, the probability occurring at each points in the space is represented by square of the wave function. Thus the position and momentum of the particles cannot be simultaneously determined. This means that there is an uncertainty relationship between the position and momentum, then the values of mechanical quantities of the particles are only denoted by some statistical average values of the wave function, etc.. These concepts are considerably inconsistent with conventional knowledge to particles and result in a long-term controversies of one century on its essences and explanations in physics, which have not an unitary conclusion up to now. Therefore, we feel especially perplexity to the uncertainty relationship, which is a persistent ailment in quantum physics we may say. Whether is the uncertainty relationship an intrinsic feature of microscopic particles or caused by the quantum measurement? This problem is not clear as yet[8-12]. Therefore, to clarify the essence of uncertainty relationship is a most challenging problem in physics. Obviously, it is closely related to elementary features of microscopic particles.

As known, the uncertainty relation in the quantum mechanics can be obtained from [1-6]

$$I(\xi)=\int\left|\left(\xi\Delta\hat{A}+i\Delta\hat{B}\right)\psi(\vec{r},t)\right|^2 d\vec{r}\geq 0 \tag{140}$$

or

$$\bar{\hat{F}}(\xi)=\int\psi^*(\vec{r},t)\hat{F}\left[\hat{A}(\vec{r},t),\hat{B}(\vec{r},t)\right]\psi(\vec{r},t)\,d\vec{r}$$

In the coordinate representation, $\vec{A}$ and $\vec{B}$ are operators of two physical quantities, for example, position and momentum, or energy and time, and satisfy the commutation relation $\left[\hat{A},\hat{B}\right]=i\hat{C}$, $\psi(x,t)$ and $\psi^*(x,t)$ are wave functions of the microscopic particle satisfying the linear Schro- dinger equation(7) and its conjugate equation, respectively, $\hat{F}=\left(\Delta A\xi+\Delta B\right)^2$, ($\Delta\hat{A}=\hat{A}$-$\bar{A}$, $\Delta\hat{B}=\hat{B}$-$\bar{B}$, $\bar{A}$ and $\bar{B}$ are the average values of the physical quantities in the state denoted by $\psi(x,t)$), is an operator of physical quantity related to $\bar{A}$ and $\bar{B}$, ξ is a real parameter.

After some simplifications, we can get

$$I=\bar{F}=\overline{\Delta\hat{A}^2}\xi^2+2\overline{\Delta\hat{A}\Delta\hat{B}}\xi+\overline{\Delta B^2}\geq 0$$

or

$$\overline{\Delta\hat{A}^2}\xi^2+\bar{C}\xi+\overline{\Delta\hat{B}^2}\geq 0 \tag{141}$$

Using mathematical identities, this can be written as

$$\overline{\Delta\hat{A}^2}\,\overline{\Delta\hat{B}^2} \geq \frac{\bar{\hat{C}}^2}{4} \tag{142}$$

This is the uncertainty relation in the quantum mechanics. From the above derivation we see that the uncertainty relation was obtained based on the fundamental hypotheses of linear quantum mechanics, including properties of operators of the mechanical quantities, the state of particle represented by the wave function, which satisfies the linear Schrödinger equation (7), the concept of average values of mechanical quantities and the commutation relations and eigenequation of operators. Therefore, we can conclude that the uncertainty relation Eq. (142) is a necessary result of the quantum mechanics. Since the quantum mechanics only describes the wave nature of microscopic particles, the uncertainty relation is a result of the wave feature of microscopic particles, and it inherits the wave nature of microscopic particles. This is why its coordinate and momentum cannot be determined simultaneously. This is an essential interpretation for the uncertainty relation Eq. (142) in quantum mechanics. It is not related to measurement, but closely related to the quantum mechanics. In other words, if quantum mechanics could correctly describe the states of microscopic particles, then the uncertainty relation should also reflect the peculiarities of microscopic particles.

Equation (141) can be written in the following form:

$$\hat{\hat{F}} = \overline{\Delta\hat{A}^2}\left(\xi + \frac{\overline{\Delta\hat{A}\Delta\hat{B}}}{\overline{\Delta\hat{\hat{A}}^2}}\right)^2 + \overline{\Delta B^2} - \frac{\left(\overline{\Delta\hat{A}\Delta\hat{B}}\right)^2}{\overline{\overline{\Delta A^2}}} \geq 0$$

or

$$\overline{\Delta\hat{A}^2}\left(\xi + \frac{\bar{\hat{C}}}{4\overline{\overline{\Delta\hat{\hat{A}}^2}}}\right)^2 + \overline{\Delta\hat{B}^2} - \frac{\left(\bar{\hat{C}}\right)^2}{4\overline{\overline{\Delta A^2}}} \geq 0 \tag{143}$$

This shows that $\overline{\Delta\hat{A}^2} \neq 0$, if $\left(\overline{\Delta\hat{A}\Delta\hat{B}}\right)^2$ or $\bar{\hat{C}}^2/4$ is not zero, else, we cannot obtain Eq.(142) and $\overline{\Delta\hat{A}^2}\,\overline{\Delta\hat{B}^2} > \overline{\left(\Delta A\Delta B\right)^2}$ because when $\overline{\Delta\hat{A}^2} = 0$, Eq. (143) does not hold. Therefore, $\overline{\left(\Delta\hat{A}^2\right)} \neq 0$ is a necessary condition for the uncertainty relation Eq. (142), $\overline{\Delta\hat{A}^2}$ can approach zero, but cannot be equal to zero. Therefore, in the quantum mechanics, the right uncertainty relation should take the form [18]:

$$\overline{\Delta\hat{A}^2\Delta\hat{B}^2} > \frac{\left(\bar{\hat{C}}^2\right)^2}{4} \tag{144}$$

ii. *The new uncertainty relation of microscopic particles described by nonlinear SchrÖdinger equation*

We now return to study the uncertainty relation of microscopic particles described nonlinear SchrÖdinger equation (20). In such a case the microscopic particles is a soliton and has a wave- corpuscle duality. Thus we have the reasons to believe that the uncertainty relation in this case should be different from that in the quantum theory given above.

We now derive this relation for position and momentum of a microscopic particle depicted by the nonlinear SchrÖdinger Equation (26) with a solution, ϕ_s, as given in Eq.(34), which is now represented by[25-26]

$$\phi_s(x',t') = 2\sqrt{2}\eta\,\mathrm{sech}\{2\eta(x' - x_0') - 8\eta\xi t'\}\exp\{2i\xi x' - i4(\xi^2 - \eta^2)t' + i\theta\} \tag{145}$$

where $x' = x\sqrt{2m}/\hbar, t' = t/\hbar$, $2\sqrt{2}\eta$ is the amplitude, $2\sqrt{2}\xi$ denotes the velocity, θ is a constant. The function $\phi_s\left(x',t'\right)$ is a square integral function localized at $x_0' = 0$ in the coordinate space. If the microscopic particle is localized at $x_0' \neq 0$. The Fourier transform of this function is

$$\phi_s\left(p,t'\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_s\left(x',t'\right)e^{-ipx'} \tag{146}$$

It shows that $\phi_s\left(p,t^{'}\right)$ is localized at p in momentum space. For Eq.(146), the Fourier transform is explicitly given by

$$\phi_s\left(p,t^{'}\right)=\sqrt{\frac{\pi}{2}}\operatorname{sech}\left[\frac{\pi}{4\sqrt{2\eta}}\left(p-2\sqrt{2}\xi\right)\right]\exp\{i4(\eta^2+\xi^2-p\xi/2\sqrt{2})t^{'}-i\left(p-2\sqrt{2}\xi\right)x_0^{'}+i\theta\} \tag{147}$$

The results in Eqs. (146) and (147) show that the microscopic particle is localized not only in position space in the shape of soliton, but also in the momentum space in a soliton. For convenience, we introduce the normalization coefficient B_0 in Eqs. (145) and (147), then obviously $B_o^2=\frac{1}{4\sqrt{2}}\eta$, the position of the certain of mass of the microscopic particle, $\left\langle x^{'}\right\rangle$, and its square, $\left\langle x'^2\right\rangle, at\ \ t'=0$ are given by

$$\left\langle x'\right\rangle=\int_{-\infty}^{\infty}dx'\left|\phi_s\left(x'\right)\right|^2,\quad \left\langle x'^2\right\rangle=\int_{-\infty}^{\infty}dx'x'^2\left|\phi_s\left(x'\right)\right|^2. \tag{148}$$

We can thus find that

$$\left\langle x'\right\rangle=4\sqrt{2}\eta A_0^2x_0^{'},\quad \left\langle x'^2\right\rangle=\frac{A_0^2\pi^2}{12\sqrt{2}\eta}+4\sqrt{2}A_0^2\eta x_0^{'2} \tag{149}$$

respectively. Similarly, the momentum of the center of mass of the microscopic particle, $\left\langle p\right\rangle$, and its square , $\left\langle p^2\right\rangle$, are given by

$$\left\langle p\right\rangle=\int_{-\infty}^{\infty}p\left|\hat{\phi}_s\left(p\right)\right|^2dp,\quad \left\langle p^2\right\rangle=\int_{-\infty}^{\infty}p^2\left|\hat{\phi}_s\left(p\right)\right|^2dp \tag{150}$$

which yield

$$\left\langle p\right\rangle=16A_0^2\eta\xi,\quad \left\langle p^2\right\rangle=\frac{32\sqrt{2}}{3}A_0^2\eta^3+32\sqrt{2}A_0^2\eta\xi^3 \tag{151}$$

The standard deviations of position $\Delta x'=\sqrt{\left\langle x'^2\right\rangle-\left\langle x'\right\rangle^2}$ and momentum $\Delta p=\sqrt{\left\langle p^2\right\rangle-\left\langle p\right\rangle^2}$ are given by

$$\begin{aligned}\left(\Delta x'\right)^2&=A_0^2\left[\frac{\pi^2}{12\eta}+4\eta x_0^{'2}\left(1-4\sqrt{2}\eta A_0^2\right)\right]=\frac{\pi^2}{96\eta^2},\\ \left(\Delta p\right)^2&=32\sqrt{2}A_0^2\left[\frac{1}{3}\eta^3+\eta\xi^3\left(1-4\sqrt{2}\eta A_0^2\right)\right]=\frac{8}{3}\eta^2,\end{aligned} \tag{152}$$

respectively. Thus we obtain the uncertainty relation between position and momentum for the microscopic particle depicted by nonlinear SchrÖdinger equation in Eq.(26)

$$\Delta x'\Delta p=\frac{\pi}{6} \tag{153}$$

This result is not related to the features of the microscopic particle (soliton) depicted by the nonlinear SchrÖdinger equation because Eq. (153) has nothing to do with characteristic parameters of the nonlinear SchrÖdinger equation. π in Eq. (153) comes from of the integral coefficient $1/\sqrt{2\pi}$. For a quantized microscopic particle, π in Eq. (153) should be replaced by $\pi\hbar$, because Eq. (147) is replaced by

$$\phi_s\left(p,t'\right)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}dx'\phi_s\left(x',t'\right)e^{-ipx'/\hbar}. \tag{154}$$

The corresponding uncertainty relation of the quantum microscopic particle is given by

$$\Delta x \Delta p = \frac{\pi \hbar}{6} = \frac{h}{12} \tag{155}$$

The uncertainty relation in Eq.(155) or Eq.(153) differ from the $\Delta x \Delta p > h/2$ in the quantum mechanics Eq.(144). However, the minimum value $\Delta x \Delta p = h/2$ has not been both obtained from the solutions of linear SchrÖdinger equation and observed in practical quantum mechanical systems up to now except for the coherent and squeezed states of microscopic particles.

