Certain Results on Bicomplex Topologies and their Comparison

By Akhil Prakash & Prabhat Kumar
Dr. B.R. Ambedkar University

Abstract- In this paper, we have investigated the relation between idempotent order and norm topological structures. We have discussed about the relation between real order topology and idempotent order topology and we have also established the relation between real order topology and norm topology.

Keywords: real order topology, complex order topology, idempotent order topology, norm topology, comparison.

GJSFR-F Classification : MSC 2010: 54A10, 30G35
Certain Results on Bicomplex Topologies and their Comparison

Akhil Prakash & Prabhat Kumar

Abstract: In this paper, we have investigated the relation between idempotent order and norm topological structures. We have discussed about the relation between real order topology and idempotent order topology and we have also established the relation between real order topology and norm topology.

Keywords: real order topology, complex order topology, idempotent order topology, norm topology, comparison.

I. Introduction

In 1892, Corrado Segre (1860-1924) published a paper [6] in which he treated an infinite set of Algebras whose elements he called bicomplex numbers, tricomplex numbers, ..., n-complex numbers. A bicomplex number is an element of the form 

\[(x_1 + i_1 x_2) + i_2 (x_3 + i_1 x_4),\]

where \(x_1, \ldots, x_4\) are real numbers, \(i_1^2 = -1\) and \(i_1 i_2 = i_2 i_1\).

Segre showed that every bicomplex number \(z_1 + i_2 z_2\) can be represented as the complex combination

\[(z_1 - i_1 z_2) \left[ \frac{1 + i_1 i_2}{2} \right] + (z_1 + i_1 z_2) \left[ \frac{1 - i_1 i_2}{2} \right].\]

Srivastava [8] introduced the notations \(\xi^1\) and \(\xi^2\) for the idempotent components of the bicomplex number \(\xi = z_1 + i_2 z_2\), so that

\[\xi = \xi^1 \cdot \frac{1 + i_1 i_2}{2} + \xi^2 \cdot \frac{1 - i_1 i_2}{2}\]

Michiji Futagawa seems to have been the first to consider the theory of functions of a bicomplex variable [1, 2] in 1928 and 1932. The hyper complex system of Ringleb [5] is more general than the Algebras; he showed in 1933 that Futagawa system is a special case of his own.


Throughout, the symbols \(\mathbb{C}_2\), \(\mathbb{C}_1\), \(\mathbb{C}_0\) denote the set of all bicomplex, complex and real numbers respectively.
In $\mathbb{C}_2$ besides 0 and 1- there are exactly two non-trivial idempotent elements denoted as $e_1$ and $e_2$ and defined as

$$e_1 = \frac{1+i_1i_2}{2} \quad \text{and} \quad e_2 = \frac{1-i_1i_2}{2}$$

Obviously $(e_1)^n = e_1$, $(e_2)^n = e_2$

$$e_1 + e_2 = 1, \quad e_1e_2 = 0$$

Every bicomplex number $\xi$ has unique idempotent representation as complex combination of $e_1$ and $e_2$ as follows

$$\xi = z_1 + i_2z_2 = (z_1-i_1z_2)e_1 + (z_1+i_1z_2)e_2$$

The complex numbers $(z_1-i_1z_2)$ and $(z_1+i_1z_2)$ are called idempotent component of $\xi$, and are denoted by $^1\xi$ and $^2\xi$ respectively (cf. Srivastava [8]).

Thus $\xi = ^1\xi e_1 + ^2\xi e_2$

**a) $h_1$, $h_2$ image and Cartesian idempotent set**

The $h_1$ and $h_2$ image of a set $X$ are denoted as $^1X$ and $^2X$ respectively and defined as

$$h_1(X) = ^1X = \{z: ze_1 + we_2 \in X\} = \{^1\xi : \xi \in X\}$$

$$h_2(X) = ^2X = \{w: ze_1 + we_2 \in X\} = \{^2\xi : \xi \in X\}$$

The Cartesian idempotent product of $^1X$ and $^2X$ is the set which is the subset of $\mathbb{C}_2$ and denoted as $^1X \times_e^2X$ and defined as

$$^1X \times_e^2X = \{ze_1 + we_2 : z \in ^1X, w \in ^2X\}$$

If $X = ^1X \times_e^2X$ then $X$ is said to be Cartesian idempotent set (cf. Srivastava [8]).

**II. Certain Results from Topologies on Bicomplex Space**

**a) Norm, Complex and Idempotent topologies on $\mathbb{C}_2$[9].**

**2.1.1 Norm topology**

The norm of a bicomplex number $\xi = z_1 + i_2z_2 = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = ^1\xi e_1 + ^2\xi e_2$ is defined as

$$\|\xi\| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}$$

$$= (|z_1|^2 + |z_2|^2)^{1/2}$$

$$= \sqrt{|z_1|^2 + |z_2|^2}$$

Since $\mathbb{C}_2$ is modified normed algebra w.r.t this norm therefore for $\delta > 0$, the $\delta$-ball centered at $x$ is the set

$$B(x, \delta) = \{y \in \mathbb{C}_2 : \|x - y\| < \delta\} \quad \text{of all points} \quad \text{y whose distance from} \quad \text{x is less than} \quad \delta,$$

it is called the $\delta$-ball centered at $x$.

The collection $'B_N'$ of all $\delta$-balls $B(x, \delta)$, for $x \in \mathbb{C}_2$ and $\delta > 0$ is a basis for a topology on $\mathbb{C}_2$. The topology generated by $B_N$ is called norm topology on $\mathbb{C}_2$ and denoted by $T_N$.  

2.1.2 Theorem
If $X$ is a Cartesian idempotent set in $\mathbb{C}_2$ then $X$ is open (w.r.t. norm topology) if and only if $^1X$ and $^2X$ are open in complex plane (cf. Price [3]).

2.1.3 Complex topology
The norm of a complex number $'z'$ is defined as $\|z\| = |z|$

Since $\mathbb{C}_1$ is a normed algebra w.r.t. this norm therefore the collection $'B'$ of all circular disk $S(z,\delta)$, $z \in \mathbb{C}_1$ and $\delta > 0$ will be a basis for a topology on $\mathbb{C}_1$, where $S(z,\delta) = \{w \in \mathbb{C}_1: |z - w| < \delta\}$ Therefore

$B \times B = \{S_1 \times S_2: S_1, S_2 \in B\}$ will be a basis for some topology on $\mathbb{C}_1 \times \mathbb{C}_1$. Since $\mathbb{C}_2 \cong \mathbb{C}_1 \times \mathbb{C}_1$ therefore $B_c = \{S_1 \times_c S_2: S_1, S_2 \in B\}$ will be a basis for some topology on $\mathbb{C}_2$, where

$S_1 \times_c S_2 = \{\eta = w_1 + i_2 w_2: w_1 \in S_1, w_2 \in S_2\}$

If $S_1 = S_1(z_1, r_1)$ and $S_2 = S_2(z_2, r_2)$ Then $S_1 \times_c S_2 = S_1(z_1, r_1) \times_c S_2(z_2, r_2)$

$= \{\eta = w_1 + i_2 w_2: |z_1 - w_1| < r_1, |z_2 - w_2| < r_2\}$

The set $S_1(z_1, r_1) \times_c S_2(z_2, r_2)$ is denoted by $C(\xi = z_1 + i_2 z_2; r_1, r_2)$ and this set $C(\xi = z_1 + i_2 z_2; r_1, r_2)$ is called open complex discus centered at $\xi$ and associated radii $r_1$ and $r_2$ Therefore

$C(\xi = z_1 + i_2 z_2; r_1, r_2) = \{\eta = w_1 + i_2 w_2: |z_1 - w_1| < r_1, |z_2 - w_2| < r_2\}$

Thus $B_c = \text{Set of all open complex discus}$

$= \{C(\xi; r_1, r_2): \xi \in \mathbb{C}_2$ and $r_1, r_2 > 0\}$

The topology generated by $'B_c'$ is called complex topology and denoted by $\mathcal{T}_c$.

