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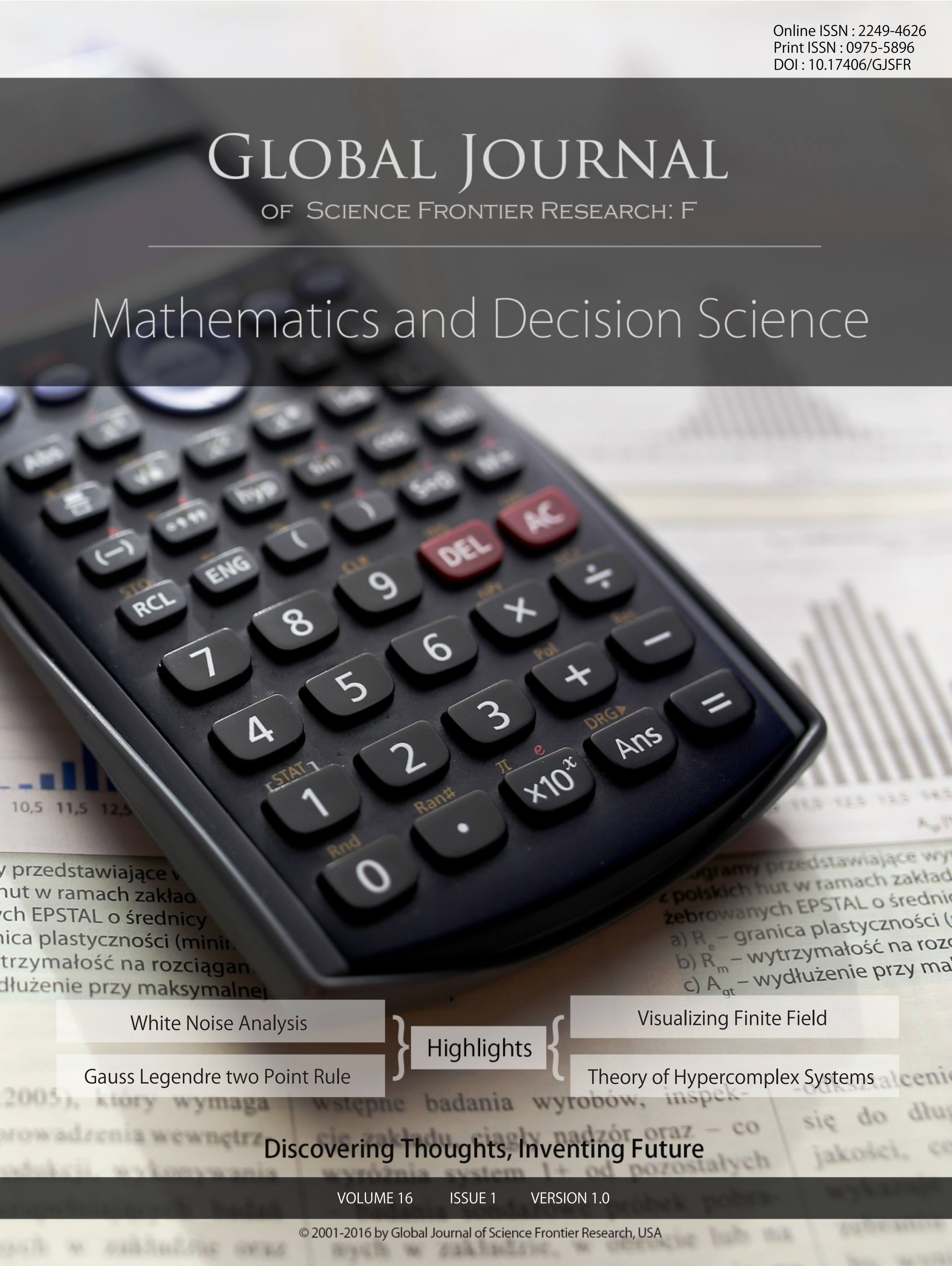

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Mathematics and Decision Science

Highlights

- White Noise Analysis
- Gauss Legendre two Point Rule
- Visualizing Finite Field
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Discovering Thoughts, Inventing Future

VOLUME 16 ISSUE 1 VERSION 1.0

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Lyapunov-Type Inequality for Fractional order Difference Equations

By Meihong Cui, Jingxu Xin, Xianmei Huang & Chengmin Houx

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Abstract- In this paper, we obtain some Lyapunov-type inequalities for a class of fractional order difference equations with homogeneous boundary value conditions, the results of this paper are new and generalize some early results in the literature.

Keywords: lyapunov-type inequalities, fractional order difference equations, boundary value problem.

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1. S.S. Cheng, *Lyapunov inequalities for differential and difference equations*, Fasc. Math. 23(1991) 25-41.

Lyapunov-Type Inequality for Fractional order Difference Equations

Meihong Cui [α], Jingxu Xin [σ], Xianmei Huang [ρ] & Chengmin Houx [ω]

Abstract- In this paper, we obtain some Lyapunov-type inequalities for a class of fractional order difference equations with homogeneous boundary value conditions, the results of this paper are new and generalize some early results in the literature.

Keywords: *lyapunov-type inequalities, fractional order difference equations, boundary value problem.*

I. INTRODUCTION

Lyapunov inequality plays an important role both in the study of various properties of solutions of differential and difference equations such as oscillation theory, disconjugacy and eigenvalue problems and in the application in many directions of mathematics research areas. There have been many proofs and generalizations as well as improvements in this topic. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey paper [1] by Cheng. For other related references, we refer the reader to the few papers [2-9]. In 1983, Cheng [10] first obtained the following Lyapunov inequality:

$$\mathfrak{F}(b-a)\sum_{n=a}^{b-2} q(n) \geq 4, \tag{1.1}$$

where $a, b \in \mathbf{Z}, q(n)$ is a non-negative function defined on the set $\{a, a+1, \cdots, b\}$ and

$$\mathfrak{F}(m) = \begin{cases} \frac{m^2-1}{m}, & \text{if } m-1 \text{ is even}, \\ m, & \text{if } m-1 \text{ is odd}, \end{cases} \tag{1.2}$$

if the second-order difference equation

$$\Delta^2 x(n) + q(n)x(n+1) = 0, \tag{1.3}$$

has a real solution $x(n)$ such that

$$x(a) = x(b) = 0, x(n) \not\equiv 0, n \in \mathbf{Z}[a, b], \tag{1.4}$$

where and in the sequel, $a, b \in \mathbf{N}, \mathbf{Z}[a, b] = \{a, a+1, a+2, ..., b-1, b\}; c, d \in \mathbf{R}, d = c+N, N \in \mathbf{N}, [c, d]_{\mathbf{N}_c} = \{c, c+1, c+2, ..., d-1, d\}$. The constant 4 in (1.1) cannot be replaced by a larger number, either. For more discrete Lyapunov-type inequalities, we refer the reader to Cheng [11-

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13], Clark and Hinton [14,15], Guseinov and Kaymakcalan [16], Lin and Yang [17] and Zhang and Tang [18].

In 2012, Zhang and Tang [19] considered the following even order difference equation

$$\Delta^{2k}x(n)+(-1)^{k-1}q(n)x(n+1)=0, \tag{1.5}$$

under the following boundary conditions:

$$\Delta^{2i}x(a)=\Delta^{2i}x(b)=0, i=0,1,...,k-1; x(n)\not\equiv 0, n\in\mathbf{Z}[a,b]. \tag{1.6}$$

They obtained the following main result:

Theorem A. Assume that $k\in\mathbf{N}$ *and* $q(n)$ *is a real-valued function on* $\mathbf{Z}$*. If (1.5) has a solution* $x(n)\not\equiv 0$ *satisfying the boundary value conditions (1.6), then*

$$\sum_{n=a}^{b-1}[|q(n)|(n-a+1)(b-n-1)]\geq\frac{2^{3(k-1)}}{(b-a)^{2k-3}}. \tag{1.7}$$

In this paper, we will consider the following fractional order difference equation

$$\Delta^{\nu}_{\nu-2k+a}x(n)+(-1)^{k-1}q(n)x(n+\nu-2k+1)=0, \tag{1.8}$$

where $k\in\mathbf{N}, n\in\mathbf{Z}, \nu\in\mathbf{R}^{+}$ satisfy $2k-1<\nu\leq 2k$ and $q(n)$ is a real-valued function defined on $\mathbf{Z}$. In this work, we establish two discrete Lyapunov-type inequalities for (1.8) under the following boundary conditions:

$$x(a+\nu-2k)=x(b+\nu-2k)=0, \tag{1.9}$$

$$\Delta^{\nu-2i}_{a+\nu-2k}x(n)|_{n=a}=\Delta^{\nu-2i}_{a+\nu-2k}x(n)|_{n=b}=0, i=1,...,k-1; x(n+\nu-2k)\not\equiv 0, n\in\mathbf{Z}[a,b]. \tag{1.10}$$

When $\nu=2, k=1$, (1.8)(1.9) reduce to (1.3)(1.4). When $\nu=2k$, (1.8)(1.9) (1.10) reduce to (1.5)(1.6).

II. Preliminaries

In this section, we collect some basic definitions and lemmas for manipulating discrete fractional operators. These can be found in the reference [20].

For any integer β, let $\mathbf{N}_{\beta}=\{\beta,\beta+1,\beta+2,...\}$. We define $t^{\underline{\nu}}:=\frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}}=0$.

Deffinition 2.1. The ν-th fractional sum of f for $\nu>0$ is defined by

$$\Delta_a^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{\underline{\nu-1}}f(s),$$

for $t\in\mathbf{N}_{a+\nu}$. Also, we define the trivial sum by $\Delta_a^{-0}f(t):=f(t)$, for $t\in N_a$. We also define the ν-th fractional difference for $\nu>0$ by $\Delta_a^{\nu}f(t):=\Delta^{N}\Delta_a^{\nu-N}f(t)$,where $t\in\mathbf{N}_{a+N-\nu}$ and $N\in\mathbf{N}$ is chosen so that $0\leq N-1<\nu\leq N$.

Lemma 2.2. Let $f:\mathbf{N}_a\longrightarrow\mathbf{R}$ *be given and suppose* $k\in\mathbf{N}_0$ *and* $\nu>0$*. Then for* $t\in\mathbf{N}_{a+\nu}$*,*

$$\Delta_a^{-\nu}\Delta^{k}f(t)=\Delta_a^{k-\nu}f(t)-\sum_{j=0}^{k-1}\frac{\Delta^{j}f(a)}{\Gamma(\nu-k+j+1)}(t-a)^{\underline{\nu-k+j}}.$$

Ref

19. Q. Zhang, X.H. Tang, *Lyapunov-type inequalities for even order difference equations*, Appl. Math. Lett. 25 (2012) 1830-1834.

Moreover, if $\mu > 0$ *with* $M - 1 < \mu \leq M$*, then for* $t \in \mathbf{N}_{a+M-\mu+\nu}$*,*

$$\Delta_{a+M-\mu}^{-\nu}\Delta_a^{\mu}f(t) = \Delta_a^{\mu-\nu}f(t) - \sum_{j=0}^{M-1}\frac{\Delta_a^{j-M+\mu}f(a+M-\mu)}{\Gamma(\nu-M+j+1)}(t-a-M+\mu)^{\underline{\nu-M+j}}.$$

III. THE MAIN RESULTS

Theorem 3.1. *Let* $\nu \in (1,2]$*. Assume that* $x(n+\nu-2)$ *is a real valued function on* $\mathbf{Z}[a,b]$ *satisfy (1.8)(1.9) and* $x(n+\nu-2) \not\equiv 0$ *for* $n \in \mathbf{Z}[a,b]$*. Then,*

$$\widetilde{\mathfrak{F}}(b-a)\sum_{n=a}^{b-2}|q(n)| \geq 1, \tag{3.1}$$

where $a, b \in \mathbf{Z}$ *and*

$$\widetilde{\mathfrak{F}}(m) = \frac{1}{\Gamma(\nu)}\begin{cases} \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}(\frac{m-1}{2}+\nu-2)^{\underline{\nu-1}}}{(m+\nu-2)^{\underline{\nu-1}}}, & \text{if}\, m-1 \text{ is even,} \\ \frac{((\frac{m}{2}+\nu-2)^{\underline{\nu-1}})^2}{(m+\nu-2)^{\underline{\nu-1}}}, & \text{if}\, m-1 \text{ is odd.} \end{cases} \tag{3.2}$$

Proof. By Lemma 2.2, we have

$$x(n) = -\Delta_a^{-\nu}q(n)x(n+\nu-1) + c_1(n-a)^{\underline{\nu-1}} + c_2(n-a)^{\underline{\nu-2}}.$$

Since $x(a+\nu-2) = x(b+\nu-2) = 0$, we get

$$x(n) = \sum_{s=a}^{b-2}G(n,s)q(s)x(s+\nu-1) = -\sum_{s=a}^{b-2}G(n,s)\Delta_{a+\nu-2}^{\nu}x(s), \tag{3.3}$$

where

$$G(n,s) = \frac{1}{\Gamma(\nu)}\begin{cases} \frac{(b+\nu-s-3)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}} - (n-s-1)^{\underline{\nu-1}}, & a \leq s < n-\nu+1 \leq b-2, \\ \frac{(b+\nu-s-3)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}}, & a \leq n-\nu+1 \leq s \leq b-2. \end{cases} \tag{3.4}$$

One can see that $\Delta_s G(n,s) < 0$, for $a \leq n-\nu+1 \leq s \leq b-2$ and $\Delta_s G(n,s) > 0$, for $a \leq s < n-\nu+1 \leq b-2$. Indeed, for $a \leq s < n-\nu+1 \leq b-2$, we have $\Delta_s G(n,s) = -(\nu-1)\frac{(b+\nu-s-4)^{\underline{\nu-2}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}} + (\nu-1)(n-s-2)^{\underline{\nu-2}}$.

