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Existence of Fixed Points of a Pair of Self Maps under Weak Generalized Geraghty Contractions in Complete Partially Ordered Partial b - Metric Spaces

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Existence of Fixed Points of a Pair of Self Maps under Weak Generalized Geraghty Contractions in Complete Partially Ordered Partial b - Metric Spaces

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1. INTRODUCTION AND PRELIMINARIES

Most of the fixed point theorems in nonlinear analysis usually start with Banach [9] contraction principle. A huge amount of literature is witnessed on applications, generalizations and extensions of this principle carried out by several authors in different directions like weakening the hypothesis and considering different mappings. But all the generalizations may not be from this principle. In 1989, Bakktin [8] introduced the concept of a b - metric space as a generalization of a metric space. In 1993, Czerwik[11] extended many results related to the b - metric space. In 1994, Matthews [20] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill [28] generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [35] generalized both the concepts of b - metric and partial metric space by introducing the notation of partial b - metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [16,35,23,32,38]. Some authors [4,20,24,30,31] obtained some fixed point theorems in b - metric spaces . After that some authors proved $\alpha - \psi$ versions of certain fixed point theorems in different types of metric

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spaces [3,16]. Recently Samet et.al [29] and Jalal Hassanzadeasl [14] obtained fixed point theorems for $\alpha - \psi$ contractive mappings . Mustafa [24] gave a generalization of Banach contraction principle in complete ordered partial b - metric space by using the notion of a generalized $\alpha - \psi$ weakly contractive mapping. Babu et.al[5] proved coupled fixed point theorems by using (α, φ, β) - weak generalized Geraghty contraction. In 2012, Mohammad Mursaleen et.al [22] proved coupled fixed point theorems for $\alpha - \psi$ contractive type mappings in partially ordered metric spaces. In this paper we extend the concepts of G. V. R. Babu. et.al.[6] to complete partially ordered partial b- metric space with coefficient $s \geq 1$ and obtain sufficient conditions for the existence of fixed points of weak generalized Geraghty contractions in a complete partially ordered partial b - metric space with coefficient $s \geq 1$. A supporting example is also given. Further an open problem is also given at the end of this paper. Shukla [35] introduced the notation of a partial b - metric space as follows.

Definition 1.1. (S.Shukla [35]) Let X be a non empty set and let $s \geq 1$ be a given real number. A function $p : X \times X \rightarrow [0, \infty)$ is called a partial b - metric if for all $x, y, z \in X$ the following conditions are satisfied.

- (i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$
- (ii) $p(x, x) \leq p(x, y)$
- (iii) $p(x, y) = p(y, x)$
- (iv) $p(x, y) \leq s\{p(x, z) + p(z, y)\} - p(z, z)$

The pair (X, p) is called a partial b - metric space. The number $s \geq 1$ is called a coefficient of (X, p) .

Definition 1.2. (Z.Mustafa.et.al.[24]) A sequence $\{x_n\}$ in a partial b - metric space (X, p) is said to be:

- (i) convergent to a point $x \in X$ if $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$
- (ii) a Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and is finite
- (iii) a partial b - metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = \lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$.

Definition 1.3. (E.Karapinar. et.al [18]) Let (X, \leq) be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing, if $x_n \leq x_{n+1} \forall n \in \mathbb{N}$

Definition 1.4. (Z.Mustafa.et.al.[24]) A triple (X, \leq, p) is called an ordered partial b - metric space if (X, \leq) is a partially ordered set and p is a partial b - metric on X . In Sastry et.al[30], the notion of a partially ordered partial metric space is introduced.

For definiteness sake Sastry et.al[30](Definition 2.1) adopting the definition 1.1 for partial metric and defined the triple (X, \leq, p) as partially ordered partial metric space. A partially ordered partial metric space (X, \leq, p) is said to be complete if every Cauchy sequence is convergent.

Notation: The following notation is used throughout this paper.

(X, \leq, p) be a complete partially ordered partial b - metric space with coefficient $s \geq 1$ and we write it as X . Let $T : X \rightarrow X$ be a self map of X and let $Fix(T)$ denote the set of all fixed points of T . We denote $\Omega = \{\beta : (0, \infty) \rightarrow [0, 1) / \beta(t_n) \rightarrow 1 \Rightarrow t_n \rightarrow 0\}$, and that $\Phi_s = \{\varphi : [0, \infty) \rightarrow [0, \infty) / \varphi \text{ is non-decreasing, continuous, } \lim_{t \rightarrow r^+} \varphi(t) < \frac{r}{s} \text{ and } \varphi(t) = 0 \Leftrightarrow t = 0\}$. We call the elements of Φ_s as altering distance functions.

Further, we use the following notation: for any sequences $\{a_n\}$ and $\{b_n\}$ in X with $p_n = p(a_n, b_n) \neq 0$,

$$\text{we write } \Delta_n = \frac{p(T(a_n), T(b_n))}{p_n} \text{ and } \Delta_n^\varphi = \frac{\varphi(sp(T(a_n), T(b_n)))}{\varphi(p_n)} \forall n,$$

We denote the set of all real numbers by \mathbb{R} , the set of all nonnegative reals by \mathbb{R}^+ and the set of all natural numbers by \mathbb{N} .

Definition 1.5. (I.Beg and A.R.Butt [10]) Let (X, \leq) be a partially ordered set and $S, T : X \rightarrow X$ be such that $Sx \leq TSx$ and $Tx \leq STx \forall x \in X$. Then S and T are said to be weakly increasing mappings.

Definition 1.6. (B.Samet et.al.[29]) Let $T : X \rightarrow X$ be a self map and $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. Then T is said to be α - admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Definition 1.7. (JalalHassanzadeasl., [14]) Let $T, S : X \rightarrow X$, and let $\alpha : X \times X \rightarrow [0, +\infty)$. We say that T, S are coupled α - admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Sy) \geq 1$ and $\alpha(Sx, Ty) \geq 1$ for all $x, y \in X$

Definition 1.8. (E. Karapinar. et.al. [18]) An α - admissible map T is said to be triangular α - admissible if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$.

Lemma 1.9. (E.Karapinar. et.al.[18]) Let $T : X \rightarrow X$ be triangular α - admissible map. Assume that there exists $x_1 \in X \ni \alpha(x_1, Tx_1) \geq 1$. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n, n = 0, 1, 2, \dots$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

For more details and examples on α - admissible and coupled α - admissible maps, one can refer [17], [18] and [29].

The following lemma can be easily established.

Definition 1.10. (S. H. Cho. et.al.[13]) Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow \mathbb{R}$ be a function. A map $T : X \rightarrow X$ is called an α - Geraghty type contraction if there exists $\beta \in \Omega$ such that $\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$ for all $x, y \in X$. (1.10.1)

Definition 1.11. (G. V. R. Babu. et.al.[6]) Let (X, d) be a metric space and $T : X \rightarrow X$ be a self map. If there exist $\alpha : X \times X \rightarrow \mathbb{R}, \varphi \in \Phi$ and $L \geq 0$ such that

$$\alpha(x, y)\varphi((d(Tx, Ty))) < \varphi((M(x, y))) + L.N(x, y) \tag{1.11.1}$$

for all $x, y \in X, x \neq y$ where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\},$$

and $N(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$, then we say that T is an almost generalized $\alpha -$ contractive map with respect to an altering distance function φ .

Definition 1.12. (G. V. R. Babu. et.al.[6]) Let (X, d) be a metric space let $T : X \rightarrow X$ be a self map. If $\exists \alpha : X \times X \rightarrow \mathbb{R}, \beta \in \Omega, \varphi \in \Phi$ and $L \geq 0 \ni \alpha(x, y)\varphi((d(Tx, Ty))) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y)) + L.N(x, y)$

$$\tag{1.12.1}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\},$$

$$N(x, y) = \min\{d(x, Tx), d(x, Ty), d(y, Tx)\},$$

then we say that T is (α, φ, β) - weak generalized Geraghty contractive map.

The following theorems are established in (K.P.R,Sastry et.al.[32]).

