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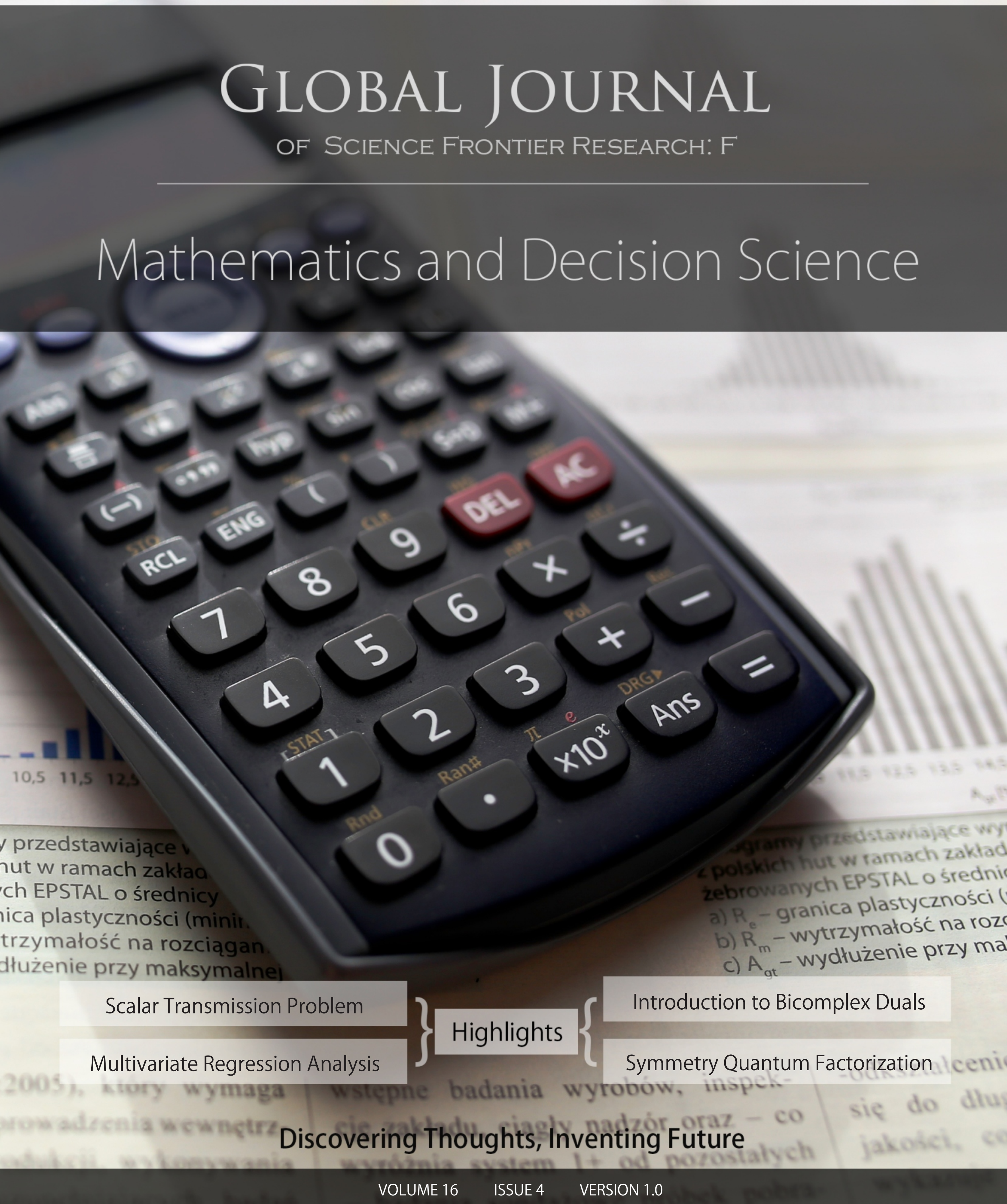

Global Journal

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Mathematics and Decision Science

Highlights

- Scalar Transmission Problem
- Multivariate Regression Analysis
- Introduction to Bicomplex Duals
- Symmetry Quantum Factorization

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Multi-Criteria Optimization Problems with Non-Smooth Functions

By H. S. Faruque Alam
University of Chittagong

Abstract- In this paper, we deal with non-smooth multi criteria optimization problem. Throughout the text we consider the functions which are non-differentiable. A few special derivatives have been deployed to obtain first order optimality conditions. We presented the generalized form of optimality conditions for multi-objective optimization problem when classical derivatives failed to apply. We illustrate these optimality conditions by means of suitable examples.

Keywords: *non smooth function, multi-criteria optimization, optimality condition.*

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Multi-Criteria Optimization Problems with Non-Smooth Functions

H. S. Faruque Alam

Abstract- In this paper, we deal with non-smooth multi criteria optimization problem. Throughout the text we consider the functions which are non-differentiable. A few special derivatives have been deployed to obtain first order optimality conditions. We presented the generalized form of optimality conditions for multi-objective optimization problem when classical derivatives failed to apply. We illustrate these optimality conditions by means of suitable examples.

Keywords: non smooth function, multi-criteria optimization, optimality condition.

I. Introduction

We intend the term "non-smooth" to refer to certain situations in which smoothness (differentiability) of the data is not necessarily postulated. If we want to use the first order necessary condition, we have to require that the function under discussion actually has a derivative. Recent research is concerned with problems that do not have this property. So we need several alternative concepts of generalized directional derivatives that allow to establish an analysis for non-smooth functions. First order non-smooth necessary optimality conditions for single and multi-objective optimization problems, have been provided by many authors. Among those we may refer to Bigi [1], Clarke [2], Stein [3], Yang [4,5] etc. Of these, Preda [6] established the necessary conditions for semi differentiable function, where the Lagrange multipliers associated with each of the objective function are positive.

In section 3, we review and prove first order optimality conditions for non-smooth optimization problems. We have extended to multi objective non-smooth optimization the approach introduced by O. Stein [3] for scalar objective optimization. To deal with multi objective optimization a number of approaches have been proposed to develop a necessary optimality conditions with example. Also have discussed some review result bases on G. Bigi's [1] work.

II. Preliminaries

In this section, we introduce some notations and definitions, which are used throughout the paper. Let E_n be n-dimensional Euclidean space.

For $\mathbf{x},\mathbf{y} \in E_n$, we use the following conventions.

$$\mathbf{x} \geqq \mathbf{y}, \quad \text{iff } x_i \geqq y_i, \quad i=1,\dots,n,$$

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$$\mathbf{x} \geq \mathbf{y}, \quad \text{iff}\, \mathbf{x} \geqq \mathbf{y} \quad \text{and} \quad \mathbf{x} \neq \mathbf{y},$$

$$\mathbf{x} > \mathbf{y}, \quad \text{iff}\, x_i > y_i \; \text{i=1,…,n},$$

At first, we consider the following multi objective optimization problem **P**: $\min \mathbf{f}(\mathbf{x})$, subject to the conditions that the minimizing point (or vector) $\bar{\mathbf{x}}$ should lie in the set X:

$$\bar{\mathbf{x}} \in \mathrm{X} = \left\{\mathbf{x} \in \mathrm{E}_n \middle| \mathbf{g}(\mathbf{x}) \leqq 0, \mathbf{h}(\mathbf{x}) = 0\right\}$$

Let, $f:\mathrm{E_n} \to \mathrm{E_1}$, $g:\mathrm{E_n} \to \mathrm{E_m}$ and $h:\mathrm{E_n} \to \mathrm{E_p}$ be vector-valued functions defined by $\mathbf{f}(\mathbf{x}) \equiv (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_l(\mathbf{x}))$, $\mathbf{g}(\mathbf{x}) \equiv (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_m(\mathbf{x}))$ and $\mathbf{h}(\mathbf{x}) \equiv (h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots, h_p(\mathbf{x}))$ where $f_i:\mathrm{E_n} \to \mathrm{E_1}$ for i=1,..,l, $g_j:\mathrm{E_n} \to \mathrm{E_1}$ for j=1,…,m and $h_k:\mathrm{E_n} \to \mathrm{E_1}$ for k=1,…,p. Assume that $I(\bar{\mathbf{x}}) = \{j : g_j(\bar{\mathbf{x}}) = 0\}$ for j=1,…,m.

III. First Order Optimality Conditions

Extensions of optimality conditions to non-smooth optimization problems can be found in the literature under different assumptions on the directional differentiability of the functions involved. Preda and Yang [4,6] use directional differentiability in the Gateaux differentiable sense; G. Bigi and Castalenni and Papalardo [1] use directional derivatives in the Dini-Hadamard sense; Kuntz and Scholtes [7,8] use quasi differentiability i.e. Dini directional differentiability where the directional derivative can be written as the difference of two sub linear functions; Clarke [2] assumes Clarke directional differentiability.

In contrast to these approaches, the chapter does not make any directional differentiability assumptions, but we just present a few component-wise optimality criteria based on Dini-Hadamard derivatives.

Definition 3.1 An efficient point $\bar{\mathbf{x}}$ for **P** is said to be of order one, if a bounded open neighbourhood N can be chosen such that there exists a positive constant c with

$$\mathbf{f}(\mathbf{x}) \geqq \mathbf{f}(\bar{\mathbf{x}}) + c\|\mathbf{x_n} - \bar{\mathbf{x}}\|, \text{ for all } \mathbf{x} \in \mathrm{X} \cap \mathrm{N}.$$

Also we define the inner tangent cone.

Definition3.2

$$\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}}) \equiv \left\{\begin{array}{l}\mathbf{d} \in \mathrm{E}_n \middle| \exists \bar{t} > 0 \text{ such that } \bar{\mathbf{x}} + \mathbf{t}\bar{\mathbf{d}} \in \mathrm{X}, \\ \text{for all } \mathrm{t} \in (0,\bar{\mathrm{t}}), \bar{\mathbf{d}} \in N(\mathbf{d})\end{array}\right\}$$

The following proposition cites some of the Laurent results [3,9,10] about the basic properties of these tangent cones, which will be important in the sequel.

Proposition 3.1 [See 3, 11, 12]. Let $\bar{\mathbf{x}} \in \mathrm{X} \subset \mathrm{E_n}$. Then

i) $\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}})$ and $\mathrm{T}(\mathrm{X};\bar{\mathrm{x}})$ are open and closed cones, respectively

ii) $\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}}) \subset \mathrm{T}(\mathrm{X};\bar{\mathrm{x}})$

iii) $\mathrm{T}^*(\mathrm{X};\bar{\mathrm{x}})^c = \mathrm{T}(\mathrm{X}^c;\bar{\mathrm{x}})$

Thus, we start recalling the following classical definitions of upper and lower directional derivatives of f at $\bar{\mathbf{x}}$ in the direction $\boldsymbol{d}$ in the Hadamard sense (see [2,3])

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Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

Definitions 3.3 The upper Hadamard derivative of $f : E_n \to E_1$ at $\bar{\mathbf{x}}$ in the direction $\boldsymbol{d}$ is

$$F^+ f(\bar{\mathbf{x}}, \mathbf{d}) = \lim_{d \to \bar{d}} \sup_{t \to 0^+} \frac{f(\bar{\mathbf{x}} + t\mathbf{d}) - f(\bar{\mathbf{x}})}{t}$$

Definitions 3.4 The lower Hadamard derivative of $f : E_n \to E_1$ at $\bar{\mathbf{x}}$ in the direction $\boldsymbol{d}$ is

Notes

$$F^- f(\bar{\mathbf{x}}, \mathbf{d}) = \lim_{d \to \bar{d}} \inf_{t \to 0^+} \frac{f(\bar{\mathbf{x}} + t\mathbf{d}) - f(\bar{\mathbf{x}})}{t}$$

It is easy to check that, for each $\mathbf{d} \neq \mathbf{0}$, we have $F^+ f(\bar{\mathbf{x}}, \mathbf{d}) = F^- f(\bar{\mathbf{x}}, \mathbf{d}) = \nabla f(\bar{\mathbf{x}})^{\mathbf{T}} \mathbf{d}$, whenever *f* is differentiable at $\bar{\mathbf{x}}$.
In this case, we write

$$Ff(\bar{\mathbf{x}}, \mathbf{d}) = \lim_{d \to \bar{d}, t \to 0^+} \frac{f(\bar{\mathbf{x}} + t\mathbf{d}) - f(\bar{\mathbf{x}})}{t}$$

Thus we can rely on the Hadamard derivatives of the components of the objective function *f* to study optimality for problem **P**.

Theorem 3.1

i) If $\bar{\mathbf{x}} \in X$ is an efficient solution of P then for any direction $\mathbf{d} \in T(X; \bar{x})$

the system $$F^+ f_i(\bar{\mathbf{x}})^{\mathbf{T}} \mathbf{d} < 0, i = 1, 2, ..., l \qquad \ldots \ldots \ldots \qquad (3.1)$$

has no solution $\mathbf{d} \in E_n$.

ii) If $\bar{\mathbf{x}} \in X$ is an efficient solution of order one for **P** then for any direction $\mathbf{d} \in T(X; \bar{x}) \setminus \{\mathbf{0}\}$ the system

$$F^+ f_i(\bar{\mathbf{x}})^{\mathbf{T}} \mathbf{d} \leqq 0, i = 1, 2, ..., l \qquad \ldots \ldots \ldots \qquad (3.2)$$

has no solution $\mathbf{d} \in E_n$.

iii) If $\bar{\mathbf{x}} \in X$ is an efficient solution of order one for **P** then for any direction $\mathbf{d} \in T(X; \bar{x}) \setminus \{\mathbf{0}\}$ the system $$F^- f_i(\bar{\mathbf{x}})^{\mathbf{T}} \mathbf{d} \leqq 0, i = 1, 2, ..., l \qquad \ldots \ldots \ldots \qquad (3.3)$$

has no solution $\mathbf{d} \in E_n$.

iv) Let $\boldsymbol{f}$ be directionally differentiable. If $\bar{\mathbf{x}} \in X$ is an efficient solution of order one for **P** then for any direction $\mathbf{d} \in T(X; \bar{x}) \setminus \{\mathbf{0}\}$

the system $$Ff_i(\bar{\mathbf{x}})^{\mathbf{T}} \mathbf{d} \leqq 0, i = 1, 2, ..., l \qquad \ldots \ldots \ldots \qquad (3.4)$$

has no solution $\mathbf{d} \in E_n$.

i) Let $\mathbf{d} \in T(X; \bar{x})$, that is $\mathbf{d} = \lim_{n \to \infty} t_n (\mathbf{x_n} - \bar{\mathbf{x}})$, where $t_n > 0$, $\mathbf{x}_n \in X$ for each n, and $\lim_{n \to \infty} \mathbf{x_n} = \bar{\mathbf{x}}$.

Since $\bar{\mathbf{x}}$ is an efficient solution, so there is no point $\mathbf{x}_n \in X$, where $\mathbf{f}(\mathbf{x}_n) \leq \mathbf{f}(\bar{\mathbf{x}})$

Then for all n we have

$$f_i(\mathbf{x}_n) = f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \leq f_i(\bar{\mathbf{x}}),\ i = 1,2,\ldots,l$$

for sufficiently large n. Consequently, it holds that

$$\lim_{d \to \bar{d}} \sup_{\frac{1}{t_n} \to 0^+} \frac{f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d}\right) - f_i(\bar{\mathbf{x}})}{\frac{1}{t_n}} \leq 0,\quad i = 1,2,\ldots,l$$

$\Rightarrow F^+ f_i(\bar{\mathbf{x}}, \mathbf{d}) \leq 0,\quad i = 1,2,\ldots,l$

which implies (3.1) has no solution.

ii) Let $\mathbf{d} \in \mathrm{T}(\mathrm{X}; \bar{\mathrm{x}}) \setminus \{\mathbf{0}\}$, that is $\mathbf{d} = \lim_{n\to\infty} t_n(\mathbf{x_n} - \bar{\mathbf{x}})$, where $t_n > 0$, $\mathbf{x}_n \in \mathrm{X}$ for each n, and $\lim_{n\to\infty} \mathbf{x_n} = \bar{\mathbf{x}}$ with $\mathbf{d} \neq \mathbf{0}$. We find a c such that

$$t_n f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{-}\ f_i(\bar{\mathbf{x}}) \geqq c t_n \|\mathbf{x_n} - \bar{\mathbf{x}}\|$$

for sufficiently large $n \in \mathrm{N}$, since $\bar{\mathbf{x}}$ is an efficient solution of order one, we have

$\limsup_{n\to\infty} \frac{1}{1/t_n} f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{-}\ f_i(\bar{\mathbf{x}}) \geqq c\|\mathbf{d}\| > 0$, as $\mathbf{d} \neq \mathbf{0}$. Thus, we have

$$F^+ f_i(\bar{\mathbf{x}})^{\mathrm{T}} \mathbf{d} > 0,\quad \mathrm{i} = 1, 2, \ldots, l$$

which implies (3.2) has no solution.

iii) Suppose that $\bar{\mathbf{x}}$ is not an efficient solution of order one. Let $\mathbf{d} \in \mathrm{T}(\mathrm{X}; \bar{\mathrm{x}}) \setminus \{\mathbf{0}\}$, that is $\mathbf{d} = \lim_{n\to\infty} t_n(\mathbf{x_n} - \bar{\mathbf{x}})$, where $t_n > 0$, $\mathbf{x}_n \in \mathrm{X}$ for each n, and $\lim_{n\to\infty} \mathbf{x_n} = \bar{\mathbf{x}}$ with $\mathbf{d} \neq \mathbf{0}$ and $\|\mathbf{d}\| = 1$.

Since $\bar{\mathbf{x}}$ is not an efficient solution of order one. Then, there exist sequences $\{c_n\}$ with $c_n \to 0$.

$$\frac{1}{1/t_n} f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{-}\ f_i(\bar{\mathbf{x}}) < c_n t_n \|\mathbf{x_n} - \bar{\mathbf{x}}\|$$

For sufficiently large $n \in \mathrm{N}$,

$\liminf_{n\to\infty} \frac{1}{1/t_n} f_i\left(\bar{\mathbf{x}} + \frac{1}{t_n}\mathbf{d_n}\right) \text{-}\ f_i(\bar{\mathbf{x}}) < 0$, as $\|\mathbf{d}\| = 1$ and $c_n \to 0$. Thus, we have

$$F^- f_i(\bar{\mathbf{x}})^{\mathrm{T}} \mathbf{d} < 0,\quad \mathrm{i} = 1, 2, \ldots, l$$

which implies (3.3) has no solution.

This follows directly from parts (ii) and (iii).

G. Bigi has achieved similar result of 3.1(i) by deriving the following Theorem 3.2.

Theorem 3.2 If $\bar{\mathbf{x}} \in \mathrm{X}$ is an efficient solution of $\mathbf{P}$ then

Notes

$$\max_{\forall i} F^{+} f_i(\bar{\mathbf{x}}, \mathbf{d}) \geq 0 \qquad \dots \quad \dots \qquad (3.5)$$

holds for any $\mathbf{d} \in T(X; \bar{\mathbf{x}})$

Proof: see [1].

Ref

It would be reasonable to try to replace the upper Dini-Hadamard derivatives with the lower ones for at least some components of *f* since a stronger condition would be achieved; G. Bigi [1] has given the following example and shows that it is not possible to substitute even one of them with the corresponding lower derivative.

Example 3.1 (G. Bigi [1]) Consider problem

$X = \{x \in E_2^{+} : x_1 \geq x_2^2 \geq x_1^4\}$ and $f(x_1, x_2) = (x_2 - x_1, x_1 - x_2)$ if $\mathbf{x} \in X$ and $f(x_1, x_2) = (-x_1 - x_2, -x_1 - x_2)$ if $\mathbf{x} \notin X$.

Thus, $\bar{x} = (0,0)$ is a vector minimum point and $F^{+} f_1(\bar{x}; w) = w_2 - w_1 = -F^{+} f_2(\bar{x}; w)$ so that (3.5) holds.