Thus, $\Delta_s G(n,s) > 0$, if and only if

$$\frac{(n-s-2)^{\underline{\nu-2}}(b+\nu-a-2)^{\underline{\nu-1}}}{(n-a)^{\underline{\nu-1}}(b+\nu-s-4)^{\underline{\nu-2}}} > 1.$$

The inequality follows from the fact that $t^{\underline{\alpha}}$ is increasing and $t^{\underline{-\alpha}}$ is decreasing if $0 < \alpha \leq 1$. Note that $G(n, b-2) > 0$ and

$$\begin{aligned} G(n,a) &= \frac{1}{\Gamma(\nu)}\left(\frac{(b+\nu-a-3)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}} - (n-a-1)^{\underline{\nu-1}}\right) \\ &= \frac{1}{\Gamma(\nu)}\frac{\Gamma(n-a)}{\Gamma(n-a-\nu+1)}\left(\frac{(b-a-1)(n-a)}{(b-a+\nu-2)(n-a+1-\nu)} - 1\right) \\ &= \frac{1}{\Gamma(\nu)}\frac{\Gamma(n-a)}{\Gamma(n-a-\nu+1)}\frac{(\nu-1)(b-n+\nu-2)}{(b-a+\nu-2)(n-a+1-\nu)} > 0, \end{aligned}$$

we can get

$$G(n,s)>0,\ \text{for}\ (n,s)\in[a+\nu-1,b+\nu-3]_{\mathbf{N}_{a+\nu-1}}\times\mathbf{Z}[a,b-2].$$

Thus

$$\max_{s\in\mathbf{Z}[a,b-2]}G(n,s)=G(n,n-\nu+1),n\in[a+\nu-1,b+\nu-3]_{\mathbf{N}_{a+\nu-1}}.$$

$$|x(n)|\leq\frac{(b+2\nu-n-4)^{\underline{\nu-1}}(n-a)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=a}^{b-2}|\Delta_{a+\nu-2}^{\nu}x(s)|,n\in[a+\nu-1,b+\nu-3]_{\mathbf{N}_{a+\nu-1}}$$

and

$$|x(n+\nu-1)|\leq\frac{(b+\nu-n-3)^{\underline{\nu-1}}(n+\nu-a-1)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)}\sum_{s=a}^{b-2}|\Delta_{a+\nu-2}^{\nu}x(s)|,n\in\mathbf{Z}[a,b-2].\quad(3.5)$$

Next, we denote $g(n)=(b+\nu-n-3)^{\underline{\nu-1}}(n+\nu-a-1)^{\underline{\nu-1}}$, then by computing, we have

$$\Delta g(n)=\frac{\Gamma(b+\nu-n-3)\Gamma(n+\nu-a)}{\Gamma(b-n-1)\Gamma(n-a+2)}(\nu-1)(b-2n+a-3).$$

If $b-a-1$ is even, let $\Delta g(n)=0$, then $n=\frac{b-a-1}{2}+a-1$.

$$\max_{n\in\mathbf{Z}[a,b-2]}g(n)=g(\frac{b-a-1}{2}+a-1)=(\frac{b-a+1+\nu-2}{2})^{\underline{\nu-1}}(\frac{b-a-1+\nu-2}{2})^{\underline{\nu-1}}.\quad(3.6)$$

If $b-a-1$ is odd, we obtain $\Delta g(n)>0$, for $n<\frac{b-a-1}{2}+a-1$; $\Delta g(n)<0$, for $n>\frac{b-a-1}{2}+a-1$. Therefore,

$$\max_{n\in\mathbf{Z}[a,b-2]}g(n)=\max\{g(\frac{b-a}{2}+a-1),g(\frac{b-a}{2}+a-2)\}.$$

Since

$$g(\frac{b-a}{2}+a-1)=g(\frac{b+a}{2}-1)=((\frac{b-a}{2}+\nu-2)^{\underline{\nu-1}})^2,$$

$$g(\frac{b-a}{2}+a-2)=g(\frac{b+a}{2}-2)=(\frac{b-a}{2}+\nu-3)^{\underline{\nu-1}}(\frac{b-a}{2}+\nu-1)^{\underline{\nu-1}},$$

$$\begin{aligned}g(\frac{b-a}{2}+a-1)-g(\frac{b-a}{2}+a-2)&=((x+\nu-2)^{\underline{\nu-1}})^2-(x+\nu-3)^{\underline{\nu-1}}(x+\nu-1)^{\underline{\nu-1}}\\&=\frac{(x+\nu-2)\Gamma^2(x+\nu-2)}{x(x-1)^2\Gamma(x-1)}(\nu-1)>0,\end{aligned}$$

where, $x=\frac{b-a}{2}$, we see that

$$\max_{n\in\mathbf{Z}[a,b-2]}g(n)=g(\frac{b-a}{2}+a-1)=((\frac{b-a}{2}+\nu-2)^{\underline{\nu-1}})^2.\quad(3.7)$$

Note that (3.5) and by (3.6)(3.7), we obtain

$$|x(n+\nu-1)|\leq\widetilde{\mathfrak{F}}(b-a)\sum_{s=a}^{b-2}|\Delta_a^{\nu}x(s)|=\widetilde{\mathfrak{F}}(b-a)\sum_{s=a}^{b-2}|q(s)||x(s+\nu-1)|.$$

Furthermore,

$$\sum_{n=a}^{b-2}|q(n)||x(n+\nu-1)| \leq \widetilde{\mathfrak{F}}(b-a)\sum_{s=a}^{b-2}|q(n)|\sum_{s=a}^{b-2}|q(s)||x(s+\nu-1)|,$$

it shows that (3.1) holds.

Remark 3.2. The inequality (3.1) is sharp. Indeed, since

$$\begin{aligned}
\Delta\widetilde{\mathfrak{F}}(m) &= \frac{1}{\Gamma(\nu)}\left(\frac{((\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}})^2}{(m+\nu-1)^{\underline{\nu-1}}} - \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}(\frac{m-1}{2}+\nu-2)^{\underline{\nu-1}}}{(m+\nu-2)^{\underline{\nu-1}}}\right) \\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}} \\
&\quad \times\left((\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-2)^{\underline{\nu-1}} - (\frac{m-1}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}\right) \\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}} \\
&\quad \times\left(\frac{\Gamma(\frac{m+1}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m+1}{2})\Gamma(m)} - \frac{\Gamma(\frac{m-1}{2}+\nu-1)\Gamma(m+\nu)}{\Gamma(\frac{m-1}{2})\Gamma(m+1)}\right) \\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\frac{\Gamma(\frac{m-1}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m-1}{2})\Gamma(m)}(\frac{2(\nu-1)}{m-1}-\frac{\nu-1}{m})>0
\end{aligned}$$

for $m-1$ is even,

$$\begin{aligned}
\Delta\widetilde{\mathfrak{F}}(m) &= \frac{1}{\Gamma(\nu)}\left(\frac{(\frac{m+2}{2}+\nu-2)^{\underline{\nu-1}}(\frac{m}{2}+\nu-2)^{\underline{\nu-1}}}{(m+\nu-1)^{\underline{\nu-1}}} - \frac{((\frac{m}{2}+\nu-2)^{\underline{\nu-1}})^2}{(m+\nu-2)^{\underline{\nu-1}}}\right) \\
&= \frac{(\frac{m}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}} \\
&\quad \times\left((\frac{m+2}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-2)^{\underline{\nu-1}} - (\frac{m}{2}+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}\right) \\
&= \frac{(\frac{m}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\left(\frac{\Gamma(\frac{m+2}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m+2}{2})\Gamma(m)} - \frac{\Gamma(\frac{m}{2}+\nu-1)\Gamma(m+\nu)}{\Gamma(\frac{m}{2})\Gamma(m+1)}\right) \\
&= \frac{(\frac{m+1}{2}+\nu-2)^{\underline{\nu-1}}}{\Gamma(\nu)(m+\nu-2)^{\underline{\nu-1}}(m+\nu-1)^{\underline{\nu-1}}}\frac{\Gamma(\frac{m}{2}+\nu-1)\Gamma(m+\nu-1)}{\Gamma(\frac{m}{2})\Gamma(m)}\frac{\nu-1}{m} > 0
\end{aligned}$$

for $m-1$ is odd. We see that $\widetilde{\mathfrak{F}}$ is a strictly increasing function of m. If

$$\widetilde{\mathfrak{F}}(b-a)\sum_{n=a}^{b-2}|q(n)| < 1,$$

then (1.8) cannot have a nontrivial solution $x(n+\nu-1)$ defined on the set $\mathbf{Z}[c,d]$ which satisfies $x(c+\nu-2) = x(d+\nu-2) = 0$, where $a-1 \leq c-1 < d+1 \leq b+1, \nu \in (1,2]$. Otherwise,

$$\sum_{n=a}^{b-2}|q(n)| \geq \sum_{n=c}^{d-2}|q(n)| \geq \frac{1}{\widetilde{\mathfrak{F}}(d-c)} \geq \frac{1}{\widetilde{\mathfrak{F}}(b-a)} > \sum_{n=a}^{b-2}|q(n)|$$

which is a contradiction.

Remark 3.3. In the Theorem 3.1, if $\nu = 2$ and $q(n)$ is a non-negative function defined on the set $\mathbf{Z}[a,b]$, then $\widetilde{\mathfrak{F}}(m) = \mathfrak{F}(m)$, i.e., (1.1) holds. Therefore Theorem 3.1 contains the result of reference [10].

Theorem 3.4. Let $\nu \in (1,2]$. Assume that $x(n+\nu-2)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.8)(1.9) and $x(n+\nu-2) \not\equiv 0$ for $\mathbf{Z}[a,b]$. Then,

$$|x(n+\nu-1)| \le \frac{(b+\nu-n-3)^{\underline{\nu-1}}(n+\nu-a-1)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=a}^{b-2} |\Delta^{\nu}_{a+\nu-2}x(s)|, \tag{3.8}$$

$$\sum_{n=a}^{b-2} |x(n+\nu-1)| \le \sum_{n=a}^{b-2} \frac{(b+\nu-n-3)^{\underline{\nu-1}})(n+\nu-a-1)^{\underline{\nu-1}}}{(b+\nu-a-2)^{\underline{\nu-1}}\Gamma(\nu)} \sum_{s=a}^{b-2} |\Delta^{\nu}_{a+\nu-2}x(s)|, \tag{3.9}$$

and

$$\sum_{n=a}^{b-2} |x(n+\nu-1)|^2 \le \sum_{n=a}^{b-2}\sum_{s=a}^{b-2} [G(n+\nu-1,s)]^2 \sum_{s=a}^{b-2} |\Delta^{\nu}_{a+\nu-2}x(s)|^2. \tag{3.10}$$

Proof. By (3.5), we can immediately get (3.8) and (3.9). Using Cauchy inequality and by (3.3), we may easily see that (3.10) holds.

Corollary 3.5. Let $\nu = 2$. Assume that $x(n)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.3)(1.4) and $x(n) \not\equiv 0$ for $\mathbf{Z}[a,b]$. Then,

$$|x(n)| \le \frac{(b-n)(n-a)}{(b-a)} \sum_{s=a}^{b-2} |\Delta^2 x(s)|, \tag{3.11}$$

$$\sum_{n=a}^{b-2} |x(n)| \le \sum_{n=a}^{b-2}\sum_{s=a}^{b-2} G(n,s)|\Delta^2 x(s)| \le \frac{(b-a)^2}{8} \sum_{s=a}^{b-2} |\Delta^2 x(s)|. \tag{3.12}$$

Proof.

Since

$$G(n,s) = \frac{1}{b-a}\begin{cases} (s+1-a)(b-n), & a \le s < n-1 \le b-2, \\ (b-s-1)(n-a), & a \le n-1 \le s \le b-2, \end{cases}$$

it follows from above form that

$$|x(n)| \le \sum_{s=a}^{b-2} G(n,s)|\Delta^2 x(s)| \le \frac{(n-a)(b-n)}{b-a} \sum_{s=a}^{b-2} |\Delta^2 x(s)|.$$

Furthermore

$$\begin{aligned}
\sum_{n=a}^{b-2} |x(n)| &\le \sum_{n=a}^{b-2}\sum_{s=a}^{b-2} G(n,s)|\Delta^2 x(s)| \\
&\le \sum_{s=a}^{b-2}\sum_{n=a}^{b-1} G(n,s)|\Delta^2 x(s)| \\
&\le \sum_{s=a}^{b-2} \Big(\sum_{n=a}^{s+1} G(n,s) + \sum_{n=s+2}^{b-1} G(n,s)\Big)|\Delta^2 x(s)| \\
&= \frac{1}{b-a} \sum_{s=a}^{b-2} \big((b-s-1)(1+2+\cdots+(s-a+1)\big)
\end{aligned}$$

Ref

10. S. S. Cheng, *A discrete analogue of the inequality of Lyapunov*, Hokkaido Math. J. 12 (1983) 105-112.

$$
\begin{aligned}
&+(s+1-a)(1+2+\cdots+(b-s-2)))|\Delta^2 x(s)| \\
=\ & \frac{1}{b-a}\sum_{s=a}^{b-2}((b-s-1)(1+2+\cdots+s-a) \\
&+(s+1-a)(1+2+\cdots+b-s-1))|\Delta^2 x(s)| \\
=\ & \frac{1}{2}\sum_{s=a}^{b-2}(b-s-1)(s+1-a)|\Delta^2 x(s)| \\
\leq\ & \frac{(b-a)^2}{8}\sum_{s=a}^{b-2}|\Delta^2 x(s)|.
\end{aligned}
$$

Theorem 3.6. *Let $\nu \in (2k-1, 2k]$. Assume that $x(n+\nu-2)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.8)-(1.10) and $x(n+\nu-2) \not\equiv 0$ for $n \in \mathbf{Z}[a,b]$. Then,*

$$
\sum_{n=a}^{b-2}|q(n)|(b+\nu-2k-n-1)^{\underline{\nu-2k+1}}(n+\nu-2k+1-a)^{\underline{\nu-2k+1}} \geq (\frac{2^{3(k-1)}}{(b-a)^2})^{k-1}(b+\nu-2k-a)^{\underline{\nu-2k+1}}. \tag{3.13}
$$

Proof. Assume that $x(n+\nu-2)$ is a real valued function on $\mathbf{Z}[a,b]$ satisfy (1.8)-(1.10) and $x(n+\nu-2) \not\equiv 0$ for $n \in \mathbf{Z}[a,b]$. Since conditions (1.10), by Corollary 3.3, we have

$$
\begin{aligned}
\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu-2(k-1)}x(s)| \ \leq\ & \frac{(b-a)^2}{8}\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu-2(k-1)+2}x(s)| \\
\leq\ & (\frac{(b-a)^2}{8})^2\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu-2(k-3)}x(s)| \\
\leq\ & \cdots \\
\leq\ & (\frac{(b-a)^2}{8})^{k-1}\sum_{n=a}^{b-2}|\Delta_{\nu-2k+a}^{\nu}x(s)|. \qquad (3.14)
\end{aligned}
$$

Denote $\beta = \nu - 2(k-1)$, then $\beta \in (1,2], \Delta_{\nu-2k+a}^{\nu-2(k-1)}x(n) = \Delta_{\beta-2+a}^{\beta}x(n)$. By Theorem 3.4, we have

$$
|x(n+\beta-1)| \leq \frac{(b+\beta-s-3)^{\underline{\beta-1}}(n+\beta-a-1)^{\underline{\beta-1}}}{(b+\beta-a-2)^{\underline{\beta-1}}}\sum_{s=a}^{b-2}|\Delta_{a+\beta-2}^{\beta}x(s)|,
$$

combine (3.14), we get

$$
\begin{aligned}
|x(n+\nu-2k+1)| \ \leq\ & (\frac{(b-a)^2}{8})^{k-1}\frac{(b+\beta-s-3)^{\underline{\beta-1}}(n+\beta-a-1)^{\underline{\beta-1}}}{(b+\beta-a-2)^{\underline{\beta-1}}}\sum_{s=a}^{b-2}|\Delta_{a+\nu-2k}^{\nu}x(s)| \\
=\ & (\frac{(b-a)^2}{8})^{k-1}\frac{(b+\nu-2k-n-1)^{\underline{\nu-2k+1}}(n+\nu-2k-a+1)^{\underline{\nu-2k+1}}}{(b+\nu-2k-a)^{\underline{\nu-2k+1}}} \\
&\times\sum_{s=a}^{b-2}|q(s)x(s+\nu-2k+1)|.
\end{aligned}
$$

$$\sum_{n=a}^{b-2} q(n)|x(n+\nu-2k+1)| \leq \sum_{n=a}^{b-2} q(n)(\frac{(b-a)^2}{8})^{k-1}\frac{(b+\nu-2k-n-1)^{\underline{\nu-2k+1}}(n+\nu-2k-a+1)^{\underline{\nu-2k+1}}}{(b+\nu-2k-a)^{\underline{\nu-2k+1}}}$$

$$\times \sum_{s=a}^{b-2} |q(s)x(s+\nu-2k+1)|.$$

it follows that (3.13) holds.

