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An a Priori Estimate for a Scalar Transmission Problem of the Laplacian in $\mathbb{R}^3$

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1. A. Kufner, Weighted Sobolev spaces, John Wiley and Sons, (1985).

An a Priori Estimate for a Scalar Transmission Problem of the Laplacian in $\mathbb{R}^3$

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I. The Scalar Transmission Problem

Let Ω_- be a bounded region in $\mathbb{R}^3$ and $\Omega_+ = \mathbb{R}^3 \setminus \overline{\Omega_-}$. Let $\Sigma = \partial\Omega_- = \partial\Omega_+$ the interface is of class C^∞, see figure 1. Throughout this work, $\mathfrak{D}$ denote the space consisting of all C^∞-functions with compact support and $\mathfrak{D}'$ is the topological dual space of $\mathfrak{D}$ (space of distributions).

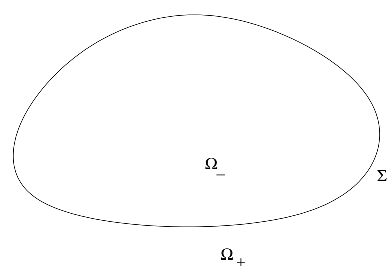

Figure 1 : Region of the problem

Consider the basic weight

$$\ell(r) = \sqrt{1+r^2}, \tag{1}$$

with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, for $\mathbf{x} = (x_1, x_2, x_3)$, is the distance of the origin. For any scalar function $u = u(x_1, x_2, x_3)$, we define the laplace and grad operator of u by

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2},$$

and

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right).$$

Due to the unboundedness of the exterior domain $A = \Omega_+$, the transmission problem is based on the weighted Sobolev spaces, also known as the Beppo-Levi spaces (see [**1**], [**2**]), these spaces were introduced and studied by Hanouzet in [**3**].
For any multi-index α in $\mathbb{N}^3$, we denote by ∂^α the differential operator of order α:

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$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}, \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

Then, for all m in $\mathbb{N}$ and all k in $\mathbb{Z}$, we define the weighted Sobolev space:

$$\mathbb{W}_k^m(\Omega^{is}) := \left\{ v \in \mathfrak{D}'(\Omega^{is}) \mid \forall \alpha \in \mathbb{N}^3, \; 0 \le |\alpha| \le m, \; \ell(r)^{|\alpha|-m+k}\partial^\alpha v \in L^2(\Omega^{is}) \right\}, \quad (2)$$

which is a Hilbert space for the norm:

$$\|v\|_{\mathbb{W}_k^m(\Omega^{is})} = \left\{ \sum_{|\alpha|=0}^{m} \|\ell(r)^{|\alpha|-m+k}\partial^\alpha v\|^2_{L^2(\Omega^{is})} \right\}^{\frac{1}{2}}.$$

And a wide range of basic elliptic problems were solved in these spaces by Giroire in [**4**],

$$\mathbb{W}_0^1(A) = \left\{ u \in \mathfrak{D}'(A) \mid (\ell(r))^{-1}u \in L^2(A), \nabla u \in \mathbf{L}^2(A) \right\} \quad (3)$$

and

$$\mathbb{W}_1^2(A) = \left\{ u \in \mathfrak{D}'(A) \,\middle|\, \frac{u}{\ell(r)} \in L^2(A), \nabla u \in \mathbf{L}^2(A), \ell(r)\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(A), 1 \le i,j \le 3 \right\} \quad (4)$$

They are reflexive Banach spaces equipped, respectively, with natural norms:

$$\|u\|_{\mathbb{W}_0^1(A)} = \left(\|(\ell(r))^{-1}u\|^2_{L^2(A)} + \|\nabla u\|^2_{\mathbf{L}^2(A)} \right)^{\frac{1}{2}}, \quad (5)$$

and

$$\|u\|_{\mathbb{W}_1^2(A)} = \left(\left\| \frac{u}{\ell(r)} \right\|^2_{L^2(A)} + \|\nabla u\|^2_{\mathbf{L}^2(A)} + \sum_{1\le i,j\le 3} \left\| \ell(r)\frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(A)} \right)^{\frac{1}{2}}$$