Theorem 1.13. (K.P.R,Sastry et.al.[32]) Let T be a self map on a complete partially ordered partial $b -$ metric space (X, \leq, p) with coefficient $s \geq 1$. Let $\alpha : X \times X \rightarrow \mathbb{R}$ be a continuous function and

$$\alpha(x, x) > 1 \forall x \in X.$$

Assume that there exists $\varphi \in \Phi_s$ such that

$$\alpha(x, y)\varphi(sp(Tx, Ty)) < \varphi(M(x, y)) \tag{1.13.1}$$

for all $x, y \in X, p(x, y) \neq 0$ where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Tx, y)]\}$$

Further, assume that

(i) T is triangular $\alpha -$ admissible, and

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$

and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \dots$

(iii) for any two sequences $\{a_n\}$ and $\{b_n\}$ in X with $p_n = p(a_n, b_n) \neq 0$, we have that

$$\Delta_n^\varphi \rightarrow 1 \Rightarrow \varphi(p_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then $\{x_n\}$ is a Cauchy sequence. And if $x_n \rightarrow z$ then z is a fixed point T in X . Further if y, z are fixed point of T in X , then either $\alpha(y, z) < 1$ or $y = z$.

Corollary 1.14. (K.P.R,Sastry et.al.[32]) Let T be a self map on a complete partially ordered partial $b -$ metric space X . Let $\alpha : X \times X \rightarrow \mathbb{R}$ be a continuous function.

Assume that there exists $\varphi \in \Phi_s$ such that

$$\alpha(x, y)\varphi(sp(Tx, Ty)) < \varphi(M(x, y)) \tag{1.14.1}$$

for all $x, y \in X, p(x, y) \neq 0$ where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Tx, y)]\}$$

Further, assume that

(i) T is α - triangular admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$

and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \dots$

(iii) for any two sequences $\{a_n\}$ and $\{b_n\}$ of X with $p_n = p(a_n, b_n) \neq 0$, we have that $\Delta_n^\varphi \rightarrow 1 \Rightarrow \varphi(p_n) \rightarrow 0$ as $n \rightarrow \infty$ (1.14.2)

Then the sequence $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ converges to z and $\alpha(z, z) > 1$. Then z is a fixed point of T in X .

Theorem 1.15. (K.P.R,Sastry et.al.[32]) Let (X, \leq, p) be a complete partially ordered partial b - metric space with coefficient $s \geq 1$, and $T : X \rightarrow X$ be a self map. Let $\alpha : X \times X \rightarrow \mathbb{R}$ be a continuous function and $\beta \in \Omega, \varphi \in \Phi_s$.

Suppose the following conditions hold:

(i) T is (α, φ, β) - weak generalized Geraghty contraction map i.e.,

$$\alpha(x, y)\varphi(sp(Tx, Ty)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y)) \forall x, y \in X, p(x, y) \neq 0 \quad (1.15.1)$$

where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Tx, y)]\}$$

(ii) T is triangular α - admissible, and

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$

and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \dots$

Then $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ converges to x

and $\alpha(x, x) > 1$. Then x is a fixed point of T in X

These results are extended to partially ordered partial b - metric space for a pair of self maps in the next session.

II. MAIN RESULT

In this section we continue our study to extend the concepts of G. V. R. Babu. et.al.[6] to complete partially ordered partial b- metric space with coefficient $s \geq 1$ and obtain sufficient conditions for the existence of fixed points of weak generalized Geraghty contractions in a complete partially ordered partial b- metric space with coefficient $s \geq 1$. A supporting example is given. Further an open problem is also given at the end of this paper.

We begin this section with the following definition.

Definition 2.1. (K.P.R,Sastry et.al.[30]) Suppose (X, \leq) is a partially ordered set and p is a partial b - metric on X as in definition 1.1 with coefficient $s \geq 1$. Then we say that the triplet (X, \leq, p) is a partially ordered partial b - metric space. Notions of convergence of a sequence and Cauchy sequence are as in definition 1.2. A partially ordered partial b - metric space (X, \leq, p) is said to be complete if every Cauchy sequence in X is convergent .

Now we state the following useful lemmas, whose proofs can be found in Sastry. et. al.[30].

Lemma 2.2. Let (X, \leq, p) be a complete partially ordered partial b - metric space. Let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0$. Suppose $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} x_n = y$

Then (a) $\lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_n, y) = p(x, y)$ and hence $x = y$

(b) (i) $\{x_n\}$ is a Cauchy sequence $\Rightarrow \lim_{m, n \rightarrow \infty} p(x_m, x_n) = 0$.

(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \exists \epsilon > 0$ and sequences $\{m_k\}, \{n_k\} \ni m_k > n_k > k \in \mathbb{N}; p(x_{n_k}, x_{m_k}) > \epsilon$ and $p(x_{n_k}, x_{m_k-1}) \leq \epsilon$

Lemma 2.3. (a) $p(x, y) = 0 \Rightarrow x = y$

(b) $p(x_{n_k-1}, x_{m_k-1}) \neq 0$

Proof. : Suppose $p(x_{n_k-1}, x_{m_k-1}) = 0$

$\Rightarrow x_{n_k-1} = x_{m_k-1}$

$\Rightarrow Tx_{n_k-1} = Tx_{m_k-1}$

$\therefore x_{n_k} = x_{m_k}$

Now $\epsilon < p(x_{n_k}, x_{m_k}) = p(x_{n_k}, x_{n_k}) < p(x_{n_k}, x_{n_k-1})$

Allowing $k \rightarrow \infty$

$\epsilon \leq \lim_{k \rightarrow \infty} p(x_{n_k}, x_{n_k-1}) = 0$, a contradiction.

$\therefore p(x_{n_k-1}, x_{m_k-1}) \neq 0$

Lemma 2.4. If $\varphi \in \Phi_s$ then

(i) $\lim_{n \rightarrow \infty} \varphi^n(t) = 0 \quad \forall t > 0$

(ii) $\varphi(t) < \frac{t}{s} \quad \forall t > 0$ where $s \geq 1$ is the coefficient of (X, p)

Definition 2.5. Let (X, \leq, p) be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $T : X \rightarrow X$ be a self map. If there exist $\alpha : X \times X \rightarrow \mathbb{R}, \beta \in \Omega, \varphi \in \Phi_s$ such that

$$\alpha(x, y)\varphi(sp(Tx, Sy)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y)) \quad (2.5.1)$$

for all $x, y \in X$, where

$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Tx, y)]\}$. Then we say that T is (α, φ, β) - weak generalized Geraghty contractive map.

Notation: The following notation is used throughout this paper.

(X, \leq, p) be a complete partially ordered partial b - metric space with coefficient $s \geq 1$ and we write it as X . Let $S, T : X \rightarrow X$ be a self map of X and let $Fix(T)$ denotes the set of all fixed points of S and T . We denote $\Omega = \{\beta : (0, \infty) \rightarrow [0, 1)/\beta(t_n) \rightarrow 1 \Rightarrow t_n \rightarrow 0\}$, and that $\Phi_s = \{\varphi : [0, \infty) \rightarrow [0, \infty)/\varphi$ is non-decreasing, continuous, $\lim_{t \rightarrow r^+} \varphi(t) < \frac{r}{s}$ and $\varphi(t) = 0 \Leftrightarrow t = 0\}$. We call the elements of Φ_s as altering distance functions.

Further, we use the following notation: for any sequences $\{a_n\}$ and $\{b_n\}$ in X with $p_n = p(a_n, b_n) \neq 0$, we write $\Delta_n = \frac{p(S(a_n), T(b_n))}{p_n}$ and $\Delta_n^\varphi = \frac{\varphi(sp(S(a_n), T(b_n)))}{\varphi(p_n)} \forall n$, We denote the set of all real numbers by \mathbb{R} , the set of all nonnegative reals by \mathbb{R}^+ and the set of all natural numbers by \mathbb{N} .