Since $F^{-} f_i(\bar{x}; w) = -w_1$ whenever $w_2 = 0$ and $F^{-} f_i(\bar{x}; w) = -w_2$ whenever $w_1 = 0$, then we have

$$\max\{F^{+} f_1(\bar{x}; w), F^{-} f_2(\bar{x}; w)\} = -w_1 \langle 0$$

For any nonzero $(w_1, 0) \in T(X, \bar{x}) = E_2^{+}$ and $\max\{F^{-} f_1(\bar{x}; w), F^{+} f_2(\bar{x}; w)\} = -w_2 \langle 0$ for any nonzero $(0, w_2) \in T(X, \bar{x})$.

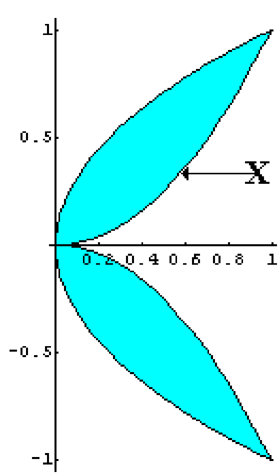

Fig. 1

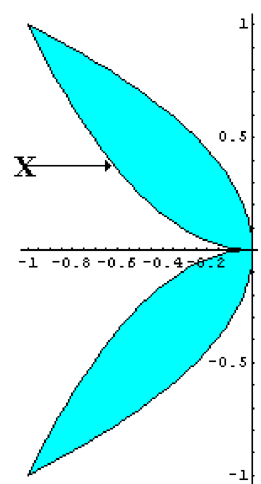

Fig. 2

Now we deduce the following necessary optimality condition where we use lower derivative.

Theorem 3.3 If $\bar{\mathbf{x}} \in X$ is an efficient solution of **P** then for any direction $\mathbf{d} \in T^{*}(X; \bar{x})$, the system

$$F^{-} f_i(\bar{\mathbf{x}})^{\mathrm{T}} \mathbf{d} < 0, \mathrm{i} = 1, 2, \dots, l \qquad \dots \quad \dots \quad \dots \qquad (3.6)$$

has no solution $\mathbf{d} \in E_n$.

Now we shall see the sufficiency of Theorem 3.1(i).

Theorem 3.4 Let $\overline{\mathbf{x}} \in \mathrm{X}$. If for any direction $\mathbf{d} \in \mathrm{T}(\mathrm{X}; \overline{\mathrm{x}})$ the system

$$F^{+} f_i(\overline{\mathbf{x}})^{\mathbf{T}} \mathbf{d} < 0, \mathrm{i} = 1, 2, \ldots, l \qquad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

has no solution $\mathbf{d} \in \mathrm{E}_n$ then $\overline{\mathbf{x}}$ is an efficient solution of **P**.

The above Theorem is no longer true, as the following example shows.

Example 3.2 Consider the problem

$$\min\{x_1, x_2\} \quad \text{and} \quad X = \{x \in E_2 \mid x_1^4 \le x_2^2 \le -x_1\}$$

It is easily verified that:

i) $F = \{F^{+} f_i(\overline{\mathbf{x}})^{\mathbf{T}} \mathbf{d} < 0, \mathrm{i} = 1, 2\}$

$= \{(d_1, d_2)^T \in \mathrm{E}_2 \mid \mathrm{d}_1 < 0, \mathrm{d}_2 < 0\}$.

ii) Clearly $\mathrm{F} \cap \mathrm{T} \neq \phi$

iii) $x_0 = (0,1)$ is not an efficient solution to the problem.

iv) Figure 2.

IV. Conclusion

Combining the result of [6], we generalized first order optimality conditions for non-differentiable functions. We illustrated the obtained results by means of two suitable examples. The results can be developed for the optimization problem where the functions are Lipschitz continuous. We leave it as our future work.

References Références Referencias

1. Bigi, G., (2003), Optimality and Lagrangian Regularity in Vector Optimization, Ph.D.
2. Clarke, F. H., (1983), Optimization and Nonsmooth Analysis, *Wiley*, New York.
3. Stein O., (2004), On Constraint Qualifications in Nonsmooth Optimization, *Journal of Optimization Theory and Applications,*Vol. 109, pp. 431-435.
4. Yang, X. Q., (1998), A Generalized Upper Dini-Derivative in Vector Optimization, Optimization, Vol.43, pp339-351.
5. Yang, X.Q. and Jeyakumar, V., (1992), Generalized SecondΩorder directional derivatives and optimization with C^{11} functions, Optimization, Vol.26, pp. 165Ω185.
6. Preda, V. and Chitescu, I., (1999), On Constraint Qualification in Multiobjective Optimization Problems: Semidifferentiable Case, *Journal of Optimization Theory and Applications,* Vol. 100,pp. 417-433.
7. Kuntz L., and Scholtes, S., (1994), A Nonsmooth Variant of the Mangasarian-Fromovitz Constraint Qualification,*Journal of Optimization Theory and Applications,* Vol. 82, pp. 59-75.
8. Kuntz L., and Scholtes, S., (1993), Constraint Qualifications is in Quasi differentiable Optimization*, Mathematical Prgramming,* Vol. 60, pp. 339-347.
9. Laurent, P. J.,(1972), Approximation et Optimization, Hermann, *Paris*, France.
10. Tiel, J. V.,(1984), Convex Analysis an Introductory Text, *John Wiley and Sons*, New York.

Ref

11. Bigi, G. and Castellani, M., (2000), Second Order Optimality Conditions for Differentiable Multiobjective Problems, *RAIRO, Operations Research*, Vol.34, pp. 411-426.
12. Bazaraa, M. S., Sherali, H. D. and Shetty, C. M., (1993), Nonlinear programming (Second Edition), *John Wiley and Sons*, New York.

Notes

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Yet Another New Proof of Feuerbach's Theorem

By Dasari Naga Vijay Krishna

Abstract- In this article we give a new proof of the celebrated theorem of Feuerbach.

Keywords: *feuerbach's theorem, stewart's theorem, nine point center, nine point circle.*

GJSFR-F Classification : MSC 2010: 11J83, 11D41

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Notes

Yet Another New Proof of Feuerbach's Theorem

Dasari Naga Vijay Krishna

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Keywords: feuerbach's theorem, stewart's theorem, nine point center, nine point circle.

I. Introduction

The Feuerbach's Theorem states that "*The nine-point circle of a triangle is tangent internally to the in circle and externally to each of the excircles*".

Feuerbach's fame is as a geometer who discovered the nine point circle of a triangle. This is sometimes called the Euler circle but this incorrectly attributes the result. Feuerbach also proved that the nine point circle touches the inscribed and three escribed circles of the triangle. These results appear in his 1822 paper, and it is on the strength of this one paper that Feuerbach's fame is based. He wrote in that paper:-

The circle which passes through the feet of the altitudes of a triangle touches all four of the circles which are tangent to the three sides of the triangle; it is internally tangent to the inscribed circle and externally tangent to each of the circles which touch the sides of the triangle externally.

The nine point circle which is described here had also been described in work of Brianchon and Poncelet the year before Feuerbach's paper appeared. However John Sturgeon Mackay notes in [4] that Feuerbach gave:-

... the first enunciation of that interesting property of the nine point circle

namely that "it is internally tangent to the inscribed circle and externally tangent to each of the circles which touch the sides of the triangle externally." The point where the incircle and the nine point circle touch is now called the Feuerbach point.

In this short paper we deal with an elementary concise proof for this celebrated theorem.

II. Notation and Background

Let ABC be a non equilateral triangle. We denote its side-lengths by a, b, c, its semi perimeter by $s=\frac{1}{2}(a+b+c)$, and its area by Δ. Its *classical centers* are the circum center S, the in center I, the centroid G, and the orthocenter O. The nine-point center

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N is the midpoint of SO and the center of the nine-point circle, which passes through the side-midpoints A', B', C' and the feet of the three altitudes. The Euler Line Theorem states that G lies on SO with OG : GS = 2 : 1 and ON : NG : GS = 3 : 1 : 2. We write I_1, I_2, I_3 for the excenters opposite A, B, C, respectively, these are points where one internal angle bisector meets two external angle bisectors. Like I, the points I_1, I_2, I_3 are equidistant from the lines AB, BC, and CA, and thus are centers of three circles each tangent to the three lines. These are the excircles. The *classical radii* are the circum radius R (= SA = SB = SC), the in radius r, and the exradii r_1, r_2, r_3.

The following formulas are well known

(a) $\Delta=\frac{abc}{4R}=rs=r_1(s-a)=r_2(s-b)=r_3(s-c)=\sqrt{s(s-a)(s-b)(s-c)}$

(b) $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, $r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, $r_2=4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$,

$r_3=4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(c) $r_1+r_2+r_3-r=4R$, $r+r_2+r_3-r_1=4R\cos A$, $r+r_1+r_3-r_2=4R\cos \mathrm{B}$,

$r+r_1+r_2-r_3=4R\cos\mathrm{C}$

II. Basic Lemma's

Lemma -1

If A, B and C are the angles of the triangle ABC then

(1) $\cos A+\cos B+\cos C=1+\frac{r}{R}$

(2) $\cos B+\cos C-\cos A=\frac{r_1}{R}-1$

(3) $\cos A+\cos C-\cos B=\frac{r_2}{R}-1$

(4) $\cos A+\cos B-\cos C=\frac{r_3}{R}-1$

Proof:

Using the formula (c) and by little algebra, we can arrive at the conclusions (1), (2), (3) and (4)

Lemma -2

If N is the center of Nine point circle of the triangle ABC then its radius is $\frac{R}{2}$

Proof:

Clearly Nine point circle is acts as the circum circle of the medial triangle, so the radius of nine point circle is the circum radius of medial triangle.

It is well known that by the midpoints theorem if a, b and c are the sides of the reference(given) triangle and Δ is its area then a/2, b/2 and c/2 are the sides of its medial triangle whose area is $\frac{\Delta}{4}$.

If R^1 is the radius of nine point circle then using (a) we have

Notes

$$R' = \frac{\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)}{4\left(\frac{\Delta}{4}\right)} = \frac{\frac{4R\Delta}{8}}{\Delta} = \frac{R}{2}$$

Notes

Hence proved

Lemma -3

If N is the nine point center of the triangle ABC then

$$AN = \frac{1}{2}\sqrt{R^2 + c^2 + b^2 - a^2} = \frac{1}{2}\sqrt{R^2 + 2bc\cos A}$$

$$BN = \frac{1}{2}\sqrt{R^2 + c^2 + a^2 - b^2} = \frac{1}{2}\sqrt{R^2 + 2ac\cos \mathrm{B}}$$

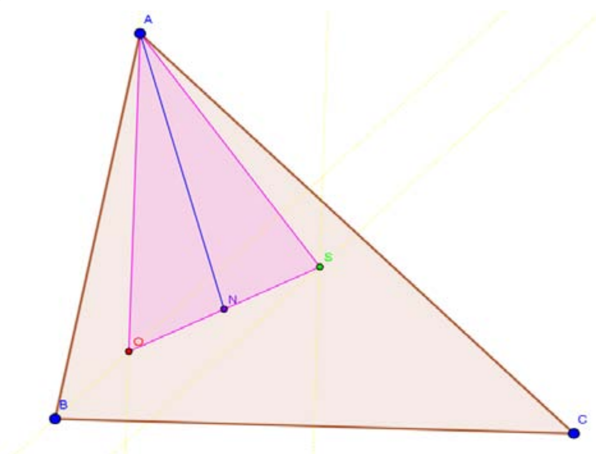
$$CN = \frac{1}{2}\sqrt{R^2 + a^2 + b^2 - c^2} = \frac{1}{2}\sqrt{R^2 + 2ab\cos \mathrm{C}}$$

Proof:

We are familiar with the following results

If G is the centroid and S is the circum center then $GS^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$

Nine point center(N) lies on the Euler's Line and N acts as the mid point of the line segment formed by joining the Orthocenter(O) and Circum center(S), So AN is the median of the triangle AOS



Hence by Apollonius theorem, we have $AN = \frac{1}{2}\sqrt{2AO^2 + 2AS^2 - OS^2}$ ....(A)

Since ON : NG : GS = 3 : 1 : 2, we have $OS^2 = 9GS^2$

And also we know that $AO = 2R\cos A$, $AS = R$

By replacing AO, AS and OS in (A) and by some computation, we get

$AN = \frac{1}{2}\sqrt{R^2 + c^2 + b^2 - a^2} = \frac{1}{2}\sqrt{R^2 + 2bc\cos A}$ (by cosine and sine rule)

Similarly we can find BN and CN

Lemma -4

If I is the In center of the triangle ABC whose sides are a, b and c and M be any point in the plane of the triangle then

$$IM^2 = \frac{a\,AM^2 + b\,BM^2 + c\,CM^2 - abc}{a+b+c}$$

Proof:
The proof of above lemma can be found in [1]
Similarly we can prove that

If I_1, I_2 and I_3 are excenters of the triangle ABC whose sides are a, b and c and M be any point in the plane of the triangle then

$$I_1M^2 = \frac{-a\,AM^2 + b\,BM^2 + c\,CM^2 + abc}{b+c-a}$$

$$I_2M^2 = \frac{a\,AM^2 - b\,BM^2 + c\,CM^2 + abc}{a+c-b}$$

$$I_3M^2 = \frac{a\,AM^2 + b\,BM^2 - c\,CM^2 + abc}{a+b-c}$$

Lemma -5

If r_1 and r_2 are the radii of two non concentric circles whose centers are at a distance of d then the circles touch each other

(i) internally only when $d = |r_1 - r_2|$

(ii) externally only when $d = r_1 + r_2$

Proof:

We know that If r_1 and r_2 are the radii of two circles whose centers are at a distance of d units then the length of their direct common tangent $= \sqrt{d^2 - (r_1 - r_2)^2}$

And the length of their transverse common tangent $= \sqrt{d^2 - (r_1 + r_2)^2}$

Now if two circles touch each other internally then their length of direct common tangent is zero

So $\sqrt{d^2 - (r_1 - r_2)^2} = 0$, it implies that $d = |r_1 - r_2|$

In the similar manner, if two circles touch each other externally then their length of transverse common tangent is zero. So $\sqrt{d^2 - (r_1 + r_2)^2} = 0$, it implies that $d = r_1 + r_2$

III. Main Results

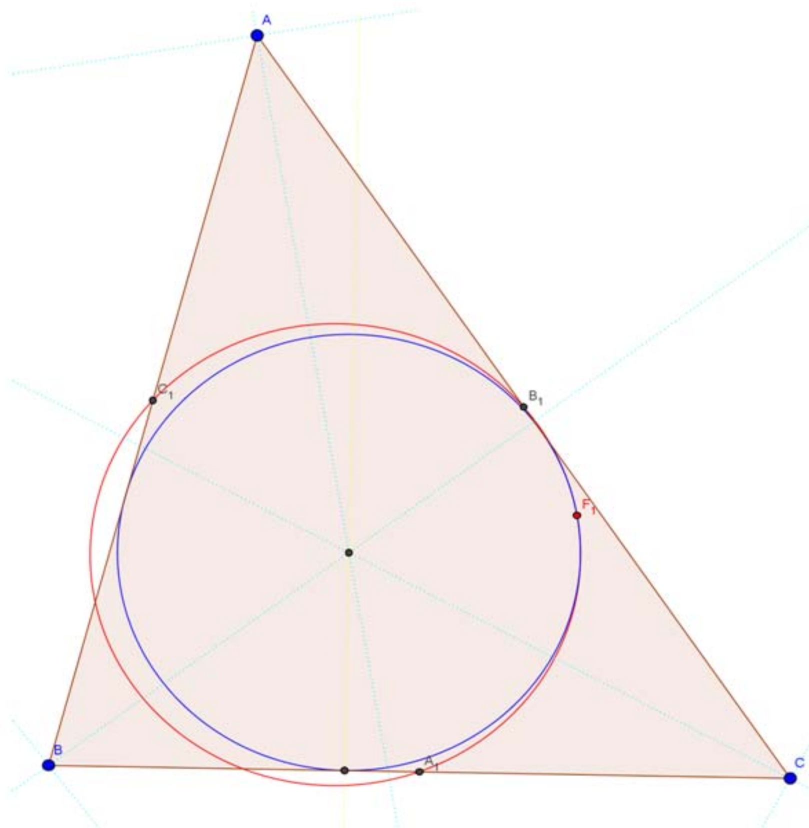
Theorem -1

Nine point circle of triangle ABC touches its in circle internally, that is if N and I are the centers of nine point circle and in circle respectively whose radii are $\frac{R}{2}$ and r then $NI = \left|\frac{R}{2} - r\right|$

Notes

Notes

Proof:



Clearly by lemma -4, for any M we have $IM^2 = \frac{a\,AM^2 + b\,BM^2 + c\,CM^2 - abc}{a+b+c}$

Now fix M as Nine point center(N),

So $$IN^2 = \frac{a\,AN^2 + b\,BN^2 + c\,CN^2 - abc}{a+b+c}$$

It can be rewritten as $$IN^2 = \frac{\sum_{a,b,c} a\left(\frac{1}{4}(R^2 + 2bc\cos A)\right) - abc}{a+b+c}$$

Using lemma -1, (a) and by some computation, we can arrive at the conclusion

$$IN^2 = (\frac{R}{2} - r)^2 \text{ (since } R \geq 2r\text{)}$$

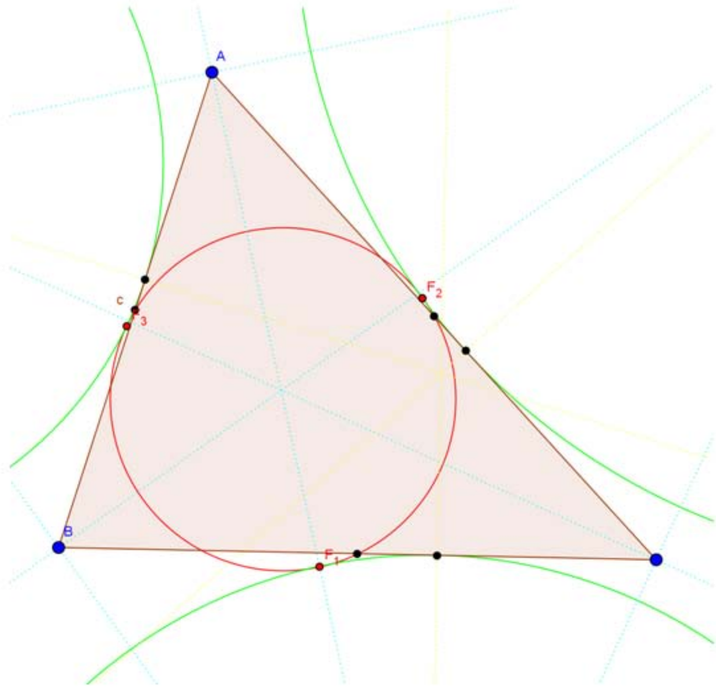
It further gives $NI = \left|\frac{R}{2} - r\right|$

Hence proved

Theorem -2

Nine point circle of triangle ABC touches their excircles externally, that is if N and I_1, I_2, I_3 are the centers of nine point circle and excircles respectively whose radii are $\frac{R}{2}$ and r_1, r_2, r_3 then $NI_1 = \frac{R}{2} + r_1$, $NI_2 = \frac{R}{2} + r_2$ and $NI_3 = \frac{R}{2} + r_3$

Proof:



Clearly by lemma -4, for any M we have $I_1M^2 = \dfrac{-a\,AM^2 + b\,BM^2 + c\,CM^2 + abc}{b+c-a}$

Now fix M as Nine point center(N),

So $$I_1N^2 = \frac{-a\,AN^2 + b\,BN^2 + c\,CN^2 + abc}{-a+b+c}$$

It can rewritten as $$I_1N^2 = \frac{\frac{R^2}{4}(-a+b+c) + \frac{abc}{2}\left(\cos B + \cos C - \cos A\right) + abc}{-a+b+c}$$

Using lemma -1, (a) and by some computation,

we can arrive at the conclusion $I_1N^2 = (\frac{R}{2} + r_1)^2$

It further gives $NI_1 = \dfrac{R}{2} + r_1$

Similarly we can prove $NI_2 = \dfrac{R}{2} + r_2$ and $NI_3 = \dfrac{R}{2} + r_3$

Hence proved

Now we are in a position to deal with the concise proof of celebrated Feuerbach's Theorem.