Corollary 3.7. *Let $\nu = 2k$. Assume that $x(n)$ is a real valued function on* $\mathbf{Z}[a,b]$ *satisfy (1.8)-(1.10) and $x(n) \not\equiv 0$ for $\mathbf{Z}[a,b]$. Then,*

$$\sum_{n=a}^{b-2} |q(n)|(b-n-1)(n+1-a) \geq \frac{2^{3(k-1)}}{(b-a)^{2k-3}}.$$

Remark 3.8. The Corollary 3.7 is the Theorem A. Therefore our results contains the main results of reference [19].

References Références Referencias

1. S.S. Cheng, *Lyapunov inequalities for differential and difference equations*, Fasc. Math. 23(1991) 25-41.
2. A.M. Liapunov, *Problme gnral de la stabilit du mouvement*, Ann. Fac. Sci. Univ. Toulouse 2 (1907) 203-407.
3. D. Cakmak, *Lyapunov-type integral inequalities for certain higher order differential equations,* Appl. Math. Comput. 216 (2010) 368-373.
4. X. He, X.H. Tang, *Lyapunov-type inequalities for even order differential equations,* Commun Pure Appl. Anal. 11 (2) (2012) 465-473.
5. L.Q. Jiang, Z. Zhou, *Lyapunov inequality for linear Hamiltonian systems on time scales,* J. Math. Anal. Appl. 310 (2005) 579-593.
6. X. Wang, *Stability criteria for linear periodic Hamiltonian systems,* J. Math. Anal. Appl. 367 (2010) 329-336.
7. X. Yang, *On Liapunov-type inequality for certain higher-order differential equations,* Appl. Math. Comput. 134 (2-3) (2003) 307-317.
8. X. Yang, *On inequalities of Lyapunov type,* Appl. Math. Comput. 134 (2-3) (2003) 293-300.
9. X. Yang, K. Lo, *Lyapunov-type inequality for a class of even-order differential equations,* Appl. Math. Comput. 215 (2010) 3884-3890.
10. S. S. Cheng, *A discrete analogue of the inequality of Lyapunov,* Hokkaido Math. J. 12 (1983) 105-112.
11. S.S. Cheng, L.Y. Hsieh, *On discrete analogue of Lyapunovs inequality: addendum,* Tamkang J. Math. 20 (1989) 333-340.
12. S.S. Cheng, *A sharp condition for the ground state of difference equation,* Appl. Anal. 34 (1989) 105-109.
13. S.S. Cheng, L.Y. Hsieh, D. Chao, *Discrete Lyapunov inequality conditions for partial difference equations,* Hokkaido Math. J. 19 (1990) 229-239.
14. S. Clark, D.B. Hinton, *Discrete Lyapunov inequalities for linear Hamiltonian systems,* Math. Inequal. Appl. 1 (1998) 201-209.
15. S. Clark, D.B. Hinton, *Discrete Lyapunov inequalities,* Dynam. Systems Appl. 8 (1999) 369-380.
16. G.Sh. Guseinov, B. Kaymakcalan, *Lyapunov inequalities for discrete linear Hamiltonian systems,* Comput. Math. Appl. 45 (2003) 1399-1416.

Ref

19. Q. Zhang, X.H. Tang, *Lyapunov-type inequalities for even order difference equations,* Appl. Math. Lett. 25 (2012) 1830-1834.

Notes

17. S.H. Lin, G.S. Yang, *On discrete analogue of Lyapunovs inequality,* Tamkang J. Math. 20(1989) 169-186.
18. Q. Zhang, X.H. Tang, *Lyapunov inequalities and stability for discrete linear Hamiltonian system,* Appl. Math. Comput. 218 (2011) 574-582.
19. Q. Zhang, X.H. Tang, *Lyapunov-type inequalities for even order difference equations,* Appl. Math. Lett. 25 (2012) 1830-1834.
20. M.T. Holm, *The Theory of Discrete Fractional Calculus: Development and Application,* DigitalCommons@University of Nebraska-Lincoln, 2011.
21. M. Unal, D. Cakmak, A. Tiryaki, *A discrete analogue of Lyapunov-type inequalities for nonlinear systems,* Comput. Math. Appl. 55 (2008) 2631-2642.

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A Construction of Non-Gaussian White Noise Analysis using the Theory of Hypercomplex Systems

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Abstract- In this paper, we present a generalization of white noise analysis to the case of non-Gaussian measures. For this purpose, we use a biorthogonal approach in which instead of the exponentials the characters of commutative hypercomplex systems are employed. Moreover, we construct the elements of Wick calculus in a non-Gaussian setting.

Keywords: white noise analysis, biorthogonal approach, hypercomplex systems, wick calculus.

GJSFR-F Classification : MSC 2010: 30G35

A Construction of Non-Gaussian White Noise Analysis using the Theory of Hypercomplex Systems

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Abstract- In this paper, we present a generalization of white noise analysis to the case of non-Gaussian measures. For this purpose, we use a biorthogonal approach in which instead of the exponentials the characters of commutative hypercomplex systems are employed. Moreover, we construct the elements of Wick calculus in a non-Gaussian setting.

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I. INTRODUCTION

White noise analysis appeared in 1975 after the work of T. Hida [25] and then has been actively developed in a series of articles and books. For a sufficiently full bibliography see [26]. As mentioned in [4], Gaussian white noise analysis can be thought of as a theory of generalized functions of infinitely many variables whose pairing with test functions from special spaces is given via the integration by the Gaussian measure. It is well known that there are several approaches to the construction of such theory of generalized functions: the Berezansky-Samoilenko approach [3] and the Hida approach [25]. In the Berezansky-Samoilenko approach, the spaces of test and generalized functions are constructed as infinite tensor products of one-dimensional spaces. The Hida approach consists in a construction of some rigging of a Fock space with subsequent application of the Wiener-Itô-Segal isomorphism to the spaces of this rigging. In most cases, investigations in white noise analysis and its generalizations are based on the Hida approach. So, the Hida approach is more convenient than Berezansky-Samoilenko approach. Recently, some authors as Okb El Bab, Zabel, Ghany and Hyder [15], Ghany [16], Ghany and Hyder [17–19], Ghany and Zakarya [20–22] and Ghany and Qurashi [23], studied some important subjects related to Gaussian white noise analysis. There exist many works investigated to white noise analysis development Works deal with the construction of spaces of test and generalized functions and operators acting in these spaces using the Wiener-Itô-Segal isomorphism and various riggings of the Fock space. For more details, see [4, 25, 28]. Works deal with the so-called Jacobi fields approach to a generalization of white noise analysis [6, 7]. In these works, the role of the Wiener-Itô-Segal isomorphism is

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Ref

25. T. Hida, *Analysis of Brownian functionals*, in Carleton Math. Lecture Notes **13**, Carleton Univ., Ottawa 1975.

played as a unitary Fourier transform which is defined by the Jacobi field, i.e., by some family of commuting selfadjoint operators that act in the Fock space and have a Jacobi structure. Works use the biorthogonal approach to a generalization of white noise analysis. In this approach, one replaces the system of Hermite polynomials, that are orthogonal with respect to the Gaussian measure, with a certain biorthogonal system. The biorthogonal approach was inspired by [12], proposed in [1] and developed in [2,8,38]. Note that in [8] was first observed that the biorthogonal approach is deep connected with the theory of hypercomplex systems.

Hypercomplex systems date back to J. Delsarte and B. M. Levitan work during the 1930s and 1940s, but the substantial development had to wait till the 1990s when Berezansky and Kondratiev put hypercomplex systems in the right setting for harmonic analysis [9]. Recently, some authors as Okb El Bab, Zabel and Ghany [37, 40, 41] studied some important subjects in hypercomplex systems.

Generalized translation operators were first introduced by Delsarte [13] as an object that generalizes the idea of translation on a group. They were systematically studied by Levitan [31–35], for some classes of generalized translation operators, he obtained generalizations of harmonic analysis, the Lie theory, the theory of almost periodic functions, the theory of group representations, etc. In fact, each hypercomplex system can be associated with a family of generalized translation operators. This correspondence can be found in [5].

This paper focuses on the connection between white noise analysis and hypercomplex systems. Precisely, we produce a generalization of white noise analysis to the case of non-Gaussian measures. For this aim, we propose a biorthogonal approach in which instead of the exponentials the characters of commutative hypercomplex systems are used. Furthermore, we construct the elements of Wick calculus, namely, we introduce a new Wick product with respect to non-Gaussian measures, the associated Hermite transform and the characterization theorem for the constructed spaces of generalized functions. In Gaussian white noise analysis such calculus has found numerous applications, in particular, in fluid mechanics and financial mathematics, see e.g. [14, 27] for more details.

This paper is organized as follows: Section 2 is devoted to provide the basic topics of hypercomplex systems with locally compact basis. In Section 3, we construct spaces of test and generalized functions by means of generalized Delsarte characters. In Section 4, we introduce the elements of Wick calculus.

II. Hypercomplex Systems with Locally Compact Basis

Let Q be a complete separable locally compact metric space of points $p, q, r, \ldots; \mathcal{B}(Q)$ is the σ-algebra of Borel subsets, and $\mathcal{B}_0(Q)$ is the subring of $\mathcal{B}(Q)$, which consists of sets with compact closure. We shall consider the Borel measures; i.e., positive regular measures on $\mathcal{B}(Q)$, finite on compact sets. We denote by $C(Q)$ the space of continuous functions on $Q, C_b(Q), C_\infty(Q)$ and $C_0(Q)$ consists respectively of bounded, tending to zero at infinity and compactly supported functions from $C(Q)$.

A hypercomplex system with the basis Q is defined by its structure measure $c(A, B, r)(A, B \in B(Q); r \in Q)$. A structure measure $c(A, B, r)$ is a Borel measure in A (respectively B) if we fix B, r (respectively A, r) which satisfies the following properties:

(I) $\forall A, B \in \mathcal{B}_0(Q)$, the function $c(A, B, r) \in C_0(Q)$,

(II) $\forall A, B \in \mathcal{B}_0(Q)$ and $s, r \in Q$, the following associativity relation holds

Ref

13. J. Delsarte, Sur certaines transformation fonctionelles relative aux equations linearies aux derivees partiels du seconde ordre, *C. R. Acad. Sci., Ser.* **17** (1938), 178-182.

$$\int_Q c(A,B,r)d_r c(E_r,C,s) = \int_Q c(B,C,r)d_r c(A,E_r,s), \quad C \in \mathcal{B}(Q). \tag{2.1}$$

(III) The structure measure is said to be commutative if

$$c(A,B,r) = c(B,A,r), \quad (A,B \in \mathcal{B}_0(Q)). \tag{2.2}$$

A measure m is said to be a multiplicative measure if

$$\int_Q c(A,B,r)dm(r) = m(A)m(B); \quad A,B \in \mathcal{B}_0(Q). \tag{2.3}$$

(IV) We will suppose the existence of a multiplicative measure. Under certain restrictions imposed on the commutative structure measure, multiplicative measure exists. (See [30]).

Ref

30. A. A. Kalyuzhnyi, A theorem on the existence of multiplicative measure, *Ukr. Math. Zh.*, **35**(1983), 369-371.

Consider the space $L_1(Q, dm(x))$ of functions on Q with respect to the multiplicative measure m. For any $\Phi, \Psi \in L_1(Q, dm(x))$, the convolution

$$\begin{aligned}(\Phi * \Psi)(r) &= \int_Q \Phi(p)dp \int_Q \Psi(q)dqc(E_p,E_q,r) \\ &= \int_Q \int_Q \Phi(p)\Psi(q)c(p,q,r)dm(p)dm(q) \\ &= \int_Q \int_Q \Phi(p)\Psi(q)dm_r(p,q) \end{aligned} \tag{2.4}$$

is well defined. (See [5]).

The space $L_1(Q, dm(x))$ with the convolution (2.4) is a Banach algebra which is commutative if (**III**) holds. This Banach algebra is called the hypercomplex system with the basis Q.

It is obvious that $c(A,B,r) = (K_A * K_B)(r)$; $A,B \in \mathcal{B}_0(Q)$ and K_A is the characteristic function of the set A.

A hypercomplex system may or may not have a unity. If a unity not included in $L_1(Q, dm(x))$, then it is convenient to join it formally to $L_1(Q, dm(x))$.

A non zero measurable and bounded almost everywhere function $Q \ni r \mapsto \chi(r) \in \mathbb{C}$ is said to be a character of the hypercomplex system $L_1(Q, dm(x))$ if $\forall A,B \in \mathcal{B}_0(Q)$

$$\int_Q c(A,B,r)\chi(r)dm(r) = \chi(A)\chi(B), \tag{2.5}$$

$$\int_C \chi(r)dm(r) = \chi(C), \quad C \in \mathcal{B}_0(Q). \tag{2.6}$$

(V) A hypercomplex system is said to be normal, if there exists an involution homomorphism $Q \ni r \mapsto r^* \in Q$ such that $m(A) = m(A^*)$ and $c(A,B,C) = c(C,B^*,A), c(A,B,C) = c(A^*,C,B), (A,B \in \mathcal{B}_0(Q))$, where

$$c(A,B,C) = \int_C c(A,B,r)dm(r) \tag{2.7}$$

(VI) A normal hypercomplex system possesses a basis unity if there exists a point $e \in Q$ such that $e^* = e$ and

$$c(A, B, e) = m(A^* \cap B), \quad A, B \in \mathcal{B}(Q). \tag{2.8}$$

We should remark that, for a normal hypercomplex system, the mapping

$$L_1(Q, dm(x)) \ni \Phi(r) \mapsto \Phi^*(r) \in L_1(Q, dm(x))$$

is an involution in the Banach algebra $L_1(Q, dm(x))$, the multiplicative measure is unique and the characters of such a system are continuous. (See [5]). A character χ of a normal hypercomplex system is said to be Hermitian if

$$\chi(r^*) = \overline{\chi(r)}, \quad (r \in Q), \tag{2.9}$$

and ordinary if it is bounded. The function $\chi(r) = 1$, $r \in Q$ is always a character; for all χ, $\chi(e) = 1$.