Therefore we can draw a conclusion that the minimum uncertainty relationship is a nonlinear effect, instead of linear effect, and a result of wave-corpuscle duality. From this result we see that when the motion of the particles satisfies $\Delta x \Delta p > h/2 \ or \ \pi/6$, the particles obey laws of motion in quantum mechanics, and the particles are some waves. When the motion of the particles satisfies $\Delta x \Delta p = h/12 \ or \ \pi/6$, the particles should be described by nonlinear SchrÖdinger equation, and have wave-corpuscle duality. If the position and momentum of the particles satisfies $\Delta x \Delta p = 0$, then this is the feature of classical particles with only a corpuscle feature. Therefore, the theory established by nonlinear SchrÖdinger equation bridges the gap between the classical and linear quantum mechanics. Therefore to study the properties of microscopic particles described by nonlinear SchrÖdinger equation has important significances in physics.

iii. *The uncertainty relations of the coherent states*

As a matter of fact, we can represent one-quantum coherent state of harmonic oscillator by [18-19]

$$\left|\alpha\right\rangle = \exp\left(\alpha \hat{b}^{+} - \alpha^{*} \hat{b}\right)\left|0\right\rangle = e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n-1}} \hat{b}^{+n} \left|0\right\rangle,$$

in the number picture, which is a coherent superposition of a large number of microscopic particles (quanta). Thus

$$\left\langle \alpha \left| \hat{x} \right| \alpha \right\rangle = \sqrt{\frac{\hbar}{2\omega m}} \left(\alpha + \alpha^{*}\right), \quad \left\langle \alpha \left| \hat{p} \right| \alpha \right\rangle = i\sqrt{\hbar m \omega} \left(\alpha - \alpha^{*}\right)$$

and

$$\left\langle \alpha \left| \hat{x}^2 \right| \alpha \right\rangle = \frac{\hbar}{2\omega m} \left(\alpha^{*2} + \alpha^2 + 2\alpha\alpha^{*} + 1\right), \quad \left\langle \alpha \left| \hat{p}^2 \right| \alpha \right\rangle = \frac{\hbar \omega m}{2} \left(\alpha^{*2} + \alpha^2 - 2\alpha\alpha^{*} - 1\right),$$

where

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega m}} \left(\hat{b} + \hat{b}^{+}\right), \quad \hat{p} = i\sqrt{\frac{\hbar \omega m}{2}} \left(\hat{b}^{+} - \hat{b}\right),$$

and $\hat{b}^{+}\left(\hat{b}\right)$ is the creation (annihilation) operator of microscopic particle (quantum), α and α^{*} are some unknown functions, ω is the frequency of the particle, m is its mass. Thus we can get

$$\left(\Delta x\right)^2 = \frac{\hbar}{2\omega m}, \quad \left(\Delta p\right)^2 = \frac{\hbar \omega m}{2}, \quad \left\langle \Delta x\right\rangle^2 \left\langle \Delta p\right\rangle^2 = \frac{h^2}{4} \tag{156}$$

$$\frac{\Delta x}{\Delta p} = \frac{1}{\omega m}, \text{ or } \Delta p = \left(\omega m\right) \Delta x$$

For the squeezed state of the microscopic particle: $\left|\beta\right\rangle = \exp\left[\beta\left(b^{+2} - b^2\right)\right]\left|0\right\rangle$, which is a two quanta coherent state, we can find that

$$\left\langle \beta \left| \Delta x^2 \right| \beta \right\rangle = \frac{\hbar}{2m\omega} e^{4\beta}, \left\langle \beta \left| \Delta p^2 \right| \beta \right\rangle = \frac{\hbar m \omega}{2} e^{-4\beta} \quad ,$$

using a similar approach as the above . Here β is the squeezed coefficient and $\left|\beta\right| < 1$. Thus,

$$\Delta x \Delta p = \frac{h}{2}, \quad \frac{\Delta x}{\Delta p} = \frac{1}{m\omega} e^{8\beta}, \text{ or } \Delta p = \Delta x \left(\omega m\right) e^{-8\beta} \tag{157}$$

This shows that the momentum of the microscopic particle (quantum) is squeezed in the two-quanta coherent state compared to that in the one-quantum coherent state.

From the above results, we see that both one-quantum and two-quanta coherent states satisfy the minimal uncertainty principle. This is the same with the above result of the microscopic particle described by nonlinear SchrÖdinger equation (20). We can conclude that a coherent state is a kind of nonlinear quantum effect, at the same time, the coherence of quanta is a nonlinear phenomenon, instead of a linear effect.

As is known, the coherent state satisfies the classical equation of motion, in which the fluctuation in the number of particles approaches zero, which is a classically steady wave. In fact, according to quantum theory, the coherent state of a harmonic oscillator at time t can be represented by

$$|\alpha,t\rangle = e^{-i\hat{H}t}|\alpha\rangle = e^{-i\hbar\omega(\hat{b}^{+}\hat{b}+1/2)t}|\alpha\rangle = e^{-i\hbar\omega t/2-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\hbar n\omega t}}{\sqrt{n!}}|n\rangle$$

$$= e^{-i\hbar\omega t/2}\left|\alpha e^{-i\hbar\omega t}\right\rangle, \quad \left(|n\rangle = \left(b^{+}\right)^n|0\rangle\right)$$

This shows that the shape of a coherent state can be retained during its motion. This is the same as that of a microscopic particle (soliton). The mean position of the particle in the time-dependent coherent state is

$$\langle\alpha,t|x|\alpha,t\rangle = \langle\alpha|e^{iHt/\hbar}xe^{-iHt/\hbar}|\alpha\rangle = \left\langle\alpha\left|x-\frac{it}{h}[x,H]+\frac{(-it)^2}{2!\hbar^2}[[x,H],H]+\cdots\right|\alpha\right\rangle$$

$$= \left\langle\alpha\left|x+\frac{pt}{m}-\frac{1}{2!}t^2\omega^2x+\cdots\right|\alpha\right\rangle = \left\langle\alpha\left|x\cos\omega t+\frac{p}{m\omega}\sin\omega t\right|\alpha\right\rangle = \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\omega t+\theta) \quad (158)$$

where $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $x+iy=\alpha$, $[x,H]=\frac{i\hbar p}{m}$, $[p,H]=-i\hbar m\omega^2 x.$

Comparing (95) with the solution of a classical harmonic oscillator

$$x=\sqrt{\frac{2E}{m\omega^2}}\cos(\omega t+\theta), \quad E=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$$

we find that they are similar with

$$E=\hbar\omega\alpha^2=\langle\alpha|H|\alpha\rangle-\langle0|H|0\rangle, \quad H=\hbar\omega\left(b^{+}b+\frac{1}{2}\right).$$

Thus, we can say that the mass center of the coherent state-packet indeed obeys the classical law of motion, which is the same as the law of motion of microscopic particles described by nonlinear SchrÖdinger equation discussed in Eqs. (111).

We can similarly obtain

$$\langle\alpha,t|p|\alpha,t\rangle=-\sqrt{2m\hbar\omega}|\alpha|\sin(\omega t+\theta), \langle\alpha,t|x^2|\alpha,t\rangle=\frac{2\hbar}{\omega m}\left[|\alpha|^2\cos^2(\omega t+\theta)+\frac{1}{4}\right],$$

$$\langle\alpha,t|p^2|\alpha,t\rangle=2m\hbar\omega\left[|\alpha|^2\sin^2(\omega t+\theta)+\frac{1}{4}\right]$$

and

$$[\Delta x(t)]^2=\frac{\hbar}{2\omega m}, \quad [\Delta p(t)]^2=\frac{1}{2}m\omega\hbar, \Delta x(t)\Delta p(t)=\frac{h}{2} \quad (159)$$

This is the same with Eq. (155). It shows that the minimal uncertainty principle for the coherent state is retained at all times, *i.e.*, the uncertainty relation does not change with time t.

The mean number of quanta in the coherent state is given by

$$\bar{n}=\langle\alpha|\hat{N}|\alpha\rangle=\langle\alpha|\hat{b}^{+}b|\alpha\rangle=\alpha^{2},\quad \langle\alpha|N^{2}|\alpha\rangle=|\alpha|^{4}+|\alpha|^{2}$$

Therefore, the fluctuation of the quantum in the coherent state is

$$\Delta n=\sqrt{\langle\alpha|\hat{N}^{2}|\alpha\rangle-\left(\langle\alpha|N^{2}|\alpha\rangle\right)^{2}}=|\alpha|.$$

which leads to

$$\frac{\Delta n}{n}=\frac{1}{|\alpha|}\ll 1.$$

It is thus obvious that the fluctuation of the quantum in the coherent state is very small. The coherent state is quite close to the feature of soliton and solitary wave.

These properties of coherent states are also similar to those of microscopic particles described by the nonlinear Schrodinger equation (20). In practice, the state of a microscopic particle described by nonlinear SchrÖdinger equation can always be represented by a coherent state, for example, the Davydov's wave functions, both $\mathrm{ID}_1>$ and $\mathrm{ID}_2>$, [55] and Pang's wavefunction of exciton-solitons in protein molecules and the wave function in acetanilide[56-61]; the wave function of proton transfer in hydrogen-bonded systems [37-41] and the BCS's wave function in superconductors[62], etc. Hence, the coherence of particles is a kind of nonlinear phenomenon that occurs only in nonlinear quantum systems. Thus it does not belong to systems described by linear quantum mechanics, because the coherent state cannot be obtained by superposition of linear waves, such as plane wave, de Broglie wave, or Bloch wave, which are solutions of the linear SchrÖdinger equation in the quantum mechanics. Therefore, the minimal uncertainty relation Eq. (155), as well as Eqs. (157) and (159), are only applicable to microscopic particles described by nonlinear SchrÖdinger equation. In other words, only microscopic particles described by nonlinear SchrÖdinger equation satisfy the minimal uncertainty principle. It reflects the wave-corpuscle duality of microscopic particles because it holds only if the duality exists.

This uncertainty principle also suggests that the position and momentum of the microscopic particle can be simultaneously determined in a certain degree. A rough estimate for the size of the uncertainty can be given. If it is required that $\phi_s(x,t)$ in Eq.(145) or $\phi_s(p,t)$ in Eq. (147) satisfies the admissibility condition *i.e.*, $\phi_s(0)\approx 0$, we choose $\xi=140,\ \eta=\sqrt{300/0.253}/2\sqrt{2}$ and $\vec{x}_0=0$ in Eq.(145) (In fact, in such a case we can get $\phi_s(0)\approx 10^{-6}$, thus the admissibility condition can be satisfied). We then get $\Delta x\approx 0.02624$ and $\Delta p\approx 19.893$, according to (154) and (155). These results show that the position and momentum of microscopic particles described by nonlinear SchrÖdinger equation can be simultaneously determined within a certain approximation.

Pang *et al*. [55-61] also calculated the uncertainty relation and quantum fluctuations and studied their properties in nonlinearly coupled electron-phonon systems based on the Holstein model by a new ansatz in Pang's new model including the correlations among one-phonon coherent and two-phonon squeezing states and polaron state. Many interesting results were obtained. The minimum uncertainty relation takes different forms in different systems which are related to the properties of the microscopic particles. Nevertheless, the minimum uncertainty relation in Eq. (155) holds for both the one-quantum coherent state and two-quanta squeezed state. These works enhanced our understanding of the significance and nature of the minimum uncertainty relation.

iv. *Quantum fluctuation effects of particles described by quantized nonlinear SchrÖdinger equation*

Finally, we determine the quantum fluctuation effect arising from the uncertainty of position and momentum of the microscopic particle described by quantized nonlinear SchrÖdinger equation. The features of quantized nonlinear SchrÖdinger equation were discussed by Lai and Haus et. al[63]. A superposition of a subclass of bound state $|n,P\rangle$, characterized by number of the boson (such as, photon or phonon), n , and the momentum of the center of the mass P, can reproduce the expectation values of the microscopic particle (soliton) in the limit where the average number of the bosons are larger; we refer to these states formed by the superposition of $|n,P\rangle$ as a fundamental soliton states. In quantum theory, the quantized dynamic equation in the second quantized picture is given by

$$i\hbar\frac{\partial}{\partial t}\hat{\phi}(x,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial u^2}\hat{\phi}(x,t)+2b\hat{\phi}^+(x,t)\hat{\phi}(x,t)\hat{\phi}(x,t) \quad (160)$$