Theorem - 3(Feuerbach, 1822)

In a nonequilateral triangle, the nine-point circle is internally tangent to the incircle and externally tangent to the three excircles.

Proof:

Theorem -1 and Theorem-2 completes the proof of Feuerbach's Theorem.

For historical details see [4] , [5], [6] and [9]

Notes

Notes

References Références Referencias

1. Dasari Naga Vijay Krishna, The New Proof of Euler's Inequality Using Spieker Center, *International Journal of Mathematics And its Applications*, Volume 3, Issue 4-E (2015), 67-73.
2. H. S. M. Coxeter, *Introduction to Geometry*, 2nd edition, John Wiley & Sons, Hoboken, N. J., 1969.
3. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, D.C., 1967.
4. J. S. MacKay, History of the nine point circle, *Proc. Edinb. Math. Soc.*, 11 (1892) 19–61.
5. Jean - Louis AYME, FEURBACH's THEOREM- A NEW PURELY SYNTHETIC PROOF.
6. K. W. Feuerbach, *Eigenschaften einiger merkw¨urdigen Punkte des geradlinigen Dreiecks und mehrerer durch sie bestimmten Linien und Figuren: Eine analytisch-trigonometrische Abhandlung*, Riegel & Wiesner, N¨urnberg, 1822.
7. K. J. Sanjana, An elementary proof of Feuerbach's Theorem, *Mathematical Notes (1924)*, 22 pp 11-12,Copyright Edinburgh Mathematical Society 1924.
8. Michael J. G. Scheer, A Simple Vector Proof of Feuerbach's Theorem, *Forum Geometricorum*, Volume 11 (2011) 205–210.
9. http://www.mat.ucm.es/catedramdeguzman/old/08sabormat/geometriatriangulo/feuerbach/guggenbuhlfeuerbach.html

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A Survey on Developments in Sequence Spaces and introduction to Bicomplex Duals

By Dr. Mamta Amol Wagh
University of Delhi

Abstract- A survey on recent developments in the duality theory of sequence spaces has been done and duality in some Bicomplex sequence spaces is introduced which gives rise to sixteen types of different duals.

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ASURVEYONDEVELOPMENTSINSEQUENCESPACESANDINTRODUCTIONTOBICOMPLEXDUALS

Strictly as per the compliance and regulations of :

Ref

10. Köthe, G & Toeplitz, O: "Lineare Raume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen", Jour. Reine angew. Math. 171, 1934, 193 – 226.

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Abstract- A survey on recent developments in the duality theory of sequence spaces has been done and duality in some Bicomplex sequence spaces is introduced which gives rise to sixteen types of different duals.

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I. Introduction

In several branches of analysis, for instance, the structural theory of topological vector spaces, Schauder basis theory, Summability theory, and the theory of functions, the study of sequence spaces occupies a very prominent position. The impact and importance of this study can be appreciated when one sees the construction of numerous examples of locally convex spaces obtained as a consequence of the dual structure displayed by several pairs of distinct sequence spaces. There is an ever increasing interest in the theory of sequence spaces that has made remarkable advances in enveloping summability theory via unified techniques effecting matrix transformations from one sequence space into another.

Thus we have several important applications of the theory of sequence spaces and therefore we attempt to present a survey on recent developments in sequence spaces and their different kinds of duals.

There are two types of dual of a sequence space, namely Algebraic dual and Topological dual. The set of all linear functionals, on a linear space V, with domain as V and range as K is denoted by $L(V, K) = V^{\#}$ and is called algebraic dual of V. If we consider the set of all continuous linear functional, then we get topological dual denoted by V^*.

From the point of view of the duality theory, the study of sequence spaces is much more profitable. Köthe & Toeplitz were the first to recogonize the problem that it is difficult to find the topological duals of sequence spaces equipped with linear topologies. To resolve it, they introduced a kind of dual, α - dual, in quite many familiar and useful sequence spaces. In the same paper [10], they also introduced another kind of dual namely β – dual which together with the given sequence space forms a nice dual system. A still more general notion of a dual, γ - dual was later

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introduced by Garling [6]. For symmetric sequence spaces there is another notion of a dual, called a δ – dual due to Garling [7] and Ruckle [28].

Let λ be a sequence space, and ω is the family of all sequences $\{x_n\}$ with $x_n \in K$, $n \geq 1$. Define α - , β - , γ- , and δ – dual, respectively as follows:

1. $\alpha - dual \quad \lambda^{\alpha} = \left\{ x : x \in \omega, \sum_{i\geq 1} |x_i y_i| < \infty, \forall y \in \lambda \right\}$

2. $\beta - dual \quad \lambda^{\beta} = \left\{ x : x \in \omega, \left| \sum_{i\geq 1} x_i y_i \right| < \infty, \forall y \in \lambda \right\}$

3. $\gamma - dual \quad \lambda^{\gamma} = \left\{ x : x \in \omega, \sup_n \left| \sum_{i=1}^{n} x_i y_i \right| < \infty, \forall y \in \lambda \right\}$

4. $\delta - dual \quad \lambda^{\delta} = \left\{ x : x \in \omega, \sum_{i\geq 1} |x_i y_{\rho(i)}| < \infty, \forall y \in \lambda \text{ and } \rho \in \pi \right\}$

Here π is the set of all permutations of N.
$\lambda^{\alpha}, \lambda^{\beta}, \lambda^{\gamma}, and\ \lambda^{\delta}$ are sequence spaces, and $\phi \subset \lambda^{\delta} \subset \lambda^{\alpha} \subset \lambda^{\beta} \subset \lambda^{\gamma}$.

Constantine G. Lascarides [11] in his paper, "A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer", has examined the Köthe – Toeplitz reflexivity of certain sequence spaces and characterized some classes of matrix transformations defined on them.

Matrix transformation: let *X, Y* be two nonempty subsets of the spaces of all complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers a_{nk} $(n, k = 1,2,\ldots)$. For every $x = (x_k) \in X$ and every integer n we write

$$A_n(x) = \sum_k a_{nk} x_k ,$$

Where the sum without limits is always taken from $k = 1$ *to* $k = \infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. we say that $A \in (X, Y)$ if and only if $Ax \in Y$ whenever $x \in X$.
The following classes of sequences were defined by Maddox [17]:

$$l(p) = \left\{ x : \sum_k |x_k|^{p_k} < \infty \right\},$$

$$l_\infty(p) = \left\{ x : \sup_k |x_k|^{p_k} < \infty \right\},$$

$$c(p) = \left\{ x : |x_k - l|^{p_k} \to 0 \ for\ some\ l \right\},$$

$$c_0(p) = \left\{ x : |x_k|^{p_k} \to 0 \right\}.$$

When all the terms of (p_k) are constant and all equal to p>0 we have $l(p) = l_p, l_\infty(p) = l_\infty, c(p) = c, and\ c_0(p) = c_0$, where l_p, l_∞, c, c_0, are respectively the spaces of p – summable, bounded, converget, and null sequences.

Ref

6. Garling, D.G.H.: "The β - and γ - duality", *Proc. Cambridge Philos. Soc. (1967), 963 – 981.*

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It was shown in [14],[15],[17], that the sets $l(p), l_\infty(p), c(p) \, and \, c_0(p)$, are linear spaces under coordinatewise addition and scalar multiplication if and only if $p \in l_\infty$.

Let E be a nonempty subset of s. Then we denote by $E^\dagger$ the generalized Köthe – Toeplitz dual of E, i.e.,

$$E^\dagger = \left\{ a : \sum_k a_k x_k \ converges \ for \ every \ x \in E \right\}$$

Some properties of dual spaces:

Lemma 1:

The Köthe – Toeplitz duality has the following properties:

(i) $E^\dagger$ is a linear subspace of s for every $E \subset s$.

(ii) $E \subset F$ implies $E^\dagger \supset F^\dagger, \forall E, F \subset s$.

(iii) $E^{\dagger\dagger} \equiv (E^\dagger)^\dagger \supset E, \forall E \subset s$.

(iv) $(\bigcup E_j)^\dagger = \bigcap E$, $for \ every \ family \{E_j\} \ with \ E_j \subset s$.

A nonempty subset E of s is said to be perfect or Köthe – Toeplitz reflexive if and only if $E^{\dagger\dagger} = E$. $E^\dagger$ is perfect for every E. If E is perfect then it is a linear space. The converse is not always true, e.g., c is a linear space with Köthe – Toeplitz dual l_1 and not perfect.

Let E(p) be any of the sets $l(p), l_\infty(p), c(p), c_0(p)$, Let $E(p;1) = E^\dagger(p), E(p;2) = E^{\dagger\dagger}(p)$ etc. Then $E(p;1) = E(p;2n+1), \forall n \geq 0$.

Köthe – Toeplitz dual of the above classes of sequences:

Lemma 2 : (I) If $0 < p_k \leq 1, \forall k$, then $l(p;1) = l_\infty(p)$

(ii) If $p_k > 1, \forall k, then \, l(p;1) = M(p)$ where

$$M(p) = \bigcup_{N \geq 1} \left\{ a : \sum_k |a_k|^{q_k} N^{-q_k} < \infty \right\}$$

With $p_k^{-1} + q_k^{-1} = 1$. For convenience we write $r_k = p_k^{-1}, s_k = q_k^{-1}$

(iii) For every $p = (p_k)$ we have

$$l_\infty(p;1) = M_\infty(p), where$$

$$M_\infty(p) = \bigcap_{N > 1} \left\{ a : \sum_k |a_k| N^{r_k} < \infty \right\}$$

(iv) Also for every $p = (p_k), c_0(p;1) = M_0(p)$ where

$$M_0(p) = \bigcup_{N > 1} \left\{ a : \sum_k |a_k| N^{-r_k} < \infty \right\}.$$

Theorem 1: For every $p = (p_k)$ we have $c(p;1) = c_0(p;1) \bigcap \gamma$ where γ is the space of all convergent series.

Ref

14. Maddox, I. J., "Paranormed sequence spaces generated by infinite matrices", Proc. Cambridge Philos. Soc. 64 , 1968, 335 – 340.

Also he has characterized the second Köthe – Toeplitz duals and discussed the reflexivity of the sets $l(p), l_\infty(p), and\ c_0(p)$,

Theorem 2: For every $p=(p_k)$ we have $c_0(p;2)=\lambda_1$ where

$$\lambda_1 = \bigcap_{N>1}\left\{y : \sup_k |y_k| N^{r_k} < \infty\right\}.$$

Theorem 3: For every $p=(p_k)$ we have $l_\infty(p;2)=\lambda_2$ where

$$\lambda_2 = \bigcup_{N>1}\left\{a : \sup_k |a_k| N^{-r_k} < \infty\right\}.$$

I.J. Maddox [12] in his paper "Generalized Köthe – Toeplitz duals" has characterized the α and β – dual spaces of generalized l_p spaces where $0<p\le\infty$. The question of when the α and β – dual spaces coincide is also considered.

Let $A=(A_k)$ denote a sequence of linear, but not necessarily bounded, operators on X into Y. Following results are given there:

Theorem 4 : Let $0<p\le 1$. Then $A\in l_p^\beta(x)$ if and only if there exists $m\in \mathrm{N}$ such that A_k is bounded, for all $k\ge m$, and

$$H = \sup_{k\ge m}\|A_k\| < \infty .$$

Theorem 5: If $0<p\le 1$ then

$$l_p^\alpha(X) = l_p^\beta(X).$$

Theorem 6: Let $1<p<\infty$.

$$M = \sum_{k=m}^{\infty}\|A_k\|^q < \infty .$$

Theorem 7: Let $1<p<\infty$. Then $A\in l_p^\beta(X)$ if and only if there exists $m\in\mathrm{N}$ such that $\mathrm{A_k}$ is bounded for all $k\ge m$, and

$$\sup \sum_{k=m}^{\infty}\left\|A_k^* f\right\|^q < \infty$$

Where the supremum is over all $f\in Y^*$ with $\|f\|\le 1$.

Theorem 8: $A\in l_\infty^\alpha(X)$ if and only if there exists $m\in\mathrm{N}$ such that A_k is bounded for all $k\ge m$, and

$$\sum_{k=m}^{\infty}\|A_k\| < \infty .$$

Theorem 9: $A\in l_\infty^\beta(X)$ if and only if there exists $m\in\mathrm{N}$ such that A_k is bounded for all $k\ge m$, and

Ref

12. Maddox, I. J. : "Generalized Köthe – Toeplitz duals", International, J. Math. & Math. Sci. 3 (3), 1980, 423 – 432.

$$\sup \sum_{k=m}^{m+n} \|A_k x_k\| < \infty$$

$$\sup \left\| \sum_{k=m}^{m+n} A_k x_k \right\| \to 0 \ (m \to \infty)$$

where the suprema are over all $n \geq 0$ and all $x_k \in X$ with $\|x_k\| \leq 1$.

Theorem 10: If $1 < p < \infty$ then there are Banach spaces X and Y such that $l_p^\alpha(X) \subset l_p^\beta(X)$ with strict inclusion.

Theorem 11: If $1 < p < \infty$ and Y is finite dimensional then for any X we have

$$l_p^\alpha(X) = l_p^\beta(X).$$

For certain values of p, and any X, the next result is the converse of previous theorem.

Theorem 12: If $2 < p < \infty$ and $l_p^\alpha(X) = l_p^\beta(X)$ then Y must be finite dimensional.

Theorem 13: Let Y be a Hilbert space and suppose $l_2^\alpha(X) = l_2^\beta(X)$. Then Y must be finite dimensional.

In this paper Maddox [13] has established relations between several notions of solidity in vector valued sequence spaces, and has introduced a generalized Köthe – Toeplitz dual space in the setting of a Banach algebra.

Topologies on a sequence space, involving β and γ – duality have been examined by Garling [6], who noted that $E^\alpha = E^\beta = E^\gamma$ when E is solid (or normal), i.e., when $x \in E$ and $|y_k| \leq |x_k|$ for all $k \in N$ together imply that $y \in E$.

For example, the space c_0 of null sequences is solid, but the space c of convergent sequences is not.

If X is a complex normed linear space, denote by B the closed unit ball of X and by *B(X)* the space of all bounded linear operators on *X*. X^* denotes the continuous dual space of *X*. Two subspaces of *s(X)* that has been considered are

$$l_\infty(X) = \left\{ x \in s(X) : \sup_k \|x_k\| < \infty \right\},$$

$$l_1(X) = \left\{ x \in s(X) : \sum_{k=1}^{\infty} \|x_k\| < \infty \right\}.$$

These spaces generalize the classical spaces l_∞ *and* l_1 which are subspaces of *s*. Consider the following statements, each of which expresses some notion of solidity for linear subspace E of *s(X)*.

(1) $x \in E$ *and* $\lambda \in l_\infty$ *imply* $\lambda x \in E$

(2) $x \in E$ *and* $|\lambda_n| \leq 1$ *imply* $\lambda x \in E$

(3) $x \in E$ *and* $|\lambda_n| = 1$ *imply* $\lambda x \in E$

(4) $x \in E$ *and* $\|y_n\| = \|x_n\|$ *imply* $y \in E$

Ref

6. Garling, D.G.H.: "The β - and γ - duality", *Proc. Cambridge Philos. Soc.* (1967), 963 – 981.

(5) $x \in E$ *and* $\|y_n\| \le \|x_n\|$ *imply* $y \in E$

(6) $x \in E$ *and* $\|A_n\| \in l_\infty$ *imply* $Ax \in E$

(7) $x \in E$ *and* $\|A_n\| \le 1$ *imply* $Ax \in E$

(8) $x \in E$ *and* $\|A_n\| = 1$ *imply* $Ax \in E$

Equivalences:

Theorem 14: In any complex linear space X the statements (1), (2) and (3) are equivalent.

Theorem 15: In any normed linear space X the statements (4), (5), (6), (7) and (8) are equivalent.

Theorem 16: In any normed linear space X any one of the statements (4) to (8) implies all of the statements (1) to (3). But (1) is equivalent to (4) if and only if X is one dimensional.

The notion of difference sequence spaces was first introduced by Kizmaz[9]. He defined the sequence spaces

$$l_\infty(\Delta) = \{x = (x_k) : \Delta x \in l_\infty\}, i.e., \left\{x = (x_k) : \sup_k \|x_k - x_{k+1}\| < \infty\right\}$$

$$c(\Delta) = \{x = (x_k) : \Delta x \in c\}, i.e. \ \{x = (x_k) : \lim \|x_k - x_{k+1}\| \ exists\}$$

$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\}, i.e. \ \{x = (x_k) : \lim \|x_k - x_{k+1}\| = 0\}$$

Where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaces with norm

$$\|x\|_1 = |x_1| + \|\Delta x\|_\infty .$$

The notion of generalized difference sequence spaces was further generalized by Et. and Colak.

Et and Colak [4] generalized the above sequence spaces to the following sequence spaces.

$$l_\infty(\Delta^m) = \{x = (x_k) : \Delta^m x \in l_\infty\},$$

$$c(\Delta^m) = \{x = (x_k) : \Delta^m x \in c\},$$

$$c_0(\Delta^m) = \{x = (x_k) : \Delta^m x \in c_0\}$$

Where, $m \in N, \Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1}), \Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v} .$$

These are Banach spaces with norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty .$$

We can see that $c_0(\Delta^m) \subset c_0(\Delta^{m+1}), c(\Delta^m) \subset c(\Delta^{m+1}), l_\infty(\Delta^m) \subset l_\infty(\Delta^{m+1})$ and

Ref

9. Kizmaz, H. : "On certain Sequence Spaces", Canad. Math. Bull. 24 (1981), 169 – 176.

$c_0(\Delta^m) \subset c(\Delta^m) \subset l_\infty(\Delta^m)$ are satisfied and strict [et and colak].

Colak [3] defined the sequence space $\Delta_v(X) = \{x = (x_k) : \Delta_v x_k \in X\}$ where $(\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and X is any sequence space, and investigated some of its topological properties.

Later Etand Esi [5] have defined the sequence spaces

$$\Delta_v^m(l_\infty), \Delta_v^m(c) \, and \, \Delta_v^m(c_0), (m \in N)$$

and some topological properties, inclusion relations of these sequence spaces have been given, their continuous and Köthe – Toeplitz duals have been computed.

Let $l_\infty, c, and\, c_0$ be the linear spaces of bounded, convergent and null sequences. $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in N = \{1,2,3,...\}$, the set of positive integers.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Define

$$\Delta_v^m(l_\infty) = \{x = (x_k) : \Delta_v^m x \in l_\infty\}$$

$$\Delta_v^m(c) = \{x = (x_k) : \Delta_v^m x \in c\}$$

$$\Delta_v^m(c_0) = \{x = (x_k) : \Delta_v^m x \in c_0\}$$

where

$$m \in N, \Delta_v^0 x = (v_k x_k), \Delta_v x_k = (v_k x_k - v_{k+1} x_{k+1}),\ \Delta_v^m x_k = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$$

and so that

$$\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$$

It is trivial that $\Delta_v^m(l_\infty)$, $\Delta_v^m(c)$, $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_v = \sum_{i=1}^{m} |x_i v_i| + \|\Delta_v^m x\|_\infty$$

Theorem 17: Let

$$U_1 = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty, \forall\, x \in X \right\} and$$

$$U_2 = \left\{ a = (a_k) : \sup_k k^{-m} |a_k v_k| < \infty \right\}$$

Then

Ref

5. Et, D. & Esi, A.: "ON Köthe – Toeplitz Duals Of Generalized Difference Sequence Spaces", Bull. Malaysian Math. Sc. Soc.(second series) 23, 2000, 25 – 32.