2.1.4 Idempotent topology
Since $B \times B = \{S_1 \times S_2: S_1, S_2 \in B\}$ is a basis for some topology on $\mathbb{C}_1 \times \mathbb{C}_1$ and $\mathbb{C}_2 \cong \mathbb{C}_1 \times \mathbb{C}_1$ therefore $B_I = \{S_1 \times_c S_2: S_1, S_2 \in B\}$ will be a basis for some topology on $\mathbb{C}_2$, Where

$S_1 \times_c S_2 = \{\eta = ^1\eta e_1 + ^2\eta e_2; ^1\eta \in S_1, ^2\eta \in S_2\}$

If $S_1 = S_1(z_1, r_1)$ and $S_2 = S_2(z_2, r_2)$ Then $S_1 \times_c S_2 = S_1(z_1, r_1) \times_c S_2(z_2, r_2)$

$= \{\eta = ^1\eta e_1 + ^2\eta e_2; |^1\eta - z_1| < r_1, |^2\eta - z_2| < r_2\}$. The set $S_1(z_1, r_1) \times_c S_2(z_2, r_2)$ is denoted by $D(\xi = z_1 e_1 + z_2 e_2; r_1, r_2)$ and this set $D(\xi = z_1 e_1 + z_2 e_2; r_1, r_2)$ is called open idempotent discus centered at $\xi$ and associated radii $r_1$ and $r_2$. Therefore $D(\xi; r_1, r_2) = \{\eta: |^1\eta - ^1\xi| < r_1, |^2\eta - ^2\xi| < r_2\}$

Thus $B_I = \text{Set of all open idempotent discus}$

$= \{D(\xi; r_1, r_2): \xi \in \mathbb{C}_2$ and $r_1, r_2 > 0\}$

The topology generated by $'B_I'$ is called idempotent topology and denoted by $\mathcal{T}_I$. 

---


© 2016 Global Journals Inc. (US)
b) **Order topology on \( \mathbb{C}_2 \)**

Singh [7] has developed certain orders on \( \mathbb{C}_2 \). He defined three types of ordering in \( \mathbb{C}_2 \), viz., Real dictionary order \(<_R \) Complex dictionary order \(<_C \) Idempotent dictionary order \(<_{ID} \). With the help of these three relations he has defined three order topologies on \( \mathbb{C}_2 \). The order topology induced by real dictionary order is called as real order topology \( T_1 \), the order topology generated by Complex dictionary order is called complex order topology \( T_1^* \) and the topology induced by Idempotent dictionary order is called idempotent order topology \( T_1^# \) on \( \mathbb{C}_2 \).

In the present paper, \( B_p, B_i \) and \( B_j^# \) denotes the basis of \( T_1, T_1^* \) and \( T_1^# \) respectively. The sets \((\xi, \eta)_R \), \((\xi, \eta)_C \) and \((\xi, \eta)_{ID} \) are the open interval with respect to Real dictionary order, Complex dictionary order and Idempotent dictionary order relation.

c) **Product and metric topology on \( \mathbb{C}_2 \)**

Singh [7] defined three product topologies on \( \mathbb{C}_2 \), viz., Real product topology \( T_2 \), Complex product topology \( T_2^* \) and Idempotent product topology \( T_2^# \). \( B_p, B_i^* \) and \( B_j^# \) denotes the basis of \( T_2, T_2^* \) and \( T_2^# \) respectively. Where

\[
B_2 = \{S_1 \times_R S_2 \times_R S_3 \times_R S_4 : S_1, S_2, S_3, S_4 \text{ are in } \tilde{B}\}
\]

\[
S_1 \times_R S_2 \times_R S_3 \times_R S_4 = \{\xi = x_1+i_1x_2+i_2x_3+i_3x_4 : x_1 \in S_1, x_2 \in S_2, x_3 \in S_3, x_4 \in S_4\}
\]

\( \tilde{B} \) is the collection all open intervals in \( \mathbb{C}_0 \)

\[
B_2^* = \{S_5 \times_c S_6 : S_5, S_6 \in B'\}
\]

\( B' \) is the collection of all open intervals in \( \mathbb{C}_1 \)

\[
S_5 \times_c S_6 = \{\xi = z_1 + i_2 z_2 : z_1 \in S_5, z_2 \in S_6\}
\]

\[
B_2^# = \{S_7 \times_c S_8 : S_7, S_8 \in B'\}
\]

\[
S_7 \times_c S_8 = \{z_1e_1+z_2e_2 : z_1 \in S_7, z_2 \in S_8\}
\]

There are three metrics on \( \mathbb{C}_2 \), viz., real metric, complex metric and idempotent metric. With the help of these three metric Singh [7] defined three metric topologies on \( \mathbb{C}_2 \). The topology generated by the real metric is known as the real metric topology \( T_3 \), the topology generated by the complex metric is called complex metric topology \( T_3^* \) and the topology generated by the idempotent metric is called idempotent metric topology \( T_3^# \).

In this present paper, \( B_p, B_i^* \) and \( B_j^# \) denotes the basis of \( T_3, T_3^* \) and \( T_3^# \) respectively.

### 2.3.1 Theorem

The norm topology \( T_N \) and idempotent topology \( T_1 \) on \( \mathbb{C}_2 \) are equivalent to each other Srivastava [9].

### 2.3.2 Theorem

The norm topology \( T_N \) and complex topology \( T_C \) on \( \mathbb{C}_2 \) are equivalent to each other Srivastava [9].
Theorems 2.3.1 and 2.3.2 imply that

2.3.3 Corollary
The complex topology $\mathcal{T}_c$ and idempotent topology $\mathcal{T}_1$ on $\mathbb{C}_2$ are equivalent.

2.3.4 Theorem
The real order topology $\mathcal{T}_1$ and real product topology $\mathcal{T}_2$ on $\mathbb{C}_2$ are equivalent to each other [7].

2.3.5 Theorem
The real product topology $\mathcal{T}_2$ and real metric topology $\mathcal{T}_3$ on $\mathbb{C}_2$ are equivalent to each other [7].

Theorems 2.3.4 and 2.3.5 imply that

2.3.6 Corollary
The real order topology $\mathcal{T}_1$ and real metric topology $\mathcal{T}_3$ on $\mathbb{C}_2$ are equivalent to each other.

2.3.7 Theorem
The complex order topology $\mathcal{T}_1^*$ and complex product topology $\mathcal{T}_2^*$ on $\mathbb{C}_2$ are equivalent to each other [7].

2.3.8 Theorem
The complex product topology $\mathcal{T}_2^*$ and complex metric topology $\mathcal{T}_3^*$ on $\mathbb{C}_2$ are equivalent to each other [7].

In view of Theorems 2.3.7 and 2.3.8, we have

2.3.9 Corollary
The complex order topology $\mathcal{T}_1^*$ and complex metric topology $\mathcal{T}_3^*$ on $\mathbb{C}_2$ are equivalent to each other.

2.3.10 Theorem
The idempotent order topology $\mathcal{T}_1^#$ and idempotent product topology $\mathcal{T}_2^#$ on $\mathbb{C}_2$ are equivalent to each other [7].

2.3.11 Theorem
The idempotent product topology $\mathcal{T}_2^#$ and idempotent metric topology $\mathcal{T}_3^#$ on $\mathbb{C}_2$ are equivalent to each other [7].

As the idempotent order topology on $\mathbb{C}_2$ and the idempotent product topology on $\mathbb{C}_2$ are equivalent to each other. Also, the idempotent product topology and idempotent metric topology are equivalent to each other, therefore we have.

2.3.12 Corollary
The idempotent order topology $\mathcal{T}_1^#$ and idempotent metric topology $\mathcal{T}_3^#$ on $\mathbb{C}_2$ are equivalent to each other.

2.3.13 Corollary
As the real dictionary ordering of the bicomplex number is same as the complex dictionary ordering of the bicomplex numbers therefore the real order topology $\mathcal{T}_1$ is equivalent to the complex order topology $\mathcal{T}_1^*$ on $\mathbb{C}_2$.

Theorems 2.3.4, 2.3.5, 2.3.7, 2.3.8 and Corollary 2.3.6, 2.3.9, 2.3.13 imply that

2.3.14 Corollary
The real order, real product, real metric, complex order, complex product and complex metric topology on $\mathbb{C}_2$ are equivalent to each other.
2.3.15 Corollary

The real dictionary ordering and complex ordering of the bicomplex numbers is different from the idempotent ordering of the bicomplex numbers therefore the idempotent order topology can be neither equivalent to the real order topology nor to the complex order topology on \( C_2 \).