We also define semi-norms

$$|u|_{\mathbb{W}_0^1(A)} = \|\nabla u\|_{\mathbf{L}^2(A)},$$

and

$$|u|_{\mathbb{W}_1^2(A)} = \left(\sum_{1\le i,j\le 3} \left\| \ell(r)\frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(A)} \right)^{\frac{1}{2}}.$$

Here $\mathbf{L}^2(A) = (L^2(A))^3$, and also we define for all m in $\mathbb{N} \cup \{0\}$ and all k in $\mathbb{Z}$

$$L^2_{m,k}(\mathbb{R}^3) := \left\{ u \in \mathbb{R} \mid \forall \alpha \in \mathbb{N}^3, \; 0 \le |\alpha| \le m, \; \ell(r)^{|\alpha|-m+k}u \in L^2(\mathbb{R}^3) \right\},$$

with the norm

$$\|u\|_{L^2_{m,k}(\mathbb{R}^3)} = \left\{ \sum_{|\alpha|=0}^{m} \|\ell(r)^{|\alpha|-m+k}u\|_{L^2(\mathbb{R}^3)} \right\}^{\frac{1}{2}}.$$

Hence

$$\mathbb{W}_0^0(\Omega_+) = L^2(\Omega_+) \quad \text{and} \quad \mathbb{W}_{-1}^0(\mathbb{R}^3) = L^2_{0,-1}(\mathbb{R}^3).$$

We set the following spaces:

$$\mathring{\mathbb{W}}_0^1(A) = \overline{\mathfrak{D}(A)}^{\|\cdot\|_{\mathbb{W}_0^1(A)}} \quad \text{and} \quad \mathring{\mathbb{W}}_1^2(A) = \overline{\mathfrak{D}(A)}^{\|\cdot\|_{\mathbb{W}_1^2(A)}}.$$

We denote by $\mathbb{W}_0^{-1}(A)$ (respectively $\mathbb{W}_1^0(A)$) the dual space of $\mathring{\mathbb{W}}_0^1(A)$ (respectively of $\mathring{\mathbb{W}}_1^2(A)$). They are spaces of distributions.
With $a(\mathbf{x}) = a_- \in \Omega_-$, $a(\mathbf{x}) = a_+ \in \Omega_+$ for constants $a\pm$, its jump $[a]_\Sigma = a_+ - a_-$, across Σ and the restriction $\varphi^+(\varphi^-)$ of a function φ to $\Omega_+(\Omega_-)$ we consider the problem:
For given

$$f \in L^2(\Omega_-) \cup \mathbb{W}_1^0(\Omega_+) \quad \text{and} \quad g \in H^{\frac{1}{2}}(\Sigma), \quad (6)$$

Ref
4. J. Giroire, Etude de quelques probl'emes aux limites ext'erieurs et r'esolution par 'equations int'egrales, PhD thesis, UPMC, Paris, France, (1987).

find $\varphi \in \mathcal{V}$, such that

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla \quad ^+} dx + a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla \quad ^-} dx = -\int_{\Omega_+ \cup \Omega_-} f \cdot \overline{\psi} dx + [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds, \quad \forall \quad \in \mathcal{V} \qquad (7)$$

with

$$\varphi \in \mathcal{V} = H_0^1(\Omega_-) \cup \mathbb{W}_0^1(\Omega_+), \;\; H_0^1(\Omega_-) = \left\{ \varphi \in H^1(\Omega_-) \;\middle|\; \int_{\Omega_-} \varphi dx = 0 \right\}, \qquad (8)$$

and φ satisfies the decay condition at infinity

$$\varphi = O\left(\frac{1}{|\mathbf{x}|}\right), \;\; \partial_{\mathbf{n}}\varphi = o\left(\frac{1}{|\mathbf{x}|^2}\right) \;\; \text{as} \;\; |\mathbf{x}| \longrightarrow \infty, \qquad (9)$$

The transmission problem (6)-(7) is elliptic. By elliptical regularity, φ has more regularity on sub-domains when the data are more regular.
We introduce

$$PH^2(\mathbb{R}^3) = \{\varphi = (\varphi^+, \varphi^-) \;\mid\; \varphi^+ \in \mathbb{W}_1^2(\Omega_+) \;\text{ and }\; \varphi^- \in H^2(\Omega_-)\},$$

with norm

$$\|\varphi\|^2_{PH^2(\mathbb{R}^3)} = \|\varphi^-\|^2_{H^2(\Omega_-)} + \|\varphi^+\|^2_{\mathbb{W}_1^2(\Omega_+)}. \qquad (10)$$

The following result is an extension of Peron's results [**5**] (for a bounded exterior domain) to an unbounded exterior domain Ω_+.