Now we state our first main result :

Theorem 2.6. Let S, T be weakly increasing self maps on a complete partially ordered partial b - metric space (X, \leq, p) with coefficient $s \geq 1$. Let $\alpha : X \times X \rightarrow \mathbb{R}$ be a continuous function and

$$\alpha(x, x) > 1 \forall x \in X.$$

Assume that there exists $\varphi \in \Phi_s$ such that

$$\alpha(x, y)\varphi(sp(Tx, Sy)) < \varphi(M(x, y)) \tag{2.6.1}$$

for all $x, y \in X, p(x, y) \neq 0$ where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Sy), \frac{1}{2s}[p(y, Tx) + p(Sy, x)]\}$$

Further, assume that

- (i) S, T are weakly increasing
- (ii) S, T are coupled and triangular α - admissible,
- (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \dots$.
- (iv) for any two sequences $\{a_n\}$ and $\{b_n\}$ in X with $p_n = p(a_n, b_n) \neq 0$, we have that $\Delta_n^\varphi \rightarrow 1 \Rightarrow \varphi(p_n) \rightarrow 0$ as $n \rightarrow \infty$.

Then $\{x_n\}$ is a Cauchy sequence. And if $x_n \rightarrow z$ then z is a common fixed point T and S in X . Further if y, z are fixed point of T in X , then either $\alpha(y, z) < 1$ or $y = z$.

Proof. We first prove that any fixed point of T is also a fixed point of S and conversely.

Let x be a fixed point of T .

$$\text{Then } Tx = x$$

$$\begin{aligned} \text{Now } M(x, x) &= \max\{p(x, x), p(Tx, x), p(Sx, x), \frac{1}{2}[p(Tx, x) + p(Sx, x)]\} \\ &= p(Sx, x) \end{aligned}$$

$$\begin{aligned} \therefore \varphi(p(x, Sx)) &= \varphi(d(Tx, Sx)) \\ &\leq \alpha(x, x)\varphi(d(Tx, Sx)) \\ &< \varphi(M(x, x)) \\ &= \varphi(p(x, Sx)) \end{aligned}$$

a contradiction, if $p(x, Sx) \neq 0$

$$\therefore p(x, Sx) = 0$$

$$\therefore \text{by lemma 2.3 } Sx = x$$

Similarly if $Sx = x$ then $Tx = x$.

Further we show that if T and S have a common fixed point then it is unique.

Let $Tx = Sx = x$ and $Ty = Sy = y$

Suppose $\alpha(x, y) < 1$ then there is nothing to prove.

To show that $x = y$. Suppose $x \neq y$

$$\begin{aligned} \text{We have } M(x, y) &= \max\{p(x, y), p(Tx, x), p(Sy, y), \frac{1}{2}[p(Tx, y) + p(Sy, x)]\} \\ &= p(x, y) \end{aligned}$$

$$\begin{aligned} \therefore \varphi(p(x, y)) &= \varphi(p(Tx, Sy)) \\ &\leq \alpha(x, y)\varphi(M(x, y)) \\ &= \alpha(x, y)\varphi(p(x, y)) < \varphi(p(x, y)), \text{ a contradiction} \\ \therefore \varphi(p(x, y)) &= 0 \Rightarrow p(x, y) = 0 \Rightarrow x = y \end{aligned}$$

Let $x_0 \in X$ and $x_{2n+1} = Tx_{2n}$;

$$x_{2n+2} = Sx_{2n+1}; n = 0, 1, 2, \dots$$

For any n suppose $x_{n+1} = x_n$

$$\begin{aligned} \text{Now } n &= 2m \\ \Rightarrow x_{2m+1} &= x_{2m} \\ \Rightarrow Tx_{2m} &= x_{2m} \\ \Rightarrow x_n &\text{ is a fixed point of } T. \\ n &= 2m + 1 \\ \Rightarrow x_{2m+2} &= x_{2m+1} \\ Sx_{2m+1} &= x_{2m+1} \\ \Rightarrow x_n &\text{ is a fixed point of } S \end{aligned}$$

\therefore For any n if $x_{n+1} = x_n$ then x_n is a common fixed point of T and S .

Hence for any n , we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$

Since S and T are weakly increasing,

$$x_1 = Tx_0 \leq STx_0 = Sx_1 = x_2 \leq TSx_1 = Tx_2 = x_3 \dots$$

$\therefore x_1 \leq x_2 \leq x_3 \leq \dots$ Thus $\{x_n\}$ is increasing.

Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$ by (iii). Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By using the α - admissibility of T , we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Sx_1) \geq 1$. Now, by mathematical induction, it is easy to see that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Let n be even and by taking $x = x_{n-1}$ and $y = x_n$ in the inequality (2.6.1), and observing that $p(x_{n-1}, x_n) \neq 0$ by lemma 2.3,

we get

$$\begin{aligned} &\varphi(p(x_n, x_{n+1})) \\ &\leq \varphi(sp(x_n, x_{n+1})) \\ &= \varphi(sp(Sx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\varphi(sp(Sx_{n-1}, Tx_n)) \\ &< \varphi(M(x_n, x_{n-1})) \end{aligned} \tag{2.6.2}$$

where

$$\begin{aligned} &M(x_n, x_{n-1}) \\ &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, Sx_{n-1}), p(x_n, Tx_n), \frac{1}{2s}[p(x_{n-1}, Tx_n) + p(x_n, Sx_{n-1})]\} \end{aligned}$$



$$\begin{aligned}
 &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\} \\
 &\leq \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[sp(x_{n-1}, x_n) + sp(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)]\} \\
 &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]\} \\
 &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}
 \end{aligned}$$

If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ for some $n \in \mathbb{N}$ (2.6.3)

then from (2.6.2) and (2.6.3), we have

$$\varphi(sp(x_n, x_{n+1})) < \varphi(M(x_{n-1}, x_n)) = \varphi(p(x_n, x_{n+1})), \text{ a contradiction.}$$

Thus, we have $M(x_{n-1}, x_n) = \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$ Similarly,

Let n be odd and by taking $x = x_{n-1}$ and $y = x_n$ in the inequality (2.6.1), and

observing that $p(x_{n-1}, x_n) \neq 0$ by lemma 2.3,

we get

$$\begin{aligned}
 &\varphi(p(x_n, x_{n+1})) \\
 &\leq \varphi(sp(x_n, x_{n+1})) \\
 &= \varphi(sp(Tx_{n-1}, Sx_n)) \\
 &\leq \alpha(x_{n-1}, x_n)\varphi(sp(Tx_{n-1}, Sx_n)) \\
 &< \varphi(M(x_{n-1}, x_n))
 \end{aligned}$$
(2.6.4)

where

$$\begin{aligned}
 &M(x_{n-1}, x_n) \\
 &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Sx_n), \frac{1}{2s}[p(x_{n-1}, Sx_n) + p(x_n, Tx_{n-1})]\} \\
 &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\} \\
 &\leq \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[sp(x_{n-1}, x_n) + sp(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)]\} \\
 &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]\} \\
 &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\}
 \end{aligned}$$

If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ for some $n \in \mathbb{N}$ (2.6.5)

then from (2.6.2) and (2.6.3), we have

$$\varphi(sp(x_n, x_{n+1})) < \varphi(M(x_{n-1}, x_n)) = \varphi(p(x_n, x_{n+1})), \text{ a contradiction.}$$

Thus, we have $M(x_{n-1}, x_n) = \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$

and hence, $p(x_n, x_{n+1}) < p(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. (2.6.6)

Thus it follows that $\{p(x_n, x_{n+1})\}$ is a non-negative, decreasing sequence of real numbers. Suppose that $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = r, r \geq 0$

Now we prove that $r = 0$.

Assume that $r > 0$.