### III. Comparison of Various Topologies on Bicomplex Space

This section is our contribution to the theory of bicomplex topology and contains some important results from topological structures on the bicomplex space. In this section, we have tried to develop some relation between various topological structures on the bicomplex space.

a) **Comparison of the idempotent order topology and norm topology on the bicomplex space**

#### 3.1.1 Lemma

The set \( (1\xi e_1 + (2\xi - i\delta)e_2, 1\xi e_1 + (2\xi + i\delta)e_2)_{ID} \) is the proper subset of \( B(\xi, r) \) where \( 0 < \delta < \sqrt{2} \) and \( r > 0 \)

**Proof**: Let suppose \( \eta \in (1\xi e_1 + (2\xi - i\delta)e_2, 1\xi e_1 + (2\xi + i\delta)e_2)_{ID} \)

\[
\Rightarrow \eta = 1\xi \text{ and } 2\xi - i\delta < \eta < 2\xi + i\delta
\]

Since \( 2\xi \in \mathbb{C}_1 \) therefore consider \( 2\xi = a + i_1b \)

where \( a, b \in \mathbb{C}_0 \)

\[
\Rightarrow \eta = a + i_1q \text{ where } b - \delta < q < b + \delta
\]

Now \( |2\xi - \eta| = |(a + i_1b) - (a + i_1q)| = |b - q| \)

Since \( b - \delta < q < b + \delta \) therefore \( |b - q| < \delta \)

\[
\Rightarrow |2\xi - \eta| < \delta
\]

Since \( 0 < \delta < \sqrt{2} \)

\[
\Rightarrow |2\xi - \eta| < \sqrt{2} r
\]

\[
\Rightarrow \frac{|2\xi - \eta|}{\sqrt{2}} < r \quad \text{...}(2)
\]

Now \( ||\xi - \eta|| = \sqrt{\frac{|1\xi - 1\eta|^2 + |2\xi - 2\eta|^2}{2}} \)

\[
= \sqrt{\frac{0 + |2\xi - 2\eta|^2}{2}} \quad \text{; since } 1\eta = 1\xi
\]
From (2), \( \|\xi - \eta\| < r \)
\[ \Rightarrow \eta \in B(\xi, r) \quad \text{...(3)} \]

Now consider an element \( \zeta \) in \( C_2 \) such that \( \zeta = \xi + r \) and \( 2\zeta = 2\xi \)

Then \( \sqrt{\frac{1\|\xi - \eta\|^2 + 2\|\xi - \eta\|^2}{2}} = \frac{r}{\sqrt{2}} < r \)

Therefore
\[ \zeta \in B(\xi, r) \quad \text{...(4)} \]

Since \( \zeta \neq \xi \)
\[ \therefore \zeta \notin (\xi + (2\xi - i\delta)e_2, \xi + (2\xi + i\delta)e_2)_{\text{id}} \quad \text{...(5)} \]

Hence the set \((\xi + (2\xi - i\delta)e_2, \xi + (2\xi + i\delta)e_2)_{\text{id}}\) is a proper subset of \( B(\xi, r) \)
where \( 0 < \delta < \sqrt{2}r \)

3.1.2 Lemma

If \((\zeta, \Psi)_{\text{id}}\) is an open interval in \( C_2 \) such that \( \zeta \neq \Psi \) then there exist no open ball \( B(\xi, r); r < \infty \) which contain the set \((\zeta, \Psi)_{\text{id}}\).

Proof- Let \( B(\xi, r) \) be an arbitrary open ball in \( C_2 \) such that \( r < \infty \)
Let \( \eta \) be the arbitrary element of \( B(\xi, r) \)
\[ \Rightarrow \eta \in B(\xi, r) \quad \text{...(6)} \]
\[ \Rightarrow \|\xi - \eta\| < r \]
\[ \Rightarrow \sqrt{\frac{1\|\xi - \eta\|^2 + 2\|\xi - \eta\|^2}{2}} < r \]
\[ \Rightarrow \|\xi - \eta\| < \sqrt{2}r, \|\xi - \eta\| < \sqrt{2}r \]
\[ \Rightarrow \eta \in D(\xi; \sqrt{2}r, \sqrt{2}r) \quad \text{...(7)} \]

From (6), (7)
Therefore
\[ B(\xi, r) \subseteq D(\xi; \sqrt{2}r, \sqrt{2}r) \quad \text{...(8)} \]

Let us consider an arbitrary open interval \((\zeta, \Psi)_{\text{id}}\) in \( C_2 \) which contain the element \( \eta \)
\[ \Rightarrow \eta \in (\zeta, \Psi)_{\text{id}} \]
\[ \Rightarrow \zeta <_{\text{id}} \eta <_{\text{id}} \Psi \]

Since \( \zeta <_{\text{id}} \eta \)
Therefore either \( 1\zeta < 1\eta \) or \( 1\zeta = 1\eta, 2\zeta < 2\eta \)
Since \( \eta <_{\text{id}} \Psi \)
Therefore either \( 1\eta < 1\Psi \) or \( 1\eta = 1\Psi, 2\eta < 2\Psi \)
Since \( 1\zeta < 1\Psi \) therefore there will be only three possibilities.
Case 1st - If \( 1\zeta < 1\eta \) and \( 1\eta < 1\Psi \)
Consider an element $y \in \mathbb{C}_2$ such that $y = \eta$ and $|2\xi - 2y|^2 > \sqrt{2}r$.

Since $y = \eta$

$$\Rightarrow \zeta < y, y < \Psi$$

$$\Rightarrow \zeta < y \text{ and } y < \Psi$$

Therefore $y \in (\zeta, \Psi)_{\text{id}}$

Since $|2\xi - 2y|^2 > \sqrt{2}r$

$$\Rightarrow y \notin D(\xi; \sqrt{2}r, \sqrt{2}r)$$

From (8), $y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta, \Psi)_{\text{id}}$ such that $y \notin B(\xi, r)$

$$\Rightarrow (\zeta, \Psi)_{\text{id}} \notin B(\xi, r)$$

Case 2nd - If $\zeta < \eta$, $\eta = \Psi$ and $\eta < \Psi$

Consider an element $y \in \mathbb{C}_2$ such that $\zeta < y < \eta$ and $|2\xi - 2y|^2 > \sqrt{2}r$

Since $\zeta < y \Rightarrow \zeta < y$

Since $y < \eta$ and $\eta = \Psi \Rightarrow y < \Psi$

$$\Rightarrow y < \Psi$$

Therefore $y \in (\zeta, \Psi)_{\text{id}}$

Since $|2\xi - 2y|^2 > \sqrt{2}r$

$$\Rightarrow y \notin D(\xi; \sqrt{2}r, \sqrt{2}r)$$

From (8), $y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta, \Psi)_{\text{id}}$ such that $y \notin B(\xi, r)$

$$\Rightarrow (\zeta, \Psi)_{\text{id}} \notin B(\xi, r)$$

Case 3rd - If $\zeta = \eta$, $\eta < \Psi$ and $\eta < \Psi$

Consider an element $y \in \mathbb{C}_2$ such that $\eta < y < \Psi$ and $|2\xi - 2y|^2 > \sqrt{2}r$

Since $\eta < y$ and $\zeta = \eta \Rightarrow \zeta < y$

$$\Rightarrow \zeta < y$$

Since $y < \Psi \Rightarrow y < \Psi$

Therefore $y \in (\zeta, \Psi)_{\text{id}}$

Since $|2\xi - 2y|^2 > \sqrt{2}r$

$$\Rightarrow y \notin D(\xi; \sqrt{2}r, \sqrt{2}r)$$

From (8), $y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta, \Psi)_{\text{id}}$ such that $y \notin B(\xi, r)$
\(\Rightarrow (\zeta, \Psi)_{\text{ID}} \not\in \mathcal{B}(\xi, r)\)

Finally the ball \(\mathcal{B}(\xi, r); r < \infty\) cannot contain any open interval \((\zeta, \Psi)_{\text{ID}}\) where \(\iota \zeta \neq \iota \Psi\)

Hence \((\zeta, \Psi)_{\text{ID}}\) cannot be contained in any ball \(\mathcal{B}(\xi, r); r < \infty\)

3.1.3 Theorem

The Idempotent order topology is strictly finer than Norm topology.

**Proof**- Let \(X = (\xi, \eta)_{\text{ID}}\) be an open interval such that \(\iota \xi = \iota \eta\)

Then \(h_1(X) = \iota \xi\) and \(h_2(X) = (\iota \xi, \iota \eta)\)

In fact \(X = \iota \xi \times \iota \eta\)

Since \(\iota \xi\) is not open in complex plane

Therefore \(X\) will not be open w.r.t. Norm topology.