Proposition 1. Let φ be a solution of the problem (7). For f and g satisfying (6) we have

$$\varphi \in PH^2(\mathbb{R}^3), \qquad (11)$$

φ solves

$$\begin{aligned}
&a_+\Delta\varphi^+ = f^+ \;\; in \;\; \Omega_+,\\
&a_-\Delta\varphi^- = f^- \;\; in \;\; \Omega_-,\\
&\varphi^+ = \varphi^-,\\
&a_+\partial_{\boldsymbol{n}}\varphi^+ - a_-\partial_{\boldsymbol{n}}\varphi^- = [a]_\Sigma \cdot g \;\; on \;\; \Sigma,\\
&\varphi = O\left(\frac{1}{|\boldsymbol{x}|}\right), \;\; \partial_{\boldsymbol{n}}\varphi = o\left(\frac{1}{|\boldsymbol{x}|^2}\right) \;\; as \;\; |\boldsymbol{x}| \longrightarrow \infty,
\end{aligned} \qquad (12)$$

where $\partial_{\boldsymbol{n}}$ denote the normal derivative.

Proof. We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_-. Let $\Omega_+ = \lim_{R\to\infty} \Omega_R$ and $\Omega_R = B_R \cap \Omega_+$, with $\partial\Omega_R = \partial B_R \cup \Sigma$, see figure 2.

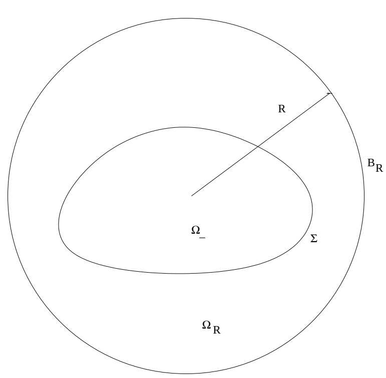

Figure 2 : The domain $\Omega_R = B_R \cap \Omega_+$

Ref

5. V. Peron, Modjelisation mathjematique de phjenom'enes jelectromagnjetiques dans des matjeriaux 'a fort contraste, PhD thesis, Universitje de Rennes I, Rennes, France, (2009).

Then, the first term in (7) is

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = \lim_{R\to\infty} a_+ \int_{\Omega_R} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx,$$

and by integration by parts in Ω_R

$$a_+ \int_{\Omega_R} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = a_+ \int_{\Omega_R} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\partial\Omega_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds$$

$$= a_+ \int_{\Omega_R} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds + a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds,$$

then, when $R \to \infty$, comes

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds + \lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds.$$

The second term in (7) by integration by parts, yields

$$a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla\psi} dx = a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx - a_- \int_{\Sigma} \partial_{\mathbf{n}}\varphi^- \cdot \overline{\psi} ds$$

then

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx + a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla\psi} dx =$$

$$= a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx + a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx +$$

$$+ \int_{\Sigma} (a_+\partial_{\mathbf{n}}\varphi^+ - a_-\partial_{\mathbf{n}}\varphi^-) \cdot \overline{\psi} ds + \lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds.$$

The right part in (7) is

$$- \int_{\Omega_+\cup\Omega_-} f \cdot \overline{\psi} dx + [a]_\Sigma \int_{\Sigma} g \cdot \overline{\psi} ds = - \int_{\Omega_+} f^+ \cdot \overline{\psi} dx - \int_{\Omega_-} f^- \cdot \overline{\psi} dx + [a]_\Sigma \int_{\Sigma} g \cdot \overline{\psi} ds,$$

then, we have

$$a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx = - \int_{\Omega_+} f^+ \cdot \overline{\psi} dx,$$

$$a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx = - \int_{\Omega_-} f^- \cdot \overline{\psi} dx,$$

$$\int_{\Sigma} (a_+\partial_{\mathbf{n}}\varphi^+ - a_-\partial_{\mathbf{n}}\varphi^-) \cdot \overline{\psi} ds = [a]_\Sigma \int_{\Sigma} g \cdot \overline{\psi} ds,$$

and

$$\lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds. = 0.$$

This implies (12), because φ satisfies (9). q.e.d.