Denote the set of all bounded Hermitian characters by X_h, i.e.,

$$X_h = \{\chi \in C_b(Q) : \chi \neq 0, \int_Q c(A, B, r)\chi(r)dm(r) = \chi(A)\chi(B), \overline{\chi(r)} = \chi(r^*)\}. \tag{2.10}$$

Let $L_1(Q, dm(x))$ be a hypercomplex system with basis Q and let Ω be a space of complex valued functions on Q. Assume that an operator valued function $Q \ni p \mapsto L_p : \Omega \to \Omega$ is given such that the function $g(p) = (L_p f)(q)$ belongs to Ω for any $f \in \Omega$ and any fixed $q \in Q$. For such hypercomplex system it is possible to introduce generalized translation operators L_p:

$$Q \times Q \ni \langle p, q\rangle \mapsto (L_p\Phi)(q) \in \mathbb{C}, \quad \Phi \in C(Q), \tag{2.11}$$

where $(L_p\Phi)(q) = \Phi(q + p)$ in case of the usual translation on $Q = \mathbb{R}$. We will suppose, in addition, that this function is separately continuous. By using the operators L_p one can rewrite the convolution (2.4) as follows:

$$\big(\Phi * \Psi\big)(p) = \int_Q \Phi(q)\big(L_{q^*}\Psi\big)(p)\, dm(q) \quad \Phi, \Psi \in L_1(Q, dm(x)). \tag{2.12}$$

A generalized character of a hypercomplex system $L_1(Q, dm(x))$ is defined to be a function $\chi \in C(Q)$ for which

$$(L_p\chi)(q) = \chi(p)\chi(q), \qquad p, q \in Q. \tag{2.13}$$

Now, we give Some examples to illustrative the concept of hypercomplex systems

Example 2.1. Let $Q = G$ be commutative, locally compact group. with unite e. Consider G is groups algebra, i.e., a set $L_1(G, m)$ of functions defined on the groups G with respect to the measure m is denoted by Haar measure $dm(p)$. Then, we can define the involution with the form

$$G \ni p \longmapsto p^* = p^{-1}, \tag{2.14}$$

where

$$\big(L_p\Phi\big)(q) = \Phi(pq), \quad (p, q \in G), \tag{2.15}$$

Ref

5. Yu. M. Berezansky and A. A. Kalyuzhnyi, *Harmonic Analysis in Hypercomplex Systems*, Naukova Dumka,Kyiv, Ukrania, 1992. (in Russian; English transl.: Kluwer: Dordrecht-Boston-London, 1996).

then the convolution in (2.12) becomes in the form

$$\big(\Phi * \Psi\big)(p) = \int\limits_{Q} \Phi(pq)\Psi(q^{*})d(q), \tag{2.16}$$

and we can define the structure measure as the following

$$C\big(A, B, r\big) = m\big(A^{-1}r \cap B\big), \tag{2.17}$$

where $A \in \mathcal{B}(G), r \in G$. Thus, we obtain a commutative hypercomplex System $L_1(G, m)$ with basis unity e.

Example 2.2. Consider the sturm-Liouvvlle equation in the quadratic axis $Q = \mathbb{R}_{+} = [0, \infty)$:

$$(Mu)(x) = -u^{''} + g(x)u = \lambda u, \tag{2.18}$$

where $g \in C(\mathbb{R}_{+})$ is a non-negative and non-increasing potential. connect with this equation the following hyperbolic equation on $\mathbb{R}^2 : M_{(x)}u = M_{(y)}v$, i.e.,

$$-\frac{\partial^2 v}{\partial x^2} + g(x)v = -\frac{\partial^2 v}{\partial y^2} + g(y)v, \quad v = v(x, y), \tag{2.19}$$

where $x, y \in \mathbb{R}$, g is the even extension of the potential to the whole axis. Equation (2.19) gives rise to a generalized translation operator $L_{(x)}$, $x \in \mathbb{R}_{+}$: by definition

$$\big(L_{(x)}f\big)(y) = \frac{v\big(x, y\big)}{v(x)v(y)}, \quad x, y \in \mathbb{R}_{+}, \tag{2.20}$$

where $v(x, y)$ is the solution of the Cauchy problem for (2.19) with initial conditions $v(x, 0) = f(x)v(x)$, $\frac{\partial v}{\partial y}(x, 0) = 0$, and $v(x) = \phi(x, 0)$, where $\phi(x, \lambda)$ is the solution of the following problem on $\mathbb{R}_{+} : M\phi = \lambda\phi$, $\phi(0, \lambda) = 1$, $\phi^{'}(0, \lambda) = 0$ $\big($both functions $f(x)$ and $v(x)$ being evenly extended to $\mathbb{R}\big)$. Now, for any $x \in \mathbb{R}_{+}$, $x^{*} = x, dm(x) = v^2(x)d(x)$, and $\chi(x) = \phi\big(x, \lambda\big)$, where $\lambda \in \mathbb{C}$ is arbitrary, the convolution for hypercomplex system with basis $Q = \mathbb{R}_{+}$ is constructed by (2.12) or (2.4) with some structure measure c.

Example 2.3. Consider the special case Example 2.2, when $g = 0$. Then, define generalized translation operators as following

$$(L_{(x)}f)(y) = \frac{1}{2}\Big(f\big(|y - x|\big) + f\big(y + x\big)\Big), \quad x, y \in Q = \mathbb{R}_{+}, \tag{2.21}$$

and

$$\phi(x, \lambda) = \cos\Big(\sqrt{\lambda}x\Big), \quad v(x) = 1, \quad x \in \mathbb{R}_{+}, \ \lambda \in \mathbb{C}. \tag{2.22}$$

In fact, there are many examples of hypercomplex systems. For more details (see [5]).

III. Spaces of Test and Generalized Functions

The classical approach to white noise analysis [4, 25, 26] consists in the construction of some rigging of the Fock space $F(H)$ and in the consequent application to the spaces of this rigging the Wiener-Itô- Segal transform I. This operator transfers the space $F(H)$ to the space $L_2(\Omega', d\gamma(x))$ and the spaces of rigging to some spaces of test and generalized functions. The pairing between test and generalized functions is given by the scalar product in $L_2(\Omega', d\gamma(x))$. Here, Ω' is the dual space of a real nuclear space Ω and γ is a Gaussian measure on Ω'.

Ref

4. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral methods in infinite dimensional analysis*, vol. 1, 2, Kluwer, Dordrecht 1995.

In the model one-dimensional case the space $F(H)$ is equal to the space l_2 of complex-valued sequences $\varphi = (\varphi_0, \varphi_1, ...), \Omega' = \mathbb{R}$, and the Wiener-Itô- Segal transform has the form

$$l_2 \ni \varphi = (\varphi_0, \varphi_1, ...) \mapsto (I\varphi)(x) = \sum_{n=0}^{\infty} \varphi_n H_n(x) \in L_2(\mathbb{R}, d\gamma(x)), \ x \in \mathbb{R}, \tag{3.1}$$

where $\left(H_n(x)\right)_{n=0}^{\infty}$ is the sequence of orthonormal Hermite polynomials with respect to γ. The operator I transfers the rigging of l_2 to that of $L_2(\mathbb{R}, d\gamma(x))$, which gives a model of white noise analysis.

The generalization of white noise analysis proposed in [1], consists in the following. Let ρ be a probability measure on $\mathbb{R}$ (instead of the Gaussian measure γ) and let $(P_n(x), Q_n(x))_{n=0}^{\infty}$ $(x \in \mathbb{R})$ be the biorthogonal system of functions constructed in some canonical way. The biorthogonality means

$$\int_{\mathbb{R}} P_n(x) Q_m(x) d\rho(x) = \delta_{n,m}.\ n!, \quad , \quad m, n \in \mathbb{Z}_+. \tag{3.2}$$

Instead of the functions P_n, we take the Appell polynomials connected with the measure ρ, i.e. the coefficients of the power expansion with respect to $\lambda \in \mathbb{C}$

$$\widehat{\rho}(\lambda) = \int_{\mathbb{R}} e^{\lambda x} d\rho(x), \qquad \frac{e^{\lambda x}}{\widehat{\rho}(\lambda)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} P_n(x), \quad x \in \mathbb{R}. \tag{3.3}$$

The functions Q_n are defined by

$$Q_n(x) = \left(\left(\left(\frac{d}{dx} \right)^* \right)^n 1 \right)(x), \tag{3.4}$$

where the adjoint operator is considered in the $L_2(\mathbb{R}, d\rho(x))$ sense.

The operator I is defined now just as above, but with changing H_n by P_n. This operator transfers the rigging of l_2 to a rigging of $L_2(\mathbb{R}, d\rho(x))$, which gives some theory of generalized functions on $\mathbb{R}$ with pairing determinated by $L_2(\mathbb{R}, d\rho(x))$.

In this section, we suggest a generalization of the above mentioned construction in which instead of the function $e^{\lambda x}$ the character of some pretty arbitrary, commutative hypercomplex system is used. Such a generalization gives the possibility of constructing a lot of spaces of generalized functions connected with different examples of hypercomplex system.

Consider a subclass of the above hypercomplex systems for which the set of generalized characters is in one-to-one correspondence with the complex plane $\mathbb{C}$: $\chi \longleftrightarrow \lambda \in \mathbb{C}$; denote this character by $\chi(x, \lambda)$. We assume that $\chi(x, 0) = 1$ $(x \in Q)$, i.e., the unit character corresponds to $\lambda = 0$. We suppose the function $Q \times \mathbb{C} \ni \langle x, \lambda \rangle \mapsto \chi(x, \lambda) \in \mathbb{C}$ is continuous, and the function $\mathbb{C} \ni \lambda \mapsto \chi(x, \lambda) \in \mathbb{C}$ is an entire for each $x \in Q$. Thus, for each $x \in Q$ the following expansion holds

$$\chi(x, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \chi_n(x), \quad \lambda \in \mathbb{C}, \tag{3.5}$$

where the coefficients $\chi_n \in C(Q)$ called the Delsarte characters [6]. It is possible, of course, to give a direct definition of the Delsarte characters if we will rewrite Eq.(2.13) in terms of χ_n.

Our purpose is now to construct the expansions of functions on Q using the Delsarte characters and to introduce the corresponding spaces of test and generalized functions.

Ref

6. Yu. M. Berezansky, A generalization of white noise analysis by means of theory of hypergroups, *Reports on Mathematical Physics* **38** (1996), 289-300.

Let ρ be a fixed Borel probability measure ($\rho(Q)=1$), positive on open sets, which is not connected directly with the multiplicative measure m . We suppose that χ_n's belong to $L_2(Q, d\rho(x))$, satisfy the estimate

$$\exists\, C > 0 : \|\chi_n\|_{L_2(Q,d\rho(x))} = \left(\int_Q |\chi_n(x)|^2 d\rho(x) \right)^2 \leq C^n n!, \quad n \in \mathbb{Z}_+. \tag{3.6}$$

(explaining that always $\|\chi_n\|_{L_2(Q,d\rho(x))} = 1$), are linearly independent and the system $(\chi_n)_n^\infty$ is total in $L_2(Q, d\rho(x))$.

Note that for a large class of hypercomplex systems (the Taylor-Delsarte hypercomplex system) vectors χ_n are automatically linearly independent. If in the estimate (3.6) instead of $n!$ the quantity $(n!)^\epsilon$ with fixed ($\epsilon < 1$) stands or if supp ρ is a compact set, then also automatically the system $(\chi_n)_n^\infty$ is total. In general we will understand the above mentioned conditions as some conditions on the measure ρ.

Now, we use the simple constructions connected with Hilbert rigging [4].

Let us consider a rigging of a Hilbert space H_0 with positive and negative spaces H_+ and H_- as following

$$H_- \supseteq H_0 \supseteq H_+. \tag{3.7}$$

Let $V : H_- \longrightarrow H_+$ be the canonical isometry transferring H_- onto H_+ [4]. A biorthogonal basis $(p_n, q_n)_{n=0}^\infty$ in the space H_0 is to be understood as sequences $(p_n)_{n=0}^\infty \subset H_+$ and $(q_n = V^{-1}p_n)_{n=0}^\infty \subset H_-$, where the first one is an orthogonal basis in H_+ and therefore, the second one is an orthogonal basis in H_-.

Note that these systems of vectors p_n and q_n are biorthogonal (with respect to H_o):

$$(p_n, q_n)_{H_0} = \delta_{n,m} h_n, \quad h_n = \|p_n\|^2_{H_+} = \|q_n\|^2_{H_-}, \quad n, m \in \mathbb{Z}_+, \tag{3.8}$$

$$\forall \varphi \in H_+, \quad \varphi = \sum_{n=0}^{\infty} \varphi_n p_n, \quad \varphi_n = (\varphi, q_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^{\infty} |\varphi_n|^2 h_n = \|\varphi\|^2_{H_+} < \infty,$$

$$\forall \xi \in H_-, \quad \xi = \sum_{n=0}^{\infty} \xi_n q_n, \quad \xi_n = (\xi, p_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^{\infty} |\xi_n|^2 h_n = \|\xi\|^2_{H_-} < \infty,$$

$$(\xi, \varphi)_{H_0} = \sum_{n=0}^{\infty} \xi_n \overline{\varphi_n} h_n. \tag{3.9}$$

This definition is justified by the following.

Let $(p_n)_{n=0}^\infty$ be an arbitrary total sequence of vectors p_n of a Hilbert space H_0. It is easy to prove that such sequence $(h_n)_{n=0}^\infty$ of positive numbers h_n exists for which the set of test functions

$$H_+ = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \, p_n \mid \varphi_n \in \mathbb{C} : \|\varphi\|^2_{H_+} = \sum_{n=0}^{\infty} |\varphi_n|^2 \, h_n < \infty \right\}, \tag{3.10}$$

with the corresponding scalar product is the positive space with respect to H_0 and therefore the chain (3.7) exists. Note that it is necessary to assume in addition the fulfilment of the following necessary and sufficient condition on $(p_n)_{n=0}^\infty$: an arbitrary sequence $(\varphi^{(i)})_{i=0}^\infty$ of vectors $\varphi^{(i)} \in H_+$ with finite sequences of coordinates $\varphi_n^{(i)}$ which is fundamental in H_+ and converges to 0 in H_0 must converge to 0 in H_+. This condition will always be fulfilled in our case.

Ref

4. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral methods in infinite dimensional analysis*, vol. 1, 2, Kluwer, Dordrecht 1995.