Now by (2.6.2)

$$\begin{aligned}
 &\varphi(p(x_n, x_{n+1})) \\
 &\leq \varphi(sp(x_n, x_{n+1})) \\
 &< \varphi(M(x_{n-1}, x_n)) \\
 &= \varphi(p(x_{n-1}, x_n)) \text{ for all } n \in \mathbb{N}
 \end{aligned}$$

On taking limits as $n \rightarrow \infty$,

we have,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \varphi(p(x_n, x_{n+1})) \\ & \leq \lim_{n \rightarrow \infty} \varphi(sp(x_n, x_{n+1})) \\ & \leq \lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n)) \\ & \Rightarrow \varphi(r) \leq \varphi(sr) \leq \varphi(r) \\ & \Rightarrow \varphi(r) = \varphi(sr) \end{aligned}$$

Let n be odd. By choosing $a_n = x_n, b_n = x_{n+1}$,

$$\begin{aligned} \Delta_n^\varphi &= \frac{\varphi(sp(T(x_{n-1}), S(x_n)))}{\varphi(p(x_{n-1}, x_n))} \\ \therefore \lim_{n \rightarrow \infty} \Delta_n^\varphi &= \lim_{n \rightarrow \infty} \frac{\varphi(sp(x_n, x_{n+1}))}{\varphi(p(x_{n-1}, x_n))} \\ &= \frac{\lim_{n \rightarrow \infty} \varphi(sp(x_n, x_{n+1}))}{\lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n))} \\ &= \frac{\varphi(s \lim_{n \rightarrow \infty} p(x_n, x_{n+1}))}{\varphi(\lim_{n \rightarrow \infty} p(x_{n-1}, x_n))} \\ &= \frac{\varphi(sr)}{\varphi(r)} \\ &= 1 \text{ (since } \varphi(r) = \varphi(sr)) \end{aligned} \tag{2.6.7}$$

Hence by our assumption $\varphi(p_n) \rightarrow 0$ as $n \rightarrow \infty$ i.e.,

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n)) &= 0 \\ \Rightarrow \varphi(r) &= 0 \\ \Rightarrow r &= 0, \text{ a contradiction for our assumption } r > 0 \end{aligned}$$

Hence $r = 0$.

Similarly,

Let n be even. By choosing $a_n = x_n, b_n = x_{n+1}$,

$$\begin{aligned} \Delta_n^\varphi &= \frac{\varphi(sp(S(x_{n-1}), T(x_n)))}{\varphi(p(x_{n-1}, x_n))} \\ \therefore \lim_{n \rightarrow \infty} \Delta_n^\varphi &= \lim_{n \rightarrow \infty} \frac{\varphi(sp(x_n, x_{n+1}))}{\varphi(p(x_{n-1}, x_n))} \\ &= \frac{\lim_{n \rightarrow \infty} \varphi(sp(x_n, x_{n+1}))}{\lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n))} \\ &= \frac{\varphi(s \lim_{n \rightarrow \infty} p(x_n, x_{n+1}))}{\varphi(\lim_{n \rightarrow \infty} p(x_{n-1}, x_n))} \\ &= \frac{\varphi(sr)}{\varphi(r)} \\ &= 1 \text{ (since } \varphi(r) = \varphi(sr)) \end{aligned} \tag{2.6.8}$$

Hence by our assumption $\varphi(p_n) \rightarrow 0$ as $n \rightarrow \infty$ i.e.,

$$\begin{aligned} \lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n)) &= 0 \quad \forall n \in \mathbb{N} \\ \Rightarrow \varphi(r) &= 0 \\ \Rightarrow r &= 0, \text{ a contradiction for our assumption } r > 0 \end{aligned}$$

Hence $r = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X . Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 2.2(b), there exist some $\epsilon > 0$, and sub-sequences $\{x_{m_k}\}$

and $\{x_{n_k}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that $p(x_{m_k}, x_{n_k}) \geq \epsilon$ and $p(x_{m_k-1}, x_{n_k}) < \epsilon$ and by lemma 1.9

We have Case(i): Let m_k is odd and n_k is even

$$\therefore s\epsilon \leq sp(x_{m_k}, x_{n_k})$$

$$\Rightarrow \varphi(s\epsilon) \leq \varphi(sp(x_{m_k}, x_{n_k}))$$

$$= \varphi(sp(Tx_{m_k-1}, Sx_{n_k-1}))$$

$$\leq \alpha(x_{m_k-1}, x_{n_k-1}) \varphi(sp(Tx_{m_k-1}, Sx_{n_k-1}))$$

$$(\text{by lemma 1.9, } \alpha(x_{m_k-1}, x_{n_k-1}) \geq 1)$$

$$< \varphi(M(x_{m_k-1}, x_{n_k-1}))$$

$$(2.6.9)$$

$$\text{where } M(x_{m_k-1}, x_{n_k-1})$$

$$= \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, Sx_{n_k-1}), p(x_{m_k-1}, Tx_{m_k-1}),$$

$$\frac{1}{2s} \{p(x_{m_k-1}, Sx_{n_k-1}) + p(Tx_{m_k-1}, x_{n_k-1})\}$$

$$= \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \frac{1}{2s} \{p(x_{m_k-1}, x_{n_k}) + p(x_{m_k}, x_{n_k-1})\}$$

$$\leq \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \frac{1}{2s} \{sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) - p(x_{n_k-1}, x_{n_k-1}) + sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{m_k-1}, x_{m_k}) - p(x_{m_k-1}, x_{m_k-1})\}$$

$$\leq \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \frac{1}{2s} \{2sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) + sp(x_{m_k}, x_{m_k-1})\}$$

$$\leq p(x_{m_k-1}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1})$$

$$\leq sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) - p(x_{n_k}, x_{n_k}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1})$$

$$\leq sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1})$$

$$\therefore \varphi(s\epsilon)$$

$$\leq \varphi(sp(x_{m_k}, x_{n_k}))$$

$$< \varphi(M(x_{m_k-1}, x_{n_k-1}))$$

$$\leq \varphi(p(x_{m_k-1}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1}))$$

$$\leq \varphi(sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1}))$$

$$(2.6.10)$$

Allowing $k \rightarrow \infty$,

$$\varphi(s\epsilon) \leq \lim_{k \rightarrow \infty} \varphi(sp(x_{m_k}, x_{n_k}))$$

$$\leq \lim_{k \rightarrow \infty} \varphi(M(x_{m_k-1}, x_{n_k-1}))$$

$$\leq \lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k-1})) \leq \varphi(s\epsilon)$$

$$\therefore \lim_{k \rightarrow \infty} \varphi(p(x_{m_k}, x_{n_k}))$$

$$= \lim_{k \rightarrow \infty} \varphi(M(x_{m_k-1}, x_{n_k-1}))$$

$$= \lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k-1}))$$

$$= \varphi(s\epsilon)$$

$$(2.6.11)$$

$$\begin{aligned}
 & \therefore \lim_{n \rightarrow \infty} \Delta_n^\varphi \\
 &= \lim_{n \rightarrow \infty} \frac{\varphi(sp(Tx_{m_k-1}, Sx_{n_k-1}))}{\varphi(p_n)} \\
 &= \frac{\lim_{k \rightarrow \infty} \varphi(sp(x_{m_k}, x_{n_k}))}{\lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k-1}))} \\
 &= \frac{\varphi(s\epsilon)}{\varphi(s\epsilon)} \text{ (by (2.6.8))} \\
 &= 1.
 \end{aligned} \tag{2.6.12}$$

Hence by our assumption $\varphi(p(x_{m_k-1}, x_{n_k-1})) \rightarrow 0$ as $k \rightarrow \infty$
i.e., $\varphi(s\epsilon) = 0$

$\Rightarrow s\epsilon = 0$, a contradiction.