Next we set $a_+ = 1$, $a_- = \rho \in \mathbb{C}$, and consider:
Find $\varphi_\rho \in \mathcal{V}$, such that, for all $\quad \in \mathcal{V}$,

$$\int_{\Omega_+} \nabla\varphi_\rho^+ \cdot \overline{\nabla \quad ^+} dx + \rho \int_{\Omega_-} \nabla\varphi_\rho^- \cdot \overline{\nabla \quad ^-} dx = - \int_{\Omega_+\cup\Omega_-} f \cdot \overline{\psi} dx + (1-\rho) \int_{\Sigma} g \cdot \overline{\psi} ds, \quad (\mathbf{P}_\rho)$$

with f and g satisfying (6) independent of ρ and φ satisfying (9).
We construct a mapping $\rho \longmapsto \varphi_\rho$ where φ_ρ solves $(\mathbf{P}_\rho)$ and consider its behavior when $|\rho| \to \infty$.
We assume

Notes

$$\int_{\Omega_+\cup\Omega_-} f dx = 0 \quad \text{and} \quad \int_\Sigma g ds = 0. \tag{13}$$

and show an a priori estimate for φ_ρ uniformly in ρ.
We show now that $\varphi_\rho \in \mathcal{V}$. By construction, φ_ρ is a solution of problem (12), with $a_- = \rho$, $a_+ = 1$. Especially $\varphi_\rho \in H^1(\Omega_-) \cup \mathbb{W}_0^1(\Omega_+)$. Finally $\int_{\Omega_-} \varphi_\rho^- dx = 0$, because φ_ρ^- has integral mean zero.

To complete the proof of Proposition 1 let as now to prove the following a priori estimate. Its application gives the assertion of Proposition 1.

Ref

7. G. Caloz and M. Dauge and V. Peron, Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism, Journal of Mathematical Analysis and applications. 370 (2) (2010), 555–572.

II. A Priori Estimate

The main result for this work is to show a priori estimate in PH^2 uniformly in ρ for a solution $\varphi_\rho \in \mathcal{V}$ of $(\mathbf{P}_\rho)$; that is the following theorem ([**6**, Teorema 3]).

Theorem 1. Assuming (6) and (13), there exists a constant $\rho_0 > 0$ such that for all $\rho \in \{\vec{z} \in \mathbb{C} | |\vec{z}| \geq \rho_0\}$, problem $(\boldsymbol{P}_\rho)$ has a solution $\varphi_\rho \in PH^2(\mathbb{R}^3)$ with

$$\|\varphi_\rho\|_{PH^2(\mathbb{R}^3)} \leq C_{\rho_0}(\|f^-\|_{L^2(\Omega_-)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}), \tag{14}$$

where $C_{\rho_0} > 0$ is independent of ρ, f and g.

The proof of Theorem 1 follows the same steps as the approach in [**5**, **7**] and is given via the following steps.

First we expand φ_ρ in a power series in ρ^{-1}.

$$\varphi_\rho = \begin{cases} \sum_{n=0}^{\infty} \varphi_n^+ \rho^{-n}, & \text{in} \quad \Omega_+, \\ \sum_{n=0}^{\infty} \varphi_n^- \rho^{-n}, & \text{in} \quad \Omega_-. \end{cases} \tag{15}$$

We show that these series converge in the norm in the space PH^2 to a solution of problem $(\mathbf{P}_\rho)$.

Inserting (15) in (12) and identifying terms of like powers of ρ^{-1} we obtain a family of problems independent of ρ, coupled by their conditions on Σ, and the decay condition at infinity. Then by simple calculation we obtain:

$$\begin{aligned} \Delta\varphi_0^- &= 0, \quad \text{in} \quad \Omega_-, \\ \partial_{\mathbf{n}}\varphi_0^- &= g, \quad \text{on} \quad \Sigma, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \Delta\varphi_0^+ &= f^+, \quad \text{in} \quad \Omega_+, \\ \varphi_0^+ &= \varphi_0^-, \quad \text{on} \quad \Sigma, \end{aligned} \tag{17}$$