The operators $\hat{\phi}(x,t)$ and $\hat{\phi}^+(x,t)$ are the annihilation and creation operators of microscopic particles at a "point " *x* and "time " *t*, they satisfy the commutation relation:

$$[\hat{\phi}(x'',t),\hat{\phi}^+(x,t)]=\delta(x-x''),[\hat{\phi}(x'',t),\hat{\phi}(x,t)]=[\hat{\phi}^+(x'',t),\hat{\phi}^+(x,t)]=0 \quad (161)$$

The corresponding quantum Hamiltonian is given by

$$H=\frac{\hbar^2}{2m}\int \hat{\phi}_x^+(x,t)\hat{\phi}_x(x,t)dx+b\int\hat{\phi}^+(x,t)\hat{\phi}^+(x,t)\hat{\phi}(x,t)\hat{\phi}(x,t)dx \quad (162)$$

In the Schrodinger picture, the time evolution of the system is described by

$$i\hbar\frac{d}{dt}|\Phi\rangle=\hat{H}_s|\Phi\rangle \quad (163)$$

with the commutation relation:

$$[\hat{\phi}(x''),\hat{\phi}^+(x)]=\delta(x-x''),[\hat{\phi}(x''),\hat{\phi}(x)]=[\hat{\phi}^+(x''),\hat{\phi}^+(x)]=0 \quad (164)$$

where $\hat{\phi}(x)$ and $\hat{\phi}^+(x)$ are the field operators in the SchrÖdinger representation. The corresponding quantum Hamiltonian is given by

$$\hat{H}_s=\frac{\hbar^2}{2m}\int \hat{\phi}_x^+(x)\hat{\phi}_x(x)dx+b\int\hat{\phi}^+(x)\hat{\phi}^+(x)\hat{\phi}(x)\hat{\phi}(x)dx \quad (165)$$

The many-particle state $|\Phi\rangle$ can be built up from the n-quantum states given by

$$|\Phi\rangle=\sum_n a_n\int\frac{1}{\sqrt{n!}}f_n(x_1...,x_n,t)\hat{\phi}^+(x_1)...\hat{\phi}^+(x_n)dx_1...dx_n|0\rangle \quad (166)$$

The quantum theory based on Eq.(166) describes an ensemble of bosons interacting via a $\delta-$potential. Note that $\hat{H}$ preserves both the particle number.

$$\hat{N}=\int\hat{\phi}^+(x)\hat{\phi}(x)dx \quad (167)$$

and the total momentum

$$\hat{P}=i\frac{\hbar}{2}\int\left[\frac{\partial}{\partial x}\hat{\phi}^+(x)\hat{\phi}(x)-\hat{\phi}^+(x)\frac{\partial}{\partial x}\hat{\phi}(x)\right]d \quad (168)$$

Lai *et al*.[55] proved that the boson number and momentum operator commute, so that common eigenstates of $\hat{H}$, $\hat{P}$ and $\hat{N}$ exist in such a case. In the case of a negative b, the interaction between the bosons is attractive and Hamiltonian Eq. (160) has bound states. A subset of these bound states is characterized solely by the eigenvalues of $\hat{N}$ and $\hat{P}$:

$$f_{n,p}=N_{n,}\exp\left(ip\sum_{j=1}^{\infty}x_j+\frac{b}{2}\sum_{1\le i,j<n}^{\infty}|x_i-x_j|\right), \quad (169)$$

where $$N_n = \sqrt{\frac{(n-1)!|b|^{n-1}}{2\pi}}.$$

Thus

$$f_n(x_1,\cdots,x_n,t) = \int dp g_n(p) f_n p(x_1,\cdots,x_n,t) e^{-iE(n,p)t}, \tag{170}$$

where $$g_n = \sqrt{g(p)} e^{-inpx_0}, \text{ and } g(p) = \frac{e\phi\left\{-(p-p_0)^2 \Big/ \left|2(\Delta p)^2\right|\right\}}{\sqrt{2\pi(\Delta p)^2}}$$

Using $f_{n,p}$ given in Eq. (169), we find that $|n,P\rangle$ decays exponentially with separation between an arbitrary pair of bosons. It describes an soliton moving with n-quantum, momentum $P=\hbar np$ and energy $E(n,p)=np^2-|b|^2(n^2-1)n/12$. By construction, the quantum number p in this wave function is related to the momentum of the center of mass of the n interacting bosons, which is now defined as

$$\hat{X} = \lim_{\varepsilon\to 0}\int x\hat{\phi}^+(x)\hat{\phi}(x)dx\left(\varepsilon+\hat{N}\right)^{-1} \tag{171}$$

with $$\left[\hat{X},\hat{P}\right] = i\hbar$$

The limit of $\varepsilon \to 0$ is introduced to regularize the position operator for the vacuum state.

We are interested in the quantum fluctuations of Eqs. (167), (168) and (169) for a state $|\Phi(t)\rangle$ with a large average Boson number and a well-defined mean field. Kartner and Boiven[64] decomposed these operators in its mean values and a remainder which is responsible for the quantum fluctuations.

$$\hat{\phi}(x) = \left\langle \psi'(0)\left|\hat{\phi}^+(x)\right|\psi(0)\right\rangle + \hat{\phi}_1(x), \left[\hat{\phi}_1(x),\hat{\phi}_1^+(x')\right] = \delta(x-x'), \left[\hat{\phi}_1(x),\hat{\phi}_1^+(x')\right] = 0 \tag{172}$$

Since the field operator $\hat{\phi}$ is time independent in the Schrodinger representation, we can then choose $t=0$ for definiteness. Inserting Eq.(172) into Eqs.(167), (168) and (171) and neglecting terms of second and higher order in the noise operator, Kartner et al[64]. obtained that

$$\hat{N} = n_0 + \Delta\hat{n}, n_0 = \int dx\left(\left\langle\hat{\phi}^+(x)\right\rangle\left\langle\hat{\phi}(x)\right\rangle\right), \Delta\hat{n} = \int dx\left(\left\langle\hat{\phi}^+(x)\right\rangle\hat{\phi}_1(x)\right) + c.c.,$$

$$\hat{P} = \hbar n_0 P_0 + \hbar n_0 \Delta P, P_0 = \frac{i}{n_0}\int dx\left\langle\hat{\phi}_x^+(x)\right\rangle\left\langle\hat{\phi}(x)\right\rangle, \Delta P = \frac{i}{n_0}\int dx\left\langle\hat{\phi}_x^+(x)\right\rangle\hat{\phi}_1(x) + c.c.,$$

$$\hat{X} = x_0\left(1-\frac{\Delta\hat{n}}{n_0}\right) + \Delta\hat{x}, x_0 = \frac{1}{n_0}\int dxx\left\langle\hat{\phi}^+(x)\right\rangle\left\langle\hat{\phi}(x)\right\rangle, \Delta\hat{x} = \frac{1}{n_0}\int dxx\left\langle\hat{\phi}^+(x)\right\rangle\hat{\phi}(x) + c.c.$$

where $\Delta\hat{x}$ is the deviation from the mean value of the position operator, $\Delta\hat{n}, \Delta\hat{p}$, and $\Delta\hat{x}$ are linear in the noise operator. Because the third- and fourth-order correlators of $\hat{\phi}_1$ and $\hat{\phi}_1^+$ are very small, they can be neglected in the limit of large n_0. Note that $\Delta\hat{n}, \Delta\hat{p}$, and $\Delta\hat{x}$ are all quadratures of the noise operator with $\Delta\hat{p}$ and $\Delta\hat{x}$ being conjugate variables. To complete this set, we introduce a quadrature variable conjugate to $\Delta\hat{n}$,

$$\Delta\hat{\theta} = \frac{1}{n_0}\int dx\left\{i\left[\hat{\phi}^+(x) + x\left\langle\hat{\phi}_x^+(x)\right\rangle\right] - p_0 x\left\langle\hat{\phi}_x^+(x)\right\rangle\right\}\hat{\phi}_1(x) + c.c.$$

As is known, if the propagation distance is not too large, the mean value of the particle is given to the first order by the classical soliton solution

$$\left\langle \hat{\phi}(x)\right\rangle = \phi_{0,n_0}(x,t)\left[1+O\left(\frac{1}{n_0}\right)\right]$$

with

$$\phi_{0,n_0}(x,t)=\frac{n_0\sqrt{|b|}}{2}\exp\left[i\Omega_{nl}-ip_0^2t+ip_0(x-x_0)+i\theta_0\right]\times\operatorname{sech}\left[\frac{n_0|b|}{2}(x-x_0-2p_0t)\right], \tag{173}$$

and the nonlinear phase shift $\Omega_{nl}=n_0^2|b|^2t/4$. If $p_0=x_0=\theta_0=0$, we obtain the following for the fluctuation operators in the Heisenberg picture,

$$\Delta\hat{n}(t)=\int dx\left[f_{-n}(x)^*F'_{nl}+c.c\right],\Delta\hat{\theta}(t)=\int dx\left[f_{-\theta}(x)^*F'_{nl}+c.c\right],$$

$$\Delta\hat{p}(t)=\int dx\left[f_{-p}(x)^*F'_{nl}+c.c\right],\Delta x(t)=\int dx\left[f_{-x}(x)^*F'_{nl}+c.c\right],$$

with $$F'_{nl}=e^{i\Omega_{nl}}\hat{\phi}_1(x,t),$$

and the set of adjoint functions

$$f_{-n}(x)=\frac{n_0\sqrt{|b|}}{2}\operatorname{sech}(x_{n_0}),\ f_{-\theta}(x)=\frac{i\sqrt{|b|}}{2}\left[\operatorname{sech}(x_{n_0})+x_{n_0}\frac{d}{dx_{n_0}}\operatorname{sech}(x_{n_0})\right],$$

$$f_{-p}(x)=-\frac{in_0\sqrt{|b|^3}}{4}\frac{d}{dx_{n_0}}\operatorname{sech}(x_{n_0}),\ f_{-x}(x)=\frac{1}{n_0\sqrt{|b|}}x_{n_0}\operatorname{sech}(x_{n_0}),$$

where $x_{n_0}=\frac{1}{2}n_0|b|x$

For a coherent state defined by

$$\hat{\phi}(x)\left|\Phi_{0,n_0}\right\rangle=\phi_{0,n_0}(x)\left|\Phi_{0,n_0}\right\rangle,\ \ \hat{\phi}_1(x)\left|\phi_{0,n_0}\right\rangle=0$$

where

$$\left|\Phi_{0,n_0}\right\rangle=\exp\left\{\int d\left[\phi_{0,n_0}x(x)\hat{\phi}^+(x)-\phi^*_{0,n_0}(x)\hat{\phi}^+(x)\right]\right\}|0\rangle$$

ϕ_{0,n_0} has been given by Eq. (173). Kartner *et. al.* further obtained that

$$\left\langle\Delta\hat{n}_0^2\right\rangle=n_0,\ \left\langle\Delta\hat{\theta}_0^2\right\rangle=\frac{0.6075}{n_0},\ \left\langle\Delta\hat{p}_0^2\right\rangle=\frac{1}{3n_0\tau_0^2},\ \ \left\langle\Delta\hat{x}_0^2\right\rangle=\frac{1.645\tau_0^2}{2n_0},$$

where $\tau_0^2=2/n_0|b|$ is the width of the microscopic particle. The uncertainty products of Boson number and phase, momentum and position are, respectively,

$$\left\langle\Delta\hat{n}_0^2\right\rangle\left\langle\Delta\hat{\theta}_0^2\right\rangle=0.6075\geq0.25,n_0^2\left\langle\Delta\hat{p}_0^2\right\rangle\left\langle\Delta\hat{x}_0^2\right\rangle=0.27\geq0.25,$$

Here the quantum fluctuation of the coherent state is white, i.e.,

$$\left\langle\hat{\phi}_1(x)\hat{\phi}_1(y)\right\rangle=\left\langle\hat{\phi}_1^+(x)\hat{\phi}_1(y)\right\rangle=0$$

However, the quantum fluctuation of the particle cannot be written because the particle interaction results in correlations between them. Thus, Kätner and Boivin[64] assumed a fundamental soliton state with a Poissonian distribution for the boson number $p = \frac{n_0^n}{n!} e^{-n_0}$ and a Gaussian distribution for the momentum Eq.(170) with a width $\langle \Delta p_0^2 \rangle = n_0 |b|^2 / 4\mu$, where μ is a parameter of the order of unity compared to n_0. They finally obtained the minimum uncertainty values:

$$\langle \Delta\hat{\theta}_0^2 \rangle = \frac{0.25}{\langle \Delta\hat{n}^2 \rangle} = \frac{0.25}{n_0}\left[1+O\left(\frac{1}{n_0}\right)\right], \text{and } \langle \Delta\hat{x}_0^2 \rangle = \frac{0.25}{\langle n_0^2 \rangle} = \frac{0.25\mu\tau_0^2}{n_0}\left[1+O\left(\frac{1}{n_0}\right)\right]$$

up to order $1/n_0$ for the corresponding initial fluctuations in microscopic particle phase and timing. Thus, at t=0 the fundamental soliton with the given Boson number and momentum distributions is a minimum uncertainty state in the four collective variables, the Boson number, phase, momentum and position, up to the terms of O($1/n_0$), which are of the form [57,26-27]

$$\langle \Delta\hat{n}_0{}^2 \rangle \langle \Delta\hat{\theta}_0^2 \rangle = 0.25\left[1+O\left(\frac{1}{n_0}\right)\right], \text{ and } n_0^2 \langle \Delta p_0^2 \rangle \langle \Delta\hat{x}_0^2 \rangle = 0.25\left[1+O\left(\frac{1}{n_0}\right)\right] \quad (174)$$

These are the uncertainty relations arising from the quantum fluctuations of microscopic particles described by quantized nonlinear quantum SchrÖdinger equation. They are the same as Eqs.(155)-(157). This means that the uncertainty relation of the particles takes the minimum values in such a case.