$(i)\ \left(\Delta_v^m(l_\infty)\right)^\alpha = \left(\Delta_v^m(c)\right)^\alpha = \left(\Delta_v^m(c_0)\right)^\alpha = U_1$

$(ii)\ \left(\Delta_v^m(l_\infty)\right)^{\alpha\alpha} = \left(\Delta_v^m(c)\right)^{\alpha\alpha} = \left(\Delta_v^m(c_0)\right)^{\alpha\alpha} = U_2$

Corollary 1: $\left(\Delta_v^m(l_\infty)\right), \left(\Delta_v^m(c)\right)$ *and* $\left(\Delta_v^m(c_0)\right)$ are not perfect.

Corollary 2: If we take $(v_k)=(1,1,1....)$ and $m=1$ in theorem 17, then we obtain for $X=l_\infty$ *or* c

$$(i)\ (\Delta^m(X))^\alpha = \left\{a=(a_k): \sum_k k^{-m}|a_k| < \infty\right\},$$

$$(ii)\ (\Delta^m(X))^{\alpha\alpha} = \left\{a=(a_k): \sup_k k^{-m}|a_k| < \infty\right\},$$

$$(iii)\ (\Delta^m(X))^\alpha = \left\{a=(a_k): \sum_k k^{-m}\left|a_k v_k^{-1}\right| < \infty\right\}.$$

Corollary 3: If we take $v = (k^m)$ in theorem 17, then we obtain

$(i)\ \left(\Delta_v^m(l_\infty)\right)^\alpha = \left(\Delta_v^m(c)\right)^\alpha = \left(\Delta_v^m(c_0)\right)^\alpha = l_1$

$(ii)\ \left(\Delta_v^m(l_\infty)\right)^{\alpha\alpha} = \left(\Delta_v^m(c)\right)^{\alpha\alpha} = \left(\Delta_v^m(c_0)\right)^{\alpha\alpha} = l_\infty$

Prabhat Chandra and Binod Chandra Tripathy [2] introduced the concept of η - dual of sequence spaces. Further they established some results involving the perfectness of different sequence spaces relative to η - dual.

Denote by σ the space of all eventually alternating sequences i.e., if $(x_k)\in\sigma$, then there exists $k_0 \in N$ such that $x_k = -x_{k+1}, \forall k > k_0$. It is well known that $bv_0 = bv \cap c_0$.

Definition: Let E be a non – empty subset of ω and $r \geq 1$, then the η - dual of E is defined as

$$E^\eta = \left\{(y_k)\in\omega : (x_k y_k)\in l_r \ \ for\ all\ (x_k)\in E\right\}.$$

A non – empty subset E of ω is said to be perfect or η – reflexive if $E^{\eta\eta} = E$. Taking r = 1 in the above definition we get the α – dual of E.

Theorem 18: $l_r^\eta = l_\infty, l_\infty^\eta = l_r$ and the spaces l_r *and* l_∞ are perfect spaces.

Theorem 19: $\sigma^\eta = l_r$ and σ is not perfect.

Theorem 20: Let $p > r \geq 1$, then $l_p^\eta = l_q$ where $p^{-1} + q^{-1} = r^{-1}$. The space l_p is perfect.

Theorem 21: $c_0^\eta = c^\eta = l_r$ and the sequence spaces c_0 *and* c are not perfect.

Theorem 22: $(bv)^\eta = l_r = (bv_0)^\eta$ and the spaces bv and bv_0 are not perfect.

T. Balasubramanian and A. Pandiarani [1] have given an account of some of the main developments which has occurred since Robinson's [19] paper of 1950.

A sequence $x=(x_k)$ is said to be an entire sequence if $\lim_{k\to\infty}|x_k|^{1/k} = 0$.

Ref

2. Chandra, P. & Tripathy, B.C. : "On Generalized Köthe – Toeplitz Duals Of Some Sequence Spaces", Indian J. pure appl. Math., 33 (8), 2002, 1301- 1306.

Notes

A sequence $x=(x_k)$ is said to be an analytic sequence if $-|x_k|^{1/k}$ " $= 0$.

Let Γ *and* $\wedge$ respectively denote the linear space of all entire and analytic sequences. G_λ denote the linear space of all sequences $x=(x_k)$ such that

$$\sum \lambda_k^2 |x_k|^2 < \infty, \lambda_k \text{ is fixed}, \lambda_k > 0, \text{and } \frac{\lambda_{k+1}}{\lambda_k} \to 1 \text{ as } k \to \infty .$$

If $(X, \| \,\|)$ is any Banach space over C then we define

$$\Gamma(X) = \left\{ x=(x_k) : \lim_{k\to\infty} \|x_k\|^{1/k} < \infty \right\}$$

$$\wedge(X) = \left\{ x=(x_k) : \sup_k \|x_k\|^{1/k} < \infty \right\}$$

$$G_\lambda(X) = \left\{ x=(x_k) : \sum \lambda_k^2 \|x_k\|^2 < \infty \right\}$$

Where (λ_k) *is fixed sequence of real numbers and* $\frac{\lambda_{k+1}}{\lambda_k} \to 1 \text{ as } k \to \infty$

Theorem 23: $G_\lambda(X) \subset \Gamma(X) \subset \wedge(X)$.

Suppose in general that *(A_k)* is a sequence of linear but not necessarily bounded operators A_k mapping a Banach space X into a Banach space Y.

Theorem 24: $(A_k) \in \Gamma^{\beta}(X)$ if and only if there exist $m \in N$ such that

(i) $(A_k) \in B(X,Y)$

(ii) $\sup_{k\ge m} \|A_k\|^{1/k} < \infty$

Theorem 25: $(A_k) \in \wedge^{\beta}(X)$ if and only if

(i) $(A_k) \in \Gamma^{\beta}(X)$ and

(ii) $\|R_n\|^{1/n} \to 0 \text{ as } n \to \infty$.

Theorem 26: $(A_k) \in G_\lambda^{\beta}(X)$ if and only if there exists $m \in N$ such that

(i) $(A_k) \in B(X,Y)$ *for all* $k \ge m$ and

(ii) $\sum_{k=m}^{\infty} \frac{1}{\lambda^2} \|A_k\|^2 < \infty$.

Theorem 27: $(A_k) \in \Gamma^{\alpha}(X)$ if and only if there exists $m \in N$ such that

(i) $(A_k) \in B(X,Y)$

(ii) $\left(\sum_{k=m}^{\infty} \|A_k\| \right)^{1/k} < \infty$.

It is clear from theorem 27 that these conditions are also necessary and sufficient for $(A_k) \in \wedge^{\beta}(X)$

Hence $\Gamma^{\alpha}(X) = \wedge^{\alpha}(X)$.

Zeren and Bektas [29] introduced and studied the new sequence spaces

$[V,\lambda,F,p,q,u]_0(\Delta_v^m)$, $[V,\lambda,F,p,q,u]_1(\Delta_v^m)$ and $[V,\lambda,F,p,q,u]_\infty(\Delta_v^m)$

which are generalized difference sequence spaces defined by a sequence of moduli in locally convex Hausdorff topological linear space.

II. Recent Developments Done in Bicomplex Köthe – Toeplitz Duals of Some Bicomplex Sequence Spaces

These are some of the recent developments in the Köthe – Toeplitz duality theory of sequence spaces. We in our research work are studying the spaces of bicomplex sequences and, which is the most recent generalization of complex sequences. Köthe – Toeplitz duals of these spaces are also studied and some important results are found

About C_2:

Bicomplex Numbers were introduced by Corrado Segre (1860 – 1924) in 1892. He published a paper [20] in which he defined an infinite set of algebras and gave the concept of multicomplex numbers. For the sake of brevity, we have confined ourselves to the bicomplex version of his theory. The spaceof bicomplex numbers is the first in an infinite sequence of multicomplex spaces. Price [18] may be referred to study more about bicomplex space.

The set of bicomplex numbers is denoted by C_2 and defined as follows:

$$C_2 = \{ x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : x_1, x_2, x_3, x_4 \in C_0 \}$$

or equivalently as

$$C_2 = \{ z_1 + i_2 z_2 : z_1, z_2 \in C_1 \} ;$$

where $i_1^2 = i_2^2 = -1$; $i_1 i_2 = i_2 i_1$, and C_0 and C_1 denote the sets of real and complex numbers, respectively.

The binary compositions of addition and scalar multiplication on C_2 are defined coordinate wise and the multiplication in C_2 is defined term by term. With these binary compositions, C_2 becomes a commutative algebra with identity.

Algebraic structure of C_2 differs from that of C_1 in many respects. Few of them are mentioned below:

1. Non-invertible elements exist in C_2.
2. Non-trivial idempotent elements exist in C_2.
3. Non-trivial zero divisors exist in C_2.

If ω is the family of all sequences $\xi = (\xi_k)$ with $\xi_k \in C_2$, k ≥ 1 . Where C_2 is the space of bicomplex numbers.

$$1.\ \alpha - dual \quad \lambda^{\alpha} = \left\{ \xi : \xi \in \omega', \sum_{i\geq 1} \left\| \xi_i . \eta_i \right\| < \infty, \forall y \in \lambda \right\}$$

$$2.\ \beta - dual \quad \lambda^{\beta} = \left\{ \xi : \xi \in \omega', \left\| \sum_{i\geq 1} \xi_i . \eta_i \right\| < \infty, \forall y \in \lambda \right\}$$

Ref

18. Price, G. B., "An Introduction to Multicomplex Spaces And functions", Marcel Dekker, Inc., 1991.

3. $\gamma - dual \quad \lambda^{\gamma} = \left\{ \xi : \xi \in \omega', \sup_{n} \left\| \sum_{i \geq 1} \xi_i . \eta_i \right\| < \infty, \forall y \in \lambda \right\}$

4. $\delta - dual \quad \lambda^{\delta} = \left\{ \xi : \xi \in \omega', \sum_{i \geq 1} \left\| \xi_i . \eta_{\rho(i)} \right\| < \infty, \forall y \in \lambda \text{ and } \rho \in \pi \right\}$

Ref

22. Srivastava, Rajiv K., Srivastava, Naveen K., "On a class of Entire Bicomple sequences", South East Asian J. Math. & Math. Sc. 5(3), 2007, 47 – 68.

Various types of duals such as α - , β - , γ- , and δ – dual, can be defined on the so constructed bicomplex sequence spaces. In this way we have 16 types of duals for bicomplex sequence spaces viz.

α α – dual, *α* β – dual, *α* γ – dual, *α δ* – dual

β *α* – dual, ββ – dual, βγ – dual, β *δ* – dual

γ *α* – dual, γβ – dual, γγ – dual,γ *δ* – dual

δ δ – dual, *δ* β – dual , *δ* γ – dual, *δ δ* – dual

In [23] and [24] a functional analytic study of some bicomplex sequence spaces is done. In 2007, Srivastava and Srivastava [22] defined and studied a class B of bicomplex sequences associated with the functions which are holomorphic in the bicomplex space C_2, i.e., bicomplex entire functions. In [23], the study of Srivastava and Srivastava is furthered. The structure of the spectrum of an element in B has been formulated. Two subclasses B′ and B′′ have been defined and studied with a functional analytic viewpoint. Another class B* is also defined which has been shown to be a subalgebra of B which is not a normed subalgebra of B. However B* has been furnished with a Hilbert Space structure. In [24], certain properties of a particular class B′ of bicomplex sequences associated with the bicomplex functions which are holomorphic in the bicomplex space C_2, is investigated with a functional analytic view point. B′ has been provided with a modified Gelfand algebraic structure and it has been proved that B′ is an algebra ideal which is not a maximal ideal of B. Invertible and quasi invertible elements in B′ have been studied. A characterization of zero divisors is given and a sufficient condition for an element to be topological zero divisor has been derived. Algebra homomorphism between B and B′ has been investigated.In [25], [26] bicomplex duals of sequence spaces studied in [23], [24] are defined and analysed. [27] gives a study on some bicomplex modules. We are still working on duals and modules of these bicomplex sequence spaces and some interesting new results are expected in further study.

I shall always be thankful to Prof. Rajiv K. Srivastava for work done in this paper.

References Références Referencias

1. Balasubramanian T. & Pandiarani A. : "Duals of vector sequence spaces", Int. Journal of Math. Analysis, 3 (16), 2009, 779 – 785.
2. Chandra, P. & Tripathy, B.C. : "On Generalized Köthe – Toeplitz Duals Of Some Sequence Spaces", Indian J. pure appl. Math., 33 (8), 2002, 1301- 1306.
3. Colak R. "On some Generalized Sequence Spaces", Commun. Fac. Sci. Univ. Ank. Series A_1, 38, 1689, 35 – 46.
4. Colak R. & Et. M, "On some Generalized Difference sequence spaces", Soochow Journal of Math., 21, 1995, 377 – 386.
5. Et, D. & Esi, A.: "ON Köthe – Toeplitz Duals Of Generalized Difference Sequence Spaces", Bull. Malaysian Math. Sc. Soc.(second series) 23, 2000, 25 – 32.

6. Garling, D.G.H.: "The β - and γ - duality", *Proc. Cambridge Philos. Soc. (1967), 963 – 981.*
7. Garling, D.G.H.: "On symmetric sequence spaces", *Proc. London Math. Soc. 16 (3), 1966, 85 – 106.*
8. Kamthan, P.K & Gupta, M: " Sequence spaces and Series", *Marcel Dekker, Inc. New York and Bassel.*
9. Kizmaz, H. : "On certain Sequence Spaces", Canad. Math. Bull. 24 (1981), 169 – 176.
10. Köthe, G & Toeplitz, O: "Lineare Raume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen", Jour. Reine angew. Math. 171, 1934 , 193 – 226.
11. Lascarides, G. Constantine: "A Study of Certain sequence spaces of Maddox and A generalization of a theorem of Iyer",Pacific Journal of Mathematics, 38 (2), 1971.
12. Maddox, I. J. : "Generalized Köthe – Toeplitz duals", International, J. Math. & Math. Sci. 3 (3), 1980, 423 – 432.
13. Maddox, I. J. : "Solidity in Sequence Spaces", Revista Mathematica de la, Universidad Complutense de Madrid, 4 (2), 1991, 185 – 192.
14. Maddox, I. J., "Paranormed sequence spaces generated by infinite matrices", Proc. Cambridge Philos. Soc. 64 , 1968, 335 – 340.
15. Maddox, I. J., "Continuous and Köthe – Toeplitz duals of certain sequence spaces.", Proc. Cambridge Philos. Soc. 65 , 1969, 431 – 435,
16. Maddox, I. J., "Some properties of paranormed sequence spaces", J. London Math. Soc., (2), 1, 1969, 316 – 322.
17. Maddox, I. J., "Spaces of strongly summable sequences", Quaterly J. Math. Oxford, 2(18),1967, 345 – 355.
18. Price, G. B., "An Introduction to Multicomplex Spaces And functions", Marcel Dekker, Inc., 1991.
19. Robinson, A. R., "On functional transformations and summability", Proc. London Math. Soc., 52, 1995, 132 – 160.
20. Segre C., "Le Rappresentazioni Reali Delle forme complesse e Gli Enti Iperalgebrici", Math. Ann., 40, 1892, 413-467.
21. Srivastava, Rajiv K., "Certain topological aspects of Bicomplex space", Bull. Pure & Appl. Math. 2(2), 2008, 222-234.
22. Srivastava, Rajiv K., Srivastava, Naveen K., "On a class of Entire Bicomple sequences", South East Asian J. Math. & Math. Sc. 5(3), 2007, 47 – 68.
23. Wagh Mamta A., "On certain spaces of Bicomplex sequences ", Inter. J. Phy. Chem. and Math. fund.(IJPCMF), 7(1), 2014, 1 – 6.
24. Wagh Mamta A., "On a Class of Bicomplex Sequences", Inter. J. Trends in Math. & Stats., 3(5), 2014, 158 – 171.
25. Wagh, Mamta A. & Kumar, S., "Köthe – Toeplitz Duals of Certain Bicomplex Sequence Spaces", Inter. J. Math. & Comp. App. Research(IJMCAR), 4(3), 2014, 87–98.
26. Wagh, Mamta A. & Kumar, S., "On certain Bicomplex Duals", Global J. Sc. & Frontier Research (GJSFR), 14(6), 2014, 17 – 23.

Notes

27. Wagh, Mamta A., "On some Bicomplex Modules", to appear in Global J. Sc. & Frontier Research GJSFR, 16(3), 2016.
28. W. Ruckle, "Topologies on sequence spaces", Pacific Jour. Math. 42(1), 1972, 235-249.
29. Zeren S. & Bektas A. Cigdema. "Generalized difference sequence spaces defined by a sequence of moduli", Kragujevac Journal of Mathematics, 36(1), 2012, 83 – 91.

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Super-Symmetry Quantum Factorization Of Radial Schrodinger Equation for the Doubly An-Harmonic Oscillator Via Transformation to Bi-Confluent Heun's Differential Operators

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Abstract- Radial Schrodinger equation for the doubly An-Harmonic oscillator is considered and the transformation to Bi-Confluent equation done in [13] are considered. Super-symmetry method of factorization is applied to the transformed equation in obtaining solutions of the Schrodinger equation. Partner potentials and super-potentials are obtained.

Keywords: heun's, bi-confluent, schrodinger equation, super-symmetry, factorization.

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12. M. Suzuki, E. Takasugi and H. Umetsu Prof. Thor. Phys. 100, 491 (1998).

Super-Symmetry Quantum Factorization of Radial Schrodinger Equation for the Doubly An-Harmonic Oscillator Via Transformation to Bi-Confluent Heun's Differential Operators

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Abstract- Radial Schrodinger equation for the doubly An-Harmonic oscillator is considered and the transformation to Bi-Confluent equation done in [13] are considered. Super-symmetry method of factorization is applied to the transformed equation in obtaining solutions of the Schrodinger equation. Partner potentials and super-potentials are obtained.

Keywords: *heun's, bi-confluent, schrodinger equation, super-symmetry, factorization.*

I. Introduction

Heuns differential equation and its confluent forms have been subject of many investigations in last years due to a large number of their applications in mathematical physics and quantum mechanics [13, 9]. They indeed play a central role in a number of physical problems, like quasi-exactly solvable systems [10], higher dimensional correlated systems [11], Kerr-de Sitter black holes [12], Calogero-Moser-Sytherland system [14], finite lattice Bethe-ansatz systems [15], etc. Besides, this equation appears as a natural generalization of the hypergeometric equation and its special cases including the Gauss hypergeometric, confluent hypergeometric, Mathieu, Ince, Lame, Bessel, Legendre, Laguerre equation, etc.

The general second order Heun's differential equation (GHE) can be written, in canonical form, as follows [6]

\begin{equation} $$\mathcal{D}^2\mathcal{Y} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\mathcal{D}\mathcal{Y} + \frac{(\alpha\beta x - q)}{x\,(x-1)(x-a)}\mathcal{Y} = 0, \tag{I.1}$$
\end{equation}

Where $ $\mathcal{D} = \frac{d}{dx}, \{\alpha, \beta, \gamma, \delta, \epsilon, a, q\}\ (a \neq 0,1)$ $ are parameters, generally complex and arbitrary, linked by the Fuschian constraint $ $\{\alpha + \beta + 1 = \gamma + \delta + \epsilon.$ $ This equation has four regular singular points at $ $\{0,1,a,\infty\},$ $ with the exponents of these singularities being respectively, $ $\{0,1,-\gamma\}, \{0,1-\delta\}, \{0,1-\epsilon\},$ $ and $ $\{\alpha, \beta\}.$ $ The equation (I.1) can be transformed into the following other multi-parameters equations one of which is (14): Bi-confluent Heun's equation (BHE) (see page 131 of [13]).

\begin{equation} $$\mathcal{D}^2\mathcal{Y} + \left(\frac{\alpha+1}{x} - \beta - 2x\right)\mathcal{D}y + \left(\gamma - \alpha - z^2 - \frac{\beta+(\alpha+1)\beta}{2x}\right)\mathcal{Y} = 0, \tag{I.2}$$
\end{equation}

This equation has many important features in obtaining solutions to Schrodinger equations.
Consider the Schrodinger equations.