(By Theorem-2.1.2)

Since \(X\) is open w.r.t. Idempotent order topology therefore Idempotent order topology and norm topology are not equivalent and Norm topology cannot be finer than Idempotent order topology

Now we want to show for all open ball \(\mathcal{B}(\xi, r)\) and for all \(\eta \in \mathcal{B}(\xi, r)\) then there exist \((\zeta, \Psi)_{\text{ID}}\) such that \(\eta \in (\zeta, \Psi)_{\text{ID}} \subseteq \mathcal{B}(\xi, r)\)

Let us consider an arbitrary ball \(\mathcal{B}(\xi, r)\) and consider an arbitrary element \(\eta\) of \(\mathcal{B}(\xi, r)\)

\(\Rightarrow \eta \in \mathcal{B}(\xi, r)\)

1st Method- Since \(\eta \in \mathcal{B}(\xi, r)\) then there exist a ball \(\mathcal{B}(\eta, s); s > 0\) such that \(\mathcal{B}(\eta, s) \subseteq \mathcal{B}(\xi, r)\)

From Lemma-3.1.1,

\((\iota \eta_1 + (\iota \eta - i \delta) \eta_2)_{\text{ID}}, (\iota \eta_1 + (\iota \eta + i \delta) \eta_2)_{\text{ID}}\) will be the proper subset of \(\mathcal{B}(\eta, s)\)

Therefore for \(\eta \in \mathcal{B}(\xi, r)\) we have a set \((\iota \eta_1 + (\iota \eta - i \delta) \eta_2, (\iota \eta_1 + (\iota \eta + i \delta) \eta_2)_{\text{ID}}\) such that \(\eta \in (\iota \eta_1 + (\iota \eta - i \delta) \eta_2, (\iota \eta_1 + (\iota \eta + i \delta) \eta_2)_{\text{ID}}\) and

\((\iota \eta_1 + (\iota \eta - i \delta) \eta_2, (\iota \eta_1 + (\iota \eta + i \delta) \eta_2)_{\text{ID}}\) is the subset of \(\mathcal{B}(\xi, r)\)

2nd Method- Since \(\eta \in \mathcal{B}(\xi, r)\)

\[\Rightarrow \sqrt{\frac{1_{\xi - 1 \eta}^2 + 2_{\xi - 2 \eta}^2}{2}} < r\]

\[\Rightarrow |_{\xi - 1 \eta}^2 + |_{\xi - 2 \eta}^2 < 2r^2\]

Let \(|_{\xi - 1 \eta}^2 = d_1^2, |_{\xi - 2 \eta}^2 = d_2^2\)

\[\Rightarrow d_1^2 + d_2^2 < 2r^2\]

There exist \(d_3 > 0\) such that \(d_2 < d_3\) and \(d_1^2 + d_3^2 < 2r^2\)

Consider a set \(Q = (\iota \eta_1 + (\iota \eta - i \delta) \eta_2, (\iota \eta_1 + (\iota \eta + i \delta) \eta_2)_{\text{ID}}\) where \(\delta = d_3 - d_2 > 0\)
Obviously $\eta \in \mathbb{Q}$

Let $y \in \mathbb{Q}$

\[ \Rightarrow y = \eta \text{ and } (\bar{y} - i) < (\eta + i) \]

Let $\eta = a + i\beta$

Then $\bar{y} = a + i\beta$ where $b - \delta < \beta < b + \delta$

Now $|y^2 - \eta| \leq |\bar{y} - \eta| + |y - \eta|$

\[ \Rightarrow |y^2 - \eta| \leq d_2 + |\beta - \delta| \]

\[ \Rightarrow |y^2 - \eta| \leq d_2 \pm (\beta - \delta) \quad \ldots (10) \]

Since $b - \delta < \beta < b + \delta$ and $\delta = d_2 \pm d_2$

Therefore $\pm (\beta - \delta) < d_2 - d_2$

From (10), $|y^2 - \eta| < d_2$

Since $|\eta^2 - y| = d_1$

Therefore $|\eta^2 - y|^2 + |y^2 - \eta|^2 < d_1^2 + d_2^2 < 2r^2$

\[ \Rightarrow y \in B(\xi, r) \quad \ldots (11) \]

From (9), (11) $Q \subseteq B(\xi, r)$

Therefore for $\eta \in B(\xi, r)$ we have a open interval $Q$ w.r.t. idempotent ordering such that $\eta \in Q \subseteq B(\xi, r)$

Hence it proves that Idempotent order topology is strictly finer than Norm topology.

**Theorem 3.1.3** together with Theorems 2.3.1, 2.3.2, 2.3.10, 2.3.11 and Corollary 2.3.3, 2.3.12 generate a new corollary which states that

**3.1.4 Corollary**

The topology $\mathbb{T}_1^\#$ (and therefore $\mathbb{T}_2^\#$ and $\mathbb{T}_3^\#$) on $\mathbb{C}_2$ is strictly finer than the topology $\mathbb{T}_N$ (and therefore $\mathbb{T}_1$ and $\mathbb{T}_C$) on $\mathbb{C}_2$.

**b) Comparison of the idempotent order topology and real order topology on the bicomplex space**

**3.2.1 Theorem**

The Real order topology and Idempotent order topology are not comparable.

**Proof** Consider an open interval ‘A’ w.r.t. real ordering such that $A = (\xi, \eta)_r$ where

$\xi = 1 + 2i_1 + 4i_2 + 7i_1i_2$ and $\eta = 2 + 2i_1 + 3i_2 + 4i_1i_2$

Consider an element $x$ in $\mathbb{C}_2$ such that $x = 1 + 2i_1 + 5i_2 + 7i_1i_2$ therefore $\xi \prec_R x$ and $x \prec_R \eta$

$\Rightarrow x \in (\xi, \eta)_r$

Let us consider arbitrary open interval $(\zeta, \Psi)_ID$ in $\mathbb{C}_2$ (w.r.t. Idempotent ordering) which contain the element $x$. 
i.e. \( x \in (\zeta, \Psi) \Rightarrow \zeta < x, x < \Psi \)

Since \( \zeta < x \)

Therefore either \( \zeta < x \) or \( \zeta = x \), \( \zeta < x \)

Since \( x < \Psi \)

Therefore either \( x < \Psi \) or \( x = \Psi \), \( x < \Psi \)

Hence there will be four possibilities.

Case A- If \( \zeta < x \) and \( x < \Psi \)

Since \( x = 1 + 2i + 5i_2 + 7i_3 \)

Therefore \( \zeta < x = 8 - 3i + 2i_1 \)

Consider an element \( y \) in \( \mathbb{C}_2 \) such that \( y = 6 + 2i + 5i_2 + 2i_3 \)

Therefore \( \zeta < y \) and \( y < x \)

Therefore \( \zeta < y \) \( \Rightarrow \) \( y \in (\zeta, \Psi) \)

Since \( \xi < y \) and \( \eta < y \)

\( \Rightarrow y \in (\xi, \eta) \) \( \ldots \) (12)

From (12), (13) \( (\zeta, \Psi) \not\in (\xi, \eta) \)

Case B-If \( \zeta < x \) and \( x < \Psi \)

Consider an element \( y \) in \( \mathbb{C}_2 \) such that \( \zeta < y \) \( \Rightarrow y \)

Since \( y = 1 + x \), \( y < x \)

Therefore \( \zeta < y \) \( \Rightarrow y \in (\zeta, \Psi) \)

Since \( \xi < y \) and \( \eta < y \)

\( \Rightarrow y \in (\xi, \eta) \) \( \ldots \) (13)

If \( \zeta < x \) and \( y < x \)

\( \Rightarrow \) Therefore there are only two possibilities either \( y = 8 + 3i_1 \) where \( \delta < -3 \) or \( \zeta = a + i \), \( a < 8 \)

If \( y = 8 + 3i_1 \) where \( \delta < -3 \)

Then \( y \) will be in the form \( y = a_1 + i_1 a_2 + i_2 a_3 + i_3 a_4 \)

Where \( a_1 + a_i = 8, a_2 + a_3 < -3 \)

Choose \( a_1 = 7, a_3 = 1 \) and choose \( a_2 \) and \( a_3 \) in such a way that \( a_2 + a_3 < -3 \)

Therefore \( y = 7 + i_1 a_2 + i_2 a_3 + i_3 a_4 \)

\( \Rightarrow \xi < y \), \( \eta < y \)

\( \Rightarrow y \in (\xi, \eta) \) \( \ldots \) (14)

If \( y = a + i \) where \( a < 8 \)

Then \( y \) will be in the form \( y = b_1 + i_1 b_2 + i_2 b_3 + i_3 i_2 b_4 \), \( b_1 + b_4 < 8 \)

Choose \( b_1 = 8 \) and \( b_4 = -C \) where \( C > 0 \)

Therefore \( y = 8 + i_1 b_2 + i_2 b_3 + i_3 i_2 C \)
⇒ \( y \notin (\xi, \eta)_R \) \hspace{1cm} \text{...(16)}

From (12), (14) \( (\zeta, \Psi)_{ID} \not\subseteq (\xi, \eta)_R \)

Case C- If \( ^1\zeta = \ _1x, \ ^2\zeta<^2x \) and \( ^1x<^1\Psi \)

Consider an element \( y \) in \( \mathbb{C}_2 \) such that \( ^1x<^1y<^1\Psi \)

Since \( ^1y<^1\Psi \Rightarrow y<_{ID}\Psi \)

Since \( ^1\zeta = ^1x \) and \( ^1x<^1y \) \( \Rightarrow ^1\zeta<_{ID}y \)

\( \Rightarrow y \in (\zeta, \Psi)_{ID} \) \hspace{1cm} \text{...(17)}

Since \( ^1x = 8-3i \) and \( ^1x<^1y \)

Therefore there are only two possibilities either \( ^1y = 8+\delta i \) where \( \delta > -3 \) or \( ^1y = a + i_1b \) where \( a > 8, b \in \mathbb{C}_0 \)