Finally, we conclude that the uncertainty relation of the microscopic particles described by the nonlinear quantum SchrÖdinger equation regardless whether a state is coherent or squeezed, a system is classical or quantum.

h) The features of reflection and transmission of microscopic particles at interfaces and its wave behavior

As mentioned above, microscopic particles described nonlinear SchrÖdinger equation (20) have also the wave property, in addition to the corpuscle property. This wave feature can be conjectured from the following reasons.

(1) Eqs. (20)－(23) are wave equations and their solutions, Eqs.(34) and (50)- (59) are solitary waves having the features of traveling waves. A solitary wave consists of a carrier wave and an envelope wave, has certain amplitude, width, velocity, frequency, wavevector, and so on, and satisfies the principles of superposition of waves, although the latter are different when compared with classical waves or the de Broglie waves in the quantum mechanics.

(2) The solitary waves have reflection, transmission, scattering, diffraction and tunneling effects, just as that of classical waves or the de Broglie waves in the quantum mechanics. At present, we study the reflection and transmission of the microscopic particles at an interface.

The propagation of microscopic particles (solitons) in a nonlinear nonuniform media is different from that in uniform media. The nonuniformity can be due to a physical confining structure or two nonlinear materials being juxtaposed. One could expect that a portion of microscopic particles that was incident upon such an interface from one side would be reflected and a portion would be transmitted to the other side due to its wave feature. Lonngren *et al.* [65-66] observed the reflection and transmission of microscopic particles (solitons) in a plasma consisting of a positive ion and a negative ion interface, and numerically simulated the phenomena at the interface of two nonlinear materials. To illustrate the rules of reflection and transmission of microscopic particles, we discuss here the work of Lonngren *et al.* [65-66]

Lonngren *et al.* [65] simulated numerically the behaviors of microscopic particles described by nonlinear SchrÖdinger equation (20). They found that the signal had the property of a soliton. These results are in agreement with numerical investigations of similar problems by Aceves et al [67]. A sequence of pictures obtained by Lonngren *et al.* [65] at uniform temporal increments of the spatial evolution of the signal are shown in Fig. 9. From this figure, we note that the incident microscopic particles propagating toward the interface between the two nonlinear media splits into a reflected and transmitted soliton at the interface. From the numerical values used in producing the figure, the relative amplitudes of the incident, the reflected and the transmitted solitons can be deduced.

They assumed that the energy that is carried by the incident microscopic particle is all transfer- red to either the transmitted or reflected microscopic particle and none is lost through radiation. Thus

$$E_{\text{inc}} = E_{\text{ref}} + E_{\text{trans}} \quad (175)$$

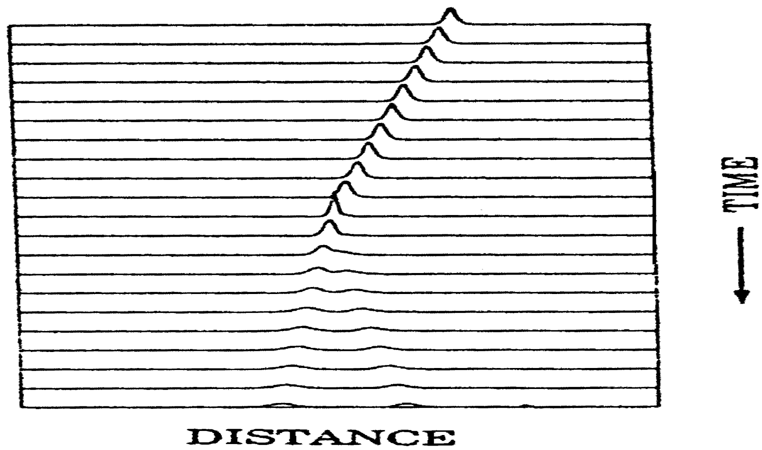

Fig. 10: Simulation results showing the collision and scattering of an incident microscopic particles described by Eq.(20) (top) onto an interface. The peak nonlinear refractive index change is 0.67% of the linear refractive index for the incident microscopic particles and the linear offset between the two regions is also 0.67%.

Lonngren *et al.* gave approximately the energy of the microscopic particle by

$$E_j = \frac{A_j^2}{Z_c} W_j ,$$

where the subscript j refers to the incident, reflected or transmitted microscopic particles. The amplitude of the microscopic particle is A_j and its width is W_j. The characteristic impedance of a particle is given by Z_c. Hence, Eq. (176) can be written as

$$\frac{A_{\text{inc}}^2}{Z_{\text{cI}}} W_{\text{inc}} = \frac{A_{\text{ref}}^2}{Z_{\text{cI}}} W_{\text{ref}} + \frac{A_{\text{trans}}^2}{Z_{\text{cII}}} W_{\text{trans}} \qquad (176)$$

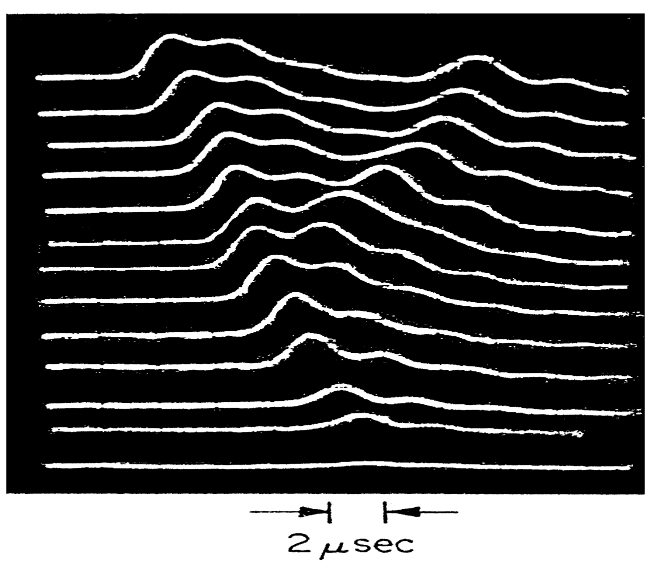

Fig. 11: Sequence of the signals detected as the probe is moved in 2 mm increments from 30 to 6 mm in front of the reflector. The incident and reflected KdV solitons coalesce at the point of reflection, which is approximately 16 mm in front of the reflector. A transmitted soliton is observed closer to the disc. The amplitude scale at 8 and 6 mm is increased by 2 from the previous traces

Since $A_j W_j = \text{constant}$ for the microscopic particle described nonlinear SchrÖdinger equation (26) (see Eq. (130)-(131) in which B_j is replaced by A_j), we obtain the following relation between the reflection coefficient R=$A_{\text{ref}}/A_{\text{inc}}$ and the transmission coefficient $T = A_{\text{trans}} / A_{\text{inc}}$

$$1 = R + \frac{Z_{cI}}{Z_{cII}} T \tag{177}$$

for the microscopic particle described by nonlinear SchrÖdinger equation (26).

To verify further this idea, Lonngrel *et al.*[44] conducted experiments with KdV soliton. They found that the detected signal had the characteristics of a KdV soliton. Lonngrel *et al.*[68] showed a sequence of pictures taken using a small probe at equal spatial increments starting initially in a homogeneous plasma sheath adjacent to a perturbing biased object, as shown in Fig.10. From this figure, we see that the probe first detects the incident soliton and some time later the reflected soliton. The signals are observed, as expected, to coalesce together as the probe passed through the point where the soliton was actually reflected. Beyond this point which was at the location where the density started to decrease in the steady-state sheath, a transmitted soliton was observed. From Fig.10, the relative amplitudes of incident, the reflected and the transmitted solitons can be deduced, which was done by the author.

For the KdV solitons, there is also $A_j W_j^2 = \text{constant}$ (see Eq.(138)) . Thus, for the KdVsoliton, we have

$$1 = R^{3/2} + \frac{Z_{eI}}{Z_{eII}} T^{3/2}$$

The relations between the reflection and the transmission coefficients for the microscopic particle described by nonlinear Scrodinger equation (26) and KdV soliton are shown in Fig.11, with the ratio of characteristic impedances set to one. The experimental results on KdV solitons and results of the numerical simulation of microscopic particle described by nonlinear Scrodinger equation (26) are also given in this figure. The computed data are shown using triangles. Good agreement between the analytic results and simulation results can be seen. The oscillatory deviation from the analytic result is due to the presence of radiation modes in addition to the soliton modes. The interference between these two types of modes results in the oscillation in the

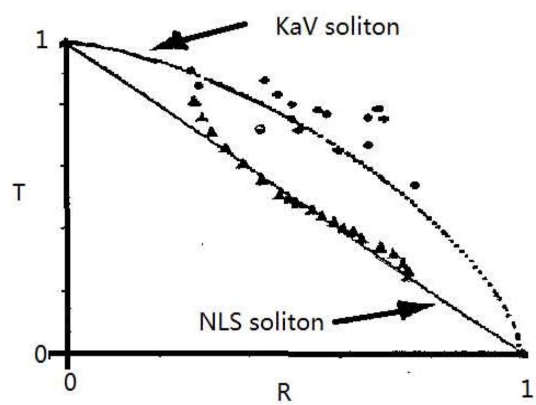

Fig. 12: The relationship between the reflection and transmission coefficients of a microscopic particle (soliton)given in Eq.(178). The solid circles are results from the laboratory experiment on KdV solitons and the hollow circle is Y. Nishida's result. The solid triangles are Lonngren *et al.*'s numerical results for the microscopic particle described by nonlinear SchrÖdinger equation (20)

soliton amplitude. In the asymptotic limit, the radiation will spread and damp the oscillation, and result in the reflection –transmission coefficient curve falling on the analytic curve.

The above rule of propagation of the microscopic particles described by nonlinear SchrÖdinger equation is different from that of linear waves in classical physics. Lonngren *et al*.[68] found that a linear wave obeyed the following relation:

$$1 = R^2 + \frac{Z_{eI}}{Z_{eII}} T^2 \tag{178}$$

This can be also derived from Eq.(177), by assuming the linear waves. The width of the incident, reflected and transmitted pulses W_j will be the same. For the linear waves

$$R = \frac{Z_{cII} - Z_{cI}}{Z_{cII} + Z_{cI}} \text{,and } T = \frac{2Z_{cII}}{Z_{cII} + Z_{cI}}$$

Equation (179) is satisfied. Obviously, equation (179) is different from Eq. (178). This shows clearly that the microscopic particles described by the nonlinear SchrÖdinger equation have a wave feature, but it is different from that of linear classical waves and the de Broglie waves in quantum mechanics.

i) The properties of eigenvalue problem of microscopic particles described by nonlinear SchrÖdinger equation

i. *The eigenenergy spectrum of the Hamiltonian of the nonlinear systems*

In the quantum mechanics, because the Hamiltonian of the systems is independent of the state wavefunction of the particle, the eigenenergy spectrum of the Hamiltonian operator of the systems can be easily obtained from its eigenequation, H$\left|\psi(x,t)\right\rangle$ =E$\left|\psi(x,t)\right\rangle$, where $\left|\psi(x,t)\right\rangle$ is its eigenwave-function in coordinate or particle number representation. It also is just a time-independent linear SchrÖdinger equation in the coordinate representation.