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\begin{equation}

$$\mathcal{Y}''(x_1) + \{\mathrm{E} - \mu x_1^2 - \lambda_1^4 - \eta x_1^6\}\mathcal{Y}(x) = 0, \quad \text{\end{equation}} \tag{I.3}$$

Is transformed into multi-parameters equation Bi-confluent Heun differential equation via the transformation

\begin{equation} $\mathcal{Y}(x_1) = x^{\frac{1}{2}} e - \left(\frac{1}{4}\alpha C \propto_1^4 + \frac{1}{2}\beta C \propto_1^2\right)\xi(x), x_1^2 = \left(\frac{2}{\alpha_C}\right)^{\frac{1}{2}} x$ \end{equation}

The Bi-confluent equation obtained was

\begin{equation}

$$x\xi(x)'' + \left\{\frac{3}{2} + \beta_C\left(\frac{2}{\alpha_C}\right)^{\frac{1}{2}} x - 2x^2\right\}\xi'(x) + \left\{\frac{1}{2\alpha_C}\left({\beta^2}_c - 3\alpha_C - \mu\right)x + \frac{1}{4}\left(\frac{2}{\alpha_C}\right)^{\frac{1}{2}}(\mathrm{E} + 3\beta_c)\right\}\xi(x) = 0$$

\end{equation} (I.4)

with

$$\alpha = \frac{1}{2}, \beta = \beta_c\left(\frac{2}{\alpha_C}\right)^{\frac{1}{2}}, \gamma = \frac{1}{2\alpha_C}\left(\beta_c^2 - 3\alpha_c - \mu\right) + \frac{3}{2}, \delta = -\frac{E}{(2\alpha_c)^{\frac{1}{2}}}$$

Comparing equation (I.2) and (I.4), the following parameter relations were deduced in page 201 of [13],

$$\alpha = 2(l+1)\alpha_F^{\frac{1}{2}} - 1;\ \beta = \frac{2\beta F}{\alpha_F^{\frac{1}{2}}};\ \gamma = \frac{\epsilon}{\alpha F} + 2(l+1)\left(\alpha_F^{\frac{1}{2}} - 1\right);\ \delta = \frac{2}{\alpha_F^{\frac{1}{2}}}\left\{-a + 2\beta_F(l+1)\left(1 - \alpha_F^{\frac{1}{2}}\right)\right\}$$

At this point, it is necessary to emphasize that the super-symmetry method of factorization of the Bi-confluent Heun's differential equation can be use to obtain solutions of the named Schrodinger equation. To do this we briefly discuss the SUSY method of factorization.

In recent work [4], the concept of factorization method, super-symmetry quantum mechanics (SUSY QM) and shape invariant techniques have been extended to Sturn-Liouville (SL) equations to solve Schrödinger equations. In the present work this concept shall be extended to the Bi-confluent Heun's differential equation.

II. Factorization of BH Operator

a) General method of factorization of SL operators

In this section, we extend to BH the general method of factorization of SL operators developed in the previous work [4] to construct new solvable potentials for Heun's operators. For a matter of convenience we first briefly recall the results of [4].

b) Brief Review of general method of factorization of SL Operator

Following [4], the concept of factorization method was extended to SL equation SUSY QM and shape invariance were widely developed to solve Schrödinger equations and reviewed. Consider the one-dimensional second order differential equation

$$\mathrm{H}\Phi = \xi\Phi', \ \Phi\Phi' \in \mathrm{AC_{loc}}\,(]\mathrm{a\,b},[), \tag{2.1}$$

where

$$\mathrm{H} = -\sigma(x)\frac{d^2}{dx^2} - \tau(x)\frac{d}{dx} + V(x). \tag{2.2}$$

ξ is a constants, $\sigma(x), \tau(x)$ and $V(x)$ are real function defined on the open interval, $(]\mathrm{a\,b},[) \subseteq \mathrm{R}$ and $\mathrm{AC_{loc}}(\mathrm{a,b})$ is the set of local absolute continuous functions given by

$$AC_{\mathrm{loc}}(\mathrm{a,b}) = \{f \in AC[\alpha_1\beta_1],\ \forall[\alpha_1\beta_1] \subset (\mathrm{a,b}), [\alpha_1\beta_1]\ \text{compact}\}, \tag{2.3}$$

$$AC[\alpha_1\beta_1] = \left\{f \in C[\alpha_1\beta_1], f(\mathrm{x}) = f(\alpha_1) + \int_{\alpha_1}^{\mathrm{x}} g(t)dt,\ g \in L^1[\alpha_1\beta_1]\right\}. \tag{2.4}$$

Ref
4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005)

The suitable Hilbert space $\mathbb{H} = L^2\left([a,b], \rho(x)dx\right)$ with the inner product defined by means of non-negative weight function $\rho(x)$ on $]a\,b, [$:

$$\langle u, v\rangle = \int_a^b \bar{u}(x)v(x)p(x)dx, \qquad u(x), v(x) \in \mathbb{H}, \tag{2.5}$$

Where $\bar{u}$ is the complex conjugate of u. The domain of H will be examined below. Choosing the weight function $\rho(x)$ such that the Pearson equation.

$$[\sigma(x)\rho(x)]' = \tau(x)p(x), \tag{2.6}$$

Is satisfied. The differential equation (2.1) can be reduced to the self-adjoint form [4]

$$[\sigma(x)p(x)\Psi'(x)]' - [V-\xi] \quad \Psi(x)p(x) = 0, \tag{2.7}$$

and the operator (2.2) can be written in the equivalent form of SL operators [4]

$$H = \frac{1}{\rho(x)}\left(-\frac{d}{dx}\rho(x)\frac{d}{dx} + q(x)\right), \tag{2.8}$$

Where $p(x) = \sigma(x)\rho(x)$ *and* $q(x)V(x)p(x)$. Eq. (2.5), together with the following boundary condition:

$$\rho(x)\sigma(x)\left[\bar{u}(x)v'(x) - \bar{u}'(x)v(x)\right]\Big|_a^b = 0, \ \forall u, v \in \mathbb{H}, \tag{2.9}$$

Is called *Sturm-Liouville system* [4]. The boundary condition ensures the self-adjointness of the operator H. Since we want the operator to be self-adjoint, we take on the Hilbert space H as

$$\mathcal{D}(H) = \{u \in \mathbb{H}, u, pu' \in AC_{\text{loc}}(\text{a}, \text{b}), Hu \in \mathbb{H}\},$$

$$p(x)\left[\bar{u}(x)v'(x) - \bar{u}'(x)v(x)\right]\Big|_a^b = 0, \ \forall u, v \in \mathbb{H}. \tag{2.10}$$

It is clear that $\mathcal{D}(H)$ is dense in $\mathbb{H}$ since $C_0^\infty(]a\,b, [) \subset \mathcal{D}(H)$. by requiring:

(i) $p \in AC_{\text{loc}}(]a\,b, [), \rho' \in L^2_{loc}(]a\,b, [), \rho^{-1} \in L^2_{loc}(]a\,b, [)$ positive and real-valued;

(ii) $q \in L^2_{loc}(]a\,b, [)$, be real-valued;

(iii) $\in L^1_{loc}(]a\,b, [), p^{-1} \in L^1_{loc}(]a\,b, [)$, positive and real valued. Then, the operator (H, $\mathcal{D}(H)$) is self adjoint [4].

The purpose of this section is to introduce a factorization model with an annihilator operator of the form

$$A = \kappa\left[\frac{d}{dx} + W(x)\right], \tag{2.11}$$

with domain:

$$\mathcal{D}(A) = \{u \in \mathbb{H}, \kappa u' + \kappa W u \in \mathbb{H}\}, \tag{2.12}$$

where κ and W are continuous functions on $(]a\,b, [)$. We infer that $\mathcal{D}(A)$ dense in $\mathbb{H}$ since $H^{1,2}\left((]a\,b, [), p(x)dx\right)$ is dence in $\mathbb{H}$since $H^{1,2}\left((]a\,b, [), p(x)dx\right) \subset \mathcal{D}(A)$ where $H^{m,n}(\Omega)$ is a sobolev space of indices $\{m, n\}$. The operator A is closed in $\mathbb{H}$. The adjoint operator A^+ is given by [4].

$$\mathcal{D}(A^+) = \{v \in \mathbb{H}\}|\exists \bar{v} \in \mathbb{H}\}: \quad \langle Au, v\rangle = \langle Au, \bar{v}\rangle, \forall u, \in \mathcal{D}(A), A^+v = \bar{v}. \tag{2.13}$$

The explicit expression of (A^+) is given through the following theorem

Theorem 2.1 [4] Suppose the following boundary condition

$$\kappa(x)\rho(x)u(x)v(x)\Big|_a^b = 0, \forall u \in \mathcal{D}(A) \text{ and } v\ \mathcal{D}(A^+), \tag{2.14}$$

Ref

4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005).

is verified. then the operator A^+ can be written as

$$A^\dagger = \kappa(x)\left[-\frac{d}{dx} + W(x) + \mu(x)\right], \tag{2.15}$$

where $\mu(x)$ is a real continuous function defined by $\mu(x) = \frac{d}{dx}\ln[\kappa(x)\rho(x)]$.

Let H_1 and H_2 be the product operators A^+A and AA^+, respectively,

$$H_1 = A^+A, \qquad H_2 = AA^+, \tag{2.16}$$

with the corresponding domains

$$\mathcal{D}\,(H_1) = \{u \in \mathcal{D}(A), v = Au \in \mathcal{D}(A^\dagger) \text{ and } A^\dagger v \in \mathbb{H}\},$$
$$\mathcal{D}\,(H_2) = \{u \in \mathcal{D}(A^\dagger), v = A^\dagger u \in \mathcal{D}(A) \text{ and } Av \in \mathbb{H}\}. \tag{2.17}$$

Remark that

$$H^{1,2}(]\mathrm{a}\ \mathrm{b}[, p(x)dx) \subset \mathcal{D}(A) \subset \mathcal{D}(A^+),$$

$$\mathcal{D}\,(H_1), \mathcal{D}\,(H_2) \supset H^{2,2}(]\mathrm{a}\ \mathrm{b}[, \rho(x)dx).$$

We infer the $\mathcal{D}\,(H_1)$ and $\mathcal{D}\,(H_2)$ are dense in $\mathbb{H}$. Furthermore, the following theorem gives the additional conditions to subject to the function of κ and the potentials V so that the operator H factorizes in terms of A and A^+ .

Theorem 2.2 *[4] Suppose that*

(i) κ *and* μ *are related to* σ *and* τ *as:*

$$\kappa^2 = \sigma; \quad \kappa(\kappa^1 - \kappa\mu) = \tau; \tag{2.18}$$

(ii) *the potential function V is related to the W by the Riccati type equation*

$$V - \xi_0 = \sigma\,(W^2 - W) - \tau W'. \tag{2.19}$$

Then the operators $H_{1,2}$ *are self-adjoint and*

$$H_1 = A^+A = H - \xi_0 = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + \sigma\,(W^2 - W') - \tau\,W',$$

$$H_2 = AA^+ = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + \sigma\,(W^2 - W') + (\tau - \sigma')W + \kappa(\kappa\mu)'. \tag{2.20}$$

Let us remark that the condition $\kappa(\kappa^1 - \kappa\mu) = \tau$ of (2.19) can be done from the Pearson equation (2.6) and the constraint $\kappa^2 = \sigma$. The quantity α can also be expressed as $\alpha = \frac{\kappa^1}{\kappa} - \frac{\tau}{\sigma}$. by means of the operation A and $A^\dagger$ we can form a superalgebra as follows;

$$\{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i = H_{ss}\delta_{ij}, \qquad [H_{ss}Q_i] = 0; \qquad i,j = 1,2,$$

Where $Q_1 = (Q^+ + Q^-)/\sqrt{2}$ and $Q_2 = (Q^+ + Q^-)/i\sqrt{2}$

With $Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$, $Q^- = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$, $\qquad H_{ss} = \begin{pmatrix} A^+A & 0 \\ 0 & AA^+ \end{pmatrix}$. (2.21)

We can rewrite the operators $H_{1,2}$ as

$$H_1 = A^+A = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + V_1 \text{ and } H_2 = AA^+ = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + V_2 \tag{2.22}$$

Ref

4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005)

Where

$$V_1 = \sigma\,(W^2 - W') - \tau W, \quad V_2 = \sigma\,(W^2 - W') - (\tau - \sigma')W + \kappa(\kappa\mu)'. \tag{2.23}$$

It clearly appears the SUSY QM is extended to SL operators. We design here that the operators $H_{1,}H_2$ as SUSY partner. V_1, V_2 are SUSY partner potentials. The expression [4]

$$H_{ss} = [-D^2 + W^2(x)]I_2 + W'(x)\sigma_3, \tag{2.24}$$

which gives the superalgebra in terms of the superpotentials takes here the form

$$H_{ss} = \quad -[\sigma\frac{d^2}{dx^2} - \tau(x)\frac{d}{dx} - \sigma W^2 + \tau W - \frac{1}{2}(\sigma' W + \kappa(\kappa\mu)')]I_2$$

$$[\sigma W' + \frac{1}{2}(\sigma' W + (\kappa\mu)']\sigma_{3,}] \tag{2.25}$$

Where $\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the Pauli spin matrix and I_2 designs the 2×2 identity matrix. The equation (2.23) are the Riccati equation relating the partner potentials to the superpotentials. We denote, for $n \geq 0$, and $\xi_n^{(1)}$ the energy eigenfunctions and eigenvalues of H_1, respectively and$\Phi_n^{(2)}$ and $\xi_n^{(1)}$of H_2, respectively.

The pair of SL operators $H_{1,2}$ satisfies the interwining relation

$$H_2 A = AH, \quad H_1 A^+ = A^+ H_{2,} \tag{2.26}$$

and their states are related by SUSY transformations

$$H_2\left(A\Phi_n^{(1)}\right) = \xi_n^{(1)}\left(A\Phi_n^{(1)}\right), \qquad H_1\left(A\Phi_n^{(2)}\right) = \xi_n^{(2)}\left(A\Phi_n^{(2)}\right). \tag{2.27}$$

It is then straight forward to show that the eigenvalues of H_1 and H_2 are positive definite $\xi_n^{(1,2)} \geq 0$ and isospectra, i.e. they have almost the same energy eigenvalues, except for the ground state energy of H_1. Their energy spectra are related by the same equations.

$$\xi_n = \xi_n^{(1)} + \xi_{0,}\ \Psi_n = \Psi_n^{(1)}, \quad \xi_n^{(2)} = \xi_{n+1,}^{(1)} \qquad \Psi_n^{(2)} = [\xi_{n+1}^{(1)}]^{-1/2} A \Psi_{n+1}^{(1)},$$
$$\Psi_{n+1}^{(2)} = [\xi_n^{(1)}]^{-1/2} A^+ \Psi_n^{(2)}, \qquad n = 0,1,2, \ldots \tag{2.28}$$

We remark that the SL operator (2.2) is related to the Schrodinger-type operator. If we make the chance of variable such that $x = x(t)$ such that $\frac{d}{dx} = \kappa(x(t))$ and define the new function

$$\Psi_n(t) = \sqrt{(\kappa(x(t))\rho x(t)\phi_n(x(t))} \tag{2.29}$$

Then equation (2.1) turns to an equation of the Schrodinger type

$$\frac{d^2}{dt^2}\Psi_n(t) + V(t)\,\Psi_n = \xi_n \Psi_n(t), \tag{2.30}$$

Where

$$V(t) = \left[V(x) + \frac{\tau(x)\,((\kappa(x)p(x))' + \kappa^2(x)(\kappa(x)\rho(x))''}{2\kappa(x)\rho(x)} - \frac{3\kappa^2(x)((\kappa(x)\rho(x))')^2}{4(\kappa(x)\rho(x))^2}\right]_{x=x(t)} \tag{2.31}$$

We shall denote various forms of corresponding superpotentials and partner-potentials of BHE by $BHEW_j$ and $BHEV_j$ and $j = 1, \ldots k$. The integer k depends on the number of the corresponding solutions of the Bi-confluent Heun's equation.

Ref

4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005)

c) Factorization of Bi-confluent Heun's Differential Operator (BH)

The second order differential operator s corresponding to BHE reads as

$$H^{BHE} = -xD^2 - (1+\alpha-\beta x-2x^2)D - \left((\gamma-\alpha-2)x - {}^1/_2\,[\delta+\beta(1+\alpha)]\right), \qquad (2.32)$$

Having the following factorization characteristics.

(1) $\sigma = x, \kappa^2 = x$ which impies $\kappa = \pm\sqrt{x}$,

(2) $\tau = (1+\alpha-\beta x-2x^2)$,

(3) $V = -[(\gamma-\alpha-2)x - {}^{[\delta+\beta(1+\alpha)]}/_2]$,

(4) $\mu = \frac{\beta x+2x^2-\alpha-{}^1/_2}{x}$

The operator H factorizes into two first order differentials operators $A = \kappa(x)\,(x)(D+W(x))$and $A^+ = \kappa(x)(D+W(x)+\mu)$. The operator H, also could be expressed I terms of $H_{1,2}$ as

$$H_1 = -\sigma(x)D^2 - \tau(x)D + V_1(x), \quad H_2 = -\sigma(x)D^2 - \tau(x)D + V_2(x),$$

$$V_1(x) = \sigma(W^2 - W') - \tau W, \quad V_2(x) = \sigma(W^2 - W') - (\tau - \sigma')W + \kappa(\kappa\alpha)'. \qquad (2.33)$$

As accustomed, we set $V_1 = V - E_0$, with $E_0 = 0$ and $W = -z'(x)/z(x)$ into the Riccati equation of V_1 we obtain the original BHE equation given below

$$xD^2z + (1+\alpha-\beta x-2x^2)Dz + ((\gamma-\alpha-2)x - {}^1/_2\,[\delta+\beta(1+\alpha)]z = 0. \qquad (2.34)$$

Also for BHE, from results in [13](pages 203-206), Eq. (2.34) has 16 forms of solutions corresponding to the super-potential $BHEW_j$; $j = 1,2\ldots,16,$ where

$$BHEW_j(x) = -(ln_{zj}(x))'. \qquad (3.35)$$

The $zj\,(x)$ are the solutions of (2.34). one of which is

$$z_1(x) = N(\alpha,\beta,\gamma,\delta;x), \quad z_2(x) = x^{-\alpha}N(-\alpha,\beta,\gamma,\delta;x),$$

when α is not a relative integer. Here $N(\alpha,\beta,\gamma,\delta;x) = \sum_{n\ge0}\frac{A_n}{(1+\alpha)n}\frac{x^n}{n!}$, where $A_0 = 1$; $A_1 = \frac{1}{2}\left(\delta+\beta(1+\alpha)\right)$ and

$$A_{n+2} = \left\{(n+1)\beta + \tfrac{1}{2}\left(\delta+\beta(1+\alpha)\right)\right\}A_{n+1} - (n-1)(n+1+\alpha)(\gamma-2-\alpha-2n)A_n$$

The associated partner potentials reads as, for $j = 1,2\ldots,16,$

$$BHEV_{2j} = x(BHEW_j^2 + BHEW_j^1 + (\alpha-\beta x-2x^2)BHEW_j + \frac{{}^1/_2+2x^2+\alpha}{\sqrt{x}}\,. \qquad (2.36)$$

Thus, the BH operator factorizes as, for $j = 1,2\ldots16, \varepsilon = \pm1,$

$$A = \varepsilon\sqrt{x}\left(D + BHEW_j(x)\right),$$

$$A = \sqrt{x}\left(-D + BHEW_j(x) + \frac{{}^1/_2+2x^2+\alpha}{\sqrt{x}}\right)$$

Ref

13. A. Ronveaux, (ed.) Heun's Differential equation (Oxford university press, Oxford, 1995)

Haven derive the solution of (2.34), these solutions are related to equation (1.4) with parameter relation as stated above. These relation gives the solution of the equation (1.3). a super-potentials and partner potentials are obtained. It should be emphasizes at this stage that the partner-potentials and Super-potentials are not shape invariant.