Case(ii): Let m_k is odd and n_k is odd

$$\begin{aligned}
 \therefore \varphi(sp(x_{m_k}, x_{n_k+1})) &\leq \alpha(x_{m_k-1}, x_{n_k}) \varphi(sp(Tx_{m_k-1}, Sx_{n_k})) \\
 &< \varphi(M(x_{m_k-1}, x_{n_k}))
 \end{aligned}$$

$$\begin{aligned}
 & \text{where } M(x_{m_k-1}, x_{n_k}) \\
 &= \max[p(x_{m_k-1}, x_{n_k}), p(x_{m_k-1}, Tx_{m_k-1}), p(x_{n_k}, Sx_{n_k}), \frac{1}{2s} [\{p(Tx_{m_k-1}, x_{n_k}) + p(x_{m_k-1}, Sx_{n_k})\}]] \\
 &= \max[p(x_{m_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), p(x_{n_k}, x_{n_k+1}), \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}]] \\
 &= p(x_{m_k-1}, x_{n_k}) \quad \text{or} \quad \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}]
 \end{aligned}$$

Suppose $M(x_{m_k-1}, x_{n_k}) = p(x_{m_k-1}, x_{n_k}) < \epsilon$

$$\begin{aligned}
 \text{But } \epsilon &\leq p(x_{m_k}, x_{n_k}) \leq sp(x_{m_k}, x_{n_k+1}) + sp(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1}) \\
 &\leq sp(x_{m_k}, x_{n_k+1}) + s\eta \quad \text{where } \eta > 0 \ni p(x_{n_k+1}, x_{n_k}) < \eta
 \end{aligned} \tag{2.6.13}$$

$$\Rightarrow \epsilon - s\eta \leq sp(x_{m_k}, x_{n_k+1}) \tag{2.6.14}$$

Since φ is non decreasing

$$\begin{aligned}
 \therefore \varphi(\epsilon - s\eta) &\leq \varphi(sp(x_{m_k}, x_{n_k+1})) \\
 &< \varphi(p(x_{m_k-1}, x_{n_k})) < \varphi(\epsilon)
 \end{aligned} \tag{2.6.15}$$

As φ is continuous and $\eta \rightarrow 0$ as $k \rightarrow \infty$, we get

$$\begin{aligned}
 \varphi(\epsilon) &\leq \lim_{k \rightarrow \infty} \varphi(sp(x_{m_k}, x_{n_k+1})) \leq \lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k})) \leq \varphi(\epsilon) \\
 \therefore \lim_{k \rightarrow \infty} \varphi(sp(x_{m_k}, x_{n_k+1})) &= \varphi(\epsilon) = \lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k}))
 \end{aligned}$$

Suppose $M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}]$

On the other hand

$$\begin{aligned}
 & p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1}) \\
 &\leq sp(x_{m_k}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) - p(x_{n_k-1}, x_{n_k-1}) + sp(x_{m_k+1}, x_{m_k}) \\
 &\quad + sp(x_{m_k}, x_{n_k-1}) - p(x_{m_k}, x_{m_k}) \\
 &\leq sp(x_{m_k}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) + sp(x_{m_k}, x_{n_k-1}) + sp(x_{m_k+1}, x_{m_k}) \\
 &\leq 2sp(x_{m_k}, x_{n_k-1}) + 2s\eta \leq 2s\epsilon + 2s\eta
 \end{aligned}$$

where $p(x_{m_k+1}, x_{m_k}) \leq \eta$ and $p(x_{n_k}, x_{n_k-1}) \leq \eta$ for some $\eta > 0$ for large k

$$\therefore \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}] \leq \epsilon + \eta \tag{2.6.16}$$

Therefore,

$$M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}] \leq \epsilon + \eta$$

\therefore From (2.6.13), (2.6.15) and (2.6.16)

$$\begin{aligned} \varphi(\epsilon - s\eta) &\leq \varphi(sp(x_{m_k}, x_{n_k+1})) \\ &\leq \varphi(M(x_{m_k-1}, x_{n_k})) \\ &\leq \varphi(\epsilon + \eta) \end{aligned}$$

As φ is continuous and $\eta \rightarrow 0$ as $k \rightarrow \infty$, we get

$$\varphi(sp(x_{m_k}, x_{n_k+1})) = \varphi(\epsilon)$$

$$\begin{aligned} &\therefore \lim_{n \rightarrow \infty} \Delta_n^\varphi \\ &= \lim_{n \rightarrow \infty} \frac{\varphi(sp(Tx_{m_k-1}, Sx_{n_k}))}{\varphi(p_n)} \\ &= \lim_{k \rightarrow \infty} \frac{\varphi(sp(x_{m_k}, x_{n_k+1}))}{\lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k}))} \\ &= \frac{\varphi(\epsilon)}{\varphi(\epsilon)} \text{ (by (2.7.8))} \\ &= 1. \end{aligned} \tag{2.6.17}$$

Hence by our assumption $\varphi(p(x_{m_k-1}, x_{n_k-1})) \rightarrow 0$ as $k \rightarrow \infty$

i.e., $\varphi(\epsilon) = 0$

$\Rightarrow \epsilon = 0$, a contradiction.

Similarly the other two cases can be discussed.

$\therefore \{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. Now, we show that z is a fixed point of T . We consider

$$\begin{aligned} &\varphi(sp(x_{2n+1}, Tz)) \\ &= \varphi(sp(Sx_{2n}, Tz)) \\ &\leq \alpha(x_{2n}, z)\varphi(sp(Sx_{2n}, Tz)) \text{ (since } \alpha \text{ is continuous and } \alpha(z, z) > 1) \\ &< \varphi(M(x_{2n}, z)) \end{aligned} \tag{2.6.18}$$

But $M(x_{2n}, z)$

$$\begin{aligned} &= \max\{p(x_{2n}, z), p(x_{2n}, x_{2n+1}), p(z, Tz), \frac{1}{2s}[p(x_{2n+1}, z) + p(x_{2n}, Tz)]\} \\ &= p(z, Tz) \text{ for large } n \end{aligned}$$

$$\therefore \varphi(sp(x_{2n+1}, Tz)) \leq \alpha(x_{2n}, z)\varphi(sp(Sx_{2n}, Tz)) < \varphi(p(z, Tz))$$

Suppose $\varphi(sp(z, Tz)) \neq 0$

Dividing through out by $\varphi(sp(z, Tz))$

$$\frac{\varphi(sp(x_{2n+1}, Tz))}{\varphi(sp(z, Tz))} \leq \alpha(x_{2n}, z) \left\{ \frac{\varphi(sp(x_{2n}, Tz))}{\varphi(sp(z, Tz))} \right\} < \frac{\varphi(p(z, Tz))}{\varphi(sp(z, Tz))} \leq 1$$

On letting $n \rightarrow \infty$,

we get $1 \leq \alpha(z, z) \leq 1 \Rightarrow \alpha(z, z) = 1$, a contradiction

$$\therefore \varphi(sp(z, Tz)) = 0 \Rightarrow p(z, Tz) = 0$$

Then by lemma 2.3, $z = Tz$.

$\therefore z$ is a fixed point of T in X .

Similarly $\varphi(sp(x_{2n}, Sz))$ is to be considered to show z is a fixed point of S in X .

Hence z is a common fixed point of S and T in X .

Suppose y, z and $y \neq z$ are common fixed points of S and T in X

$$\text{Then } Ty = Sy = y, Tz = Sz = z \tag{2.6.19}$$

Suppose $\alpha(y, z) < 1$ then there is nothing to prove.

Suppose $\alpha(y, z) \geq 1$ and $p(y, z) \neq 0$

Now $\varphi(sp(Sy, Tz))$

$$\begin{aligned} &\leq \alpha(y, z)\varphi(sp(Sy, Tz)) \\ &< \varphi(M(y, z)) \end{aligned} \tag{2.6.20}$$

But $M(y, z)$

$$= \max\{p(y, z), p(y, Sy), p(z, Tz), \frac{1}{2s}[p(y, Tz) + p(Sy, z)]\}$$

$$= p(y, z)$$

$$\therefore \varphi(sp(Sy, Tz)) \leq \alpha(y, z)\varphi(sp(Sy, Tz)) < \varphi(p(y, z))$$

which is a contradiction

$$\therefore p(y, z) = 0$$

Then by lemma 2.3, $y = z$.

The following corollary can be easily established.