and for $k \in \mathbb{N}$ with the Kronecker symbol $\delta_{k,1}$

$$\begin{aligned} \Delta\varphi_k^- &= \delta_{k,1} f^-, \quad \text{in} \quad \Omega_-, \\ \partial_{\mathbf{n}}\varphi_k^- &= -\delta_{k,1} g + \partial_{\mathbf{n}}\varphi_{k-1}^+, \quad \text{on} \quad \Sigma, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \Delta\varphi_k^+ &= 0, \quad \text{in} \quad \Omega_+, \\ \varphi_k^+ &= \varphi_k^-, \quad \text{on} \quad \Sigma, \end{aligned} \tag{19}$$

and the condition at infinity

$$\varphi_\rho = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \partial_{\mathbf{n}}\varphi_\rho = o\left(\frac{1}{|\mathbf{x}|^2}\right) \quad \text{as} \quad |\mathbf{x}| \longrightarrow \infty, \tag{20}$$

We construct every term successively φ_n^- and φ_n^+, by beginning in φ_0^- and φ_0^+.
Let us assume that $\{\varphi_k^-\}_{k=0}^{n-1}$ and $\{\varphi_k^+\}_{k=0}^{n-1}$ are known. Then, problem (18) defines a unique φ_n^-. Its trace on Σ is inserted in (19) as Dirichlet data to determine the external part φ_n^+.

The Neumann problem (16) has a unique solution $\varphi_0^- \in H^1(\Omega_-)$ if $\int_{\Omega_-} \varphi_0^- dx = 0$. We remember that we have the compatibility condition $\int_\Sigma g ds = 0$. Also, by elliptic regularity, $\varphi_0^- \in H^2(\Omega_-)$ and there is a constant $C_N > 0$, independent of ρ, such that (see [**8**, Theorem 2.5.2])

$$\|\varphi_0^-\|_{H^2(\Omega_-)} \leq C_N \|g\|_{H^{\frac{1}{2}}(\Sigma)}. \tag{21}$$

We are interested in φ_0^+ in (17). Problem (17) has a unique solution (see [**4**, Chapter 2]), $\varphi_0^+ \in \mathbb{W}_0^1(\Omega_+)$. Also, by elliptic regularity and since $\varphi_0^- \in H^2(\Omega_-)$, $\varphi_0^+ \in \mathbb{W}_1^2(\Omega_+)$ and there is a constant $C_{DN} > 0$ independent of ρ, such that (see [**2**, Theorem 6])

$$\|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}(\|\varphi_0^-\|_{H^2(\Omega_-)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)}). \tag{22}$$

Now that (20) guaranties that $\varphi_0^+ \in \mathbb{W}_0^1(\Omega_+)$ and not only in $\mathbb{W}_0^1(\Omega_+)\diagdown\mathbb{R}$. Similarly we can deal with (18) and (19). Since φ_ρ satisfies the decay condition at infinity, φ_ρ can not behave like a constant. Therefore the constraints (23) are not necessary.
Next we show that the Neumann problem (18) is compatible.
For $k = 1$, is necessary to prove that

$$\int_{\Omega_-} f^- dx + \int_\Sigma (-g + \partial_{\mathbf{n}}\varphi_0^+) ds = 0. \tag{23}$$

According to (17) and (20)

$$\begin{aligned} \Delta\varphi_0^+ &= f^+, && \text{in} \quad \Omega_+, \\ \varphi_0^+ &= \varphi_0^-, && \text{on} \quad \Sigma, \\ \partial_{\mathbf{n}}\varphi_0^+ &= o\left(\frac{1}{|\mathbf{x}|^2}\right), && \text{as} \quad |\mathbf{x}| \longrightarrow \infty. \end{aligned} \tag{24}$$