However, for nonlinear SchrÖdinger equation (20), which can represent as $i\hbar\frac{\partial\phi}{\partial t}=\hat{H}(\phi)\phi$, where $\hat{H}(\phi)=-\frac{\hbar^2}{2m}\nabla^2-b\left|\phi\right|^2+V(r,t)$ is the Hamiltonian operator of the system,but corresponding eigenequation and eigenvalues can be obtained through inserting Eq.(60) into Eq.(20), it is of the form

$$\hat{H}(\varphi)\varphi=E\varphi \text{ or } E\varphi=-\frac{\hbar^2}{2m}\nabla^2\varphi(r)+V(r)\varphi(r)-b\left|\varphi\right|^2\varphi \tag{179}$$

where the Hamiltonian operator is now represented by

$$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2+V(r)-b\left|\varphi\right|^2=\hat{H}_0-b\rho(r) \tag{180}$$

Where $\hat{H}_0=-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r}),\quad \rho=\left|\varphi\right|^2$ or $\left|\phi(\vec{r},t)\right|^2=\rho(\vec{r},t)$, E and $\phi(x,t)$ are just the eigenvalue and eigenfunction of the Hamiltonian operator, respectively. Its distinction with that in quantum mechanics is that the Hamiltonian operator is dependent on the wave function of the particles, thus the eigenvalues and eigenfunction cannot be obtained in accordance with above same method in the quantum mechanics. If life-multiplying both sides of Eq.(180) by φ^* and integrating it with respect to x we can find the eigenenergy of the Hamiltonian operator, which can denote as

$$E=\int[-\frac{\hbar^2}{2m}\varphi^*\nabla^2\varphi(x)+V(x)\varphi^*\varphi^2-b\left|\varphi\right|^2\varphi^*\varphi]dx$$

$$=\int[\frac{\hbar^2}{2m}\left|\nabla\varphi\right|^2+V(x)\left|\varphi\right|^2-b\left|\varphi\right|^2\left|\varphi\right|^2]dx$$

This is just Eq.(72). Therefore we can determine the eigenfunctions and eigenvalues of the Hamiltonian operator from Eqs.(180) and Eq.(72), respectively. However, the eigenfunctions and eigenvalues found from this way are only ones of a single particle.

In fact, From Eq.(20) we can also the energy spectra of many particles or many model of motion. In such a case we ever translate Eq.(180) or the above Hamiltonian operator into the particle number representation from coordinate representation to find the eigenfunctions and eigenvalues of the Hamiltonian operator of the systems. We often use the latter to find the eigenenergy of the Hamiltonian operator of the systems.

We know that the wave function of a microscopic particle can be quantized by the creation and annihilation operators of the particle in the second quantum representation. Then the Hamiltonian of a system described by the wave function $\phi(x,t)$ can be quantized by introducing creation and annihilation operators in the particle number representation or second quantization representation. Thus, we can calculate the eigenenergy spectrum by using the eigenequation of the quantum Hamiltonian and corresponding wavevector in number representation. For convenience, we express the nonlinear SchrÖdinger equation (20) in the following discrete form:

$$i\hbar\frac{\partial\phi_j}{\partial t}=-\frac{\hbar}{2mr_0^2}(\phi_{j+1}-2\phi_j+\phi_{j-1})-b\,|\,\phi_j\,|^2\,\phi_j+V(j,t)\phi,(j=1,2,3,...,J) \tag{181}$$

in a lattice field, where r_0 is a spacing between two neighboring lattice points, j labels the discrete lattice points, J is total number of lattice points in the lattice field in the system. The vector form of the above equation in the lattice field is

$$[i\hbar\frac{\partial}{\partial t}-\frac{\hbar^2}{mr_0^2}-V(j,t)]\bar{\phi}=-\varepsilon M\bar{\phi}-bdiag.(|\phi_1|^2,|\phi_2|^2 ...|\phi_\alpha|^2)\bar{\phi}, \tag{182}$$

where $\bar{\phi}(x,t)$ is the column vector, $\bar{\phi}(x,t)$ =Col.(ϕ_1, ϕ_2 ,... ϕ_α), whose complex components, equation (183) is a vector nonlinear SchrÖdinger equation with α modes of motion. In Eq. (183), b is a nonlinear parameter and α is a number of motion modes that exist in the systems. M=[M_{nl}]is an $\alpha\times\alpha$ real symmetric dispersion matrix, $\varepsilon=\hbar^2/2mr_0^2$. Here, n and l are integers denoting the modes of motion. The Hamiltonian and the particle number corresponding to Eq. (183), respectively, are

$$H=\sum_{N=1}^{\alpha}\left(\hbar\omega_0|\phi_n|^2-\frac{1}{2}b|\phi_n|^4\right)-\varepsilon\sum_{n\neq l}^{\alpha}M_{nl}\phi_n\phi_l \text{, and } N=\sum_{N=L}^{\alpha}|\phi_n|^2 \tag{183}$$

where $\hbar\omega_0=\hbar^2/2mr_0^2+V(j,t)$.

We have assumed that $V(j,t)$ are independent of j and t. In the canonical second quantization theory, the complex amplitude (ϕ_n^* and ϕ_n) become boson creation and annihilation operators ($\hat{B}_n^+$ and $\hat{B}_n$) in the number representation. If $|m_n>$ is an eigenfunction of a particular mode, then

$$\hat{B}_n^+|m_n\rangle=\sqrt{m_n+1}\,|m_n+1>,\hat{B}_n\,|m_n>=\sqrt{m_n}\,|m_n-1> \text{and } \hat{B}_n\,|0>=0.$$

Since no particular ordering is specified in Eq.(184) thus we use the averages:

$$|\phi|^2\rightarrow\frac{1}{2}(\hat{B}_n^+\hat{B}_n+\hat{B}_n\hat{B}_n^+)$$

and

$$|\phi_n|^4\rightarrow\frac{1}{6}(\hat{B}_n^+\hat{B}_n^+\hat{B}_n\hat{B}_n+\hat{B}_n^+\hat{B}_n\hat{B}_n^+\hat{B}_n+\hat{B}_n^+\hat{B}_n\hat{B}_n\hat{B}_n^++\hat{B}_n\hat{B}_n^+\hat{B}_n\hat{B}_n^++\hat{B}_n\hat{B}_n\hat{B}_n^+\hat{B}_n^++\hat{B}_n\hat{B}_n^+\hat{B}_n^+\hat{B}_n)$$

with the Boson commutation rule $\hat{B}_n\hat{B}_n^+-\hat{B}_n^+\hat{B}_n=1$, the Eq. (184) then becomes

$$\hat{H}=\sum_{n=1}^{\alpha}[(\hbar\omega_0-\frac{1}{2}b)(\hat{B}_n^+\hat{B}_n+\frac{1}{2})-\frac{1}{2}\,b\hat{B}_n^+\hat{B}_n\hat{B}_n^+\hat{B}_n]-\varepsilon\sum_{n\neq l}^{\alpha}M_{nl}\hat{B}_n^+\hat{B}_l \tag{184}$$

$$\hat{N}=\sum_{n=1}^{\alpha}(\hat{B}_n^+\hat{B}_n^++\frac{1}{2}) \tag{185}$$

From now on, we will use the notation [m_1, m_2,... m_α] to denote the products of number states$|m_1>|m_2>\ldots|m_\alpha>$. Thus, stationary states of the vector nonlinear SchrÖdinger equation (114) must be eigenfunctions of both $\hat{N}$ and $\hat{H}$. Consider an m-quantum state (i.e., the nth excited level, m= $m_1+m_2+\ldots m_j$), with m< α . An eigenfunction of $\hat{N}$ can be established as

$$|\phi_m\rangle=C_1\text{[m,0,0,...,0]}+\ldots+C_2\text{[0,m,0,0,...,0]}+\ldots+C_i\text{[0,0,0,..., m ,]}+\ldots+C_{i+1}\text{[m-1,m,0,0,...,0]}+\ldots+C_p\text{[0,0,0,...,0,1,1...,1]}. \tag{186}$$

The number of terms in Eq.(117) is equal to the number of ways that m quanta can be placed on α sites, which is given by P= $\frac{(m+\alpha-1)}{m!(\alpha-1)!}$. The wave function $|\phi_m>$ in Eq.(187) is an eigenfunction of $\hat{N}$ for any values of the $C_\alpha^{'}$. Thus, we are free to choose these coefficients so that

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$$\hat{H}\,|\phi_m\rangle = E|\phi_m\rangle. \tag{187}$$

Equation (188) requires that the column vector C=Col.($C_1, C_2, \ldots C_p$) satisfies the matrix equation:

$$|\,H - IE\,|C = 0 \tag{188}$$

where H is a $p\times p$ symmetric matrix with real elements. I is a $p\times p$ identity matrix, E is just the eigenenergy. Eq. (188) is an eigenvalue equation of quantum Hamiltonian operator (185) of the systems. We can find the eigenenergy spectra E_m of the systems from Eq. (189) for given parameters, ε, ω_0 , and b. Scott *et al.* [69-71] and Pang *et al.* [72-79] used this method to calculate the energy-spectra of vibrational excitations (quanta) in many nonlinear systems, for example, small molecules or organic molecular crystals and biomolecules. These results can be compared with the experimental data.

ii. *The eigenvalue problem of the nonlinear SchrÖdinger equation and its properties*

In the quantum mechanics we know that the time-independent linear SchrÖdinger equation is an eigenequation of the Hamiltonian operator in the coordinate representation. However, we do not know the meaning of the eigenvalue problem of the nonlinear SchrÖdinger equation, which is therefore a new problem. This problem comes from the Lax method. According to this method, for any nonlinear equation, $\frac{\partial}{\partial t}\phi(\vec{r},t) = K(\phi(\vec{r},t))$, where $K(\phi(\vec{r},t))$ is a nonlinear operator. If $K(\phi(\vec{r},t))$ is related to two linear operators $\hat{L}$ and $\hat{B}$, which depend on ϕ and satisfy the Lax operator equation:

$$i\hat{L}_{t'} = \hat{B}\hat{L} - \hat{L}\hat{B} = [\hat{B}, \hat{L}] \tag{189}$$

where $t' = t/\hbar$, then the eigenvalue E, which does not vary with time, and eigenfunction ψ of the nonlinear equation is determined by the eigenequation of operator $\hat{L}$ as follows

$$\hat{L}\psi = \lambda\psi\,; \qquad i\psi_{t'} = \hat{B}\psi \tag{190}$$

where E= λ. Thus, the eigenvector and eigenvalue of nonlinear systems are determined by the eigenvector and eigenvalue of the above linear operators. In general, concerning any types of nonlinear equation, the corresponding linear eigenequation and time-independent eigenvalue can always be found from the Lax equation. For the nonlinear SchrÖdinger equation (26), the two linear operators $\hat{L}$ and $\hat{B}$ are determined by[25-26]

$$\hat{L} = \begin{pmatrix} 1+s & 0 \\ 0 & 1-s \end{pmatrix}\frac{\partial}{\partial x'} + \begin{pmatrix} 0 & \phi^* \\ \phi & 0 \end{pmatrix},$$

$$\hat{B} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial^2}{\partial x'^2} + \begin{pmatrix} |\phi|^2/(1+s) & i\phi_{x'} \\ -i\phi_{x'} & -|\phi|^2/(1-s) \end{pmatrix} \tag{191}$$

where $s^2 = 1 - 2/b$, $x' = x\sqrt{2m/\hbar^2}$. Thus the eigenvalue of Eq.(26) is determined by

$$\hat{L}\psi = \lambda\psi\,, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{192}$$

Its corresponding solution can be found by use of inverse-scattering or another method.