If \( ^1y = 8+\delta i \) where \( \delta > -3 \)

Then \( y \) will be in the form \( y = a_1+i_1a_2+i_2a_3+i_1i_2a_4 \)

Where \( a_1+a_4 = 8, a_2-a_3 > -3 \)

Choose \( a_1=7, a_4=1 \) and choose \( a_2 \) and \( a_3 \) in such a way that \( a_2-a_3 > -3 \)

Therefore \( y = 7+i_1a_2+i_2a_3+i_1i_2 \)

\( \Rightarrow ^1\zeta<_{R} y, \ \eta<_{R} y \)

\( \Rightarrow y \in (\zeta, \eta)_R \) \hspace{1cm} \text{...(18)}

From (17), (18) \( (\zeta, \Psi)_{ID} \not\subseteq (\xi, \eta)_R \)

If \( ^1y = a + i_1b \) where \( a > 8, b \in \mathbb{C}_0 \)

Then \( y \) will be in the form \( y = b_1+i_1b_2+i_2b_3+i_1i_2b_4 \), Where \( b_1+b_4 > 8 \)

Choose \( b_1 = 8 \) and \( b_4 = C \) where \( C > 0 \)

Therefore \( y = 8+i_1b_2+i_2b_3-i_1i_2C \)

\( \Rightarrow y \in (\zeta, \eta)_R \) \hspace{1cm} \text{...(19)}

From (17), (19) \( (\zeta, \Psi)_{ID} \not\subseteq (\xi, \eta)_R \)

Case D- If \( ^1\zeta = ^1x, \ ^2\zeta<^2x \) and \( ^1x = ^1\Psi, \ ^2x<^2\Psi \)

Consider an element \( y \) in \( \mathbb{C}_2 \) such that

\( ^1y = ^1x \) and \( ^2\zeta<^2y<^2\Psi \)

\( \Rightarrow ^1\zeta<_{ID} y, \ y<_{ID}^2\Psi \)

\( \Rightarrow y \in (\zeta, \Psi)_{ID} \) \hspace{1cm} \text{...(20)}

If \( Z = a+i_1b \) is a complex number then the region of all complex number which is greater than \( Z \) or less than \( Z \) is defined as follows.

Figure-1 shows the region of all complex numbers which is greater than \( Z \) and figure-2 shows the region of all complex numbers which is less than \( Z \).
Since \( z^2 = -6 + 7i \) and \( z < x < \Psi \)
Here there are four possibilities.

Possibility 1st-If \( z \) is on the axis QT except Q point and \( \Psi \) is on the axis QS except Q point.

Since \( y = x \) and \( z < y < \Psi \)
Therefore \( y = -8 - 3i \) and \( y = -6 + i \) where \( b > 7 \) or \( b < 7 \).

We will consider only \( b < 7 \).

Then \( y = 1 + (2 - \delta)i_1 + (5 - \delta)i_2 + 7i_1i_2 \) where \( \delta > 0 \)

\[ \Rightarrow y < R\xi \]

\[ \Rightarrow y \notin (\xi, \eta)_R \] \hspace{1cm} \ldots(21)

From (20), (21) \( (\zeta, \Psi)_{ID} \notin (\xi, \eta)_R \)

Possibility 2\textsuperscript{nd} - If \( \zeta \) is situated in the region \( QSRT \) except \( ST \) axis and \( \Psi \) is on the axis \( QS \) except \( Q \) point.

Since \( y = x \) and \( \zeta < y < \Psi \)

Therefore \( y = -8 - 3i \)

Consider \( y = a + i \), where \( a < -6, b \in \mathbb{C}_0 \)

Then \( y = (1 - \delta_1) + (2 - \delta_2)i_1 + (5 - \delta_2)i_2 + (7 + \delta_1)i_1i_2 \)

where \( \delta_1 > 0 \) and \( \delta_2 \in \mathbb{C}_0 \)

\[ \Rightarrow y < R\xi \]

\[ \Rightarrow y \notin (\xi, \eta)_R \] \hspace{1cm} \ldots(22)

From (20), (22) \( (\zeta, \Psi)_{ID} \notin (\xi, \eta)_R \)

Possibility 3\textsuperscript{rd} - If \( \Psi \) is situated in the region \( QSPT \) except \( ST \) axis and \( \zeta \) is on the axis \( QT \) except \( Q \) point.
Since \(1^y = 1^x\) and \(2^\zeta < 2^y < 2^\Psi\)
Therefore \(1^y = 8 - 3i\)
Consider \(2^y\) in such a way that \(2^\zeta < 2^y < 2^\Psi\)
Therefore \(2^y = -6 + i^b\) where \(b < 7\)

\[
\Rightarrow y = 1 + (2 - \delta)i_1 + (5 - \delta)i_2 + 7i_1i_2 \quad \text{where} \quad \delta > 0
\]

\[
\Rightarrow y < 7\xi
\]

\[
\Rightarrow y \in (\xi, \eta)_R
\]

From (20), (23) \((\zeta, \Psi)_R \not\subseteq (\xi, \eta)_R\)

Possibility 4th-If \(2^\Psi\) is situated in the region \(QSPT\) except \(ST\) axis and \(2^\zeta\) is situated in the region \(QSRT\) except \(ST\) axis.

\(2^y = -6 + 5i\) satisfies the condition \(2^\zeta < 2^y < 2^\Psi\)
Therefore on choosing \(2^y = -6 + 5i\)
Since \(1^y = 8 - 3i\) then \(y = 1 + i_1 + 4i_2 + 7i_1i_2\)

\[
\Rightarrow y < 7\xi
\]

From (20), (24) \((\zeta, \Psi)_R \not\subseteq (\xi, \eta)_R\)
Finally all open interval (w.r.t. idempotent ordering) which contain the element x they cannot be subset of the set A.

Since A is open w.r.t. Real order topology therefore it shows that Idempotent order topology is not finer than Real order topology.

Now consider an open interval ‘B’ w.r.t. idempotent ordering such that B = (α, β)_{ID}

Where \( \alpha = (6-i_1)e_1+(-2+5i_1)e_2 \) and \( \beta = (8-2i_1)e_1+(-6+6i_1)e_2 \)

Consider an element p in \( \mathbb{C}_2 \) such that \( p=2+2.5i_1+3.5i_2+4i_1i_2 \)

\[ \Rightarrow 1p = (6 - i_1), \quad 2p = (-2 + 6i_1) \]

Since \( 1\alpha = 1p \) and \( 2\alpha < 2p \Rightarrow \alpha < _{ID} p \)

Let us consider arbitrary open interval \((\phi, \Psi)_{R}\) in \( \mathbb{C}_2 \) (w.r.t. Real ordering) which contain the element p.

\[ i.e. p \in (\phi, \Psi)_{R} \Rightarrow \phi < _{R} p, \quad p < _{R} \Psi \]

Let \( \phi = x_1+i_1x_2+i_2x_3+i_1i_2x_4 \) and \( \Psi = y_1+i_1y_2+i_2y_3+i_1i_2y_4 \)

Since \( p=2+2.5i_1+3.5i_2+4i_1i_2 \) and \( \phi < _{R} p, \quad p < _{R} \Psi \)

Therefore there are 16 cases.

Case-(i) If \( x_1 < 2 \) and \( y_1 > 2 \)

\[ \Rightarrow \phi = (2-\epsilon)+i_1x_2+i_2x_3+i_1i_2x_4 \]

\[ \Psi = (2+\delta)+i_1y_2+i_2y_3+i_1i_2y_4 \] where \( \epsilon, \delta > 0 \)

Consider an element A in \( \mathbb{C}_2 \) such that

\[ A = 2+2.5i_1+3.5i_2+9i_1i_2 \]

\[ \Rightarrow A \in (\phi, \Psi)_{R} \quad \text{...(25)} \]

Since \( 1A = (11-i_1) \)

\[ \Rightarrow 1\alpha < 1A \text{ and } 1\beta < 1A \]

\[ \Rightarrow \alpha < _{ID} A, \quad \beta < _{ID} A \]

\[ \Rightarrow A \in (\alpha, \beta)_{ID} \quad \text{...(26)} \]

From (25), (26) \( (\phi, \Psi)_{R} \notin (\alpha, \beta)_{ID} \)

Case-(ii) If \( x_1 = 2, \quad x_2 < 2.5 \) and \( y_1 > 2 \)

\[ \Rightarrow \phi = 2+(2.5-\epsilon)i_1+i_2x_3+i_1i_2x_4 \]

\[ \Psi = (2+\delta)+i_1y_2+i_2y_3+i_1i_2y_4 \] where \( \epsilon, \delta > 0 \)

Also in this situation \( A \in (\phi, \Psi)_{R} \quad \text{...(27)} \)