We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_- (see figure 2). Then for the bounded domain $\Omega_+ \cap B_R$, integrating by part in $(24)_1$ gives

$$\int_{\Omega_+\cap B_R} f^+ \overline{\ \ }^{+} dx = \int_{\Omega_+\cap B_R} \Delta\varphi_0^+ \overline{\ \ }^{+} dx$$

$$= \int_{\Omega_+\cap B_R} \nabla\varphi_0^+ \cdot \overline{\nabla \ \ }^{+} dx + \int_{\partial(\Omega_+\cap B_R)} \overline{\ \ }^{+} \cdot \partial_{\mathbf{n}}\varphi_0^+ ds,$$

for $\ \equiv 1$ yields

$$\int_{\Omega_+\cap B_R} f^+ dx = \int_{\partial(\Omega_+\cap B_R)} \partial_{\mathbf{n}}\varphi_0^+ ds$$

and $\partial(\Omega_+ \cap B_R) = \partial B_R \cup \Sigma$, then

$$\int_{\Omega_+\cap B_R} f^+ dx = \int_{\partial B_R} \partial_{\mathbf{n}}\varphi_0^+ ds + \int_\Sigma \partial_{\mathbf{n}}\varphi_0^+ ds$$

$$= \int_{\partial B_R} o\left(\frac{1}{R^2}\right) ds + \int_\Sigma \partial_{\mathbf{n}}\varphi_0^+ ds$$

Ref

4. J. Giroire, Etude de quelques problèmes aux limites extérieurs et résolution par équations intégrales, PhD thesis, UPMC, Paris, France, (1987).

$$= o\left(\frac{1}{R^2}\right) R^2 + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds,$$

then

$$\int_{\Omega_+} f^+ dx = o(1) + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds, \quad \text{as} \quad R \longrightarrow \infty,$$

then

Notes

$$\int_{\Omega_+} f^+ dx = \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds.$$

Under the hypothesis (13)

$$\int_\Sigma g ds = 0, \quad \text{and} \quad \int_{\mathbb{R}^3} f dx = 0,$$

then

$$\int_{\Omega_+} f^+ dx = -\int_{\Omega_-} f^- dx,$$

the compatibility condition (23) is deducted.
For $k \geq 2$, we assume that the term φ_{k-1}^+ is constructed. It is necessary to show that

$$\int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds = 0. \tag{25}$$

According to (19) and (20)

$$\begin{aligned} &\Delta \varphi_{k-1}^+ = 0, && \text{in} \quad \Omega_+, \\ &\varphi_{k-1}^+ = \varphi_{k-1}^-, && \text{on} \quad \Sigma, \\ &\partial_{\mathbf{n}} \varphi_{k-1}^+ = o\left(\frac{1}{|\mathbf{x}|^2}\right), && \text{as} \quad |\mathbf{x}| \longrightarrow \infty. \end{aligned} \tag{26}$$

Again we choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_-. Then for the bounded domain $\Omega_+ \cap B_R$, integrating by part in $(26)_1$ gives

$$0 = \int_{\Omega_+ \cap B_R} \Delta \varphi_{k-1}^+ \overline{\quad}^{+} dx = \int_{\Omega_+ \cap B_R} \nabla \varphi_{k-1}^+ \cdot \overline{\nabla \quad}^{+} dx + \int_{\partial(\Omega_+ \cap B_R)} \overline{\quad}^{+} \cdot \partial_{\mathbf{n}} \varphi_{k-1}^+ ds,$$

for $\quad \equiv 1$ yields

$$0 = \int_{\partial(\Omega_+ \cap B_R)} \partial_{\mathbf{n}} \varphi_{k-1}^+ ds$$

and $\partial(\Omega_+ \cap B_R) = \partial B_R \cup \Sigma$, then

$$0 = \int_{\partial B_R} \partial_{\mathbf{n}} \varphi_{k-1}^+ ds + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds$$

$$= \int_{\partial B_R} o\left(\frac{1}{R^2}\right) ds + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds$$

$$= o\left(\frac{1}{R^2}\right) R^2 + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds,$$

then

$$0 = o(1) + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds, \quad \text{as} \quad R \longrightarrow \infty,$$

then

$$0 = \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds,$$

then (25) is deducted.

Consequently, the Neumann problem (18) admits a solution $\varphi_k^- \in H^1(\Omega_-)$, which is unique under condition $\int_{\Omega_-} \varphi_k^- dx = 0$ (see [**8**, Theorem 2.5.10]). Also, $\varphi_k^- \in H^2(\Omega_-)$ and (see [**5**, **7**])

$$\|\varphi_k^-\|_{H^2(\Omega_-)} \leq C_N[\delta_k^1(\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}) + \|\partial_{\mathbf{n}}\varphi_{k-1}^+\|_{H^{\frac{1}{2}}(\Sigma)}]. \tag{27}$$

Finally, problem (19) has a unique solution $\varphi_k^+ \in \mathbb{W}_0^1(\Omega_+)$ (see [**4**, Chapter 2] and the estimate (see [**8**, Theorem 2.5.14])

$$\|\varphi_k^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_k^-\|_{H^2(\Omega_-)}. \tag{28}$$

Next, we demonstrate the convergence in $PH^2(\mathbb{R}^3)$ of the series (15) for large $|\rho|$.