According to this way the eigenequation corresponding to the nonlinear SchrÖdinger equation (26) and the Galilei invariance are represented by the linear Zakharov-Shabat equation [25-26]]:

$$i\psi_{x'} + \Phi\psi = \lambda\sigma_3\psi \tag{193}$$

This is an eigenequation of eigenfunction ψ with an eigenvalue λ and potential Φ , where,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Phi = \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix} \tag{194}$$

where ϕ satisfies Eq.(26). It evolves with time according to Eq. (191). However, what are the properties of the eigenvalue problems determined by these equations? This deserves further discussion.

As is known, the eigenequation is invariant under the Galilei transformation. As a matter of fact, if we substitute the following Galilei transformation:

$$\phi'(\tilde{x},\tilde{t}) = e^{ivx'-iv^2/2}\phi(x',t'), \tilde{x} = x' - vt', \tilde{t} = t' \tag{195}$$

into Eq. (125), then Φ is transformed into

$$\Phi'(\tilde{x}) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \Phi(x') \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{196}$$

where $\theta = vx' - \frac{1}{2}v^2 t' + \theta_0$, here θ_0 is an arbitrary constant. If the eigenfunction $\psi(x')$ is transformed as

$$\psi'(\tilde{x}) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \psi(x') \tag{197}$$

then Eq. (194) becomes

$$i\psi'_{x'} + \Phi'\psi' = (\lambda - \frac{v}{2})\sigma_3\psi' \tag{198}$$

It is clear that in the reference frame that is moving with the velocity v, the eigenvalue is reduced to v/2 compared with that in the rest frame. It shows that the velocity of the microscopic particle is given by $2\Re(\lambda_k)$. When θ is constant, i.e., $\theta = \theta_0$, the eigenvalue is unchanged because v=0. This implies that the nonlinear SchrÖdinger equation (26) is invariant under the gauge transformation, $\phi' = e^{i\theta_0}\phi(x')$.

Satsuma and Yajima [80] studied the eigenfunction of Eq. (194) and its properties, where the eigenfunction satisfied the boundary condition, $\psi = 0$ at $|x| \to \infty$. The eigenvalues and the corresponding eigenfunctions were denoted by $\lambda_1, \lambda_2, ..., \lambda_N and \psi_1, \psi_2, ..., \psi_N$. respectively. For a given eigenfunction , $\psi_n(x')$, equation (194) reads

$$i\frac{d\psi_n(x')}{dx'} + \Phi(x')\psi_n(x') = \lambda_n \sigma_3 \psi_n(x'), n = 1, 2, ..., N \tag{199}$$

$\psi(x')$ was expressed in terms of Pauli's spin matrices σ_1 and σ_2 ,

$$\psi(x') = \Re[\psi(x')]\sigma_1 - \Im[\psi(x')]\sigma_2 \tag{200}$$

Multiplying Eq. (200) by σ_2 from the left and taking the transpose of the resulting equation, we get

$$-i\frac{d\psi_m^T}{dx'}\sigma_2 - \psi_m^T \Phi^* \sigma_2 = i\lambda_m \psi_m^T \sigma_1 \tag{201}$$

where the superscript T denotes transpose. Multiplying the above equation by ψ_n from right and Eq. (199) by $\psi_m^T \sigma_2$ from the left and subtracting one from the other, Satsuma and Yajima[80] obtained the following equation

$$(\lambda_n - \lambda_m)\int_{-\infty}^{\infty} \psi_m^T \sigma_1 \psi_n dx' = 0$$

The boundary conditions, $\psi_n, \psi_m \to 0$ as $|x'| \to \infty$, were used in obtaining the above equation. The following orthonormal condition was then derived:

$$\int_{-\infty}^{\infty} \psi_m^T \sigma_1 \psi_n dx' = \delta_{nm} \tag{202}$$

Satsuma and Yajima further demonstrated that Eq. (200) has the following symmetry properties.

(1) If $\phi(x')$ satisfies $\phi(-x') = \phi^*(x')$, then replacing x' by - x' in Eq. (200) and multiplying it by σ_2 from left , we can get

$$i\frac{d}{dx'}[\sigma_2\psi_n(-x')] + \Phi(x')[\sigma_2\psi_n(-x')] = \lambda_n\sigma_3[\sigma_2\psi_n(-x')]$$

Since $\sigma_2\psi_n(-x')$ is also an eigenfunction associated with λ_n, its behavior resembles that of $\psi_n(x')$ in the asymptotic region, i.e., $\sigma_2\psi_n(-x') \to 0$ as $|x'| \to \infty$, thus ψ_n has the following symmetry

$$\sigma_2\psi_n(-x') = \delta\ \psi_n(x')\text{, or } \psi_n(-x') = \delta\ \sigma_2\psi_n(-x')\text{, } (\delta = \pm 1)$$

Therefore, if $\phi(-x') = \phi^*(x')$, then $\psi(x')$ satisfies the symmetry property $\psi_n(-x') = \delta\ \sigma_1\psi_n(-x')$ with $\delta = \pm 1$. This can easily be verified by replacing σ_1 with σ_2 in the above derivations.

(2) If $\phi(x')$ is a symmetric (or antisymmetric) function of x', i.e., $\phi(-x') = \pm\phi(x')$, then $\psi_n^{(s)}(x') = \sigma_1\psi*(-x')$ is the eigenfunction belonging to the eigenvalue $-\lambda_n^*$, and $\psi'^{(a)}_n(x') = \sigma_2\psi*(-x')$ is the eigenfunction belonging to the eigenvalue λ_n^*. The suffix s (or a) to the eigenfunction ψ'_n indicates that ϕ is symmetric (or antisymmetric). Since $\phi(-x') = \phi\ (x')$, replacing x' with - x' in Eq. (130) and taking complex conjugate, we get

$$i\frac{d}{dx'}[\sigma_1\psi*(-x')] + \psi(x')[\sigma_1\psi*(-x')] = -\lambda_n\sigma_3[\sigma_1\psi^*{}_n(-x')]$$

Compared with Eq. (200), the above equation implies that $-\lambda_n^*$ is also an eigenvalue and the associated eigenfunction $\psi_n^s(x')$ is just $\sigma_1\psi^*{}_n(-x')$, with the arbitrary constant. For $\phi(-x') = -\phi\ (x')$, the same conclusion is obtained by replacing σ_1 with σ_2 in the above derivations.

These symmetry properties are useful in providing a general view of the solution of Eq. (26) with V(x,t)= A(ϕ)=0 . As is known, the real part of the eigenvalue, ξ_n, corresponds to the velocity of a soliton and the imaginary part, η_n, the amplitude. Then, if $\phi(x', t'=0)$, whose initial value has the symmetry $\phi(x', t'=0), = \pm\ \phi(-x', t'=0)$, breaks into the series of solutions, the decay is bisymmetric, corresponding to the eigenvalues $-\lambda_n^*$. If $\phi(x')$ is real, the above symmetry property yields

$$\psi'^{(s)}_n(-x') = \sigma_1[-\delta\sigma_2\psi_n^*(-x')] = \delta\sigma_2\psi'^{(s)}_n(x')$$

$$\psi'^{(a)}_n(-x') = \sigma_2[-\delta\sigma_1\psi_n^*(-x')] = \delta\sigma_1\psi'^{(a)}_n(x')$$

i.e., $\psi'^{(s)}_n(x')$ has the same parity as $\psi_n(x')$, while $\psi'^{(a)}_n(x')$ has the opposite one. When $\phi(-x') = -\phi\ (x')$ and λ_n is pure imaginary ($\lambda_n = -\lambda_n^*$), the eigenvalues corresponding to the positive and negative parity eigenfunctions degenerate.

(3) If $\phi(x')$ is real, but not antisymmetric, then the eigenvalue λ_n is pure imaginary, i.e., $\Re(\lambda_n)$=0. From Eq. (200) and its Hermitian conjugate, Satsuma et.al[68] found that

$$\Re(\lambda_n)\langle n|\sigma_2|n\rangle = \langle n|\Im[\phi(x')]\sigma_3|n\rangle \tag{203}$$

with

$$<\mathrm{m}|\,\sigma_2\,|\mathrm{n}>=\int_{-\infty}^{\infty}\psi_m^+\sigma_2\psi_n dx' \tag{204}$$

where $[\Phi,\sigma_1]=2i\Im(\phi)\sigma_3$ was used. We see from Eq. (204) that $\Re(\lambda_n)$ vanishes if ϕ is real and $<\mathrm{m}|\,\sigma_2\,|\mathrm{n}>\neq 0$. When ϕ is a real and an antisymmetric function of x', the symmetry property (I) gives

$$<\mathrm{m}|\,\sigma_2\,|\mathrm{n}>=\delta^2\int_{-\infty}^{\infty}\psi_m^+(-x')\sigma_1\sigma_2\sigma_1\psi_n(-x')dx' = -<|\,\sigma_2\,|>$$

Thus $<\mathrm{n}|\,\sigma_2\,|\mathrm{n}>=0$.

(4) If the initial value takes the form of $\phi=e^{ivx'}R(x')$, where $R(x')$ is a real, but not antisymmetric function of x', then all the eigenvalues have the common real part, -v/2. This can be easily shown by the Galilei transformation. In fact, when $\phi(x',t'=0)=e^{ivx'}R(x')$, the solution does not decay to the series of solitons moving with the different velocities, but form a bound state. In this case, the real parts are common to all the eigenvalues, i.e., the relative velocities of the solitons vanish.

(5) If ϕ is a real non-antisymmertric function of x', it can be shown that

$$\psi_n^*(x')=i\delta\sigma_3\psi_n(x') \tag{205}$$

where $\delta=\pm 1$. Because $\Re(\lambda_n)=0$, from the complex conjugate of Eq. (200), one can get $\psi_n^*(x')\propto\sigma_3\psi$. Substituting Eq. (194) into the normalization condition Eq. (202), one then has $\delta=\pm 1$. If the eigenvalue of Eq. (124) is real, i.e., $\lambda=\xi$ is real, then

$$i\frac{d\psi}{dx'}+\Phi\psi=\xi\sigma_3\psi \tag{206}$$

and the adjoint function of $\psi,\overline{\psi}=i\sigma_2\psi^*$, is also a solution of Eq. (207), i.e.,

$$i\frac{d\overline{\psi}}{dx'}+\Phi\overline{\psi}=\xi\sigma_3\overline{\psi}$$

From this and Eq. (207), Satsuma and Yajima obtained the following

$$\frac{d}{dx'}(\psi^+\psi)=\frac{d}{dx'}(\overline{\psi}^+\psi)=\frac{d}{dx'}(\psi^+\overline{\psi})=\frac{d}{dx'}(\overline{\psi}^+\overline{\psi})=0 \tag{207}$$

Using the above boundary conditions, they found that the solutions of Eq.(194) $\psi_1(x',\xi),\psi_2(x',\xi)$, and $\overline{\psi}_2(x',\xi)$ satisfy the following relations.

$$\psi_1^+\psi_1=\psi_2^+\psi_2=\overline{\psi}_2^+\overline{\psi}_2=1,\overline{\psi}_2^+\psi_2=\psi_2^+\overline{\psi}_2=0$$

From $\psi_1=a(\xi)\overline{\psi}_2+b(\xi)\psi_2$, we get a= $\overline{\psi}_2^+\psi_1$ and b= $\overline{\psi}_2^+\psi_1$, where $\psi_1=(x',\xi)=\begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi x'}$, as $x'=-\infty$ and $\psi_2=(x',\xi)=\begin{pmatrix}0\\1\end{pmatrix}e^{+i\xi x'}$, $\overline{\psi}_2(x',\xi)=\begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi x'}$ as $x'=\infty$. As pointed out earlier, if a real (not antisymmetric) initial value is considered, the microscopic particle does not decay into moving solitons, but forms a bound states of solitons pulsating with the proper frequency. Sataúma and Yajima developed a perturbation approach to investigate the conditions for the solutions to evolve and decay into moving solitons.