Corollary 2.7. *Let T be a self map on a complete partially ordered partial b - metric space X . Let $\alpha : X \times X \rightarrow \mathbb{R}$ be a continuous function.*

Assume that there exists $\varphi \in \Phi_s$ such that

$$\alpha(x, y)\varphi(sp(Tx, Ty)) < \varphi(M(x, y)) \tag{2.7.1}$$

for all $x, y \in X, p(x, y) \neq 0$ where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Tx, y)]\}$$

Further, assume that

(i) T is α - triangular admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$

and set $x_n = Tx_{n-1}$ for $n = 1, 2, 3, \dots$

(iii) for any two sequences $\{a_n\}$ and $\{b_n\}$ of X with $p_n = p(a_n, b_n) \neq 0$, we have that

$$\Delta_n^\varphi \rightarrow 1 \Rightarrow \varphi(p_n) \rightarrow 0 \text{ as } n \rightarrow \infty \tag{2.7.2}$$

Then the sequence $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ converges to z and $\alpha(z, z) > 1$. Then z is a fixed point of T in X .

In the following, we prove the existence of fixed points of (α, φ, β) - weak generalized Geraghty contraction type maps in a complete partially ordered partial b - metric space.

Theorem 2.8. *Let (X, \leq, p) be a complete partially ordered partial b - metric space with coefficient $s \geq 1$, and $S, T : X \rightarrow X$ are weakly increasing self maps on X . Let*

$\alpha : X \times X \rightarrow \mathbb{R}$ be a continuous function and $\beta \in \Omega, \varphi \in \Phi_s$.

Suppose the following conditions hold:

(i) S, T are (α, φ, β) - weak generalized Geraghty contraction map i.e.,

$$\alpha(x, y)\varphi(sp(Tx, Sy)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y)) \forall x, y \in X, p(x, y) \neq 0 \tag{2.8.1}$$

where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Sy), \frac{1}{2s}[p(x, Sy) + p(Tx, y)]\}$$

(ii) S, T are coupled and triangular α - admissible,

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$

Then $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ converges to x and $\alpha(x, x) > 1$. Then x is a fixed point of T in X .

Proof. As in theorem 2.6, let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$ by (iii). If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = Tx_n$ and hence x_n is a fixed point of T or S . Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By using the α - admissibility of T , we have $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(Tx_0, Sx_1) \geq 1$. Now, by mathematical induction, it is easy to see that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Let n be even and by taking $x = x_{n-1}$ and $y = x_n$ in the inequality (2.8.1), and observing that $p(x_{n-1}, x_n) \neq 0$ by lemma 2.3,

we get

$$\begin{aligned} & \varphi(sp(x_n, x_{n+1})) \\ &= \varphi(sp(Sx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\varphi(sp(Sx_{n-1}, Tx_n)) \\ &\leq \beta(\varphi(M(x_{n-1}, x_n)))\varphi(M(x_{n-1}, x_n)) \text{ (since } p(x_n, x_{n+1}) \neq 0 \forall n) \\ &< \varphi(M(x_{n-1}, x_n)) \end{aligned} \tag{2.8.2}$$

where

$$\begin{aligned} & M(x_{n-1}, x_n) \\ &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, Sx_{n-1}), p(x_n, Tx_n), \frac{1}{2s}[p(x_{n-1}, Tx_n) + p(x_n, Sx_{n-1})]\} \\ &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\} \\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \end{aligned}$$

If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ for some $n \in \mathbb{N}$ (2.8.3)

then from (2.8.2) and (2.8.3), we have

$$\varphi(sp(x_n, x_{n+1})) < \varphi(M(x_{n-1}, x_n)) = \varphi(p(x_n, x_{n+1})), \text{ a contradiction.}$$

Let n be odd and by taking $x = x_{n-1}$ and $y = x_n$ in the inequality (2.8.1), and observing that $p(x_{n-1}, x_n) \neq 0$ by lemma 2.3,

we get

$$\begin{aligned} & \varphi(sp(x_n, x_{n+1})) \\ &= \varphi(sp(Tx_{n-1}, Sx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\varphi(sp(Tx_{n-1}, Sx_n)) \\ &\leq \beta(\varphi(M(x_{n-1}, x_n)))\varphi(M(x_{n-1}, x_n)) \text{ (since } p(x_n, x_{n+1}) \neq 0 \forall n) \\ &< \varphi(M(x_{n-1}, x_n)) \end{aligned} \tag{2.8.4}$$

where

$$\begin{aligned} & M(x_{n-1}, x_n) \\ &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Sx_n), \frac{1}{2s}[p(x_{n-1}, Sx_n) + p(x_n, Tx_{n-1})]\} \\ &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\} \\ &= \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \end{aligned}$$

If $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ for some $n \in \mathbb{N}$ (2.8.5)

then from (2.8.4) and (2.8.5), we have

$$\varphi(sp(x_n, x_{n+1})) < \varphi(M(x_{n-1}, x_n)) = \varphi(p(x_n, x_{n+1})), \text{ a contradiction.}$$

Thus, we have $\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$ for all $n \in N$ and hence, $p(x_n, x_{n+1}) < p(x_{n-1}, x_n)$ for all $n \in N$. (2.8.6)

Thus it follows that $\{p(x_n, x_{n+1})\}$ is a non-negative, decreasing sequence of real numbers. Suppose that $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = r, r \geq 0$

Now we prove that $r = 0$.

Assume that $r > 0$.

We have

$$\begin{aligned} & \varphi(p(x_n, x_{n+1})) \\ & \leq \varphi(sp(x_n, x_{n+1})) \\ & \leq \beta(\varphi(p(x_{n-1}, x_n)))\varphi(p(x_{n-1}, x_n)) \\ & < \varphi(p(x_{n-1}, x_n)) \end{aligned}$$

Allowing as $n \rightarrow \infty$

$$\begin{aligned} \varphi(r) &= \lim_{n \rightarrow \infty} \varphi(p(x_n, x_{n+1})) \\ &\leq \liminf_{n \rightarrow \infty} \beta(\varphi(p(x_{n-1}, x_n)))\varphi(p(x_{n-1}, x_n)) \\ &\leq \limsup_{n \rightarrow \infty} \beta(\varphi(p(x_{n-1}, x_n)))\varphi(p(x_{n-1}, x_n)) \\ &\leq \lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n)) = \varphi(r) \\ &\Rightarrow \varphi(r) \leq \liminf_{n \rightarrow \infty} \beta(\varphi(p(x_{n-1}, x_n)))\varphi(r) \\ &\leq \limsup_{n \rightarrow \infty} \beta(\varphi(p(x_{n-1}, x_n)))\varphi(r) \\ &\leq \varphi(r) \\ &\Rightarrow \varphi(r) = 0 \text{ or } \lim_{n \rightarrow \infty} \beta(\varphi(p(x_{n-1}, x_n))) = 1 \end{aligned}$$

$$\varphi(r) = 0 \text{ or } \lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n)) = 0 \text{ (since } \beta \in \Omega \text{)}$$

$$\lim_{n \rightarrow \infty} \varphi(p(x_{n-1}, x_n)) = 0$$

$\therefore \varphi(r) = 0 \Rightarrow r = 0$, a contradiction.