III. Remarks

Apart from the confinement potentials of the Schrödinger equation, this method generates other potentials such as the partner potentials and super-potentials which are not shape invariant.

Notes

References Références Referencias

1. K. Kuiten, Heun's equation and the hypergeometric equation, S.I.A.M. J. Math. Anal. 10, 3 (1979)
2. Erdelyi et al, Higher Trunscidential Fucntions, (Bateman Manuscript Project), Vols. 1 and 3, McGraw-Hill, New York, 1953, 1955
3. C. Snow, Hypergeometric and Legendre Functions with Application to Integral Cautions of Theory, Department of Commerce, National bureau of Standards, Applied Matheamtics Series, 19, United States Government Printing Office, Washinting, D.C. (1952).
4. M.N. Hounkonnou, K.S. Sodoga and E.S. Azatassou, J. Phys A: math. Gen. 38, 371 (2005)
5. G. Teiscl, Schrodinger operators, course http://www.math.univie.ac.at /- Gerald/, (1999).
6. F. W. Byron, Jr, and R. W. fuller, mathematics of classical and quantum physics (New York, Dover publications Inc, 1972).
7. S. M. Robert. The 192 solutions of the Heun's equation, Mathematics of computation. Oo.0000 (2004).
8. A. Ronveax, (ed). Heun's Differential equation (Oxford university press, Oxford, 1995)
9. A. O. Smirinov, Eliptic solutions and Heun's equations C.R.M. Proceedings and Lecture notes 32, 287 (2002).
10. N. H. Christ and T.D. Lee, Phys. Rev. D. 12, 1606 91975); D.P. jatker, C.N. Kumar and A. khare, Phys. Lett. A 142, (19829); A. Khare and B.P. Mandal, Phys Lett. A. 239, 197 (1998)
11. R. K. bhadari, A. Khare, J. Law, M.V.N. Murthy and D. Sen; J. Phys A: math. Gen. 30, 2557 (1997)
12. M. Suzuki, E. Takasugi and H. Umetsu Prof. Thor. Phys. 100, 491 (1998).
13. A. Ronveaux, (ed.) Heun's Differential equation (Oxford university press, Oxford, 1995)
14. K. Takemura, Commu. Math. Phys. 235, 467 (2003); K Takemura, J. Nonlinear Math. Phys. 11, 21 (2004).
15. P. Dorey, J. Suzuki and R. Tateo, J. Phys. A 37, 2047 (2004).
16. A. Ambramowitz and J.A. Stegun, (eds) Handbook of mathematics functions (New York, Dover, 91965).
17. J. Mathews and R. L. Walker Mathematics methods of physics. Second edition Φ 1964, 1970

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Certain Results on Bicomplex Topologies and their Comparison

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Abstract- In this paper, we have investigated the relation between idempotent order and norm topological structures. We have discussed about the relation between real order topology and idempotent order topology and we have also established the relation between real order topology and norm topology.

Keywords: real order topology, complex order topology, idempotent order topology, norm topology, comparison.

GJSFR-F Classification : *MSC 2010: 54A10, 30G35*

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Strictly as per the compliance and regulations of :

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5. Ringleb, F.: Beitrage Zur Funktionen theorie in Hyperkomplexon Systemen, I, Rend. Circ. Mat. Palermo, 57 (1933), 311-340.

Certain Results on Bicomplex Topologies and their Comparison

Akhil Prakash [α] & Prabhat Kumar [σ]

Abstract- In this paper, we have investigated the relation between idempotent order and norm topological structures. We have discussed about the relation between real order topology and idempotent order topology and we have also established the relation between real order topology and norm topology.

Keywords: real order topology, complex order topology, idempotent order topology, norm topology, comparison.

I. Introduction

In 1892, Corrado Segre (1860-1924) published a paper [6] in which he treated an infinite set of Algebras whose elements he called bicomplex numbers, tricomplex numbers,…., n-complex numbers. A bicomplex number is an element of the form $(x_1+i_1x_2)+i_2(x_3+i_1x_4)$, where $x_1,\dots, x_4$ are real numbers, $i_1^2=i_2^2=-1$ and $i_1i_2=i_2i_1$.

Segre showed that every bicomplex number $z_1+i_2z_2$ can be represented as the complex combination

$$(z_1-i_1z_2)\left[\frac{1+i_1i_2}{2}\right]+(z_1+i_1z_2)\left[\frac{1-i_1i_2}{2}\right]$$

Srivastava [8] introduced the notations $^1\xi$ and $^2\xi$ for the idempotent components of the bicomplex number $\xi = z_1+i_2z_2$, so that

$$\xi = {}^1\xi.\frac{1+i_1i_2}{2}+{}^2\xi.\frac{1-i_1i_2}{2}$$

Michiji Futagawa seems to have been the first to consider the theory of functions of a bicomplex variable [1, 2] in 1928 and 1932.

The hyper complex system of **Ringleb** [5] is more general than the Algebras; he showed in 1933 that Futagawa system is a special case of his own.

In 1953 James D. Riley published a paper [4] entitled "Contributions to theory of functions of a bicomplex variable".

Throughout, the symbols $\mathbb{C}_2$, $\mathbb{C}_1$, $\mathbb{C}_0$ denote the set of all bicomplex, complex and real numbers respectively.

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In $\mathbb{C}_2$-besides 0 and 1- there are exactly two non-trivial idempotent elements denoted as e_1 and e_2 and defined as

$$e_1 = \frac{1+i_1 i_2}{2} \text{ and } e_2 = \frac{1-i_1 i_2}{2}$$

Obviously $(e_1)^n = e_1$, $(e_2)^n = e_2$

$$e_1 + e_2 = 1,\ e_1.e_2 = 0$$

Every bicomplex number ξ has unique idempotent representation as complex combination of e_1 and e_2 as follows

$$\xi = z_1+i_2z_2 = (z_1-i_1z_2)e_1 + (z_1+i_1z_2)e_2$$

The complex numbers $(z_1-i_1z_2)$ and $(z_1+i_1z_2)$ are called idempotent component of ξ, and are denoted by ${}^1\xi$ and ${}^2\xi$ respectively *(cf. Srivastava [8]).*

Thus $\xi = {}^1\xi\ e_1 + {}^2\xi\ e_2$

a) h_1, h_2 image and Cartesian idempotent set

The h_1 and h_2 image of a set X are denoted as ${}^1X$ and ${}^2X$ respectively and defined as

$$h_1(X) = {}^1X = \{z:\ ze_1+we_2 \in X\} = \{{}^1\xi : \xi \in X\}$$

$$h_2(X) = {}^2X = \{w:\ ze_1+we_2 \in X\} = \{{}^2\xi : \xi \in X\}$$

The Cartesian idempotent product of ${}^1X$ and ${}^2X$ is the set which is the subset of $\mathbb{C}_2$ and denoted as ${}^1X \times_e {}^2X$ and defined as

$${}^1X \times_e {}^2X = \{ze_1+we_2:\ z \in {}^1X,\ w \in {}^2X\}$$

If $X = {}^1X \times_e {}^2X$ then X is said to be Cartesian idempotent set *(cf. Srivastava [8]).*

II. Certain Results from Topologies on Bicomplex Space

a) Norm, Complex and Idempotent topologies on $\mathbb{C}_2$[9].

2.1.1 Norm topology

The norm of a bicomplex number $\xi = z_1+i_2z_2 = x_1+i_1x_2+i_2x_3+i_1i_2x_4 = {}^1\xi\ e_1 + {}^2\xi\ e_2$ is defined as

$$\|\xi\| = ({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2)^{1/2}$$

$$= (|z_1|^2+|z_2|^2)^{1/2}$$

$$= \sqrt{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}}$$

Since $\mathbb{C}_2$ is modified normed algebra w.r.t this norm therefore for $\delta > 0$, the δ-ball centered at x is the set

$B(x,\delta) = \{y \in \mathbb{C}_2 : \|x - y\| < \delta\}$ of all points y whose distance from x is less than δ, it is called the δ-ball centered at x.

The collection '$\boldsymbol{B_N}$' of all δ-balls $B(x,\delta)$, for $x \in \mathbb{C}_2$ and $\delta>0$ is a basis for a topology on $\mathbb{C}_2$. The topology generated by $\boldsymbol{B_N}$ is called norm topology on $\mathbb{C}_2$ and denoted by τ_N.

Ref

9. Srivastava, Rajiv K.: Certain topological aspects of the Bicomplex space, Bull. Pure & Appl. Maths., 2(2)(2008),222-234.

2.1.2 Theorem

If X is a Cartesian idempotent set in $\mathbb{C}_2$ then X is open (w.r.t. norm topology) if and only if ${}^1X$ and ${}^2X$ are open in complexplane **(cf. Price [3])**.

2.1.3 Complex topology

The norm of a complex number 'z' is defined as $\|z\| = |z|$

Since $\mathbb{C}_1$ is a normed algebra w.r.t. this norm therefore the collection '$\boldsymbol{B}$' of all circular disk $S(z,\delta)$, $z\in\mathbb{C}_1$ and $\delta > 0$ will be a basis for a topology on $\mathbb{C}_1$, where $S(z,\delta) = \{w\in\mathbb{C}_1: |z - w| < \delta\}$Therefore

$\boldsymbol{B}\times\boldsymbol{B} = \{S_1\times S_2: S_1, S_2\in\boldsymbol{B}\}$ will be a basis for some topology on $\mathbb{C}_1\times\mathbb{C}_1$.Since $C_2\cong C_1\times C_1$ therefore $\boldsymbol{B}_C = \{S_1\times_C S_2: S_1, S_2\in\boldsymbol{B}\}$ will be a basis for some topology on $\mathbb{C}_2$, where

$$S_1\times_C S_2 = \{\eta = w_1+i_2w_2: w_1\in S_1,\ w_2\in S_2\}$$

If $S_1 = S_1(z_1,r_1)$ and $S_2 = S_2(z_2,r_2)$
Then $S_1\times_C S_2 = S_1(z_1,r_1)\times_C S_2(z_2,r_2)$

$$= \{\eta = w_1+i_2w_2: |z_1 - w_1| < r_1,\ |z_2 - w_2| < r_2\}$$

The set $S_1(z_1,r_1)\times_C S_2(z_2,r_2)$ is denoted by $C(\xi = z_1+i_2z_2;r_1,r_2)$ and this set $C(\xi = z_1+i_2z_2;\ r_1,r_2)$ is called *open complex discus* centered at ξ and associated radii r_1 and r_2.
Therefore

$$C(\xi = z_1+i_2z_2;\ r_1,\ r_2) = \{\eta = w_1+i_2w_2: |z_1 - w_1| < r_1,\ |z_2 - w_2| < r_2\}$$

Thus $\boldsymbol{B}_C$ = Set of all open complex discus

$$= \{C(\xi;\ r_1,\ r_2): \xi\in\mathbb{C}_2 \text{ and } r_1,\ r_2 > 0\}$$

The topology generated by '$\boldsymbol{B}_C$' is called complex topology and denoted by $\mathcal{T}_C$.

2.1.4 Idempotent topology

Since $\boldsymbol{B}\times\boldsymbol{B} = \{S_1\times S_2: S_1, S_2\in\boldsymbol{B}\}$ is a basis for some topology on $\mathbb{C}_1\times\mathbb{C}_1$ and $C_2\cong C_1\times C_1$ therefore $\boldsymbol{B}_I = \{S_1\times_e S_2: S_1, S_2\in\boldsymbol{B}\}$ will be a basis for some topology on $\mathbb{C}_2$, Where

$$S_1\times_e S_2 = \{\eta = {}^1\eta\ e_1 + {}^2\eta\ e_2: {}^1\eta\in S_1,\ {}^2\eta\in S_2\}$$

If $S_1 = S_1(z_1,r_1)$ and $S_2 = S_2(z_2,r_2)$
Then $S_1\times_e S_2 = S_1(z_1,r_1)\times_e S_2(z_2,r_2)$
$= \{\eta = {}^1\eta\ e_1 + {}^2\eta\ e_2: |{}^1\eta - z_1| < r_1,\ |{}^2\eta - z_2| < r_2\}$. The set $S_1(z_1,r_1)\times_e S_2(z_2,r_2)$ is denoted by $D(\xi = z_1e_1+z_2e_2;\ r_1,r_2)$ and this set $D(\xi = z_1e_1+z_2e_2;\ r_1,r_2)$ is called *open idempotent discus* centered at ξ and associated radii r_1 and r_2.
Therefore $D(\xi;\ r_1,\ r_2) = \{\eta: |{}^1\eta - {}^1\xi| < r_1, |{}^2\eta - {}^2\xi| < r_2\}$
Thus $\boldsymbol{B}_I$ = Set of all open idempotent discus

$$= \{D(\xi;\ r_1,\ r_2): \xi\in\mathbb{C}_2 \text{ and } r_1,\ r_2 > 0\}$$

The topology generated by '$\boldsymbol{B}_I$' is called idempotent topology and denoted by $\mathcal{T}_I$.

Ref

3. Price, G.B.: An introduction to multicomplex spaces and Functions, Marcel Dekker Inc., New York, 1991.

b) Order topology on $\mathbb{C}_2$

Singh [7] has developed certain orders on $\mathbb{C}_2$. He defined three types of ordering in $\mathbb{C}_2$, viz., Real dictionary order $<_{R,}$ Complex dictionary order $<_{c,}$ Idempotent dictionary order $<_{ID}$. With the help of these three relations he has defined three order topologies on $\mathbb{C}_2$. The order topology induced by real dictionary order is called as real order topology τ_1, the order topology generated by Complex dictionary order is called complex order topology τ_1^* and the topology induced by Idempotent dictionary order is called idempotent order topology $\tau_1^{\#}$ on $\mathbb{C}_2$.

In the present paper, $\boldsymbol{B_1}$, $\boldsymbol{B_1^*}$ ***and*** $\boldsymbol{B_1^{\#}}$ denotes the basis of τ_1, $\tau_1^*$and$\tau_1^{\#}$ respectively. The sets $(\xi, \eta)_R$, $(\xi, \eta)_C$ and $(\xi, \eta)_{ID}$ are the open interval with respect to Real dictionary order, Complex dictionary order and Idempotent dictionary order relation.

c) Product and metric topology on $\mathbb{C}_2$

Singh [7]defined three product topologies on $\mathbb{C}_2$,viz., Real product topology τ_2,Complex product topology τ_2^* and Idempotent product topology $\tau_2^{\#}$.
$\boldsymbol{B_2}$, $\boldsymbol{B_2^*}$ and $\boldsymbol{B_2^{\#}}$ denotes the basis of τ_2, τ_2^* and $\tau_2^{\#}$ respectively. Where

$$\boldsymbol{B_2} = \{S_1 \times_R S_2 \times_R S_3 \times_R S_4 : S_1, S_2, S_3, S_4 \text{ are in } \widetilde{\boldsymbol{B}}\}$$

$$S_1 \times_R S_2 \times_R S_3 \times_R S_4 = \{\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1 \in S_1, x_2 \in S_2, x_3 \in S_3, x_4 \in S_4\}$$

$\widetilde{\boldsymbol{B}}$ is the collection all open intervals in $\mathbb{C}_0$

$$\boldsymbol{B_2^*} = \{S_5 \times_c S_6 : S_5, S_6 \in \boldsymbol{B^*}\}$$

$\boldsymbol{B^*}$ is the collection of all open intervals in $\mathbb{C}_1$

$$S_5 \times_c S_6 = \{\xi = z_1 + i_2 z_2 : z_1 \in S_5, z_2 \in S_6\}$$

$$\boldsymbol{B_2^{\#}} = \{S_7 \times_e S_8 : S_7, S_8 \in \boldsymbol{B^*}\}$$

$$S_7 \times_e S_8 = \{z_1e_1 + z_2e_2 : z_1 \in S_7, z_2 \in S_8\}$$

There are three metrics on $\mathbb{C}_2$, viz., real metric, complex metric and idempotent metric. With the help of these three metric *Singh [7]* defined three metric topologies on $\mathbb{C}_2$.The topology generated by the real metric is known as the real metric topology τ_3, the topology generated by the complex metric is called complex metric topology τ_3^* and the topology generated by the idempotent metric is called idempotent metric topology $\tau_3^{\#}$.

In this present paper, $\boldsymbol{B_3}$, $\boldsymbol{B_3^*}$and $\boldsymbol{B_3^{\#}}$ denotes the basis of τ_3, τ_3^* and $\tau_3^{\#}$ respectively.

2.3.1 Theorem

The norm topology τ_N and idempotent topology τ_I on $\mathbb{C}_2$ are equivalent to each other *Srivastava [9]*.

2.3.2 Theorem

The norm topology τ_N and complex topology τ_C on $\mathbb{C}_2$ are equivalent to each other *Srivastava [9]*.

Ref

7. Singh, Sukhdev: Certain order topologies on the bicomplex Space,M.Phil Dissertation, Dr. B. R. Ambedkar University, Agra, 2009.

Theorems 2.3.1 and 2.3.2 imply that

2.3.3 Corollary

The complex topology τ_C and idempotent topology τ_I on $\mathbb{C}_2$ are equivalent.

2.3.4 Theorem

The real order topology τ_1 and real product topology τ_2 on $\mathbb{C}_2$ are equivalent to each other [7].

2.3.5 Theorem

The real product topology τ_2 and real metric topology τ_3 on $\mathbb{C}_2$ are equivalent to each other [7].

Theorems 2.3.4 and 2.3.5 imply that

2.3.6 Corollary

The real order topology τ_1 and real metric topology τ_3 on $\mathbb{C}_2$ are equivalent to each other.

2.3.7 Theorem

The complex order topology τ_1^* and complex product topology τ_2^* on $\mathbb{C}_2$ are equivalent to each other [7].

2.3.8 Theorem

The complex product topology τ_2^* and complex metric topology τ_3^* on $\mathbb{C}_2$ are equivalent to each other [7].

In view of Theorems 2.3.7 and 2.3.8, we have

2.3.9 Corollary

The complex order topology τ_1^* and complex metric topology τ_3^* on $\mathbb{C}_2$ are equivalent to each other.

2.3.10 Theorem

The idempotent order topology $\tau_1^\#$ and idempotent product topology $\tau_2^\#$ on $\mathbb{C}_2$ are equivalent to each other [7].

2.3.11 Theorem

The idempotent product topology $\tau_2^\#$ and idempotent metric topology $\tau_3^\#$ on $\mathbb{C}_2$ are equivalent to each other [7].

As the idempotent order topology on $\mathbb{C}_2$ and the idempotent product topology on $\mathbb{C}_2$ are equivalent to each other. Also, the idempotent product topology and idempotent metric topology are equivalent to each other, therefore we have.

2.3.12 Corollary

The idempotent order topology $\tau_1^\#$ and idempotent metric topology $\tau_3^\#$ on $\mathbb{C}_2$ are equivalent to each other.