Similarly for negative space H_- by replacement p_n by q_n, we have the set of generalized functions as follows

$$H_- = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n \, q_n \mid \xi_n \in \mathbb{C} : \|\xi\|_{H_-}^2 = \sum_{n=0}^{\infty} |\xi_n|^2 \, h_n < \infty \right\}. \tag{3.11}$$

Therefore, we can construct a rigging of the space $H_0 := L_2(Q, d\rho(x))$ of the form (3.7), (3.10) and (3.11) in which the Delsarte characters $\chi_n(x)$ will play the role of the vectors p_n. This result can be stated as follows:

Theorem 3.1. There exists a quasinuclear rigging (3.7) such that $H_0 = L_2(Q, d\rho(x))$,

$$H_{-1}^{\chi} \supseteq H_0 \supseteq H_1^{\chi}, \tag{3.12}$$

and the space $H_1^{\chi} = H_+$ *is of the form (3.10), where*

$$p_n = \chi_n, \quad h_n = (n!)^2 K^n, \quad n \in \mathbb{Z}_+, \tag{3.13}$$

($K > 1$ *is a fixed sufficiently large number), and consists of continuous functions on* Q.

The system $(\chi_n, q_n^{\chi})_{n=0}^{\infty}$, *where* $q_n^{\chi} = V^{-1}\chi_n$ *(*V *is connected with (3.12)), is a biorthogonal basis of the space* H_0. *For the vectors* q_n^{χ}, *the following representation holds:*

$$q_n^{\chi} = (n!) K^n \Theta_n, \quad \Theta_n = (D^+)^n \delta_e \in H_{-1}^{\chi}, \quad n \in \mathbb{Z}_+, \tag{3.14}$$

where Θ_n *are "Delsarte co-characters" connected with the hypercomplex system* $L_1(Q, d\rho(x))$, *and the measure* ρ. *Here* D *is a continuous operator acting on the space* H_1^{χ} *and is defined by the expression*

$$D\chi_n = n\chi_{n-1}, \quad n \in \mathbb{N}, \quad D\chi_0 = 0, \tag{3.15}$$

$D^+ : H_{-1}^{\chi} \longrightarrow H_{-1}^{\chi}$ *is its dual operator with respect to the rigging (3.12). The space* H_1^{χ} *consists of continuous functions, therefore the* δ*-function* δ_x, *lumped at the point* $x \in Q$, *exist as an element of the negative space* H_{-1}^{χ}.

It is essential to introduce the rigging of the space H_0 by means of projective and inductive limits of Hilbert spaces which are constructed by rules of type (3.10), (3.12) and (3.13). Namely, for every $q \in \mathbb{N}$ we introduce the Hilbert space of type (3.10):

$$H_q^{\chi} = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \chi_n \in H_0 : \|\varphi\|_{H_q^{\chi}}^2 = \sum_{n=0}^{\infty} |\varphi_n|^2 (n!)^2 K^{qn} < \infty \right\}. \tag{3.16}$$

Then we have the nuclear rigging:

$$(\Psi^{\chi})' \supseteq H_{-q}^{\chi} \supseteq H_0 \supseteq H_q^{\chi} \supseteq \Psi^{\chi}, \tag{3.17}$$

$$\Psi^{\chi} = \operatorname{pr} \lim_{q \in \mathbb{N}} H_q^{\chi} = \bigcap_{q \in \mathbb{N}} H_q^{\chi}, \qquad (\Psi^{\chi})' = \operatorname{ind} \lim_{q \in \mathbb{N}} H_{-q}^{\chi} = \bigcup_{q \in \mathbb{N}} H_{-q}^{\chi},$$

$$H_{-q}^{\chi} = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} : \|\xi\|_{H_{-q}^{\chi}}^2 = \sum_{n=0}^{\infty} |\xi_n|^2 (n!)^2 K^{-qn} < \infty \right\}, \tag{3.18}$$

with the action

$$(\xi,\varphi)_{H_0}=\sum_{n=0}^{\infty}\xi_n\overline{\varphi_n}(n!)^2K^{qn},\quad \varphi\in H_q^{\chi},\quad \xi\in H_{-q}^{\chi}.$$

Note that due to the first equality in (3.8) for Delsarte characters and co-characters the following biorthogonality is true:

$$(\Theta_n,\chi_n)_{H_0}=\delta_{n,m}.n!,\quad n,m\in\mathbb{Z}_+. \tag{3.19}$$

To illustrate the above result, we give the following example

Example 3.1. Let $H_0=L_2(\mathbb{R},dx)$, with respect to the Lebesgue measure dx and ordinary convolution given in Eq.(2.12) the generalized character $\chi(x,\lambda)=e^{\lambda x}$ $(\lambda\in\mathbb{C})$ and $\chi_n(x)=x^n$ $(x\in\mathbb{R},\ n\in\mathbb{Z}_+)$. Therefore, the space (3.16) consists of entire functions $\varphi(x)$ and $\varphi_n(x)$ are the Taylor coefficients of $\varphi(x)$. Formula (3.8) gives their representation as the Fourier coefficients using the scalar product $(\xi,\varphi)_{H_0}$, $(\xi\in H_{-1}^{\chi},\ \varphi\in H_1^{\chi})$. The operator D now is equal d/dx.

IV. Elements of Wick Calculus

The Wick product was first introduced by Wick [39] and used as a tool to renormalize certain infinite quantities in quantum field theory. Later on, the Wick product was considered, in a stochastic setting, by Hida and Ikeda [24]. In [11], Dobroshin and Minlos were comprehensively treated this subject both in mathematical physics and probability theory. Currently, the Wick product provides a useful concept for various applications, for example, it is important in the study of stochastic ordinary and partial differential equations (see, e.g., [27]).

In this section, under the assumption that $\|\chi\|_{H_0}^2\le C^n$ for some $C>0$, we define a new Wick product, called χ-Wick product, on the space H_{-q}^{χ}, which is constructed in Section 3. Then, we give the definition of the χ-Hermite transform and apply it to establish a characterization theorem for the space H_{-q}^{χ}.

Definition 4.1. Let $\xi=\sum_{m=0}^{\infty}\xi_m q_m^{\chi}$, $\eta=\sum_{n=0}^{\infty}\eta_n q_n^{\chi}\in H_{-q}^{\chi}$ with $\xi_m,\eta_n\in\mathbb{C}$. The χ-Wick product of ξ, η, denoted by $\xi\diamond_\chi\eta$, is defined by the formula

$$\xi\diamond_\chi\eta=\sum_{m,n=0}^{\infty}\xi_m\eta_n q_{m+n}^{\chi}. \tag{4.1}$$

It is important to show that the spaces $H_{-q}^{\chi}, H_q^{\chi}$ are closed under χ-Wick product.

Lemma 4.1. If $\xi,\eta\in H_{-q}^{\chi}$ and $\varphi,\psi\in H_q^{\chi}$, *we have*

(i) $\xi\diamond_\chi\eta\in H_{-q}^{\chi}$,

(ii) $\varphi\diamond_\chi\quad\in H_q^{\chi}$.

Proof. If $\xi=\sum\limits_{m=0}^{\infty}\xi_m q_m^{\chi}$, $\eta=\sum\limits_{n=0}^{\infty}\eta_n q_n^{\chi}\in H_{-q}^{\chi}$, then for some $q_1\in\mathbb{N}$ we have

$$\sum_{m=0}^{\infty}|\xi_m|^2K^{-q_1m}<\infty \text{ and } \sum_{n=0}^{\infty}|\eta_n|^2K^{-q_1n}<\infty. \tag{4.2}$$

Ref

39. G. C. Wick, The evaluation of the collinear matrix, *Phys. Rev.* **80** (1950), 268-272.

We note that

$$\xi \diamond_\chi \eta = \sum_{m,n=0}^{\infty} \xi_m \eta_n q^\chi_{m+n} = \sum_{l=0}^{\infty}\left(\sum_{m+n=l}^{\infty} \xi_m\eta_n\right) q^\chi_l = \sum_{l=0}^{\infty} \zeta_l q^\chi_l, \tag{4.3}$$

where $\zeta_l = \sum\limits_{m+n=l}^{\infty} \xi_m \eta_n$. With $q = q_1 + p$ we have

$$\begin{aligned}\sum_{l=0}^{\infty} |\zeta_l|^2 K^{-ql} &= \sum_{l=0}^{\infty}\left|\sum_{m+n=l}^{\infty}\xi_m\eta_n\right|^2 K^{-q_1 l}K^{-pl} \\ &\leq \sum_{l=0}^{\infty}\left(\sum_{m+n=l}^{\infty}|\xi_m|^2K^{-q_1 m}\right)\left(\sum_{m+n=l}^{\infty}|\eta_n|^2K^{-q_1 n}\right)K^{-pl} \\ &\leq \left(\sum_{l=0}^{\infty}K^{-pl}\right)\left(\sum_{m=0}^{\infty}|\xi_m|^2K^{-q_1 m}\right)\left(\sum_{n=0}^{\infty}|\eta_n|^2K^{-q_1 n}\right) \\ &< \infty,\end{aligned} \tag{4.4}$$

which proves (i). The proof of (ii) is similar. ■

The following important algebraic properties of the χ-Wick product follow directly from Definition 4.1.

Lemma 4.2. For each $\xi, \eta, \zeta \in H^\chi_{-q}$, we get

(i) $\xi \diamond_\chi \eta = \eta \diamond_\chi \xi$ *(Commutative law),*

(ii) $\xi \diamond_\chi (\eta \diamond_\chi \zeta) = (\xi \diamond_\chi \eta) \diamond_\chi \zeta$ *(Associative law),*

(iii) $\xi \diamond_\chi (\eta + \zeta) = \xi \diamond_\chi \eta + \xi \diamond_\chi \zeta$ *(Distributive law).*

Remark 4.1. According to Lemmas 4.1 and 4.2, we can conclude that the spaces H^χ_{-q} and H^χ_q form topological algebras with respect to the χ-Wick product.

As shown in Lemmas 4.1 and 4.2, the χ-Wick product satisfies all the ordinary algebraic rules for multiplication. Therefore, one can carry out calculations in much the same way as with usual products. But, there are some problems when limit operations are involved. To treat these situations it is convenient to apply a transformation, called the χ-Hermite transform, which converts χ-Wick products into ordinary (complex) products and convergence in H^χ_{-q} into bounded, pointwise convergence in a certain neighborhood of 0 in $\mathbb{C}$. The original Hermite transform, which first appeared in Lindstrøm et al. [29], has been applied by the authors in many different connections. Now, we give the definition of the χ-Hermite transform and discuss its basic properties.

Definition 4.2. Let $\xi = \sum_{n=0}^{\infty} \xi_n q^\chi_n \in H^\chi_{-q}$ with $\xi_n \in \mathbb{C}$. Then, the χ-Hermite transform of ξ, denoted by $\mathcal{H}_\chi \xi$, is defined by

$$\mathcal{H}_\chi \xi(z) = \sum_{n=0}^{\infty} \xi_n z^n \in \mathbb{C} \quad \text{(when convergent)}. \tag{4.5}$$

In the following we define for $0 < M, q < \infty$ the neighborhoods $\mathbb{O}_q(M)$ of zero in $\mathbb{C}$ by

$$\mathbb{O}_q(M) = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} |z^n|^2 K^{qn} < M^2 \right\}. \tag{4.6}$$

Ref

29. T. Lindstrom, B. Oksendal and J. Uboe, Stochastic differential equations involving positive noise. In M. Barlow and N. Bingham (editors): *Stochastic Analysis.* Cambridge University Press: Cambridge, MA, (1991), 261-303.

It is easy to see that

$$q \leq p,\ \ N \leq M \Rightarrow \mathbb{O}_p(N) \subseteq \mathbb{O}_q(M). \tag{4.7}$$

Note that if $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}, z \in \mathbb{O}_q(M)$ for some $0 < M, q < \infty$, we have the estimate

$$\begin{aligned}\sum_{n=0}^{\infty} |\xi_n||z^n| &= \sum_{n=0}^{\infty} |\xi_n||z^n| K^{-\frac{qn}{2}} K^{\frac{qn}{2}} \\ &\leq \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn}\right)^{\frac{1}{2}} \\ &< M\left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn}\right)^{\frac{1}{2}} \\ &< \infty. \end{aligned} \tag{4.8}$$

The conclusion above can be stated as follows:

Proposition 4.1. If $\xi \in H_{-q}^{\chi}$, then $\mathcal{H}_{\chi}\xi$ converges for all $z \in \mathbb{O}_q(M)$ for all $q, M < \infty$.

A useful property of the χ-Hermite transform is that it converts the χ-Wick product into ordinary (complex) product.

Proposition 4.2. If $\xi, \eta \in H_{-q}^{\chi}$, then

$$\mathcal{H}_{\chi}(\xi \diamond_{\chi} \eta)(z) = \mathcal{H}_{\chi}\xi(z).\mathcal{H}_{\chi}\eta(z). \tag{4.9}$$

for all z such that $\mathcal{H}_{\chi}\xi$ and $\mathcal{H}_{\chi}\eta$ exist.

Proof. The proof is an immediate consequence of Definitions 4.1 and 4.2. ■

Let $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}$, with $\xi_n \in \mathbb{R}$. Then, the number $\xi_0 = \mathcal{H}_{\chi}\xi(0) \in \mathbb{R}$ is called the generalized expectation of ξ and is denoted by $\mathbb{E}(\xi)$. Suppose that $V \ni z \mapsto f(z) \in \mathbb{C}$ is an analytic function, where V is a neighborhood of $\mathbb{E}(\xi)$. Assume that the Taylor series of f around $\mathbb{E}(\xi)$ has coefficients in $\mathbb{R}$. Then, the χ-Wick version $f^{\diamond_{\chi}}$ of f is defined by

$$H_{-q}^{\chi} \ni \xi \mapsto f^{\diamond_{\chi}}(\xi) = \mathcal{H}^{-1}\big(f \circ \mathcal{H}_{\chi}(\xi)\big) \in H_{-q}^{\chi}. \tag{4.10}$$

Example 4.1. If the function $f : \mathbb{C} \to \mathbb{C}$ is entire, then $f^{\diamond_{\chi}}$ is defined for all $\xi \in H_{-q}^{\chi}$. For example, the χ-Wick exponential is defined by

$$\exp^{\diamond_{\chi}}(\xi) = \sum_{j=0}^{\infty} \frac{1}{j!} \xi^{\diamond_{\chi} n}. \tag{4.11}$$

Using χ-Hermite transform we see that χ-Wick exponential has the same algebraic properties as the usual exponential. For instance,

$$\exp^{\diamond_{\chi}}(\xi + \eta) = \exp^{\diamond_{\chi}}(\xi) \diamond_{\chi} \exp^{\diamond_{\chi}}(\eta),\ \ \xi, \eta \in H_{-q}^{\chi}. \tag{4.12}$$

From Proposition 4.1, we can deduce that χ-Hermite transform of any $\xi \in H_{-q}^{\chi}$ is a complex-valued analytic function on $\mathbb{O}_q(M)$ for all $q, M < \infty$. Moreover, the converse of this deduction is true, i.e., every complex-valued analytic function on $\mathbb{O}_q(M)$ (for some $q, M < \infty$) is the χ-Hermite transform of some element in H_{-q}^{χ}. To prove this, we need the following two auxiliary results.