Therefore from (26), (27) \( (\phi, \Psi)_{R} \notin (\alpha, \beta)_{ID} \)
Case-(iii) If \( x_1 = 2, x_2 = 2.5, x_3 < 3.5 \) and \( y_1 > 2 \)
\[ \Rightarrow \phi = 2 + 2.5i_1 + (3.5 - \epsilon) i_2 + i_1 i_2 x_4 \]
\[ \Rightarrow \Psi = (2 + \delta) + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4 \text{ where } \epsilon, \delta > 0 \]
\[ \Rightarrow A \in (\phi, \Psi)_R \] ... (28)

Therefore from (26), (28) \((\phi, \Psi)_R \not\in (\alpha, \beta)_{ID}\)

Case-(iv) If \( x_1 = 2, x_2 = 2.5, x_3 = 3.5, x_4 < 4 \) and \( y_1 > 2 \)
\[ \Rightarrow \phi = 2 + 2.5i_1 + 3.5i_2 + (4 - \epsilon) i_1 i_2 \]
\[ \Rightarrow \Psi = (2 + \delta) + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4 \text{ where } \epsilon, \delta > 0 \]
Since \( A = 2 + 2.5i_1 + 3.5i_2 + 9i_1 i_2 \)
\[ \Rightarrow A \in (\phi, \Psi)_R \] ... (29)

Therefore from (26), (29) \((\phi, \Psi)_R \not\in (\alpha, \beta)_{ID}\)

Case-(v) If \( x_1 < 2, y_1 = 2 \) and \( y_2 > 2.5 \)
\[ \Rightarrow \phi = (2 - \epsilon) + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \]
\[ \Rightarrow \Psi = 2 + (2.5 + \delta) i_1 + i_2 y_3 + i_1 i_2 y_4 \text{ where } \epsilon, \delta > 0 \]
Also in this situation \( A \in (\phi, \Psi)_R \) ... (30)

Therefore from (26), (30) \((\phi, \Psi)_R \not\in (\alpha, \beta)_{ID}\)

Case-(vi) If \( x_1 = 2, x_2 < 2.5, y_1 = 2 \) and \( y_2 > 2.5 \)
\[ \Rightarrow \phi = 2 + (2.5 - \epsilon) i_1 + i_2 x_3 + i_1 i_2 x_4 \]
\[ \Rightarrow \Psi = 2 + (2.5 + \delta) i_1 + i_2 y_3 + i_1 i_2 y_4 \text{ where } \epsilon, \delta > 0 \]
Also in this situation \( A \in (\phi, \Psi)_R \) ... (31)

Therefore from (26), (31) \((\phi, \Psi)_R \not\in (\alpha, \beta)_{ID}\)

Case-(vii) If \( x_1 = 2, x_2 = 2.5, x_3 < 3.5, x_4 = 2 \) and \( y_2 > 2.5 \)
\[ \Rightarrow \phi = 2 + 2.5i_1 + (3.5 - \epsilon) i_2 + i_1 i_2 x_4 \]
\[ \Rightarrow \Psi = 2 + (2.5 + \delta) i_1 + i_2 y_3 + i_1 i_2 y_4 \text{ where } \epsilon, \delta > 0 \]
\[ \Rightarrow A \in (\phi, \Psi)_R \] ... (32)

Therefore from (26), (32) \((\phi, \Psi)_R \not\in (\alpha, \beta)_{ID}\)

Case-(viii) If \( x_1 = 2, x_2 = 2.5, x_3 = 3.5, x_4 < 4, y_1 = 2 \) and \( y_2 > 2.5 \)
\[ \Rightarrow \phi = 2 + 2.5i_1 + 3.5i_2 + (4 - \epsilon) i_1 i_2 \]
\[ \Rightarrow \Psi = 2 + (2.5 + \delta) i_1 + i_2 y_3 + i_1 i_2 y_4 \text{ where } \epsilon, \delta > 0 \]
\[ \Rightarrow A \in (\phi, \Psi)_R \] ... (33)
Therefore from (26), (34) $(\phi, \Psi)_R \not\subseteq (\alpha, \beta)_{ID}$

**Case (ix)** If $x_1 < 2$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$$\Rightarrow \phi = (2-\epsilon) + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \quad \text{and}$$

$$\Psi = 2 + 2.5 i_1 + (3.5 + \delta) i_2 + i_1 i_2 y_4 \quad \text{where } \epsilon, \delta > 0$$

Since $A = 2 + 2.5 i_1 + 3.5 i_2 + 9 i_1 i_2$

$$\Rightarrow A \in (\phi, \Psi)_R$$

...(34)

Therefore from (26), (34) $(\phi, \Psi)_R \not\subseteq (\alpha, \beta)_{ID}$

**Case (x)** If $x_1 = 2$, $x_2 < 2.5$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$$\Rightarrow \phi = 2 + (2.5 - \epsilon) i_1 + i_2 x_3 + i_1 i_2 x_4 \quad \text{and}$$

$$\Psi = 2 + 2.5 i_1 + (3.5 + \delta) i_2 + i_1 i_2 y_4 \quad \text{where } \epsilon, \delta > 0$$

Also in this situation $A \in (\phi, \Psi)_R$

...(35)

Therefore from (26), (35) $(\phi, \Psi)_R \not\subseteq (\alpha, \beta)_{ID}$

**Case (xi)** If $x_1 = 2$, $x_2 = 2.5$, $x_3 < 3.5$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$$\Rightarrow \phi = 2 + 2.5 i_1 + (3.5 - \epsilon) i_2 + i_1 i_2 x_4 \quad \text{and}$$

$$\Psi = 2 + 2.5 i_1 + (3.5 + \delta) i_2 + i_1 i_2 y_4 \quad \text{where } \epsilon, \delta > 0$$

$$\Rightarrow A \in (\phi, \Psi)_R \quad \text{...(36)}$$

Therefore from (26), (36) $(\phi, \Psi)_R \not\subseteq (\alpha, \beta)_{ID}$

**Case (xii)** If $x_1 = 2$, $x_2 = 2.5$, $x_3 = 3.5$, $x_4 < 4$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$$\Rightarrow \phi = 2 + 2.5 i_1 + 3.5 i_2 + (4 - \epsilon) i_1 i_2 \quad \text{and}$$

$$\Psi = 2 + 2.5 i_1 + (3.5 + \delta) i_2 + i_1 i_2 y_4 \quad \text{where } \epsilon, \delta > 0$$

$$\Rightarrow A \in (\phi, \Psi)_R \quad \text{...(37)}$$

Therefore from (26), (37) $(\phi, \Psi)_R \not\subseteq (\alpha, \beta)_{ID}$

**Case (xiii)** If $x_1 < 2$, $y_1 = 2$, $y_2 = 2.5$, $y_3 = 3.5$ and $y_4 > 4$

$$\Rightarrow \phi = (2-\epsilon) + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \quad \text{and}$$

$$\Psi = 2 + 2.5 i_1 + 3.5 i_2 + (4 + \delta) i_1 i_2 \quad \text{where } \epsilon, \delta > 0$$

Consider an element $B$ in $\mathbb{C}_2$ such that $B = 2 + 2.5 i_1 + 3.5 i_2 + 3 i_1 i_2$

$$\Rightarrow B \in (\phi, \Psi)_R \quad \text{...(38)}$$

Since $^1B = (5-i_1)$

$$\Rightarrow ^1B < ^1\alpha, \ ^1B < ^1\beta$$

$$\Rightarrow B < _m\alpha, \ B < _m\beta$$
\[ B \varepsilon (\alpha, \beta)_{\text{ID}} \]  

From (38), (39) \((\phi, \Psi)_{\text{ID}} \not\varepsilon (\alpha, \beta)_{\text{ID}}\)

**Case-(xiv)** If \(x_1 = 2, \ x_2 < 2.5, \ y_1 = 2, \ y_2 = 2.5, y_3 = 3.5\) and \(y_4 > 4\)
\[ \Rightarrow \phi = 2 + (2.5 - \epsilon)i_1 + i_2x_3 + i_1i_2x_4 \text{ and} \]
\[ \Psi = 2 + 2.5i_1 + 3.5i_2 + (4 + \delta)i_1i_2 \text{ where } \epsilon, \delta > 0 \]

Also in this situation
\[ B \varepsilon (\phi, \Psi)_{\text{R}} \]  

Therefore from (39), (40) \((\phi, \Psi)_{\text{R}} \not\varepsilon (\alpha, \beta)_{\text{ID}}\)

**Case-(xv)** If \(x_1 = 2, \ x_2 = 2.5, \ x_3 < 3.5, \ y_1 = 2, \ y_2 = 2.5, \ y_3 = 3.5\) and \(y_4 > 4\)
\[ \Rightarrow \phi = 2 + 2.5i_1 + (3.5 - \epsilon)i_2 + i_1i_2x_4 \text{ and} \]
\[ \Psi = 2 + 2.5i_1 + 3.5i_2 + (4 + \delta)i_1i_2 \text{ where } \epsilon, \delta > 0 \]

Consider an element \(D\) in \(\mathbb{C}_2\) such that \(D = 2 + 2.5i_1 + 3.5i_2 + ai_1i_2 \) where \(4 - \epsilon < a < 4\)
\[ \Rightarrow D \varepsilon (\phi, \Psi)_{\text{R}} \]  

Since \(iD = \{(2 + a) - i_1\}\)
Since \(a < 4 \Rightarrow 2 + a < 6\)
\[ \Rightarrow iD < 'a \Rightarrow D < \text{id}a \]
\[ \Rightarrow D \varepsilon (\alpha, \beta)_{\text{ID}} \]  

Therefore from (42), (43) \((\phi, \Psi)_{\text{R}} \not\varepsilon (\alpha, \beta)_{\text{ID}}\)

Hence all open interval (w.r.t. real ordering) which contain the element ‘p’ they cannot be subset of the set ‘B’.