2.3.13 Corollary

As the real dictionary ordering of the bicomplex number is same as the complex dictionary ordering of the bicomplex numbers therefore the real order topology τ_1 is equivalent to the complex order topology τ_1^* on C_2.

Theorems 2.3.4, 2.3.5, 2.3.7, 2.3.8 and Corollary 2.3.6, 2.3.9, 2.3.13 imply that

2.3.14 Corollary

The real order, real product, real metric, complex order, complex product and complex metric topology on $\mathbb{C}_2$ are equivalent to each other.

Ref

7. Singh, Sukhdev: Certain order topologies on the bicomplex Space,M.Phil Dissertation, Dr. B. R. Ambedkar University, Agra, 2009.

2.3.15 Corollary

The real dictionary ordering and complex ordering of the bicomplex numbers is different from the idempotent ordering of the bicomplex numbers therefore the idempotent order topology can be neither equivalent to the real order topology nor to the complex order topology on C_2.

III. Comparison of Various Topologies on Bicomplex Space

This section is our contribution to the theory of bicomplex topology and contains some important results from topological structures on the bicomplex space. In this section, we have tried to develop some relation between various topological structures on the bicomplex space.

a) Comparison of the idempotent order topology and norm topology on the bicomplex space

3.1.1 Lemma

The set $({}^1\xi e_1 + ({}^2\xi - i_1\delta)e_2, {}^1\xi e_1 + ({}^2\xi + i_1\delta)e_2)_{ID}$ is the proper subset of $B(\xi, r)$ where $0 < \delta < \sqrt{2}$and $r > 0$

Proof- Let suppose

$$\eta \in ({}^1\xi e_1 + ({}^2\xi - i_1\delta)e_2, {}^1\xi e_1 + ({}^2\xi + i_1\delta)e_2)_{ID} \qquad \ldots(1)$$

$$\Rightarrow {}^1\eta = {}^1\xi \text{ and } {}^2\xi - i_1\delta < {}^2\eta < {}^2\xi + i_1\delta$$

Since ${}^2\xi \in \mathbb{C}_1$ therefore consider ${}^2\xi = a + i_1 b$

where a, b $\in \mathbb{C}_0$

$\Rightarrow {}^2\eta = a + i_1 q$ where $b - \delta < q < b + \delta$

Now $\left|{}^2\xi - {}^2\eta\right| = |(a + i_1 b) - (a + i_1 q)|$

$= \mp(b - q)$

Since $b - \delta < q < b + \delta$ therefore $\mp(b - q) < \delta$

$\Rightarrow \left|{}^2\xi - {}^2\eta\right| < \delta$

Since $0 < \delta < \sqrt{2}\, r$

$$\Rightarrow \left|{}^2\xi - {}^2\eta\right| < \sqrt{2}\, r$$

$$\Rightarrow \frac{\left|{}^2\xi - {}^2\eta\right|}{\sqrt{2}} < r \qquad \ldots(2)$$

Now $\|\xi - \eta\| = \sqrt{\dfrac{\left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2}{2}}$

$$= \sqrt{\frac{0 + \left|{}^2\xi - {}^2\eta\right|^2}{2}} \text{ ; since } {}^1\eta = {}^1\xi$$

Notes

Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

Notes

From (2), $\|\xi - \eta\| < r$

$$\Rightarrow \eta \in B(\xi, r) \qquad \ldots(3)$$

Now consider an element ζ in $\mathbb{C}_2$ such that ${}^1\zeta = {}^1\xi + r$ and ${}^2\zeta = {}^2\xi$

Then $\sqrt{\dfrac{\left|{}^1\xi - {}^1\zeta\right|^2 + \left|{}^2\xi - {}^2\zeta\right|^2}{2}} = \dfrac{r}{\sqrt{2}} < r$

Therefore $$\zeta \in B(\xi, r) \qquad \ldots(4)$$

Since ${}^1\zeta \neq {}^1\xi$

$$\therefore \zeta \notin ({}^1\xi e_1 + ({}^2\xi - i_1\delta)e_2, {}^1\xi e_1 + ({}^2\xi + i_1\delta)e_2)_{ID} \qquad \ldots(5)$$

Hence the set $({}^1\xi e_1 + ({}^2\xi - i_1\delta)e_2, {}^1\xi e_1 + ({}^2\xi + i_1\delta)e_2)_{ID}$ is a proper subset of $B(\xi, r)$ where $0 < \delta < \sqrt{2}\, r$

3.1.2 Lemma

If $(\zeta, \Psi)_{ID}$ is an open interval in $\mathbb{C}_2$ such that ${}^1\zeta \neq {}^1\Psi$ then there exist no open ball $B(\xi, r)$; $r < \infty$ which contain the set $(\zeta, \Psi)_{ID}$.

Proof- Let $B(\xi, r)$ be an arbitrary open ball in $\mathbb{C}_2$ such that $r < \infty$

Let η be the arbitrary element of $B(\xi, r)$

$$\Rightarrow \eta \in B(\xi, r) \qquad \ldots(6)$$

$$\Rightarrow \|\xi - \eta\| < r$$

$$\Rightarrow \sqrt{\frac{\left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2}{2}} < r$$

$$\Rightarrow \left|{}^1\xi - {}^1\eta\right| < \sqrt{2}\, r \,, \left|{}^2\xi - {}^2\eta\right| < \sqrt{2}\, r$$

$$\Rightarrow \eta \in D(\xi; \sqrt{2}r, \sqrt{2}r) \qquad \ldots(7)$$

From (6), (7)

Therefore $$B(\xi, r) \subseteq D(\xi; \sqrt{2}r, \sqrt{2}r) \qquad \ldots(8)$$

Let us consider an arbitrary open interval $(\zeta, \Psi)_{ID}$ in $\mathbb{C}_2$ which contain the element η

$$\Rightarrow \eta \in (\zeta, \Psi)_{ID}$$

$$\Rightarrow \zeta <_{ID} \eta <_{ID} \Psi$$

Since $\zeta <_{ID} \eta$

Therefore either ${}^1\zeta < {}^1\eta$ or ${}^1\zeta = {}^1\eta$, ${}^2\zeta < {}^2\eta$

Since $\eta <_{ID} \Psi$

Therefore either ${}^1\eta < {}^1\Psi$ or ${}^1\eta = {}^1\Psi$, ${}^2\eta < {}^2\Psi$

Since ${}^1\zeta \neq {}^1\Psi$ therefore there will be only three possibilities.

Case 1st - If ${}^1\zeta < {}^1\eta$ and ${}^1\eta < {}^1\Psi$

Consider an element $y \in \mathbb{C}_2$ such that ${}^1y = {}^1\eta$ and $\left|{}^2\xi - {}^2y\right| > \sqrt{2}r$

Since ${}^1y = {}^1\eta$

$$\Rightarrow {}^1\zeta < {}^1y,\ {}^1y < {}^1\Psi$$

$$\Rightarrow \zeta <_{ID} y \text{ and } y <_{ID} \Psi$$

Therefore $y \in (\zeta\ ,\ \Psi)_{ID}$

Since $\left|{}^2\xi - {}^2y\right| > \sqrt{2}r$

$$\Rightarrow y \notin D(\xi; \sqrt{2}r, \sqrt{2}r)$$

From (8), $y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta\ ,\Psi)_{ID}$ such that $y \notin B(\xi, r)$

$$\Rightarrow (\zeta\ ,\Psi)_{ID} \not\subset B(\xi, r)$$

Case 2nd - If ${}^1\zeta < {}^1\eta$, ${}^1\eta = {}^1\Psi$ and ${}^2\eta < {}^2\Psi$

Consider an element $y \in \mathbb{C}_2$ such that ${}^1\zeta < {}^1y < {}^1\eta$ and $\left|{}^2\xi - {}^2y\right| > \sqrt{2}r$

Since ${}^1\zeta < {}^1y \Rightarrow \zeta <_{ID} y$

Since ${}^1y < {}^1\eta$ and ${}^1\eta = {}^1\Psi \Rightarrow {}^1y < {}^1\Psi$

$$\Rightarrow y <_{ID} \Psi$$

Therefore $y \in (\zeta\ ,\ \Psi)_{ID}$

$$\text{Since } \left|{}^2\xi - {}^2y\right| > \sqrt{2}r$$

$$\Rightarrow y \notin D(\xi; \sqrt{2}r, \sqrt{2}r)$$

From (8), $y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta\ ,\ \Psi)_{ID}$ such that $y \notin B(\xi, r)$

$$\Rightarrow (\zeta\ ,\Psi)_{ID} \not\subset B(\xi, r)$$

Case 3rd - If ${}^1\zeta = {}^1\eta$, ${}^2\zeta < {}^2\eta$ and ${}^1\eta < {}^1\Psi$

Consider an element $y \in \mathbb{C}_2$ such that ${}^1\eta < {}^1y < {}^1\Psi$ and $\left|{}^2\xi - {}^2y\right| > \sqrt{2}r$

Since ${}^1\eta < {}^1y$ and ${}^1\zeta = {}^1\eta \Rightarrow {}^1\zeta < {}^1y$

$\Rightarrow \zeta <_{ID} y$

Since ${}^1y < {}^1\Psi \Rightarrow y <_{ID} \Psi$

Therefore $y \in (\zeta\ ,\ \Psi)_{ID}$

Since $\left|{}^2\xi - {}^2y\right| > \sqrt{2}r$

$$\Rightarrow y \notin D(\xi; \sqrt{2}r, \sqrt{2}r)$$

From (8), $y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta\ ,\ \Psi)_{ID}$ such that $y \notin B(\xi, r)$

Notes

$$\Rightarrow (\zeta, \Psi)_{ID} \not\subseteq B(\xi, r)$$

Finally the ball $B(\xi, r)$; $r < \infty$ cannot contain any open interval $(\zeta, \Psi)_{ID}$ where ${}^1\zeta \neq {}^1\Psi$
Hence $(\zeta, \Psi)_{ID}$ cannot be contained in any ball $B(\xi, r)$; $r < \infty$

3.1.3 Theorem

The Idempotent order topology is strictly finer than Norm topology.

Notes

Proof- Let $X = (\xi, \eta)_{ID}$ be an open interval such that ${}^1\xi = {}^1\eta$
Then $h_1(X) = {}^1X = \{{}^1\xi\}$ and $h_2(X) = {}^2X = ({}^2\xi, {}^2\eta)$
In fact $X = {}^1X \times_e {}^2X$
Since ${}^1X$ is not open in complex plane
Therefore X will not be open w.r.t. Norm topology.

(By Theorem-2.1.2)

Since X is open w.r.t. Idempotent order topology therefore Idempotent order topology and norm topology are not equivalent and Norm topology cannot be finer than Idempotent order topology

Now we want to show for all open ball $B(\xi, r)$ and for all $\eta \in B(\xi, r)$ then there exist $(\zeta, \Psi)_{ID}$ such that $\eta \in (\zeta, \Psi)_{ID} \subseteq B(\xi, r)$

Let us consider an arbitrary ball $B(\xi, r)$ and consider an arbitrary element η of $B(\xi, r)$
$\Rightarrow \eta \in B(\xi, r)$

1^{st} Method- Since $\eta \in B(\xi, r)$ then there exist a ball
$B(\eta, s)$;$s > 0$ such that $B(\eta, s) \subseteq B(\xi, r)$

From Lemma-3.1.1,
$({}^1\eta e_1 + ({}^2\eta - i_1\delta)e_2, {}^1\eta e_1 + ({}^2\eta + i_1\delta)e_2)_{ID}$ will be the proper subset of $B(\eta, s)$

Therefore for $\eta \in B(\xi, r)$ we have a set $({}^1\eta e_1 + ({}^2\eta - i_1\delta)e_2, {}^1\eta e_1 + ({}^2\eta + i_1\delta)e_2)_{ID}$ such that $\eta \in ({}^1\eta e_1 + ({}^2\eta - i_1\delta)e_2, {}^1\eta e_1 + ({}^2\eta + i_1\delta)e_2)_{ID}$ and

$({}^1\eta e_1 + ({}^2\eta - i_1\delta)e_2, {}^1\eta e_1 + ({}^2\eta + i_1\delta)e_2)_{ID}$ is the subset of $B(\xi, r)$
2^{nd} Method- Since $\eta \in B(\xi, r)$

$$\Rightarrow \sqrt{\frac{\left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2}{2}} < r$$

$$\Rightarrow \left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2 < 2r^2$$

Let $\left|{}^1\xi - {}^1\eta\right|^2 = d_1^2, \left|{}^2\xi - {}^2\eta\right|^2 = d_2^2$

$$\Rightarrow d_1^2 + d_2^2 < 2r^2$$

There exist $d_3 > 0$ such that $d_2 < d_3$ and $d_1^2 + d_3^2 < 2r^2$

Consider a set $Q = ({}^1\eta e_1 + ({}^2\eta - i_1\delta)e_2, {}^1\eta e_1 + ({}^2\eta + i_1\delta)e_2)_{ID}$ where $\delta = d_3 - d_2 > 0$

Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

Obviously $\eta \in Q$

Let $y \in Q$ ...(9)

$\Rightarrow {}^1y = {}^1\eta$ and $({}^2\eta - i_1\delta) < {}^2y < ({}^2\eta + i_1\delta)$
Let ${}^2\eta = a + i_1 b$
Then ${}^2y = a + i_1 q$ where b- $\delta < q < b + \delta$
Now $\left|{}^2\xi - {}^2y\right| \leq \left|{}^2\xi - {}^2\eta\right| + \left|{}^2\eta - {}^2y\right|$

$$\Rightarrow \left|{}^2\xi - {}^2y\right| \leq d_2 + |i_1(b - q)|$$

$$\Rightarrow \left|{}^2\xi - {}^2y\right| \leq d_2 \pm (b - q) \qquad \text{...(10)}$$

Since b- $\delta < q < b + \delta$ and $\delta = d_3 - d_2$

Therefore $\pm(b - q) < d_3 - d_2$

From (10), $\left|{}^2\xi - {}^2y\right| < d_3$

Since $\left|{}^1\xi - {}^1y\right| = d_1$

Therefore $\left|{}^1\xi - {}^1y\right|^2 + \left|{}^2\xi - {}^2y\right|^2 < d_1^2 + d_3^2 < 2r^2$

$$\Rightarrow y \in B(\xi, r) \qquad \text{...(11)}$$

From (9), (11) $Q \subseteq B(\xi, r)$

Therefore for $\eta \in B(\xi, r)$ we have a open interval Q w.r.t. idempotent ordering such that $\eta \in Q \subseteq B(\xi, r)$
Hence it proves that Idempotent order topology is strictly finer than Norm topology.

Theorem 3.1.3 together with Theorems 2.3.1, 2.3.2, 2.3.10, 2.3.11 and Corollary 2.3.3, 2.3.12 generate a new corollary which states that

3.1.4 Corollary

The topology $\mathrm{T}_1^{\#}$ (and therefore $\mathrm{T}_2^{\#}$ and $\mathrm{T}_3^{\#}$) on $\mathbb{C}_2$ is strictly finer than the topology T_N (and therefore T_I and T_C) on $\mathbb{C}_2$.

b) Comparison of the idempotent order topology and real order topology on the bicomplex space

3.2.1 Theorem

The Real order topology and Idempotent order topology are not comparable.

Proof- Consider an open interval 'A' w.r.t. real ordering such that A= $(\xi, \eta)_R$ where $\xi = 1 + 2i_1 + 4i_2 + 7i_1i_2$ and $\eta = 2 + 2i_1 + 3i_2 + 4i_1i_2$

Consider an element x in $\mathbb{C}_2$ such that $x = 1 + 2i_1 + 5i_2 + 7i_1i_2$ therefore $\xi <_R x$ and $x <_R \eta$
$\Rightarrow x \in (\xi, \eta)_R$

Let us consider arbitrary open interval $(\zeta, \Psi)_{ID}$ in $\mathbb{C}_2$ (w.r.t. Idempotent ordering) which contain the element x.

Notes

Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

Notes

i.e. $x\in(\zeta, \Psi)_{ID} \Rightarrow \zeta <_{ID} x$, $x <_{ID} \Psi$
Since $\zeta <_{ID} x$
Therefore either ${}^1\zeta < {}^1x$ or ${}^1\zeta = {}^1x$, ${}^2\zeta < {}^2x$
Since $x <_{ID} \Psi$
Therefore either ${}^1x < {}^1\Psi$ or ${}^1x = {}^1\Psi$, ${}^2x < {}^2\Psi$
Hence there will be four possibilities.

Case A- If ${}^1\zeta < {}^1x$ and ${}^1x < {}^1\Psi$
Since $x = 1+2i_1+5i_2+7i_1i_2$
Therefore ${}^1x = 8-3i_1$, ${}^2x = -6+7i_1$

Consider an element y in $\mathbb{C}_2$ such that $y = 6+2i_1+5i_2+2i_1i_2$
Therefore ${}^1y = 8-3i_1$, ${}^2y = 4+7i_1$
Since ${}^1y = {}^1x$ and ${}^1\zeta < {}^1x$, ${}^1x < {}^1\Psi$
Therefore $\zeta <_{ID} y <_{ID} \Psi$

$$\Rightarrow y \in (\zeta, \Psi)_{ID} \qquad \ldots(12)$$

Since $\xi <_R y$ and $\eta <_R y$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(13)$$

From (12), (13) $(\zeta, \Psi)_{ID} \not\subseteq (\xi, \eta)_R$
Case B-If ${}^1\zeta < {}^1x$, ${}^1x = {}^1\Psi$ and ${}^2x < {}^2\Psi$
Consider an element y in $\mathbb{C}_2$ such that ${}^1\zeta < {}^1y < {}^1x$
Since ${}^1\zeta < {}^1y \Rightarrow \zeta <_{ID} y$
Since ${}^1x = {}^1\Psi, {}^1y < {}^1x \Rightarrow y <_{ID} \Psi$

$$\Rightarrow y \in (\zeta, \Psi)_{ID} \qquad \ldots(14)$$

Since ${}^1x = 8-3i_1$ and ${}^1y < {}^1x$
$\Rightarrow$ Therefore there are only two possibilities either ${}^1y = 8 + \delta i_1$ where $\delta < -3$ or ${}^1y = a + i_1 b$ where $a < 8$, $b \in \mathbb{C}_0$
If ${}^1y = 8 + \delta i_1$ where $\delta < -3$
Then y will be in the form $y = a_1+i_1a_2+i_2a_3+i_1i_2a_4$
Where $a_1+a_4 = 8$, $a_2-a_3 < -3$
Choose $a_1=7$, $a_4=1$ and choose a_2 and a_3 in such a way that $a_2-a_3 < -3$
Therefore $y = 7+i_1a_2+i_2a_3+i_1i_2$

$$\Rightarrow \xi <_R y,\ \eta <_R y$$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(15)$$

From (14), (15) $(\zeta, \Psi)_{ID} \not\subseteq (\xi, \eta)_R$
If ${}^1y = a + i_1 b$ where $a < 8$, $b \in \mathbb{C}_0$
Then y will be in the form $y = b_1+i_1b_2+i_2b_3+i_1i_2b_4$, Where $b_1+b_4 < 8$
Choose $b_1 = 8$ and $b_4 = -C$ where $C > 0$
Therefore $y = 8+i_1b_2+i_2b_3-i_1i_2C$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(16)$$

From (12), (14) $(\zeta, \Psi)_{ID} \nsubseteq (\xi, \eta)_R$

Case C-If ${}^1\zeta = {}^1x$, ${}^2\zeta < {}^2x$ and ${}^1x < {}^1\Psi$

Consider an element y in $\mathbb{C}_2$ such that ${}^1x < {}^1y < {}^1\Psi$

Since ${}^1y < {}^1\Psi \Rightarrow y <_{ID} \Psi$

Since ${}^1\zeta = {}^1x$ and ${}^1x < {}^1y$ $\Rightarrow \zeta <_{ID} y$

$$\Rightarrow y \in (\zeta, \Psi)_{ID} \qquad \ldots(17)$$

Since ${}^1x = 8\text{-}3i_1$ and ${}^1x < {}^1y$

Therefore there are only two possibilities either ${}^1y = 8 + \delta i_1$ where $\delta > -3$ or ${}^1y = a + i_1 b$ where $a > 8$, $b \in \mathbb{C}_0$

If ${}^1y = 8 + \delta i_1$ where $\delta > -3$

Then y will be in the form $y = a_1 + i_1 a_2 + i_2 a_3 + i_1 i_2 a_4$

Where $a_1 + a_4 = 8$, $a_2 - a_3 > -3$

Choose $a_1 = 7$, $a_4 = 1$ and choose a_2 and a_3 in such a way that $a_2 - a_3 > -3$

Therefore $y = 7 + i_1 a_2 + i_2 a_3 + i_1 i_2$

$$\Rightarrow \xi <_R y, \eta <_R y$$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(18)$$

From (17), (18) $(\zeta, \Psi)_{ID} \nsubseteq (\xi, \eta)_R$

If ${}^1y = a + i_1 b$ where $a > 8$, $b \in \mathbb{C}_0$

Then y will be in the form $y = b_1 + i_1 b_2 + i_2 b_3 + i_1 i_2 b_4$, Where $b_1 + b_4 > 8$

Choose $b_1 = 8$ and $b_4 = C$ where $C > 0$

Therefore $y = 8 + i_1 b_2 + i_2 b_3 - i_1 i_2 C$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(19)$$

From (17), (19) $(\zeta, \Psi)_{ID} \nsubseteq (\xi, \eta)_R$

Case D- If ${}^1\zeta = {}^1x$, ${}^2\zeta < {}^2x$ and ${}^1x = {}^1\Psi$, ${}^2x < {}^2\Psi$

Consider an element y in $\mathbb{C}_2$ such that

$${}^1y = {}^1x \text{ and } {}^2\zeta < {}^2y < {}^2\Psi$$

$$\Rightarrow \zeta <_{ID} y,\ y <_{ID} \Psi$$

$$\Rightarrow y \in (\zeta, \Psi)_{ID} \qquad \ldots(20)$$

If $Z = a + i_1 b$ is a complex number then the region of all complex number which is greater than Z or less than Z is define as follows.