Lemma 4.3. Let $f(z) = \sum_{n=0}^{\infty} \eta_n z^n$ be an analytic function in $z \in \mathbb{C}$ such that there exists $M < \infty, C > 0$ and $\delta > 0$ such that $|f(z)| \leq M$ when $z \in \mathbb{O} := \{z \in \mathbb{C} : C|z| \leq \delta^2\}$. Then $|\eta_n z^n| \leq M$ for all $z \in \mathbb{O}$ and $n \in \mathbb{N}$.

Proof. See [27], Lemma 2.6.10.

Proposition 4.3. Let $f(z) = \sum_{n=0}^{\infty} \eta_n z^n$, $\eta_n \in \mathbb{C}$ be a formal power series in $z \in \mathbb{C}$. Suppose there exist $q, M < \infty$ and $\delta > 0$ such that $f(z)$ is convergent for $z \in \mathbb{O}_q(\delta)$ and $|f(z)| \leq M$ for all $z \in \mathbb{O}_q(\delta)$. Then

$$\sum_{n=0}^{\infty} |\eta_n z^n| \leq MA(q) \quad \text{for all } z \in \mathbb{O}_{3q}(\delta) \tag{4.13}$$

where

$$A(q) := \sum_{n=0}^{\infty} K^{-qn} < \infty \quad (\text{Note that } K > 1). \tag{4.14}$$

Proof. It is evident that $z \in \mathbb{O}_{3q}\ \delta)$ implies $K^q z \in \mathbb{O}_q\ \delta)$. According to Lemma 4.3, we get

$$\begin{aligned} \sum_{n=0}^{\infty} |\eta_n z^n| &\leq \left(\sum_{n=0}^{\infty} |\eta_n|^2 |z^n|^2 K^{qn}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} K^{-qn}\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{\infty} |\eta_n|^2 |(K^q z)^n|^2 K^{-qn}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} K^{-qn}\right)^{\frac{1}{2}} \\ &\leq M \sum_{n=0}^{\infty} K^{-qn}. \end{aligned} \tag{4.15}$$

Theorem 4.1. **(Characterization Theorem for H_{-q}^{χ})** *If $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}$, where $\xi_n \in \mathbb{C}$, then there exist $q < \infty$ and $R_q < \infty$ such that*

$$|\mathcal{H}_{\chi}\xi(z)| \leq R_q \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn}\right)^{\frac{1}{2}} \quad \forall z \in \mathbb{C}. \tag{4.16}$$

In particular, $\mathcal{H}_{\chi}\xi$ is a bounded analytic function on $\mathbb{O}_q(M)$ for all $M < \infty$. Conversely, suppose $f(z) = \sum_{n=0}^{\infty} \eta_n z^n$ is a given analytic power series of $z \in \mathbb{C}$ with $\eta_n \in \mathbb{C}$ such that there exist $q < \infty$ and $\delta > 0$, such that $f(z)$ is absolutely convergent when $z \in \mathbb{O}_q(\delta)$ and

$$\sup_{z \in \mathbb{O}_q(\delta)} |f(z)| < \infty. \tag{4.17}$$

Then, there exists a unique $\eta \in H_{-q}^{\chi}$ such that $\mathcal{H}_{\chi}\eta = f$, namely

$$\eta = \sum_{n=0}^{\infty} \eta_n q_n^{\chi}. \tag{4.18}$$

Proof. For each $z \in \mathbb{C}$, we have

$$|\mathcal{H}_{\chi}\xi(z)| \leq \sum_{n=0}^{\infty} |\xi_n||z^n| \leq \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn}\right)^{\frac{1}{2}}. \tag{4.19}$$

Since $\xi \in H_{-q}^{\chi}$, we see that $R_q^2 := \sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn} < \infty$ for all $q < \infty$.

Ref

27. H. Holden, B. Osendal, J. Uboe and T. Zhang, *Stochastic partial differential equations*, Springer Science+Business Media, LLC, (2010).

Conversely, Since $K > 1$, then $K^{-r} \in \mathbb{O}_r(\delta)$ for all $r < \infty$ and for some $\delta < \infty$. By virtue of Proposition 4.3, we have

$$\sum_{n=0}^{\infty} |\eta_n||z^n| \leq MA(q) \quad \text{for all } z \in \mathbb{O}_{3q}(\delta). \tag{4.20}$$

Hence, for $r \geq 3q$ and $z \in \mathbb{O}_{3q}(\delta)$, we get

$$\sum_{n=0}^{\infty} |\eta_n|^2 K^{-rn} \leq C \sum_{n=0}^{\infty} |\eta_n| K^{-rn} \leq C \sum_{n=0}^{\infty} |\eta_n||z^n| \leq CMA(q) < \infty. \tag{4.21}$$

where $C := \sup\{|\eta_n| : n \in \mathbb{N}\}$, and hence $\eta := \sum_{n=0}^{\infty} \eta_n q_n^{\chi} \in H_{-q}^{\chi}$, as claimed. ∎

References Références Referencias

1. S. Albeverio, Yu. G. Kondratiev and L. Streit, How to generalize white noise analysis to non-Gaussian measures. Proc. Symp. *Dynamics of Complex and Irregular Systems* (Bielefeld, 1991); Bielefeld Encount. Math. Phys., VIII, World Sci. Publ.: River Edge, NJ, (1993), 120-130.
2. S. Albeverio, Yu. L. Daletsky, Yu. G. Kondratiev and L. Streit, Non-Gaussian infinitedimensional analysis, *J. Funct. Anal.* **138** (1996), 311-350.
3. Yu. M. Berezansky and Yu. S. Samoilenko, Nuclear spaces of functions of an infinite number of variables, *Ukrainian Math. J.* **25** (1973), 723-737.
4. Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral methods in infinite dimensional analysis*, vol. 1, 2, Kluwer, Dordrecht 1995.
5. Yu. M. Berezansky and A. A. Kalyuzhnyi, *Harmonic Analysis in Hypercomplex Systems*, Naukova Dumka,Kyiv, Ukrania, 1992. (in Russian; English transl.: Kluwer: Dordrecht-Boston-London, 1996).
6. Yu. M. Berezansky, A generalization of white noise analysis by means of theory of hypergroups, *Reports on Mathematical Physics* **38** (1996), 289-300.
7. Yu. M. Berezansky, Spectral approach to white noise analysis.Proc. Symp. *Dynamics of Complex and Irregular Systems* (Bielefeld, 1991); Bielefeld Encount. Math. Phys., VIII, World Sci. Publ., River Edge, NJ, (1993), 131-140.
8. Yu. M. Berezansky and Yu. G. Kondratiev, Biorthogonal systems in hypergroups: an extension of non-Gaussian analysis, *Methods Funct. Anal. Topopology* **2** (1996), 1-50.
9. Yu. M. Berezansky and A. A. Kalyuzhnyi, Harmonic Analysis in Hypercomplex Systems, Naukova Dumka: Kiev, 1992 (in Russian; English transl.: Kluwer: Dordrecht-Boston-London, 1996).
10. W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter, Berlin 1994.
11. R. L. Dobrushin and R. L. Minlos, Polynomials in linear random functions, *Russian Math. Surveys* **32** (1977), 71-127.
12. Yu. L. Daletsky, A biorthogonal analogy of the Hermite polynomials and the inversion of the Fourier transform with respect to a non-Gaussian measure, *Funct. Anal.Appl.* **25** (1991), 138-40.
13. J. Delsarte, Sur certaines transformation fonctionelles relative aux equations linearies aux derivees partiels du seconde ordre, *C. R. Acad. Sci., Ser.* **17** (1938), 178-182.
14. R. J. Elliott and J. van der Hoek, A general fractional white noise theory and applications to finance, *Math. Finance* **13** (2003), 301-330.

15. H. A. Ghany, A. S. Okb El Bab, A. M. Zabel, and A. Hyder, The fractional coupled KdV equations: exact solutions and white noise functional approach,*Chinese Physics B*, vol. **22**, no. 8, (2013) Article ID 080501.
16. H. A. Ghany, Exact solutions for stochastic generalized Hirota-Satsuma coupled KdV equations, *Chinese Journal of Physics*, vol. **49**, no. 4, (2011) pp. 926940.
17. H. A. Ghany and A. Hyder, Local and global well-posedness of stochastic Zakharov-Kuznetsov equation *Journal of Computational Analysis and Applications,* **15** (7), 1332-1343.
18. H. A. Ghany and A. Hyder, Exact solutions for the wicktype stochastic time-fractional KdV equations, *Kuwait Journal of Science*, vol. 41, no. 1,(2014) pp. 7584.
19. H. A. Ghany and A. Hyder, Abundant solutions of Wick-type stochastic fractional 2D KdV equations, *Chinese Physics B,*, vol. **23**, no. 6,(2014) Article ID 060503.
20. H. A. Ghany and M. Zakarya, Generalized Solutions of Wick-type Stochastic KdV-Burgers Equations Using Exp-function Method *International Review of Physics,* (IREPHY) **8**(2),(2014) 38-46.
21. H. A. Ghany and M. Zakarya, Exact Solutions for Wick-type Stochastic Coupled KdV Equations *Global Journal of Science Frontier Research: F Mathematics and Decision Sciences,* **14**(2014) 56-72.
22. H. A. Ghany and M. Zakarya, Exact Traveling Wave Solutions for Wick-Type Stochastic Schamel KdV Equation *Physics Research International,* (2014) p. 1-9
23. H. A. Ghany and M. A. Qurashi, Travelling SolitaryWave Solutions for Stochastic Kadomtsev-Petviashvili Equation. *Journal of Computational Analysis and Applications,* **21** (1), (2015) p.121-131.
24. T. Hida and N. Ikeda, Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral. *Proc. Fifth Berkeley Symp.Math. Stat. Probab. II* (1965), 117-143. Univ. California Press, Berkeley.
25. T. Hida, *Analysis of Brownian functionals*, in Carleton Math. Lecture Notes **13**, Carleton Univ., Ottawa 1975.
26. T. Hida, H-H. Kuo, J. Potthoff and L. Streit, *White Noise: An infinite Dimensional Calculus*, Kluwer, Dordrecht 1993.
27. H. Holden, B. Osendal, J. Uboe and T. Zhang, *Stochastic partial differential equations,* Springer Science+Business Media, LLC, (2010).
28. H-H. Kuo, White noise theory, *Handbook of Stochastic Analysis and Applications* D. Kannan and V. Lakshmikantham (eds.) (2002), 107-158.
29. T. Lindstrom, B. Oksendal and J. Uboe, Stochastic differential equations involving positive noise. In M. Barlow and N. Bingham (editors): *Stochastic Analysis.* Cambridge University Press: Cambridge, MA, (1991), 261-303.
30. A. A. Kalyuzhnyi, A theorem on the existence of multiplicative measure, *Ukr. Math. Zh.,* **35**(1983), 369-371.
31. B. M. Levitan, Generalization of translation operation and infinite-dimensional hypercomplex systems, *Mat. Sb.* **16** (1945), 259-280.
32. B. M. Levitan, Application of generalized translation operators to linear differential equations of the second order, *Usp. Mat. Nauk* **4** (1949), 3-112.
33. B. M. Levitan, Lie theorems for generalized translation operators, *Usp. Mat. Nauk* **16** (1961), 3-30.
34. B. M. Levitan, *Generalized translation operators and some their applications*, Nauka: Moscow, 1962.
35. B. M. Levitan, *Theory of generalized translation operators*, Nauka: Moscow, 1973.

Notes

36. G. L. Litvinov, On generalized translation operators and their representations, *Tr. Sem. Vekt. Anal. MGU* **18** (1978), 345-371.
37. A. S. Okb El Bab, A. M. Zabel, H. A. Ghany, Harmonic analysis in Hypercomplex systems, *International Journal of Pure and Applied Mathematics*, **80** (2012), 739-750.
38. G. F. Us, Dual Appel systems in Poissonian analysis. *Methods Funct. Anal. Topopology* (1995), 93-108.
39. G. C. Wick, The evaluation of the collinear matrix, *Phys. Rev.* **80** (1950), 268-272.
40. A. M. Zabel and Buthinah A. Bin Dehaish, Negative definite functions on hypercomplex systems, *KYUNGPOOK Math. Journal* **46** (2006), 285-295.
41. A. M. Zabel and Buthinah A. Bin Dehaish, Ljevy Khinchin formula on commutative hypercomplex system, *KYUNGPOOK Math. Journal* **48** (2008), 559-575.

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Visualizing Finite Field of Order p^2 through Matrices

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Abstract- In this article we consider finite field of order $p^2(p \neq 2)$. We provide matrix representation of finite field of order at most $p^2 = 121$. However this article provides a technique to yield matrix field of order p^2 for every prime $p > 2$.

Keywords: finite field, matrix field, Galois field.

GJSFR-F Classification : MSC 2010: 12E20

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Visualizing Finite Field of Order p^2 through Matrices

S. K. Pandey

Abstract- In this article we consider finite field of order $p^2(p \neq 2)$. We provide matrix representation of finite field of order at most $p^2 = 121$. However this article provides a technique to yield matrix field of order p^2 for every prime $p > 2$.

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MSCS2010: 12E20.

I. INTRODUCTION

There is a finite field of order p^n for every positive prime p and positive integer n. If F is a finite field of order p and $F[x]$ is the polynomial domain defined over F then one can construct a finite field $K = \frac{F[x]}{[f(x)]}$ of order p^n, where $f(x) \in F[x]$ is an irreducible polynomial of degree n and $[f(x)]$ is the ideal generated by $f(x)$. This field K has a subfield isomorphic with the field F. One may refer [1, 2] for further details. This is the usual method of construction of finite fields.

In this article we construct finite matrix fields of order p^2 without adopting the usual method of construction of finite fields. We give finite matrix field of order at most 121. However following the approach given in this article one can construct finite matrix field of order p^2 for each $p \neq 2$.

II. FINITE MATRIX FIELD OF ORDER p^2

In [3] one can find different matrix representations of a finite field of order p. Let p be a prime number. Then $Z_p = \{0,1,2,3,4,5...p-1\}$ is a field under addition and multiplication modulo p. One matrix representation of Z_p as given in [3] is $F_p = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in Z_p \right\}$. Every field of characteristic p contains a copy of a field of order

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p. All the fields given below are finite matrix field of order p^2. One can easily see that all these fields contain F_p.

We use very simple technique to obtain matrix fields of order p^2. It is seen that $F = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in R \right\}$ is isomorphic to the field C of complex numbers. Here R stands for the field of real numbers. We notice that if we replace R by Z_p then we obtain a finite matrix field of order p^2. It may be noted that this approach does not work for $p = 2$.