Since \(B\) is open w.r.t. Idempotent order topology therefore it shows real order topology is not finer than Idempotent order topology.

Hence it proves that both topologies are not comparable.

Theorem 3.2.1, 2.3.10, 2.3.11 and corollary 2.3.12, 2.3.14 submerge together to give a new corollary which is started below.

### 3.2.2 Corollary

The topology \(T_1^\#\) (and therefore \(T_2^\#\) and \(T_3^\#\)) and the topology \(T_1\) (and therefore \(T_2, T_3, T_1^*, T_2^*\) and \(T_3^*\)) on \(\mathbb{C}_2\) are not comparable.

### c) Comparison of the real order topology and norm topology on the bicomplex space

#### 3.3.1 Lemma

The set \((\zeta, \Psi)_{\text{R}}\) is the proper subset of \(B(\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4, r)\) where  
\[ \zeta = x_1 + i_1x_2 + i_1x_3 + i_1i_2(x_4 - \epsilon), \ \Psi = x_1 + i_1x_2 + i_1x_3 + i_1i_2(x_4 + \epsilon) \]  and either \(\epsilon = \text{Min}(d_1, d_2), \ d_1^2 + d_2^2 < 2r^2\) and \(\epsilon > 0\) or \(0 < \epsilon < r\)
Proof- Let suppose
\[ \eta \in (\zeta, \Psi)_R \] ...(44)
\[ \Rightarrow \zeta <_R \eta <_R \Psi \]
\[ \Rightarrow \eta = x_1 + i_1x_2 + i_2x_3 + i_1i_2q \text{ where } (x_4 - \epsilon) < \eta < (x_4 + \epsilon) \]

Since \( \eta = x_1 + i_1x_2 + i_2x_3 + i_1i_2q \)
\[ \Rightarrow ^1\eta = (x_1 + q) + i_1(x_2 - x_3) \text{ and } ^2\eta = (x_1 - q) + i_1(x_2 + x_3) \]

Since \( \xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 \)
\[ \Rightarrow ^1\xi = (x_1 + x_4) + i_1(x_2 - x_3) \text{ and } ^2\xi = (x_1 - x_4) + i_1(x_2 + x_3) \]

Now \[ |^1\xi - \eta| = |x_4 - q| = \pm (x_4 - q) \]
Since \( (x_4 - \epsilon) < q < (x_4 + \epsilon) \) therefore \( \pm (x_4 - q) < \epsilon \)
\[ \Rightarrow |^1\xi - \eta| < \epsilon \]
Similarly \[ |^2\xi - \eta| = \pm (x_4 - q) < \epsilon \]
Since \( \epsilon = \text{Min}(d_1, d_2) \) and \( d_1^2 + d_2^2 < 2r^2 \)
Therefore \[ |^1\xi - \eta|^2 + |^2\xi - \eta|^2 < 2r^2 \]
\[ \Rightarrow \sqrt{\frac{|^1\xi - \eta|^2 + |^2\xi - \eta|^2}{2}} < r \]
Therefore \( \|\xi - \eta\| < r \)
\[ \Rightarrow \eta \in B(\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4, r) \] ...(45)

Now consider an element \( P \) in \( \mathbb{C}_2 \) such that \( ^1P = ^1\xi + r \) and \( ^2P = ^2\xi \)
Constitute \[ \sqrt{\frac{|^1\xi - P|^2 + |^2\xi - P|^2}{2}} = \frac{r}{\sqrt{2}} < r \]
Therefore \( P \in B(\xi, r) \) ...(46)

Since \( ^1\xi = (x_1 + x_4) + i_1(x_2 - x_3) \) and \( ^2\xi = (x_1 - x_4) + i_1(x_2 + x_3) \)
\[ \therefore ^1P = (x_1 + x_4 + r) + i_1(x_2 - x_3) \text{ and } ^2P = (x_1 - x_4) + i_1(x_2 + x_3) \]
\[ \therefore P = (x_1 + r/2) + i_1x_2 + i_2x_3 + i_1i_2(x_4 + r/2) \]
\[ \Rightarrow P \notin (\zeta, \Psi)_R \] ...(47)

From (44), (45), (46) and (47)

Hence the set \( (\zeta, \Psi)_R \) is the proper subset of \( B(\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4, r) \)

3.3.2 Lemma

If \( (\zeta, \Psi)_C \) is an open interval in \( \mathbb{C}_2 \) such that \( u_i \neq v_i \) or \( u_i = v_i, a_i \neq b_i \) then there exist no open ball \( B(\xi, r); r < \infty \) which contain the set \( (\zeta, \Psi)_C \).
If $(\zeta, \Psi)_r$ is an open interval in $\mathbb{C}_2$ such that $a_1 \neq b_1$ or $a_1 = b_1, \ a_2 \neq b_2$ or $a_2 = b_2, a_3 \neq b_3$ then there exist no open ball $B(\xi, r)$; $r < \infty$ which contain the set $(\zeta, \Psi)_r$.

Where $\zeta = u_1+i_2u_2 = (a_1+i_1a_2) + i_2(a_3+i_1a_4)$ and $\Psi = v_1+i_2v_2 = (b_1+i_1b_2) + i_2(b_3+i_1b_4)$

**Proof-** Let $B(\xi, r)$ be an arbitrary open ball in $\mathbb{C}_2$ such that $r < \infty$

Let suppose $\xi = z_1+i_2z_2 = (x_1+i_1x_2) + i_2(x_3+i_1x_4)$ and let $\eta = w_1+i_2w_2 = (y_1+i_1y_2) + i_2(y_3+i_1y_4)$ be the arbitrary element of $B(\xi, r)$.

\[\Rightarrow \eta \in B(\xi, r)\]
\[\Rightarrow ||\xi - \eta|| < r\]
\[\Rightarrow \sqrt{|z_1 - w_1|^2 + |z_2 - w_2|^2} < r\]

Or \[|(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2+(x_4-y_4)^2|^{1/2} < r\]
\[\Rightarrow |z_1 - w_1| < r \text{ and } |z_2 - w_2| < r\]

Or $(x_i-y_i) < r$; $i = 1, 2, 3, 4$

Let us consider an arbitrary open interval $(\zeta, \Psi)_c$ in $\mathbb{C}_2$ which contain the element $\eta$

\[\Rightarrow \eta \in (\zeta, \Psi)_c\]
\[\Rightarrow \zeta < c \eta < c \Psi\]

Since $\zeta < c \eta$

Therefore either $u_1 < w_1$ or $u_1 = w_1, u_2 < w_2$

Since $\eta < id \Psi$

Therefore either $w_1 < v_1$ or $w_1 = v_1, w_2 < v_2$

**Case-A** If $u_1 \neq v_1$, then there will be three possibilities.

**Possibility-1st**: If $u_1 < w_1$ and $w_1 < v_1$

Consider an element $y = q_1+i_2q_2 \in \mathbb{C}_2$ such that $q_1 = w_1$ and $|z_2 - q_2| > r$

Since $q_1 = w_1$

\[\Rightarrow u_1 < q_1 \text{ and } q_1 < v_1\]
\[\Rightarrow \zeta < c \ y \text{ and } y < c \Psi\]

Therefore $y \in (\zeta, \Psi)_c$

Since $|z_2 - q_2| > r$

\[\Rightarrow y \notin B(\xi, r)\]

Therefore we have an element $y \in (\zeta, \Psi)_c$ such that $y \notin B(\xi, r)$

\[\Rightarrow (\zeta, \Psi)_c \notin B(\xi, r)\]

**Possibility-2nd**: If $u_1 < w_1$, $w_1 = v_1$ and $w_2 < v_2$

Consider an element $y = q_1+i_2q_2 \in \mathbb{C}_2$ such that $u_1 < q_1 < w_1$ and $|z_2 - q_2| > r$

Since $u_1 < q_1 \Rightarrow \zeta < c \ y$

Since $q_1 < w_1$ and $w_1 = v_1$
\[ y < C \Psi \]
Therefore \( y \in (\zeta, \Psi) \).
Since \( |z_2 - q_2| > r \)
\[ \Rightarrow y \notin B(\xi, r) \]
Therefore we have an element \( y \in (\zeta, \Psi) \) such that \( y \notin B(\xi, r) \).