For the Neumann trace (see [**5**, **7**])

$$\begin{array}{rccc} \gamma_{1,\Sigma}: & \mathbb{W}_1^2(\Omega_+) & \longrightarrow & H^{\frac{1}{2}}(\Sigma), \\ & \varphi & \longmapsto & \partial_{\mathbf{n}}\varphi \end{array}$$

we have with a constant $C_1 > 0$,

$$\|\gamma_{1,\Sigma}(\varphi)\|_{H^{\frac{1}{2}}(\Sigma)} \leq C_1\|\varphi\|_{\mathbb{W}_1^2(\Omega_+)}. \tag{29}$$

We pose $\alpha = C_N C_1 C_{DN}$, where C_N and C_{DN} are the respective constants of estimates (21) and (22). With (27), (28) and (29) we show by induction

$$\begin{aligned} \|\varphi_n^-\|_{H^2(\Omega_-)} &\leq \alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)}, \\ \|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN}\cdot\alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)}. \end{aligned} \tag{30}$$

$(30)_1$ can be see as follows: For $n = 1$,

$$\|\varphi_1^-\|_{H^2(\Omega_-)} = \alpha^0\|\varphi_1^-\|_{H^2(\Omega_-)}.$$

With (27) we have for $k = 2$

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_1^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and with (29)

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)};$$

hence by (28) we have for $k = 1$

$$\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

and therefore

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N C_1 C_{DN}\|\varphi_1^-\|_{H^2(\Omega_-)} = \alpha\|\varphi_1^-\|_{H^2(\Omega_-)}.$$

We assume that $(30)_1$ is true for $k = n-1$, this is

$$\|\varphi_{n-1}^-\|_{H^2(\Omega_-)} \leq \alpha^{n-2}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

then, according to (27), for $k = n$

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_{n-1}^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (29)

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)};$$

according to (28) for $k = n-1$

$$\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_{n-1}^-\|_{H^2(\Omega_-)},$$

then

Ref

8. J. C. Nedelec, Acoustic and electromagnetic equations. Integral representations for harmonic problems, Springer-Verlag, (2001).

$$\begin{aligned}\|\varphi_n^-\|_{H^2(\Omega_-)} &\leq C_N C_1 C_{DN}\|\varphi_{n-1}^-\|_{H^2(\Omega_-)}\\ &\leq \alpha\cdot\alpha^{n-2}\|\varphi_1^-\|_{H^2(\Omega_-)}\\ &= \alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)},\end{aligned}$$

then $(30)_1$ is true for all n.

Notes

$(30)_2$ can be see as follows: According to (28) for $k=1$

$$\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

and for $k=2$

$$\|\varphi_2^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_2^-\|_{H^2(\Omega_-)}.$$

According to (27) for $k=2$

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_1^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (29)

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)},$$

then

$$\begin{aligned}\|\varphi_2^+\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN}C_N C_1\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)}\\ &\leq C_{DN}\cdot\alpha\|\varphi_1^-\|_{H^2(\Omega_-)}.\end{aligned}$$

We assume that $(30)_2$ is true for $k=n-1$, this is

$$\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\cdot\alpha^{n-2}\|\varphi_1^-\|_{H^2(\Omega_-)}$$

then, according to (28), for $k=n$

$$\|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_n^-\|_{H^2(\Omega_-)},$$

and according to (27) for $k=n$

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_{n-1}^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (29)

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)},$$

then

$$\begin{aligned}\|\varphi_n^-\|_{H^2(\Omega_-)} &\leq C_N C_1 C_{DN}\cdot\alpha^{n-2}\|\varphi_1^-\|_{H^2(\Omega_-)}\\ &= \alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)},\end{aligned}$$

then

$$\|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\cdot\alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

then $(30)_2$ is true for all n.