Notes

a) *Finite Matrix Field Of Order* 9

$$M_9 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \right\}$$

This is a finite field of order 9 under addition and multiplication of matrices modulo 3. It is a field of characteristic 3 and it contains $F_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$.

b) *Finite Matrix Field Of Order* 25

$$M_{25} = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}, \\ \begin{pmatrix} 4 & 1 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 2 & 3 \end{pmatrix}, \\ \begin{pmatrix} 3 & 2 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix} \end{matrix} \right\}$$

This is a finite field of order 25 under addition and multiplication of matrices modulo 5. One can see that it is a field of characteristic 5 and it contains

$$F_5 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \right\}.$$

c) *Finite Matrix Field Of Order* 49

$$M_{49} = \left\{ \begin{matrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 4 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 6 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 2 & 2 \end{pmatrix}, \\ \begin{pmatrix} 2 & 2 \\ 5 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 6 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}, \\ \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 5 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 6 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 6 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 5 & 4 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 3 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 5 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 6 & 4 \\ 3 & 6 \end{pmatrix}, \\ \begin{pmatrix} 6 & 3 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 5 \\ 2 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 2 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 1 \\ 6 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 6 \\ 1 & 6 \end{pmatrix} \end{matrix} \right\}$$

This is a finite field of order 49 under addition and multiplication of matrices modulo 7. One can see that it is a field of characteristic 7 and it contains

$$F_7=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}3&0\\0&3\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},\begin{pmatrix}5&0\\0&5\end{pmatrix},\begin{pmatrix}6&0\\0&6\end{pmatrix}\right\}.$$

d) *Finite Matrix Field Of Order* 121

$$M_{121}=\left\{\begin{array}{l}
\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}3&0\\0&3\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},\begin{pmatrix}5&0\\0&5\end{pmatrix},\begin{pmatrix}6&0\\0&6\end{pmatrix},\begin{pmatrix}7&0\\0&7\end{pmatrix},\begin{pmatrix}8&0\\0&8\end{pmatrix},\begin{pmatrix}9&0\\0&9\end{pmatrix},\begin{pmatrix}10&0\\0&10\end{pmatrix}, \\
\begin{pmatrix}0&10\\1&0\end{pmatrix},\begin{pmatrix}0&1\\10&0\end{pmatrix},\begin{pmatrix}0&9\\2&0\end{pmatrix},\begin{pmatrix}0&2\\9&0\end{pmatrix},\begin{pmatrix}0&8\\3&0\end{pmatrix},\begin{pmatrix}0&3\\8&0\end{pmatrix},\begin{pmatrix}0&4\\7&0\end{pmatrix},\begin{pmatrix}0&7\\4&0\end{pmatrix},\begin{pmatrix}0&6\\5&0\end{pmatrix},\begin{pmatrix}0&5\\6&0\end{pmatrix},\begin{pmatrix}1&10\\1&1\end{pmatrix}, \\
\begin{pmatrix}1&1\\10&1\end{pmatrix},\begin{pmatrix}1&2\\9&1\end{pmatrix},\begin{pmatrix}1&9\\2&1\end{pmatrix},\begin{pmatrix}1&8\\3&1\end{pmatrix},\begin{pmatrix}1&3\\8&1\end{pmatrix},\begin{pmatrix}1&4\\7&1\end{pmatrix},\begin{pmatrix}1&7\\4&1\end{pmatrix},\begin{pmatrix}1&6\\5&1\end{pmatrix},\begin{pmatrix}1&5\\6&1\end{pmatrix},\begin{pmatrix}2&10\\1&2\end{pmatrix},\begin{pmatrix}2&1\\10&2\end{pmatrix}, \\
\begin{pmatrix}2&9\\2&2\end{pmatrix}\begin{pmatrix}2&2\\9&2\end{pmatrix},\begin{pmatrix}2&8\\3&2\end{pmatrix},\begin{pmatrix}2&3\\8&2\end{pmatrix},\begin{pmatrix}2&7\\4&2\end{pmatrix},\begin{pmatrix}2&4\\7&2\end{pmatrix},\begin{pmatrix}2&6\\5&2\end{pmatrix},\begin{pmatrix}2&5\\6&2\end{pmatrix},\begin{pmatrix}3&10\\1&3\end{pmatrix},\begin{pmatrix}3&1\\10&3\end{pmatrix},\begin{pmatrix}3&9\\2&3\end{pmatrix}, \\
\begin{pmatrix}3&2\\9&3\end{pmatrix},\begin{pmatrix}3&8\\3&3\end{pmatrix},\begin{pmatrix}3&3\\8&3\end{pmatrix},\begin{pmatrix}3&7\\4&3\end{pmatrix},\begin{pmatrix}3&4\\7&3\end{pmatrix},\begin{pmatrix}3&6\\5&3\end{pmatrix},\begin{pmatrix}3&5\\6&3\end{pmatrix},\begin{pmatrix}4&10\\1&4\end{pmatrix},\begin{pmatrix}4&1\\10&4\end{pmatrix},\begin{pmatrix}4&2\\9&4\end{pmatrix},\begin{pmatrix}4&9\\2&4\end{pmatrix}, \\
\begin{pmatrix}4&8\\3&4\end{pmatrix},\begin{pmatrix}4&3\\8&4\end{pmatrix},\begin{pmatrix}4&7\\4&4\end{pmatrix},\begin{pmatrix}4&4\\7&4\end{pmatrix},\begin{pmatrix}4&6\\5&4\end{pmatrix},\begin{pmatrix}4&5\\6&4\end{pmatrix},\begin{pmatrix}5&10\\1&5\end{pmatrix},\begin{pmatrix}5&1\\10&5\end{pmatrix},\begin{pmatrix}5&9\\2&5\end{pmatrix},\begin{pmatrix}5&2\\9&5\end{pmatrix},\begin{pmatrix}5&3\\8&5\end{pmatrix}, \\
\begin{pmatrix}5&8\\3&5\end{pmatrix},\begin{pmatrix}5&7\\4&5\end{pmatrix},\begin{pmatrix}5&4\\7&5\end{pmatrix},\begin{pmatrix}5&6\\5&5\end{pmatrix},\begin{pmatrix}5&5\\6&5\end{pmatrix},\begin{pmatrix}6&10\\1&6\end{pmatrix},\begin{pmatrix}6&1\\10&6\end{pmatrix},\begin{pmatrix}6&9\\2&6\end{pmatrix},\begin{pmatrix}6&2\\9&6\end{pmatrix},\begin{pmatrix}6&8\\3&6\end{pmatrix},\begin{pmatrix}6&3\\8&6\end{pmatrix}, \\
\begin{pmatrix}6&7\\4&6\end{pmatrix},\begin{pmatrix}6&4\\7&6\end{pmatrix},\begin{pmatrix}6&6\\5&6\end{pmatrix},\begin{pmatrix}6&5\\6&6\end{pmatrix},\begin{pmatrix}7&10\\1&7\end{pmatrix},\begin{pmatrix}7&1\\10&7\end{pmatrix},\begin{pmatrix}7&9\\2&7\end{pmatrix},\begin{pmatrix}7&2\\9&7\end{pmatrix},\begin{pmatrix}7&8\\3&7\end{pmatrix},\begin{pmatrix}7&3\\8&7\end{pmatrix},\begin{pmatrix}7&7\\4&7\end{pmatrix}, \\
\begin{pmatrix}7&4\\7&7\end{pmatrix},\begin{pmatrix}7&6\\5&7\end{pmatrix},\begin{pmatrix}7&5\\6&7\end{pmatrix},\begin{pmatrix}8&10\\1&8\end{pmatrix},\begin{pmatrix}8&1\\10&8\end{pmatrix},\begin{pmatrix}8&9\\2&8\end{pmatrix},\begin{pmatrix}8&2\\9&8\end{pmatrix},\begin{pmatrix}8&8\\3&8\end{pmatrix},\begin{pmatrix}8&3\\8&8\end{pmatrix},\begin{pmatrix}8&7\\4&8\end{pmatrix},\begin{pmatrix}8&4\\7&8\end{pmatrix}, \\
\begin{pmatrix}8&6\\5&8\end{pmatrix},\begin{pmatrix}8&5\\6&8\end{pmatrix},\begin{pmatrix}9&10\\1&9\end{pmatrix},\begin{pmatrix}9&1\\10&9\end{pmatrix},\begin{pmatrix}9&9\\2&9\end{pmatrix},\begin{pmatrix}9&2\\9&9\end{pmatrix},\begin{pmatrix}9&8\\3&9\end{pmatrix},\begin{pmatrix}9&3\\8&9\end{pmatrix},\begin{pmatrix}9&7\\4&9\end{pmatrix},\begin{pmatrix}9&4\\7&9\end{pmatrix},\begin{pmatrix}9&6\\5&9\end{pmatrix}, \\
\begin{pmatrix}9&5\\6&9\end{pmatrix},\begin{pmatrix}10&10\\1&10\end{pmatrix},\begin{pmatrix}10&1\\10&10\end{pmatrix},\begin{pmatrix}10&9\\2&10\end{pmatrix},\begin{pmatrix}10&2\\9&10\end{pmatrix},\begin{pmatrix}10&8\\3&10\end{pmatrix},\begin{pmatrix}10&3\\8&10\end{pmatrix},\begin{pmatrix}10&4\\7&10\end{pmatrix},\begin{pmatrix}10&7\\4&10\end{pmatrix}, \\
\begin{pmatrix}10&6\\5&10\end{pmatrix},\begin{pmatrix}10&5\\6&10\end{pmatrix}
\end{array}\right\}$$

This is a finite field of order 121 under addition and multiplication of matrices modulo 11. It has characteristic 11 and it contains

$$F_{11}=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}3&0\\0&3\end{pmatrix},\begin{pmatrix}4&0\\0&4\end{pmatrix},\begin{pmatrix}5&0\\0&5\end{pmatrix},\begin{pmatrix}6&0\\0&6\end{pmatrix},\begin{pmatrix}7&0\\0&7\end{pmatrix},\begin{pmatrix}8&0\\0&8\end{pmatrix},\begin{pmatrix}9&0\\0&9\end{pmatrix},\begin{pmatrix}10&0\\0&10\end{pmatrix}\right\}$$

Following same technique one can find finite matrix field of order p^2 for $p > 11$.

REFERENCES RÉFÉRENCES REFERENCIAS

1. M. Artin, Algebra, Prentice Hall of India Private Limited, New Delhi, 2000.
2. T. W. Hungerford, Algebra, Springer-India, New Delhi 2005.
3. S. K. Pandey, Matrix Field of Finite and Infinite Order, Research Journal of Pure Algebra **5** (12), 214-216, 2015.

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Mixed Quadrature Rule for Double Integrals of Simpsons $\frac{1}{3}rd$ and Gauss Legendre Two Point Rule in Two Variables

By Saumya Ranjan Jena & Subash Chandra Mishra

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GJSFR-F Classification : *MSC2010: 65D30, 65D32*

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3. Jena, S.,Dash, P., (2014)“Approximaton of real definite integrals via hybrid quadrature domain” *IJSETR*, ,3(42),.

Mixed Quadrature Rule for Double Integrals of Simpsons $\frac{1}{3}rd$ and Gauss Legendre Two Point Rule in Two Variables

Saumya Ranjan Jena [α] & Subash Chandra Mishra [σ]

Abstract- In this article we are concerned with mixed quadrature rule of higher degree of precision for double integrals with double variables. The rule is numerically tested taking some suitable texts and the error bound is determined.

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I. INTRODUCTION

A Mixed quadrature rule of higher degree of precision has been formed by [2], [3], [4], [5], [6], [7]. This rule is meant for single integral. In the same vein here we develop a mixed quadrature rule of degree of precision-7 for double integrals taking the convex combination of Simpson’s $\frac{1}{3}$rd and Gauss-Legendre-2 point rule each of degree of precision 5. This paper is designed as follows. Section -2 contains formulation of quadrature of constituent rules and the corresponding errors in-2 variables. Section-3 is devoted to construction of mixed quadrature rule. The error analysis is done in section-4. In section-5 the rule is numerically verified by taking suitable examples. The conclusions are drawn in section-6.

II. FORMULATION OF QUADRATURE RULE IN TWO VARIABLES X AND Y

For approximate evaluation of real definite integral

$$I(f)=\int_{-1}^{1}\int_{-1}^{1}f(x,y)\,dxdy \tag{2.1}$$

The Simpson’s $\frac{1}{3}$ *rd rule is*

$$I(f)\cong R_{s_{\frac{1}{3}}}(f)=\frac{1}{9}\begin{bmatrix}\{f(-1,-1)+4f(0,-1)+f(1,-1)\}\\+4\{f(-1,0)+4f(0,0)+f(1,0)\}\\+\{f(-1,1)+4f(0,1)+f(1,1)\}\end{bmatrix} \tag{2.1}$$

And the Gauss-Legendre’s two point rule is

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$$I(f) \cong R_{GL2}(f) = f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)+f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \qquad (2.3)$$

where each rule of equation (2.2) and equation (2.3) is of precision 3.

Hence

$$I(f) = R_{s\frac{1}{3}}(f) + E_{s\frac{1}{3}}(f) \qquad (2.4)$$

$$I(f) = R_{GL2}(f) + E_{GL2}(f) \qquad (2.5)$$

Where $E_{S\frac{1}{3}}(f)$ and $E_{GL2}(f)$ are error in approximating the integrals $I(f)$ by equation (2.2) and equation (2.3) respectively.