If the wave function ϕ in Eq. (124) undergoes a small change, i.e., $\phi\to\phi'=\phi+\Delta\phi$, the corresponding change in Φ is given by

$$\Delta\Phi=\begin{pmatrix}0 & \Delta\phi\\ \Delta\phi^* & 0\end{pmatrix}.$$

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Then, λ_n and ψ_n changes as $\lambda_n + \Delta\lambda_n$ and $\psi_n + \Delta\psi_n$, respectively. To the first order in the variation, Eq. (194) becomes

$$[i\frac{d}{dx'} + (\Phi - \lambda_n\sigma_3)]\Delta\psi_n + (\Delta\Phi - \Delta\lambda_n\sigma_3)\psi_n = 0$$

Multiplying the above equation by $\psi_n^T\sigma_2$ from the left and integrating with respect to x' over $(-\infty, \infty)$, we get

$$\Delta\lambda_n = -i\int_{-\infty}^{\infty}\psi_n^T\sigma_2\Delta\Phi\psi_n dx' = -\int_{-\infty}^{\infty}\psi_n^T\Re(\Delta\phi)\sigma_3\psi_n dx' + i\int_{-\infty}^{\infty}\psi_n^T\Im(\Delta\phi)\psi_n dx'$$

If ϕ is a real and non-antisymmetric function of x', Eq.(207) holds and

$$\Delta\lambda_n = \delta < n|\Im(\Delta\phi)\sigma_3|n> + i\delta < n|\Re(\Delta\phi)|n> \qquad (208)$$

Equation (209) indicates that if $< n|\Im(\Delta\phi)\sigma_3|n> \neq 0$, the perturbation $\Delta\phi$ makes the real part of the eigenvalue finite. That is, for the initial value, $\phi(x') + \Delta\phi(x')$, the solution of Eq.(26) breaks up into moving solitons with velocity $2\Re(\Delta\lambda_n)$. If ϕ is real and is either a symmetric or an antisymmetric function of x', the above symmetry properties of eigenvalues of the nonlinear Schrodiger equation (26) lead to

$$< n|\Im(\Delta\phi(x'))\sigma_3|n> = -< n|\Im(\Delta\phi(-x))\sigma_3|n>$$

Therefore, if $\Im(\Delta\phi)$ is a symmetric function, then $< n|\Im(\Delta\phi)\sigma_3|n>$ vanishes, i.e., $\Re(\Delta\lambda_n) = 0$, and the soliton bound state does not resolve into moving solitons even in the presence of the perturbation $\Delta\phi$.

Satsuma and Yajima[80] also obtained the shifts of the eigenvalues of Eq.(194) under the double-humped initial values, $\phi(x', t'=0) = \phi_0(x' - x_0') + e^{i\theta_0}\phi(x' + x_0')$, where ϕ_0 is a real and symmetric function of x', x_0' and ϕ_0 are real. The shifts of the eigenvalues were finally written as

$$\Delta\lambda_n^{\pm} = \delta[\sin\ \theta < n|\sigma_3\phi_0(x' + 2x_0')|n> \mp \sin(\frac{\theta_0}{2}) < n|\sigma_3\phi_0(x')e^{2x_0'(d/dx')}|n>] +$$

$$i\delta[\cos\theta_0 < n|\phi_0(x' + 2x_0')|n> \pm \cos(\frac{\theta_0}{2}) < n|\phi_0(x')e^{2x_0'(d/dx')})|n>]$$

where

$$-\delta\cos(\frac{\theta_0}{2}) < n|\phi_0(x')e^{2x_0'(d/dx')})|n> - i\delta\sin(\frac{\theta_0}{2}) < n|\sigma_3\phi_0(x')e^{2x_0'(d/dx')}|n>$$

$$= \int_{-\infty}^{\infty}\psi_2^{''(n)T}\sigma_2\Phi\psi_1^{''(n)}dx' = \int_{-\infty}^{\infty}\psi_1^{''(n)T}\sigma_2\Phi\psi_2^{''(n)}dx'$$

$$-\delta\cos(\theta_0) < n|\phi_0(x' + 2x_0'')|n> - i\delta\sin(\theta) < n|\sigma_3\phi_0(x' + 2x_0')|n>$$

$$= \int_{-\infty}^{\infty}\psi_1^{''(n)T}\sigma_2\Phi_2\psi_1^{''(n)}dx' = \int_{-\infty}^{\infty}\psi_2^{''(n)T}\sigma_2\Phi_1\psi_2^{''(n)}dx',$$

here

$$\Phi(x') = \Phi_1(x') + \Phi_2(x'), \Phi_1(x') = \sigma_1\phi_0(x' - x_0')$$

$$\Phi_2(x') = [(\cos(\theta_0)\sigma_1 - \sin(\theta)\sigma_2]\phi_0(x' + x_0')$$

The corresponding eigenvalue equation is given by

$$i\frac{d}{dx'}\psi_n'' + \Phi(x')\psi_n'' = \lambda_n\sigma_3\psi_n''(x')$$

The eigenfunction $\psi_n''(x')$ satisfies the following symmetry and orthogonality requirements:

$$\psi_{n\pm}''(-x') = \pm\delta[\cos(\frac{\theta_0}{2})\sigma_2 + \sin(\frac{\theta_0}{2})\sigma_1]\psi_{n\pm}''(x'), \delta \pm 1$$

$$\int_{-\infty}^{\infty} \psi_{n+}''^{T}(x')\sigma_1\psi_{n-}''(x')dx' = 0$$

When $\theta_0 = 0, \phi(x')$ is real and symmetric, $\Delta\lambda_n^{(\pm)}$ is pure imaginary, when $\theta_0 = \pi$, $\phi(x')$ is real and antisymmetric, $\Delta\lambda_n^{(\pm)}$ is real,

$$\Re[\Delta\lambda_n^{(\pm)}(\theta_0 = \pi)] = \mp\delta < n | \sigma_3\phi_0(x')e^{2x_0' (d/dx')} | n >$$

$$\Im[\Delta\lambda_n^{(\pm)}(\theta_0 = \pi)] = -\delta < n | \sigma_3\phi_0(x' + 2x_0') | n > \quad (210)$$

Thus, the solution of the nonlinear SchrÖdinger equation (26) decays into paired solitons and each pair consists of solitons with equal amplitude and moving in the opposite directions with the same speed. For arbitrary θ_0', we can see from Eq. (210) that the solution of Eq. (26) breaks up into an even number of moving solitons with different speeds and amplitudes.

From the above investigations we know that the eigenvalues and eigenequations of nonlinear Schrodiger equation are a very complicated and different properties.

III. The Nonlinear SchrÖdinger Equation is a Correct and Universal Dynamic Equation of the Microscopic Particles in all Physical Systems

a) The results brought by the using the nonlinear SchrÖdinger equation

As known, the states and properties of microscopic particles were described by the linear Schrodiner equation (7) in the quantum mechanics, but the microscopic particles have only the wave feature, not corpuscle nature in such a case. This feature is contradictory with the traditional concept of particles. At the same time, position and momentum of the particles meet also the uncertainty relation, the occurrence of particles at a point in time-space is represented by a probability, the mechanical quantities of the particles are denoted by some average values, and so on. These uncertain descriptions to the properties of the microscopic particles bring us plenty of difficulties and troubles to understand their natures and essences. At the same time, these properties of microscopic particles also correspond not with the experimental results of electronic diffraction on double seam by Davisson and Germer in 1927[8-12] and de Broglie's relation of wave-corpuscle duality[8-9]. Thus there are considerable, intense and durative controversies in physics, which elongate and continue a century. Very surprisingly, these difficulties, contradictions and controversies have not been solved up to now.

In such a case we have broken through the hypothesis of independence of Hamiltonian operator of the systems on states of microscopic particles, forsaken the traditional quantum mechanical method of average field approximation to replace real and complicated interactions among the particles or between the particle and background field and introduced further the nonlinear interaction between them into the dynamic equation of particles to build the nonlinear SchrÖdinger equation. And we used it to replace the linear SchrÖdinger equation in quantum mechanics and to study further the nature and states of microscopic particles. From this investigation we find that the states and properties of microscopic particles are considerably and essentially changed relative to those in quantum mechanics, a lot of interesting and important results are obtained from this investigation. These considerable changes are described as follows.

(1) An outstanding and obvious change is that the microscopic particles have a wave- corpuscle duality and is embedded by organic combination of envelope and carrier wave. The particle has not only wave features of certain amplitude, velocity, frequency, and wavevector, but also corpuscle natures of a determinant mass centre, size, mass, momentum and energy. This is first time to shed light theoretically on the wave-corpuscle duality of microscopic particles in quantum theory. At the same time, we proved that the wave-corpuscle duality of microscopic particles is quite stable, even though they are in an externally applied potential field.

(2) The motion of the particles satisfy the classical Newtonian law, Lagrangian equation and Hamilton equation, which exhibit the classical properties of microscopic particles.

(3) The microscopic particles have determinant mass, momentum and energy, and obey the universal conservation laws of mass, momentum, energy and angular momentum.

(4) The microscopic particles meet also the classical collision rule when they collide with each other. Although these particles are deformed in collision process, they can still retain themselves form and amplitude to move towards after collision, where a phase shift may occur.

(5) The position and momentum of the mass centre of microscopic particles are determinant, but their coordinate and momentum obey still a minimum uncertainty relation, which differs from those in quantum mechanics. This property of the particle displays its wave-corpuscle duality.

(6) We can determine that the microscopic particles possess a wave feature from its features of reflection and transmission features on the interface , but this wave feature are different from that of both linear wave and KdV solitary wave.

(7) We know from the investigations of eigenvalue problem of nonlinear SchrÖdinger equation that the eigenvalue states of microscopic particles described by the nonlinear SchrÖdinger equation have a lot of unusual features which are completely different from that in quantum mechanics, the eigenenergy spectra of Hamiltonian operator of the microscopic particles can be obtained in second quantum representation or number representation. This suggests that the natures of microscopic particles described by the nonlinear SchrÖdinger equation are in essence different from those in quantum mechanics.

The above new properties of the microscopic particles exhibit and display clearly both corpuscle and wave features which are consistent with the concepts of traditional particle and wave, respectively. Therefore the natures of microscopic particles described by the nonlinear SchrÖdinger equation (20) differ in essence from those described by the linear SchrÖdinger equation (7) and relate directly to these difficult and disputed problems as mentioned above, thus this investigation pounds considerably the quantum mechanics, its influences on quantum mechanics are crucial and considerable. Thus this research idea and results could overcome and solve the century difficulties and disputations existed in the quantum mechanics. To sum up, the influences of such a investigation on the quantum mechanics can be described as follows.

b) The essences of quantum mechanics

First influence is that we see clearly the essences of the quantum mechanics. As far as the quantum mechanics is concerned, we should confirm that its birth is a revolution of physics or science, it is the foundation of modern science, its applications acquire the great successes, especially when it was applied in hydrogen atom and molecule as well as helium atom and molecule, the theoretical results obtained are consistent with experimental data. However, it nevertheless encountered some problems and difficulties, which are embodied in not only the elementary hypothesizes of quantum mechanics but also its applications. In its hypothesizes the difficulties and controversies are the occurrence of particles at a point in time-space to be represented by a probability, the mechanical quantities of the particles to be denoted by some average values and the hypothesis of independence of Hamiltonian operator of the systems on states of microscopic particles. In the applications the difficulties come from its applications in the systems of many particles and many bodies. When the quantum mechanics is used to study the properties of motion of microscopic particles in these complicated systems, we have to utilize ever many simple and approximate method unassociated with the states of particles in virtue of different approximate methods, such as, the signal and free electronic approximations, compact -binding approximation and average field approximation, and so on, to replace some complicated and real nonlinear interaction among these particles, or between the particle and backgrounds, which could determine the essences and natures of particles, in the systems in calculation.Thus we obtained only some approximation, but not real, complete and correct, solutions in which the effects and results arising from these complicated effects and nonlinear interactions are ignored completely. Then the states and properties of particles determined by the average potential as well as the studied method are not real and correct. Therefore, we can conclude that the linear SchrÖdinger equation is a linearity of dynamic equation, can only describe the properties and states of a single microscopic particle in vacuum or the system of less body without nonlinear interaction, then the quantum mechanics is correct, but has some limitations and is only a simple, approximate and linear theory and cannot represent in truth the properties and states of motion of the microscopic particles in general and complicated systems. These are just the essence of the quantum mechanics. Therefore the quantum mechanics must develop toward.