Hence for all $\rho\in\mathbb{C}$, with $|\rho|^{-1}\alpha<1$, the series (15) converges in $\mathbb{W}_1^2(\Omega_+)$ and $H^2(\Omega_-)$, respectively.

Now we are in the position to prove Theorem 1.

We show first the estimate (14) for φ_ρ in (15). Let $\rho_0>0$, such that $\rho_0^{-1}\alpha<1$, where $\alpha = C_N C_1 C_{DN}$.

Let $\rho\in\{z\in\mathbb{C}||z|\geq\rho_0\}$. According to (30) φ_ρ converges geometrically in $PH^2(\mathbb{R}^3)$ with convergence ratio $|\rho^{-1}|\alpha$, bounded by $\rho_0^{-1}\alpha$. Hence,

$$
\begin{aligned}
\|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} &\le C_{DN}\frac{1}{1-\rho_0^{-1}\alpha}\rho_0^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)} + \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)},\\
\|\varphi_\rho^-\|_{H^2(\Omega_-)} &\le \rho_0^{-1}\frac{1}{1-\rho_0^{-1}\alpha}\|\varphi_1^-\|_{H^2(\Omega_-)} + \|\varphi_0^-\|_{H^2(\Omega_-)}
\end{aligned}
\tag{31}
$$

From $(15)_1$, $(30)_2$ and the triangular inequality, we have

$$
\begin{aligned}
\|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} &= \|\textstyle\sum_{n=0}^{\infty}\varphi_n^+\rho^{-n}\|_{\mathbb{W}_1^2(\Omega_+)}\\
&\le \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)} + \textstyle\sum_{n=1}^{\infty}\|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)}|\rho^{-n}|\\
&\le \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)} + C_{DN}\cdot\alpha^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)}\textstyle\sum_{n=1}^{\infty}|\rho^{-n}|\alpha^n,
\end{aligned}
$$

and

$$
\sum_{n=1}^{\infty}|\rho^{-n}|\alpha^n = \sum_{n=1}^{\infty}(\rho^{-1}\alpha)^n = \frac{1}{1-\rho^{-1}\alpha} \le \frac{1}{1-\rho_0^{-1}\alpha}, \tag{32}
$$

then

$$
\|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} \le C_{DN}\frac{1}{1-\rho_0^{-1}\alpha}\rho_0^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)} + \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}.
$$

Using the triangle inequality, $(15)_2$ and $(30)_1$, we have

$$
\begin{aligned}
\|\varphi_\rho^-\|_{H^2(\Omega_-)} &= \|\textstyle\sum_{n=0}^{\infty}\varphi_n^-\rho^{-n}\|_{H^2(\Omega_-)}\\
&\le \|\varphi_0^-\|_{H^2(\Omega_-)} + \textstyle\sum_{n=1}^{\infty}\|\varphi_n^-\|_{H^2(\Omega_-)}|\rho^{-n}|\\
&\le \|\varphi_0^-\|_{H^2(\Omega_-)} + \alpha^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)}\textstyle\sum_{n=1}^{\infty}|\rho^{-n}|\alpha^n,
\end{aligned}
$$

this and (32) implies $(31)_2$.
Now, from (27), for $k=1$

$$
\|\varphi_1^-\|_{H^2(\Omega_-)} \le C_N[\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|\partial_{\mathbf{n}}\varphi_0^+\|_{H^{\frac{1}{2}}(\Sigma)}], \tag{33}
$$

according to (33) and (29), get

$$
\|\varphi_1^-\|_{H^2(\Omega_-)} \le C_N[\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)} + C_1\|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}]. \tag{34}
$$

From (34), (31), (21) and (22), we have

$$
\begin{aligned}
\|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} &\le C_{DN}\frac{1}{1-\rho_0^{-1}\alpha}\rho_0^{-1}C_N[\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}\\
&+C_1C_{DN}(C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)})] + C_{DN}(C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)}),
\end{aligned}
$$

and

$$
\begin{aligned}
\|\varphi_\rho^-\|_{H^2(\Omega_-)} &\le \rho_0^{-1}\frac{1}{1-\rho_0^{-1}\alpha}C_N[\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}\\
&+C_1C_{DN}(C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)})] + C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)}.
\end{aligned}
$$

This yields the estimate (14).

Notes

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