Now assuming $f(x,y)$ to be sufficiently differentiable in $-1 \le x \le 1, -1 \le y \le 1$

We can write equation (2.1) using Maclaurin's expansion

$$I(f)=\int_{-1}^{1}\int_{-1}^{1}\left[\begin{array}{l}
f(0,0)+\{x\, f_{1,0}(0,0)+y\, f_{0,1}(0,0)\} \\
+\frac{1}{2!}\{x^2\, f_{2,0}(0,0)+2xy\, f_{1,1}(0,0)+y^2\, f_{0,2}(0,0)\} \\
+\frac{1}{3!}\{x^3\, f_{3,0}(0,0)+3x^2y\, f_{2,1}(0,0)+3xy^2\, f_{1,2}(0,0)+y^3\, f_{0,3}(0,0)\} \\
+\frac{1}{4!}\left\{\begin{array}{l} x^4\, f_{4,0}(0,0)+4x^3y\, f_{3,1}(0,0)+6x^2y^2\, f_{2,2}(0,0) \\ +4xy^3\, f_{1,3}(0,0)+y^4\, f_{0,4}(0,0)\end{array}\right\} \\
+\frac{1}{5!}\left\{\begin{array}{l} x^5\, f_{5,0}(0,0)+5x^4y\, f_{4,1}(0,0)+10x^3y^2\, f_{3,2}(0,0) \\ +10x^2y^3\, f_{2,3}(0,0)+5xy^4\, f_{1,4}(0,0)+y^5\, f_{0,5}(0,0)\end{array}\right\} \\
+\frac{1}{6!}\left\{\begin{array}{l} x^6\, f_{6,0}(0,0)+6x^5y\, f_{5,1}(0,0)+15x^4y^2\, f_{4,2}(0,0) \\ +20x^3y^3\, f_{3,3}(0,0)+15x^2y^4\, f_{2,4}(0,0) \\ +6xy^5\, f_{1,5}(0,0)+y^6\, f_{0,6}(0,0)\end{array}\right\}
\end{array}\right]dxdy \qquad (2.6)$$

Integrating equation (2.6) we have

$$\begin{aligned}I(f) &= 4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right] \\ &+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{4}{7!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\ldots.\end{aligned} \qquad (2.7)$$

Again using Maclaurin's expansion

$$\begin{aligned}f(-1,-1) &= f_{0,0}-\left[f_{1,0}(0,0)+f_{0,1}(0,0)\right]+\frac{1}{2!}\left[f_{2,0}(0,0)+2f_{1,1}(0,0)+f_{0,2}(0,0)\right] \\ &-\frac{1}{3!}\left[f_{3,0}(0,0)+3f_{2,1}(0,0)+3f_{1,2}(0,0)+f_{0,3}(0,0)\right] \\ &+\frac{1}{4!}\left[f_{4,0}(0,0)+4f_{3,1}(0,0)+6f_{2,2}(0,0)+4f_{1,3}(0,0)+f_{0,4}(0,0)\right]\end{aligned}$$

$$-\frac{1}{5!}\left[f_{5,0}(0,0)+5f_{4,1}(0,0)+10f_{3,2}(0,0)+10f_{2,3}(0,0)+5f_{1,4}(0,0)+f_{0,5}(0,0)\right]$$

$$+\frac{1}{6!}\left[f_{6,0}(0,0)+6f_{5,1}(0,0)+15f_{4,2}(0,0)+20f_{3,3}(0,0)+15f_{2,4}(0,0)+6f_{1,5}(0,0)+f_{0,6}(0,0)\right]$$

$$+.............. \tag{2.8}$$

Notes

Using equation (2.8) and $f(0,-1), f(1,-1), f(-1,0), f(1,0), f(0,1), f(-1,1)$ *and* $f(1,1)$ in equation (2.2)

$$R_{s\frac{1}{3}}(f)=4f_{0,0}(0,0)+\frac{2}{3}\left\{f_{2,0}(0,0)+f_{0,2}(0,0)\right\}+\frac{1}{18}\left\{f_{4,0}(0,0)+f_{0,4}(0,0)\right\}$$

$$+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{108}\left\{f_{4,2}(0,0)+f_{2,4}(0,0)\right\}+\frac{11}{9\times 6!}\left\{f_{6,0}(0,0)+f_{0,6}(0,0)\right\} \tag{2.9}$$

a) Error in Simpson's $\frac{1}{3}$ *rd rule:*

Using equation (2.6), equation (2.9) in equation (2.4) error associated with Simpson's $\frac{1}{3}rd$ rule is

$$E_{S\frac{1}{3}}(f)=I(f)-R_{S\frac{1}{3}}(f)$$

$$E_{S\frac{1}{3}}(f)=-\frac{1}{45}\left\{f_{4,0}(0,0)+f_{0,4}(0,0)\right\}-\frac{1}{270}\left\{f_{4,2}(0,0)+f_{2,4}(0,0)\right\}-$$

$$\frac{41}{63\times 6!}\left\{f_{6,0}(0,0)+f_{0,6}(0,0)\right\} \tag{2.10}$$

As the error contains at least fourth derivative of the integral function, it vanishes for all polynomials of degree ≤ 3 in x and y. Thus the formula becomes exact for all polynomials of degree up to 3 i.e. degree of precision of the formula is three.

Similarly using Maclaurin's expansion

We have

$$f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

$$=f(0,0)-\frac{1}{\sqrt{3}}\left[f_{1,0}(0,0)+f_{0,1}(0,0)\right]$$

$$+\frac{1}{2!}\left[\left(-\frac{1}{\sqrt{3}}\right)^2 f_{2,0}(0,0)+2\left(-\frac{1}{\sqrt{3}}\right)^2 f_{1,1}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^2 f_{0,2}(0,0)\right]$$

$$+\frac{1}{3!}\left[\left(-\frac{1}{\sqrt{3}}\right)^3 f_{3,0}(0,0)+3\left(-\frac{1}{\sqrt{3}}\right)^3 f_{2,1}(0,0)+3\left(-\frac{1}{\sqrt{3}}\right)^3 f_{1,2}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^3 f_{0,3}(0,0)\right]$$

Notes

$$+\frac{1}{4!}\left[\begin{array}{l}\left(-\frac{1}{\sqrt{3}}\right)^4 f_{4,0}(0,0)+4\left(-\frac{1}{\sqrt{3}}\right)^4 f_{3,1}(0,0)+6\left(-\frac{1}{\sqrt{3}}\right)^4 f_{2,2}(0,0)\\ +4\left(-\frac{1}{\sqrt{3}}\right)^4 f_{1,3}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^4 f_{0,4}(0,0)\end{array}\right]$$

$$+\frac{1}{5!}\left[\begin{array}{l}\left(-\frac{1}{\sqrt{3}}\right)^5 f_{5,0}(0,0)+5\left(-\frac{1}{\sqrt{3}}\right)^5 f_{4,1}(0,0)+10\left(-\frac{1}{\sqrt{3}}\right)^5 f_{3,2}(0,0)\\ +10\left(-\frac{1}{\sqrt{3}}\right)^5 f_{2,3}(0,0)+5\left(-\frac{1}{\sqrt{3}}\right)^5 f_{1,4}(0,0)+\left(-\frac{1}{\sqrt{3}}\right)^5 f_{0,5}(0,0)\end{array}\right]$$

$$+\frac{1}{6!}\left[\begin{array}{l}\left(-\frac{1}{\sqrt{3}}\right)^6 f_{6,0}(0,0)+6\left(-\frac{1}{\sqrt{3}}\right)^6 f_{5,1}(0,0)+15\left(-\frac{1}{\sqrt{3}}\right)^6 f_{4,2}(0,0)\\ +20\left(-\frac{1}{\sqrt{3}}\right)^6 f_{3,3}(0,0)+15\left(-\frac{1}{\sqrt{3}}\right)^6 f_{2,4}(0,0)\\ +6\left(-\frac{1}{\sqrt{3}}\right)^6 f_{1,5}(0,0)++\left(-\frac{1}{\sqrt{3}}\right)^6 f_{0,6}(0,0)\end{array}\right]\ldots\ldots \qquad (2.11)$$

Now *$f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, can be found out same ways as* (2.10)

Now substituting the value of

$$f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

in equation (2.3)

$$R_{GL2}(f)=4f(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{9\times 3!}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]$$

$$+\frac{20}{9\times 6!}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{1}{9}f_{2,2}(0,0)+\frac{4}{27\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] \qquad (2.12)$$

b) Error in Gauss-Legendre's two point rule

Now the error associated with the Gauss- Legendre 2-point rule is obtained substituting equation (2.6) and (2.11) in equation (2.5)

$$E_{GL2}(f)=I(f)-R_{GL2}(f)$$

$$=\frac{2}{135}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]+\frac{1}{405}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{1}{1701}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] \qquad (2.13)$$

In this case also, the error contains at least fourth derivative of the integral function. Thus the degree of the precision is 3.

III. Mixed Quadrature Rule

Now multiplying (2) by equation (2.10) and (3) by equation (2.13) and adding them, we get

$$I(f)=\frac{1}{5}\left[2R_{S\frac{1}{3}}(f)+3R_{GL2}(f)\right]+\frac{1}{5}\left[2E_{S\frac{1}{3}}(f)+3E_{GL2}(f)\right] \tag{3.1}$$

Where

$$I_{mix}(f)=R_{S\frac{1}{3}GL2}(f)+E_{S\frac{1}{3}GL2}(f) \tag{3.2}$$

Where

$$R_{S\frac{1}{3}GL2}(f)=\frac{1}{5}\left[2R_{s\frac{1}{3}}(f)+3R_{GL2}(f)\right] \tag{3.3}$$

Where $R_{S\frac{1}{3}GL2}(f)$ and $E_{S\frac{1}{3}GL2}(f)$ are mixed quadrature rule and its error obtained by Simpson's $\frac{1}{3}rd$ and Gauss-Legendre's 2 point rule respectively.

The truncation error generated by this approximation is given by

$$E_{S\frac{1}{3}GL2}=\frac{1}{5}\left[2E_{s\frac{1}{3}}(f)+3R_{GL2}(f)\right]=-\frac{1}{189\times 5!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] \tag{3.4}$$

The rule (3.3) may be called as mixed quadrature rule integrate exactly all polynomial of degree ≤ 5 in x and y.

IV. Error Analysis

a) Theorem 4.1

Let $f(x,y)$ be sufficiently differentiable function in the closed interval $[-1,1]$. The bounds of truncatinal error $E_{S\frac{1}{3}GL2}(f)$ associated with the rule $R_{S\frac{1}{3}GL2}(f)$ is given by

$$\left|E_{S\frac{1}{3}GL2}(f)\right|\cong\frac{1}{189\times 5!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]$$

Proof:

The proof obviously follows from the equation (3.4)

Theorem 4.2

The bounds for the truncational error $\left|E_{S\frac{1}{3}GL2}(f)\right|\leq\frac{2M}{225}\left|\eta_2-\eta_1\right|$

Where $\eta_1,\eta_2\in[-1,1]$

Where $M=Max\left|f_{6,0}(0,0)+f_{0,6}(0,0)\right|,\ -1\leq x\leq 1,\ -1\leq y\leq 1$

Proof:
We have

$$E_{S\frac{1}{3}}(f) = -\frac{1}{45}\left|f_{4,0}(n_1) + f_{0,4}(n_1)\right|$$

$$E_{GL2}(f) = \frac{2}{135}\left|f_{4,0}(n_2) + f_{0,4}(n_2)\right|$$

Where

$$n_1, n_2 \in [-1,1]$$

Hence

$$E_{S\frac{1}{3}GL2}(f) = \frac{1}{5}\left|2E_{s\frac{1}{3}}(f) + 3E_{GL2}(f)\right|$$

$$= \frac{2}{225}\left[f_{4,0}(n_2,0) + f_{0,4}(0,n_2) - f_{4,0}(n_1,0) - f_{0,4}(0,n_1)\right]$$

$$= \frac{2}{225}\int_{n_1}^{n_2}\int_{n_1}^{n_2}\left[f_{5,0}(x,*) + f_{0,5}(*,y)\right]$$

Hence

$$\left|E_{S\frac{1}{3}GL2}(f)\right| \le \frac{2M\left|n_1 - n_2\right|}{225}$$

Where

$$M = \max\left|f_{5,0}(x,*) + f_{0,5}(*,y)\right| \qquad -1 \le x \le 1, \quad -1 \le y \le 1$$

Which, gives only the truncation error bound on n_1, n_2 are closed to each other.

b) Corollary

The error bound for the truncated error $\left|E_{S\frac{1}{3}GL2}(f)\right| \le \frac{4M}{225}$

When $\left|n_1 - n_2\right| \le 2$ [1]

V. NUMERICAL VERIFICATION

The exact value of the integrals

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} e^{x+y}dxdy = 5.524391382167262$$

$$I_1 = \int_{-1}^{1}\int_{-1}^{1} e^{-\left(x^2+y^2\right)}dxdy = 2.230985141404134$$

$$I_3 = \int_{0}^{1}\int_{0}^{1} \frac{\sin^2(x+y)}{(x+y)}dxdy = 0.613260369981918$$

Table-1 : (Comparison of exact result with approximate result)

Exact	$R_{S\frac{1}{3}}(f)$	$R_{GL2}(f)$	$R_{S\frac{1}{3}GL2}(f)$	$E_{S\frac{1}{3}GL2}(f)$
I_1 =5.524391382167262	5.577469135802469	5.488065843621398	5.523827160493826	0.00056421673436
I_3 =2.230985141404134	2.259259259259260	2.024691358024692	2.238518518518519	0.007533377114385
I_3 =0.613260369981918	0.612749599379052	0.609491314229328	0.610769318605380	0.002491051376538

VI. Conclusion

From the Table -1, we find that

$$\left|E_{S\frac{1}{3}}(f)\right| \leq \left|E_{GL2}(f)\right| \qquad (i)$$

$$\left|E_{S\frac{1}{3}GL2}(f)\right| \leq \left|E_{S\frac{1}{3}}(f)\right| \qquad (ii)$$

From (i) and (ii)

$$\left|E_{S\frac{1}{3}GL2}(f)\right| \leq \left|E_{S\frac{1}{3}}(f)\right| \leq \left|E_{GL2}(f)\right|$$

It is evident that the mixed quadrature rule $R_{S\frac{1}{3}GL2}(f)$ of degree of precision 5 provides us better result than constituent rule $R_{S\frac{1}{3}}(f), R_{GL2}(f)$ each of degree of precision three. Hence, the mixed quadrature rule is more efficient and numerically better convergent then exact result.

References Références Referencias

1. Conte ,S. Boor, C.De. (1980), "*Elementary Numerical Analysis*", Tata Mac-Graw Hill.
2. Jena,S.R. and Dash,R.B., (2009) "Mixed quadrature of real definite integral over triangles" *PAJM*, ,3, pp.119-124.
3. Jena, S.,Dash, P., (2014)"Approximaton of real definite integrals via hybrid quadrature domain" *IJSETR*, ,3(42),.
4. Kreyszig, E.,(2012), "*Advanced Engg.Mathematics,*" Wiley India.
5. Ma . J. Rokhlin, V.,Wandzura S., (1996)" *Generalised* Gaussian quadrature rules for systems for systems of arbitrary functions", *SIAM Journal of Numerical Analysis* ; 33(3) 971-996.
6. Das R. N. and Pradhan G., (1996) "A mixed quadrature rule for approximate evaluation of real definite integral". *Int. J. Math. Educ. Sci.Technol*, , 27(2), 279-283.
7. Jena, S.R. and Dash, R.B., (2011) "Study of approximate value of real definite integral by mixed quadrature rule obtained from Richardson extrapolation", *IJCSM* 3(1),pp.47-53.

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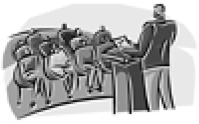

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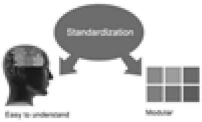

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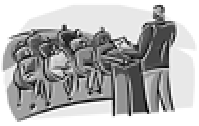

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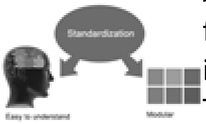

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10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

INFORMAL GUIDELINES OF RESEARCH PAPER WRITING

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

THE ADMINISTRATION RULES

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

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- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.
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- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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