c) The roots of localization of microscopic particle and the necessity developing nonlinear quantum mechanics

Second influence on the quantum mechanics is that we know clearly the basic root of no localization of microscopic particles, in other word, these difficulties and disputations in quantum mechanics, which are just that the Hamiltonian operator of the systems is too simple, composed only of kinetic and externally applied potential energy operators and depends not with wave function of states of the particles. Concretely speaking, plenty of complicated and nonlinear interactions among the particles or the particle and background field related to the states of the particles have been completely forsaken in the Hamiltonian operator of the system. Thus its dynamic equation is linear. If these nonlinear interactions are introduced into the dynamic equation of particles, then the linear SchrÖdinger equation (7) in the quantum mechanics is replaced by the nonlinear SchrÖdinger equation (20). In the latter the nonlinear interaction related to the states of particle can balance and suppress the dispersion effect of the kinetic term in the dynamic equation (7), then the effective potential of the system becomes a double-well potential from a signal well's, thus wave feature of the particle is suppressed, the shape of wave becomes the form of sech(*x*-*vt*), its mass, energy, and momentum are gathered and maintained, thus the microscopic particle is localized at x_0 and become eventually as a soliton with wave-corpuscle duality. Thus the natures of the microscopic particle are thoroughly changed in such a case. The results obtained from use of nonlinear SchrÖdinger equation (20) also verify that the natures of microscopic particles described by the nonlinear SchrÖdinger equation (20) differ in essence from those described by the linear SchrÖdinger equation (7). Thus, the basic root of localization of the microscopic particles described by nonlinear SchrÖdinger equation is just this nonlinear interactions, which suppress and cancel the dispersive effect of kinetic energy in the dynamic equation.

The third influence is that the quantum mechanics seeks the development direction, namely, it is necessary to establish and develop nonlinear quantum mechanics based on the nonlinear SchrÖdinger equation. In such a case the Hamiltonian operator of the systems must be related to the wave function of state of the microscopic particles and is nonlinear function of states of the particles. Based on these ideas and the properties of wave functions of the particles in the nonlinear SchrÖdinger equation we could establish the nonlinear quantum mechanics. To develop nonlinear quantum mechanics can promote the development of physics and can enhance and raise the knowledge and recognition to the essences of microscopic matter. This just is the most great influence on quantum mechanics [22-25].

In fact, all realistic physics systems are always composed of many particles and many bodies, hydrogen atom is a most simple system and composed also of two particles, thus the system composed only of one particle does not exist in nature. In such a case, the nonlinear interactions exist always in any realistic physics systems including the hydrogen atom[10-15]. Therefore, when the states and properties of microscopic particles in a realistic physics systems are studied by using quantum theory, we should use the nonlinear SchrÖdinger equation (20) or nonlinear quantum mechanics[22-25], instead of the linear SchrÖdinger equation (7) in quantum mechanics. Only if the coupling interaction among the particles, or between the particle and background field equal to zero or exists not, then equation(20) can degenerate to the linear SchrÖdinger equation (7). This indicates again that the linear SchrÖdinger equation in quantum mechanics is only an especial and approximate case of nonlinear SchrÖdinger equation, and can only describe the states and properties of a single particle without the nonlinear interaction.

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References Références Referencias

1. D. Bohr and J. Bub, Phys. Rev., 1935, 48: 169.
2. E. SchrÖdinger, Collected paper on wave mechanics, London: Blackie and Son, 1928.
3. E. SchrÖdinger, Proc. Cambridge Puil. Soc., 1935, 31: 555.
4. Heisenberg W., Z. Phys.,1925, 33: 879; Heisenberg W. and Euler H., Z. Phys. 1936,98:714.
5. M. Born and L. Infeld, Proc. Roy. Soc. A, 1934, 144: 425.
6. P. A. Dirac, The Principles of Quantum Mechanics, Oxford: Clarendon Press, 1967.
7. D.Bohm,Phys.Rev., 1952,85:166,180.
8. J. Potter, quantum mechanics, North-Holland publishing Co. 1973.
9. M. Jammer, The concettual development of quantum mechanics, Tomash, Los Angeles,1989.
10. J. Meconnell, quantum particle dynamics, North Holland Publishing Co.,Amsterdam, 1958.
11. L. R. Roth and A.Inomata,Fundamental questions in quantum mechanics, Gordon and Breach, New York, 1986.
12. S. Diner, D. Farque, G. Lochak, and F. Selleri, The Wave-Particle Dualism, Dordrecht: Riedel, 1984.
13. F.Selleri, quantum daradoxes and physical reality, Kluwer Academic Publishers, Dordrecht, 1990.
14. M. Ferrero and A. Van der Merwe, New Developments on Fundamental Problems in Quantum Physics, Dordrecht: Kluwer, 1997.
15. M. Ferrero and A. Van der Merwe, Fundamental Problems in Quantum Physics, Dordrecht: Kluwer, 1995.
16. L. de Broglie, Nonlinear Wave Mechanics: A Causal Interpretation, Amsterdam: Elsevier, 1960.
17. P. B. Burt, Quantum Mechanics and Nonlinear Waves, New York: Harwood Academic Publisher, 1981.
18. Pang Xiao-feng, The theory of nonlinear quantum mechanics(Chinese), Chinese Chongqing Press, Chongqing, 1994.

19. Pang Xiao-feng, Soliton Physics, Chengdu: Sichuan Science and Technology Press, 2003.
20. Go Bai. Lin. and Pang Xiao-feng, Solitons, Beijing: Chinese Science Press, 1987.
21. Pang Xiao-feng, Research and Development and of World Science and Technology, 2006, 28:11.
22. Pang Xiao-feng, Research and Development and of World Science and Technology, 2003,24:54.
23. Pang Xiao-feng, J. Low. Temp. Phys., 1985, 58: 334.
24. Pang Xiao-feng, Problems of Nonlinear Quantum Mechanics, Chengdu: Sichuan Normal University Press, 1985
25. V. E. Zakharov and A. B. Shabat, Sov. Phys JETP, 1972, 34: 62.
26. V. E. Zakharov and A. B. Shabat, Sov. Phys JETP, 1973, 37: 823.
27. Pang Xiao-feng, Chin. J. Low .Temp. Supercon., 1982, 4: 33; ibid, 1981, 3: 62.
28. Pang Xiao-feng,Chin. J. Nature, 1982, 5: 254.
29. Pang Xiao-feng, Phys. Bulletin Sin., 1983, 3: 17.
30. Pang Xiao-feng, Chin. J. Potential Science, 1985, 5: 16.
31. Pang Xiao-feng, J. Res. Met. Mat. Sin., 1986, 12: 31.
32. Pang Xiao-feng,, J. Science Exploration Sin., 1986, 4: 70.
33. Pang Xiao-feng and Cao Xianyu, Nature Science, 2007, 2: 42.
34. Pang Xiao-feng, J. Electron. Sci. and Tech. of China, 2008, 6:205.
35. H. H. Chen and C. S. Liu, Phys. Rev. Lett., 1976, 37: 693.
36. H. H. Chen, Phys. Fluids, 1978, 21: 377.
37. Pang Xiao-feng, Phys. Stat. Sol.(b) , 2003, 236: 34.
38. Pang Xiao-feng, Physica D, 2001, 154: 138.
39. Pang Xiao-feng, J. Phys. Condens. Matter, 1990, 2: 9541.
40. Pang Xiao-feng and H. J. W. Miiller-Kirsten, J. Phys. Condens. Matter, 2000, 12: 885.
41. Pang Xiao-feng and Yuan Yuan-ping., Chem. Phys. Lett., 2003, 373: 392.
42. C.Sulem and P.L.Sulem, The Nonlinrar SchrÖdinger Equation: Self-focusing and Wave Collapse, Berlin: Springer-Verlag, 1999.
43. C.Desem and P.L.Chu, Opt. Lett.,1987,12:349.
44. C.Desem and P.L.Chu,soliton-soliton interactions in optical solitons, ed. Taylor, J.R., Cambridge University Press, Cambridge, 1992.
45. B.Tan and J.P.Bord, Phys. Lett.A 1998,240:282.
46. Pang Xiao-feng, Int. J. Mod. Phys.B19(2005)4677; 20(2006)3027.
47. Pang Xiao-feng,Commun. Theor. Phys.43 (2005)367;
48. Pang Xiao-feng,Commun. Theor. Phys. 48（2007）369.
49. D. W. Aossey, S. R. Skinner, J. T. Cooney, J. E. Williams, JM. T. Gavin, D. R. Amdersen, and K. E. Lonngren, Phys. Rev. A, 1992, 45: 2606.
50. R.N.Thurston and A. M. Weiner, J. Opt. Soc. Am. B, 1991, 8: 471.
51. J. L. Cooney, M. T. Gavin, J. E. Williams, D. W. Aossey, and K. E. Lonngren, Phys. Fluids B, 1991, 3: 3277.
52. N. J. Zabuasky and M. D. Kruskal, Phys. Rev.Lett.1965, 15:240.
53. H. Ikezl, R. J. Taylor, and D. R. Baker, Phys. Rev. Lett., 1970, 25: 11.
54. G. L. Lamb, Elements of soliton theory, New York: Wiley, 1980: 118.
55. A. S. Davydov, Solitons in Molecular Systems, Dordrecht: Reidel Publishing Comp., 1985; 2nd ed, 1991.
56. Pang Xiao-feng, Phys. Rev. E, 2000, 62: 6989.
57. Pang Xiao-feng, European Phys. J. B, 2001, 19: 297.
58. Pang Xiao-feng, European Phys. J. B, 1999, 10: 415.
59. Pang Xiao-feng, Phys. Lett. A, 1999, 259: 466.
60. Pang Xiao-feng,J. Phys. Chem. Sol., 2001, 62:491.
61. Pang Xiao-feng, Science in China ser.G, 2008, 51:225.
62. L. N. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev., 1957, 108: 1175.
63. Y. Lai and H. A. Haus, Phys. Rev. A, 1989, 40: 844, 854.
64. F. X. Kartner and L. Boivin, Phys. Rev. A, 1996, 53: 454.
65. K. E. Lonngren, Plasma Phys. 1983, 25:943.
66. K. E. Lonngren, D. R. Andersen and J. L. Cooney, Phys. Lett. A, 1991, 156: 441.
67. A.B.Aceves, J.V.Mononey and A.C.Newell, Phys.Rev.A1989, 39:1809 and 1839.
68. K. E. Lonngren and A. C. Scott, Solitons in Action, New York: Academic, 1978, p15.
69. A. C. Scott and J. C. Eilbeck, Phys. Lett. A, 1986, 119: 60.
70. A. C. Scott and J. C. Eilbeck, Chem. Phys. Lett, 1986, 132: 23.
71. A. C. Scott, P. S. Lomdahl, and J. C. Eilbeck, Chem. Phys. Lett., 1985, 113: 29.

72. Pang Xiao-feng, Acta Phys. Sin., 1994, 43: 1987.
73. Pang Xiao-feng, Chin. J. Phys. Chem., 1995, 12:1062.
74. Pang Xiao-feng and X. R. Chen, Chinese Physics, 2000, 9:108.
75. Pang Xiao-feng and X. R. Chen, Commun. Theor. Phys., 2001, 35: 313.
76. Pang Xiao-feng and X. R. Chen, Commun. Theor. Phys., 2001, 35: 763.
77. Pang Xiao-feng and X. R. Chen, J. Phys. Chem. Solid., 2001, 62: 793.
78. Pang Xiao-feng and H. W. Zhang, Modern Phys. Lett. B, 2006, 20: 1923.
79. Pang Xiao-feng and X. R. Chen, Inter. J. Mod. Phys. B, 2006, 20: 2505.
80. J. Satsuma and N. Yajima, Prog. Theor. Phys. (Suppl.), 1974, 55: 284.