$$\text{Hence } r = \lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0 \tag{2.8.7}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X . Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 2.2(b), there exist some $\epsilon > 0$, and sub-sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that $p(x_{m_k}, x_{n_k}) \geq \epsilon$ and $p(x_{m_k-1}, x_{n_k}) < \epsilon$. Let m_k be odd and n_k be even

$$\begin{aligned}
 &\therefore s\epsilon \leq sp(x_{m_k}, x_{n_k}) \\
 &\Rightarrow \varphi(s\epsilon) \leq \varphi(sp(x_{m_k}, x_{n_k})) \\
 &= \varphi(sp(Tx_{m_k-1}, Sx_{n_k-1})) \\
 &\leq \alpha(x_{m_k-1}, x_{n_k-1}) \varphi(sp(Tx_{m_k-1}, Sx_{n_k-1})) \\
 &\text{(by lemma 1.9 } \alpha(x_{m_k-1}, x_{n_k-1}) \geq 1) \\
 &\leq \beta(\varphi(M(x_{m_k-1}, x_{n_k-1})))\varphi(M(x_{m_k-1}, x_{n_k-1})) \\
 &< \varphi(M(x_{m_k-1}, x_{n_k-1})) \tag{2.8.8}
 \end{aligned}$$

$$\begin{aligned}
 &\text{where } M(x_{m_k-1}, x_{n_k-1}) \\
 &= \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, Sx_{n_k-1}), p(x_{m_k-1}, Tx_{m_k-1}), \\
 &\frac{1}{2s} \{p(x_{m_k-1}, Sx_{n_k-1}) + p(Tx_{m_k-1}, x_{n_k-1})\} \\
 &= \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \frac{1}{2s} \{p(x_{m_k-1}, x_{n_k}) + p(x_{m_k}, x_{n_k-1})\} \\
 &\leq \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \frac{1}{2s} \{sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) - \\
 &p(x_{n_k-1}, x_{n_k-1}) + sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{m_k-1}, x_{m_k}) - p(x_{m_k-1}, x_{m_k-1})\} \\
 &\leq \max\{p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \frac{1}{2s} \{2sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) + \\
 &sp(x_{m_k}, x_{m_k-1})\} \\
 &\leq p(x_{m_k-1}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1}) \\
 &\leq sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) - p(x_{n_k}, x_{n_k}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1}) \\
 &\leq sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1}) \\
 &\therefore \varphi(s\epsilon) \\
 &\leq \beta(\varphi(M(x_{m_k-1}, x_{n_k-1})))\varphi(M(x_{m_k-1}, x_{n_k-1})) \\
 &\leq \varphi(M(x_{m_k-1}, x_{n_k-1})) \\
 &\leq \varphi(sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1})) \tag{2.8.9}
 \end{aligned}$$

Allowing $k \rightarrow \infty$, we get

$$\begin{aligned}
 &\varphi(s\epsilon) \\
 &\leq \liminf_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k-1}))) \varphi(M(x_{m_k-1}, x_{n_k-1})) \\
 &\leq \limsup_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k-1}))) \varphi(M(x_{m_k-1}, x_{n_k-1})) \\
 &\leq \lim_{k \rightarrow \infty} \varphi(sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1})) \\
 &\therefore \varphi(s\epsilon) \\
 &\leq \liminf_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k-1})))\varphi(s\epsilon) \\
 &\leq \limsup_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k-1})))\varphi(s\epsilon) \\
 &\leq \varphi(s\epsilon) \\
 &\therefore \text{Either } \varphi(s\epsilon) = 0 \text{ or } \lim_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k-1}))) = 1 \text{ (since } \beta \in S) \tag{2.8.10}
 \end{aligned}$$

$$\Rightarrow \varphi(s\epsilon) = 0$$

$\Rightarrow s\epsilon = 0$, a contradiction.

Let m_k be odd and n_k be odd

$$\begin{aligned}
 \therefore \varphi(sp(x_{m_k}, x_{n_k+1})) &\leq \alpha(x_{m_k-1}, x_{n_k}) \varphi(sp(Tx_{m_k-1}, Sx_{n_k})) \\
 &\leq \beta(\varphi(M(x_{m_k-1}, x_{n_k})))\varphi(M(x_{m_k-1}, x_{n_k})) \\
 &< \varphi(M(x_{m_k-1}, x_{n_k}))
 \end{aligned}$$

where $M(x_{m_k-1}, x_{n_k})$

$$\begin{aligned} &= \max\{p(x_{m_k-1}, x_{n_k}), p(x_{m_k-1}, Tx_{m_k-1}), p(x_{n_k}, Sx_{n_k}), \frac{1}{2s} [\{p(Tx_{m_k-1}, x_{n_k}) + p(x_{m_k-1}, Sx_{n_k})\}]\} \\ &= \max\{p(x_{m_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), p(x_{n_k}, x_{n_k+1}), \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}]\} \\ &= p(x_{m_k-1}, x_{n_k}) \quad \text{or} \quad \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}] \end{aligned}$$

Suppose $M(x_{m_k-1}, x_{n_k}) = p(x_{m_k-1}, x_{n_k}) < \epsilon$

$$\begin{aligned} \text{But } \epsilon &\leq p(x_{m_k}, x_{n_k}) \leq sp(x_{m_k}, x_{n_k+1}) + sp(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1}) \\ &\leq sp(x_{m_k}, x_{n_k+1}) + s\eta \quad \text{where } \eta > 0 \ni p(x_{n_k+1}, x_{n_k}) < \eta \end{aligned} \tag{2.8.11}$$

$$\Rightarrow \epsilon - s\eta \leq sp(x_{m_k}, x_{n_k+1}) \tag{2.8.12}$$

Since φ is non decreasing

$$\begin{aligned} \therefore \varphi(\epsilon - s\eta) &\leq \varphi(sp(x_{m_k}, x_{n_k+1})) \\ &< \varphi(p(x_{m_k-1}, x_{n_k})) < \varphi(\epsilon) \end{aligned} \tag{2.8.13}$$

As φ is continuous and $\eta \rightarrow 0$ as $k \rightarrow \infty$, we get

$$\varphi(\epsilon) \leq \lim_{k \rightarrow \infty} \varphi(sp(x_{m_k}, x_{n_k+1})) \leq \lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k})) \leq \varphi(\epsilon)$$

$$\therefore \lim_{k \rightarrow \infty} \varphi(sp(x_{m_k}, x_{n_k+1})) = \varphi(\epsilon) = \lim_{k \rightarrow \infty} \varphi(p(x_{m_k-1}, x_{n_k}))$$

Suppose $M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}]$

On the other hand

$$\begin{aligned} &p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1}) \\ &\leq sp(x_{m_k}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) - p(x_{n_k-1}, x_{n_k-1}) + sp(x_{m_k+1}, x_{m_k}) \\ &\quad + sp(x_{m_k}, x_{n_k-1}) - p(x_{m_k}, x_{m_k}) \\ &\leq sp(x_{m_k}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) + sp(x_{m_k}, x_{n_k-1}) + sp(x_{m_k+1}, x_{m_k}) \\ &\leq 2sp(x_{m_k}, x_{n_k-1}) + 2s\eta \leq 2s\epsilon + 2s\eta \end{aligned}$$

where $p(x_{m_k+1}, x_{m_k}) \leq \eta$ and $p(x_{n_k}, x_{n_k-1}) \leq \eta$ for some $\eta > 0$ for large k

$$\therefore \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}] \leq \epsilon + \eta \tag{2.8.16}$$

Therefore,

$$M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s} [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}] \leq \epsilon + \eta$$

\therefore From (2.8.13), (2.8.15) and (2.8.16)

$$\begin{aligned} \varphi(\epsilon - s\eta) &\leq \varphi(sp(x_{m_k}, x_{n_k+1})) \\ &\leq \varphi(M(x_{m_k-1}, x_{n_k})) \\ &\leq \varphi(\epsilon + \eta) \end{aligned}$$

As φ is continuous and $\eta \rightarrow 0$ as $k \rightarrow \infty$, we get

$$\varphi(sp(x_{m_k}, x_{n_k+1})) = \varphi(\epsilon)$$

$\therefore \varphi(\epsilon)$

$$\begin{aligned} &\leq \beta(\varphi(M(x_{m_k-1}, x_{n_k-1})))\varphi(M(x_{m_k-1}, x_{n_k-1})) \\ &\leq \varphi(M(x_{m_k-1}, x_{n_k-1})) \\ &\leq \varphi(sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1})) \end{aligned} \tag{2.8.17}$$

Allowing $k \rightarrow \infty$, we get

$$\begin{aligned}
 & \varphi(\epsilon) \\
 & \leq \liminf_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k}))) \varphi(M(x_{m_k-1}, x_{n_k})) \\
 & \leq \limsup_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k}))) \varphi(M(x_{m_k-1}, x_{n_k})) \\
 & \leq \lim_{k \rightarrow \infty} (\varphi(sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) + p(x_{m_k}, x_{m_k-1})) \\
 & \therefore \varphi(\epsilon) \\
 & \leq \liminf_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k})))\varphi(\epsilon) \\
 & \leq \limsup_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k})))\varphi(\epsilon) \\
 & \leq \varphi(\epsilon) \\
 & \therefore \text{Either } \varphi(\epsilon) = 0 \text{ or } \lim_{k \rightarrow \infty} \beta(\varphi(M(x_{m_k-1}, x_{n_k}))) = 1 \text{ (since } \beta \in S) \tag{2.8.18}
 \end{aligned}$$

$$\Rightarrow \varphi(\epsilon) = 0$$

$\Rightarrow \epsilon = 0$, a contradiction.