**Possibility-3**: If \( u_1 = w_1, u_2 < w_2 \) and \( w_1 < v_1 \)
Consider an element \( y = q_1 + i_2 q_2 \in \mathbb{C}_2 \) such that \( w_1 < q_1 < v_1 \) and \( |z_2 - q_2| > r \)
Since \( u_1 = w_1 \) and \( w_1 < q_1 \Rightarrow \zeta < C \Psi \)
Since \( q_1 < v_1 \Rightarrow y < C \Psi \)
Therefore \( y \in (\zeta, \Psi) \).
Since \( |z_2 - q_2| > r \)
\[ \Rightarrow y \notin B(\xi, r) \]
Therefore we have an element \( y \in (\zeta, \Psi) \) such that \( y \notin B(\xi, r) \).

\[ \Rightarrow (\zeta, \Psi) \notin B(\xi, r) \]

**Case-B**: If \( u_1 = v_1 \) & \( a_3 \neq b_3 \)
Since \( a_3 \neq b_3 \Rightarrow a_3 < b_3 \)
Consider an element \( s = a_1 + i_2 a_2 + i_3 c_3 + i_4 c_4 \in \mathbb{C}_2 \) such that \( a_3 < c_3 < b_3 \) and \( (x_4 - c_4) > r \)
Therefore \( s \in (\zeta, \Psi) \) and \( s \notin B(\xi, r) \)
Also in this situation \( (\zeta, \Psi) \notin B(\xi, r) \)
Finally the ball \( B(\xi, r); r < \infty \) cannot contain any open interval \( (\zeta, \Psi) \) where \( u_1 \neq v_1 \) or \( u_1 = v_1, a_3 \neq b_3 \).
Hence \( (\zeta, \Psi) \) cannot be contained in any ball \( B(\xi, r); r < \infty \).

### 3.3.3 Theorem
The Real order topology is strictly finer than Norm topology.

**Proof**: Since the Real order topology and Idempotent order topology are not comparable.[By Theorem-3.2.1]
Therefore there exist a set \( Q \subseteq \mathbb{C}_2 \) which will be open w.r.t. Real order topology and will not be open w.r.t. Idempotent order topology.
Since \( Q \) is not open w.r.t. Idempotent order topology.
Therefore, from Theorem-3.1.3
\( Q \) will not be open w.r.t. Norm topology.
Therefore we have a set \( Q \) which is open w.r.t. Real order topology and not open w.r.t. Norm topology.
Therefore Idempotent order topology and Norm topology are not equivalent and Norm topology cannot be finer than Idempotent order topology.

Now we want to show for all open ball \( B(\xi, r) \) and for all \( \eta \in B(\xi, r) \) then there exist \( (\zeta, \Psi)_r \) such that \( \eta \in (\zeta, \Psi)_r \subseteq B(\xi, r) \)
Let us consider an arbitrary ball \( B(\xi, r) \) and consider an arbitrary element \( \eta \) of \( B(\xi, r) \)
Let \( \eta = y_1 + i_1 y_2 + i_2 y_3 + i_3 y_4 \)
1st Method- Since \( \eta \in B(\xi, r) \) then there exist a ball 
\( B(\eta, s); s > 0 \) such that 
\( B(\eta, s) \subseteq B(\xi, r) \) 
Since \( \eta = y_1 + i_1y_2 + i_2y_3 + i_3y_4 \) then 
\( (\xi, \Psi)_R \) will be the proper subset of 
\( B(\eta = y_1 + i_1y_2 + i_2y_3 + i_3y_4, r) \) 
Where \( \xi = y_1 + i_1y_2 + i_2y_3 + i_3y_4 \) 
\( \Psi = y_1 + i_1y_2 + i_2y_3 + i_3y_4(\eta + \epsilon) \) and \( \epsilon = \min(d_1, d_2) \) and \( d_1^2 + d_2^2 < 2r^2 \)

[By Lemma-3.3.1]
Therefore for \( \eta \in B(\xi, r) \) we have a set \( (\xi, \Psi)_R \) such that \( \eta \in (\xi, \Psi)_R \) and \( (\xi, \Psi)_R \subseteq B(\xi, r) \).

2nd Method-Since \( \eta \in B(\xi, r) \)

\[
\Rightarrow \sqrt{\frac{|1\xi-1\eta|^2 + |2\xi-2\eta|^2}{2^2}} < r 
\Rightarrow |1\xi-1\eta|^2 + |2\xi-2\eta|^2 < 2r^2 
\]

Let \( |1\xi-1\eta|^2 = d_1^2 \cdot \frac{|2\xi-2\eta|^2}{d_2^2} = d_2^2 \) 
\( \Rightarrow d_1^2 + d_2^2 < 2r^2 \)
There exist \( d_3, d_4 > 0 \) such that \( d_1 < d_3, d_2 < d_4 \) and \( d_3^2 + d_4^2 < 2r^2 \)
Consider a set \( (\xi, \Psi)_R \) such that 
\( \xi = y_1 + i_1y_2 + i_2y_3 + i_3y_4 \) 
\( \Psi = y_1 + i_1y_2 + i_2y_3 + i_3y_4(\eta + \epsilon) \) and \( \epsilon = \min(d_3, d_4) \) 
Obviously \( \eta \in (\xi, \Psi)_R \)

Let 
\( \Rightarrow \xi < R \) \( Y < R \Psi \) 
\( \Rightarrow Y = y_1 + i_1y_2 + i_2y_3 + i_3y_4 \) where \( (y_4 - \epsilon) < a < (y_4 + \epsilon) \) 
Therefore \( ^1Y = (y_1 + a) + i_1(y_2 - y_3) \) and \( ^2Y = (y_1 - a) + i_1(y_2 + y_3) \) 
Since \( \eta = y_1 + i_1y_2 + i_2y_3 + i_3y_4 \) 
Therefore \( \eta = (y_1 + y_4) + i_1(y_2 - y_3) \) and \( \eta = (y_1 - y_4) + i_1(y_2 + y_3) \) 
Now \( |1\xi-1\eta| \leq |1\xi-1\eta| + |\eta - Y| \) 
\( \Rightarrow |1\xi-1\eta| \leq d_1 + |y_4 - a| \) 
\( \Rightarrow |1\xi-1\eta| \leq d_1 + (y_4 - a) \) 
Since \( (y_4 - \epsilon) < a < (y_4 + \epsilon) \) therefore \( \pm (y_4 - a) < \epsilon \) 
\( \Rightarrow |1\xi-1\eta| < d_1 + \epsilon \) 
Since \( \epsilon = \min(d_3, d_4) \) 
Therefore \( |1\xi-1\eta| < d_3 \) 
Similarly \( |2\xi-2\eta| < d_4 \)
Since \( d_3^2 + d_4^2 < 2r^2 \)
Therefore \( |1\xi - 1\mathbf{y}|^2 + |2\xi - 2\mathbf{y}|^2 < 2r^2 \)
\[
\Rightarrow \sqrt{\frac{|1\xi - 1\mathbf{y}|^2 + |2\xi - 2\mathbf{y}|^2}{2}} < r
\]
Therefore \( \|\xi - \mathbf{y}\| < r \)
\[
\Rightarrow \mathbf{y} \in B(\xi, r)
\]
...(49)

From (48), (49)
\[
(\zeta, \Psi)_R \subseteq B(\xi, r)
\]

Therefore for \( \eta \in B(\xi, r) \) we have a set \( (\zeta, \Psi)_R \) such that \( \eta \in (\zeta, \Psi)_R \) and \( (\zeta, \Psi)_R \subseteq B(\xi, r) \). Hence it proves that real order topology is strictly finer than Norm topology.

Compiling Theorems 3.3.3, 2.3.1, 2.3.2 and corollary 2.3.3, 2.3.14 together, result to new corollary which states that

3.3.4 Corollary

The topology \( \mathcal{T}_1 \) (and therefore \( \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_1^*, \mathcal{T}_2^*, \mathcal{T}_3^* \)) on \( \mathbb{C}_2 \) is strictly finer than the topology \( \mathcal{T}_N \) (and therefore \( \mathcal{T}_1 \) and \( \mathcal{T}_C \)) on \( \mathbb{C}_2 \).

REFERENCES