Similarly we can discuss the other two cases.

$\therefore \{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. Now, we show that z is a fixed point of T . We consider

$$\begin{aligned}
 & \varphi(sp(x_{2n}, Tz)) \\
 & = \varphi(sp(Sx_{2n-1}, Tz)) \\
 & \leq \alpha(x_{2n-1}, z)\varphi(sp(Sx_{2n-1}, Tz)) \text{ (since } \alpha \text{ is continuous and } \alpha(z, z) > 1) \\
 & \leq \beta(\varphi(M(x_{2n-1}, z)))\varphi(M(x_{2n-1}, z)) \tag{2.8.19}
 \end{aligned}$$

But $M(x_{2n-1}, z)$

$$\begin{aligned}
 & = \max\{p(x_{2n-1}, z), p(x_{2n-1}, x_{2n}), p(z, Tz), \frac{1}{2s}[p(x_{2n}, z) + p(x_{2n-1}, Tz)]\} \\
 & = p(z, Tz) \text{ for large } n
 \end{aligned}$$

$$\therefore \varphi(sp(x_{2n}, Tz)) \leq \beta(\varphi(p(z, Tz)))\varphi(p(z, Tz)) \leq \varphi(p(z, Tz))$$

On letting $n \rightarrow \infty$,

we get

$$\begin{aligned}
 & \varphi(p(z, Tz)) \leq \varphi(sp(z, Tz)) \leq \beta(\varphi(p(z, Tz)))\varphi(p(z, Tz)) \leq \varphi(p(z, Tz)) \\
 & \Rightarrow \varphi(p(z, Tz)) = 0 \text{ or } \beta(\varphi(p(z, Tz))) = 1 \\
 & \Rightarrow p(z, Tz) = 0
 \end{aligned}$$

Then by lemma 2.3, $z = Tz$.

$\therefore z$ is a fixed point of T in X . Further $\varphi(sp(x_{2n+1}, Sz))$

$$\begin{aligned}
 & = \varphi(sp(Tx_{2n}, Sz)) \\
 & \leq \alpha(x_{2n}, z)\varphi(sp(Tx_{2n}, Sz)) \text{ (since } \alpha \text{ is continuous and } \alpha(z, z) > 1) \\
 & \leq \beta(\varphi(M(x_{2n}, z)))\varphi(M(x_{2n}, z)) \tag{2.8.20}
 \end{aligned}$$

But $M(x_{2n}, z)$

$$\begin{aligned}
 & = \max\{p(x_{2n}, z), p(x_{2n}, x_{2n+1}), p(z, Sz), \frac{1}{2s}[p(x_{2n+1}, z) + p(x_{2n}, Sz)]\} \\
 & = p(z, Sz) \text{ for large } n
 \end{aligned}$$

$$\therefore \varphi(sp(x_{2n+1}, Sz)) \leq \beta(\varphi(p(z, Sz)))\varphi(p(z, Sz)) \leq \varphi(p(z, Sz))$$

On letting $n \rightarrow \infty$,

we get

$$\varphi(p(z, Sz)) \leq \varphi(sp(z, Tz)) \leq \beta(\varphi(p(z, Sz)))\varphi(p(z, Sz)) \leq \varphi(p(z, Sz))$$

$$\Rightarrow \varphi(p(z, Tz)) = 0 \text{ or } \beta(\varphi(p(z, Tz))) = 1$$

$$\Rightarrow p(z, Sz) = 0$$

Then by lemma 2.3, $z = Sz$.

$\therefore z$ is a fixed point of S in X .

Hence z is a common fixed point of S and T in X

Now we give an example in support of theorem 2.8

Example 2.9. (K.P.R.Sastry et.al.[33]) Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{10}\}$ with usual ordering.

Define

$$p(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0, 1\} \\ |x - y| & \text{if } x, y \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\} \\ 4 & \text{otherwise} \end{cases}$$

Clearly, (X, \leq, p) is a partially ordered partial b - metric space with coefficient $s = \frac{3}{4}$ (P.Kumam et.al [19])

Define $S, T : X \rightarrow X$ by

$$T1 = T\frac{1}{3} = T\frac{1}{5} = T\frac{1}{7} = T\frac{1}{9} = 0 ; T0 = T\frac{1}{2} = T\frac{1}{4} = T\frac{1}{6} = T\frac{1}{8} = T\frac{1}{10} = \frac{1}{4}$$

$$\therefore A = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\} \Rightarrow T(A) = \frac{1}{4}$$

and

$$B = \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\} \Rightarrow T(B) = 0$$

$$\therefore T(X) = \{0, \frac{1}{4}\}$$

$$\text{and } Sx = \frac{1}{4} \forall x \in X \Rightarrow S(A) = S(B) = \frac{1}{4}$$

$$\beta(t) = \begin{cases} \frac{1}{1+t} & \text{if } t \in (0, \infty) \\ 0 & \text{if } t = 0 \end{cases}$$

$$\alpha(x, y) = 2 \forall x, y \in X \text{ and } \varphi(t) = 2t \forall t \geq 0$$

For $x, y \in X$ and $p(x, y) \neq 0 \Rightarrow x \neq y$, then following are the cases

(i) For $x, y \in A \Rightarrow Sx = Ty = \frac{1}{4} \Rightarrow sp(Sx, Ty) = 0$

$$\therefore \alpha(x, y)\varphi(sp(Sx, Ty)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y))$$

(ii) For $x, y \in B \Rightarrow Sx = \frac{1}{4}, Ty = 0 \Rightarrow sp(Sx, Ty) = (\frac{8}{3})(\frac{1}{4}) = \frac{2}{3} \Rightarrow \varphi(sp(Sx, Ty)) = \frac{4}{3}$

where $M(x, y) = 4 \Rightarrow \varphi(\beta(M(x, y)))\varphi(M(x, y)) = 4(\frac{4}{5}) = \frac{16}{5}$

$$\therefore \alpha(x, y)\varphi(sp(Sx, Ty)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y))$$

(iii) For $x \in A, y \in B \Rightarrow Sx = \frac{1}{4}, Ty = 0 \Rightarrow sp(Sx, Ty) = (\frac{8}{3})(\frac{1}{4}) = \frac{2}{3}$

$$\Rightarrow \varphi(sp(Sx, Ty)) = \frac{4}{3} \text{ where } M(x, y) = 4 \Rightarrow \varphi(\beta(M(x, y)))\varphi(M(x, y)) = 4(\frac{4}{5}) = \frac{16}{5}$$

Ref

19. Kumam, P, Dung, N. V, Hang, V. T. L.: Some equivalences between cone b-metric spaces and b-metric spaces. *Abstr. Appl. Anal.* 2013 (2013), 18.

$$\therefore \alpha(x, y)\varphi(sp(Sx, Ty)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y))$$

(iv) For $x \in A, y \in B \Rightarrow Tx = Sy = \frac{1}{4} \Rightarrow sp(Tx, Sy) = 0 \Rightarrow \varphi(sp(Tx, Sy)) = 0$

$$\therefore \alpha(x, y)\varphi(sp(Tx, Sy)) \leq \beta(\varphi(M(x, y)))\varphi(M(x, y))$$

Since $T(\frac{1}{4}) = S(\frac{1}{4}) = \frac{1}{4}$ and $\alpha(\frac{1}{4}, T\frac{1}{4}) = 2 > 1$

Therefore $\frac{1}{4} \in X$ is a fixed point.

The hypothesis and conclusions of 2.8 satisfied.

Open Problem: Are the theorems 2.6 and 2.8 true for partial b - metric spaces with coefficient $s \geq 1$ when conditions on α removed?

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