Figure-1 shows the region of all complex numbers which is greater than Z and figure-2 shows the region of all complex numbers which is less than Z.

Notes

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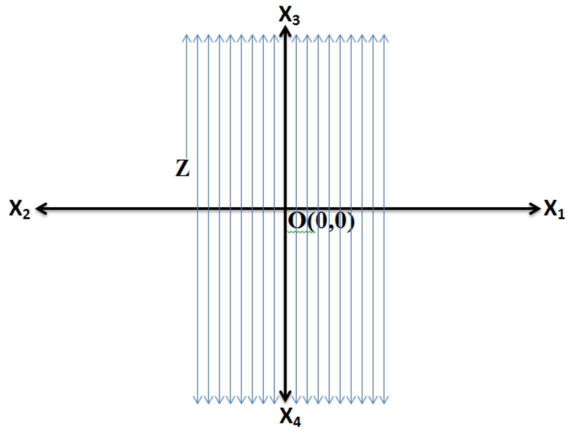

Figure-1

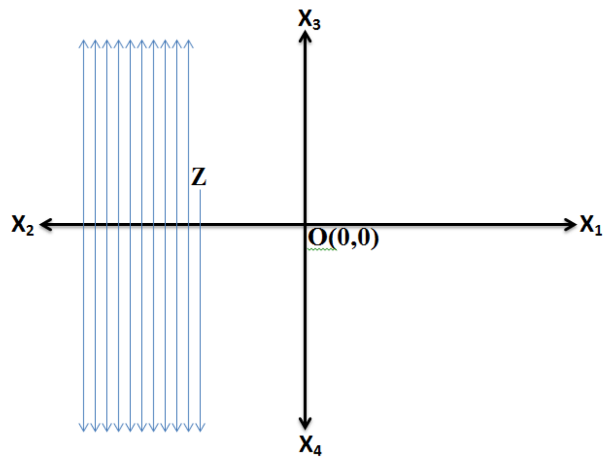

Figure-2

Since ${}^2x = -6+7i_1$ and ${}^2\zeta < {}^2x < {}^2\Psi$
Here there are four possibilities.

Possibility 1st-If ${}^2\zeta$ is on the axis **QT** except **Q** point and ${}^2\Psi$ is on the axis **QS** except **Q** point.

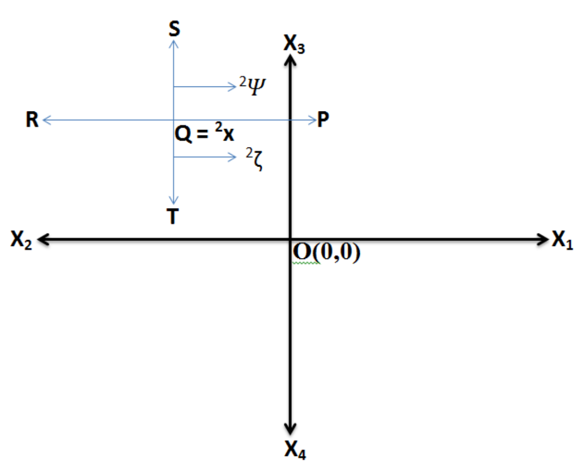

Figure-3

Since ${}^1y = {}^1x$ and ${}^2\zeta < {}^2y < {}^2\Psi$

Therefore ${}^1y = 8\text{-}3i_1$ and ${}^2y = -6+i_1b$ where $b > 7$ or $b < 7$

We will consider only $b < 7$

Then $y = 1+(2\text{-}\delta)i_1+(5\text{-}\delta)i_2+7i_1i_2$ where $\delta > 0$

$$\Rightarrow y <_R \xi$$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(21)$$

From (20), (21) $(\zeta, \Psi)_{ID} \not\subseteq (\xi, \eta)_R$

Possibility 2^{nd}-If ${}^2\zeta$ is situated in the region **QSRT** except **ST** axis and ${}^2\Psi$ is on the axis **QS** except **Q** point.

Since ${}^1y = {}^1x$ and ${}^2\zeta < {}^2y < {}^2\Psi$

Therefore ${}^1y = 8\text{-}3i_1$

Consider ${}^2y = a+i_1b$ where $a < -6$, $b \in \mathbb{C}_0$

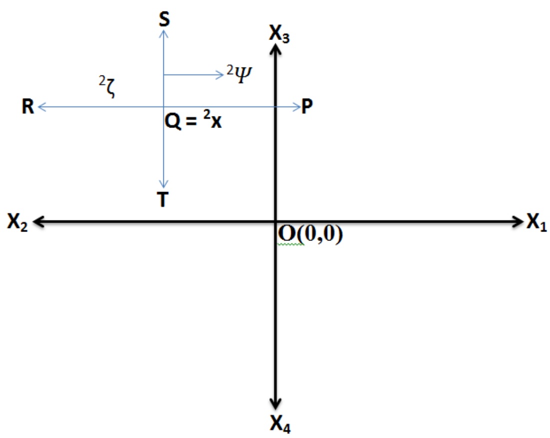

Figure-4

Then $y = (1\text{-}\ \delta_1)+(2\text{-}\delta_2)i_1+(5\text{-}\ \delta_2)i_2+(7+\ \delta_1)i_1i_2$

where $\delta_1 > 0$ and $\delta_2 \in \mathbb{C}_0$

$$\Rightarrow y <_R \xi$$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(22)$$

From (20), (22) $(\zeta, \Psi)_{ID} \not\subseteq (\xi, \eta)_R$

Possibility 3^{rd}- If ${}^2\Psi$ is situated in the region **QSPT** except **ST** axis and ${}^2\zeta$ is on the axis **QT** except **Q** point.

Notes

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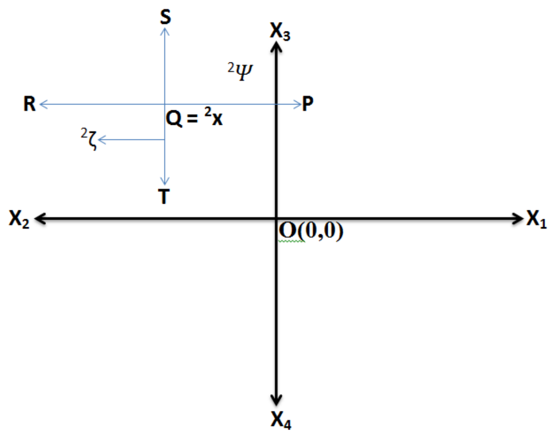

Figure-5

Since $^1y = {}^1x$ and $^2\zeta < {}^2y < {}^2\Psi$

Therefore $^1y = 8\text{-}3i_1$

Consider ^2y in such a way that $^2\zeta < {}^2y < {}^2x$

Therefore $^2y = -6 + i_1 b$ where $b < 7$

$$\Rightarrow y = 1+(2-\delta)i_1+(5-\delta)i_2+7i_1i_2 \text{ where } \delta > 0$$

$$\Rightarrow y <_R \xi$$

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(23)$$

From (20), (23) $(\zeta, \Psi)_{ID} \nsubseteq (\xi, \eta)_R$

Possibility 4th-If $^2\Psi$ is situated in the region **QSPT** except **ST** axis and $^2\zeta$ is situated in the region **QSRT** except **ST** axis.

$^2y = -6+5i_1$ satisfies the condition $^2\zeta < {}^2y < {}^2\Psi$

Therefore on choosing $^2y = -6+5i_1$

Since $^1y = 8\text{-}3i_1$ then $y = 1+i_1+4i_2+7i_1i_2$

$\Rightarrow y <_R \xi$

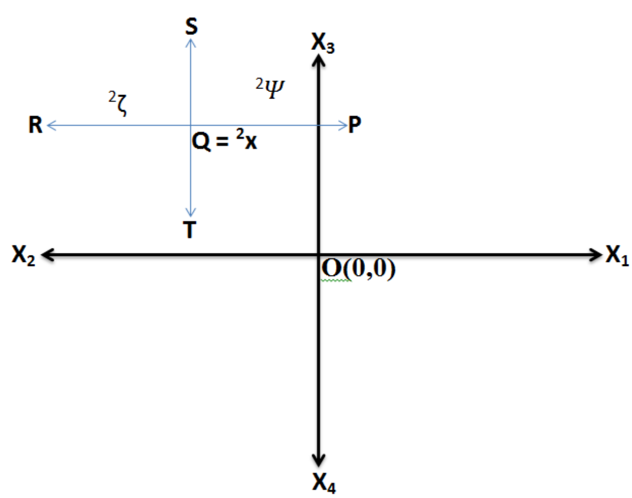

Figure-6

$$\Rightarrow y \notin (\xi, \eta)_R \qquad \ldots(24)$$

From (20), (24) $(\zeta, \Psi)_{ID} \nsubseteq (\xi, \eta)_R$

Finally all open interval (w.r.t. idempotent ordering) which contain the element x they cannot be subset of the set A.

Since A is open w.r.t. Real order topology therefore it shows that Idempotent order topology is not finer than Real order topology.

Now consider an open interval 'B' w.r.t. idempotent ordering such that $B = (\alpha, \beta)_{ID}$

Where $\alpha = (6-i_1)e_1+(-2+5i_1)e_2$ and $\beta = (8-2i_1)e_1+(-6+6i_1)e_2$

Consider an element p in $\mathbb{C}_2$ such that $p=2+2.5i_1+3.5i_2+4i_1i_2$

$$\Rightarrow {}^1p = (6 - i_1),\ {}^2p = (-2 + 6i_1)$$

Since ${}^1\alpha = {}^1p$ and ${}^2\alpha < {}^2p \Rightarrow \alpha <_{ID} p$

Since ${}^1p < {}^1\beta \Rightarrow p <_{ID} \beta$ therefore $p \in (\alpha, \beta)_{ID}$

Let us consider arbitrary open interval $(\phi, \Psi)_R$ in $\mathbb{C}_2$ (w.r.t. Real ordering) which contain the element p.

$$\text{i.e. } p \in (\phi, \Psi)_R \Rightarrow \phi <_R p,\ p <_R \Psi$$

Let $\phi = x_1+i_1x_2+i_2x_3+i_1i_2x_4$ and $\Psi = y_1+i_1y_2+i_2y_3+i_1i_2y_4$

Since $p=2+2.5i_1+3.5i_2+4i_1i_2$ and $\phi <_R p$, $p <_R \Psi$

Therefore there are 16 cases.

Case-(i) If $x_1 < 2$ and $y_1 > 2$

$\Rightarrow \phi = (2-\epsilon)+i_1x_2+i_2x_3+i_1i_2x_4$ and

$\Psi = (2+\delta)+i_1y_2+i_2y_3+i_1i_2y_4$ where $\epsilon, \delta > 0$

Consider an element A in $\mathbb{C}_2$ such that

$$A = 2+2.5i_1+3.5i_2+9i_1i_2$$

$$\Rightarrow A \in (\phi, \Psi)_R \qquad \ldots(25)$$

Since ${}^1A = (11-i_1)$

$$\Rightarrow {}^1\alpha < {}^1A \text{ and } {}^1\beta < {}^1A$$

$$\Rightarrow \alpha <_{ID} A,\ \beta <_{ID} A$$

$$\Rightarrow A \notin (\alpha, \beta)_{ID} \qquad \ldots(26)$$

From (25), (26) $(\phi, \Psi)_R \not\subseteq (\alpha, \beta)_{ID}$

Case-(ii) If $x_1=2$, $x_2<2.5$ and $y_1>2$

$\Rightarrow \phi = 2+(2.5-\epsilon)i_1+i_2x_3+i_1i_2x_4$ and

$$\Psi = (2+\delta)+i_1y_2+i_2y_3+i_1i_2y_4 \text{ where } \epsilon, \delta > 0$$

Also in this situation $A \in (\phi, \Psi)_R$ ...(27)

Therefore from (26), (27) $(\phi, \Psi)_R \not\subseteq (\alpha, \beta)_{ID}$

Notes

Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

Case-(iii) If $x_1=2$, $x_2=2.5$, $x_3<3.5$ and $y_1>2$

$\Rightarrow \phi = 2+2.5i_1+(3.5-\epsilon)i_2+i_1i_2x_4$ and

$$\Psi = (2+\delta)+i_1y_2+i_2y_3+i_1i_2y_4 \text{ where } \epsilon, \delta > 0$$

$$\Rightarrow A \in (\phi, \Psi)_R \quad \ldots(28)$$

Notes

Therefore from (26), (28) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(iv) If $x_1=2$, $x_2=2.5$, $x_3=3.5$, $x_4<4$ and $y_1>2$

$\Rightarrow \phi = 2+2.5i_1+3.5i_2+(4-\epsilon)i_1i_2$ and

$$\Psi = (2+\delta)+i_1y_2+i_2y_3+i_1i_2y_4 \text{ where } \epsilon, \delta > 0$$

$$\Rightarrow A \in (\phi, \Psi)_R \quad \ldots(29)$$

Therefore from (26), (29) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(v) If $x_1< 2$, $y_1=2$ and $y_2>2.5$

$\Rightarrow \phi = (2-\epsilon)+i_1x_2+i_2x_3+i_1i_2x_4$ and

$\Psi = 2+(2.5+\delta)i_1+i_2y_3+i_1i_2y_4$ where $\epsilon, \delta > 0$

Since $A = 2+2.5i_1+3.5i_2+9i_1i_2$

$$\Rightarrow A \in (\phi, \Psi)_R \quad \ldots(30)$$

Therefore from (26), (30) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(vi) If $x_1 = 2$, $x_2 < 2.5$, $y_1 = 2$ and $y_2 > 2.5$

$\Rightarrow \phi = 2+(2.5-\epsilon)i_1+i_2x_3+i_1i_2x_4$ and

$\Psi = 2+(2.5+\delta)i_1+i_2y_3+i_1i_2y_4$ where $\epsilon, \delta > 0$

Also in this situation $A \in (\phi, \Psi)_R$ ...(31)

Therefore from (26), (31) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(vii) If $x_1 = 2$, $x_2 = 2.5$, $x_3 < 3.5$, $y_1 = 2$ and $y_2 > 2.5$

$\Rightarrow \phi = 2+2.5i_1+(3.5-\epsilon)i_2+i_1i_2x_4$ and

$\Psi = 2+(2.5+\delta)i_1+i_2y_3+i_1i_2y_4$ where $\epsilon, \delta > 0$

$$\Rightarrow A \in (\phi, \Psi)_R \quad \ldots(32)$$

Therefore from (26), (32) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(viii) If $x_1 = 2$, $x_2 = 2.5$, $x_3 = 3.5$, $x_4 < 4$, $y_1 = 2$ and $y_2 > 2.5$

$\Rightarrow \phi = 2+2.5i_1+3.5i_2+(4-\epsilon)i_1i_2$ and

$\Psi = 2+(2.5+\delta)i_1+i_2y_3+i_1i_2y_4$ where $\epsilon, \delta > 0$

$$\Rightarrow A \in (\phi, \Psi)_R \quad \ldots(33)$$

Therefore from (26), (33) $(\phi,\Psi)_R \nsubseteq (\alpha,\beta)_{ID}$

Case-(ix) If $x_1 < 2$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$\Rightarrow \phi = (2-\epsilon)+i_1x_2+i_2x_3+i_1i_2x_4$ and

$\Psi = 2+2.5i_1+(3.5+\delta)i_2+i_1i_2y_4$ where $\epsilon, \delta > 0$

Since $A = 2+2.5i_1+3.5i_2+9i_1i_2$

$$\Rightarrow A \in (\phi,\Psi)_R \qquad \ldots(34)$$

Therefore from (26), (34) $(\phi,\Psi)_R \nsubseteq (\alpha,\beta)_{ID}$

Case-(x) If $x_1 = 2$, $x_2 < 2.5$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$\Rightarrow \phi = 2+(2.5-\epsilon)i_1+i_2x_3+i_1i_2x_4$ and

$\Psi = 2+2.5i_1+(3.5+\delta)i_2+i_1i_2y_4$ where $\epsilon, \delta > 0$

Also in this situation $\quad A \in (\phi,\Psi)_R \qquad \ldots(35)$

Therefore from (26), (35) $(\phi,\Psi)_R \nsubseteq (\alpha,\beta)_{ID}$

Case-(xi) If $x_1 = 2$, $x_2 = 2.5$, $x_3 < 3.5$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$\Rightarrow \phi = 2+2.5i_1+(3.5-\epsilon)i_2+i_1i_2x_4$ and

$\Psi = 2+2.5i_1+(3.5+\delta)i_2+i_1i_2y_4$ where $\epsilon, \delta > 0$

$$\Rightarrow A \in (\phi,\Psi)_R \qquad \ldots(36)$$

Therefore from (26), (36) $(\phi,\Psi)_R \nsubseteq (\alpha,\beta)_{ID}$

Case-(xii) If $x_1 = 2$, $x_2 = 2.5$, $x_3 = 3.5$, $x_4 < 4$, $y_1 = 2$, $y_2 = 2.5$ and $y_3 > 3.5$

$\Rightarrow \phi = 2+2.5i_1+3.5i_2+(4-\epsilon)i_1i_2$ and

$\Psi = 2+2.5i_1+(3.5+\delta)i_2+i_1i_2y_4$ where $\epsilon, \delta > 0$

$$\Rightarrow A \in (\phi,\Psi)_R \qquad \ldots(37)$$

Therefore from(26), (37) $(\phi,\Psi)_R \nsubseteq (\alpha,\beta)_{ID}$

Case-(xiii) If $x_1 < 2$, $y_1 = 2$, $y_2 = 2.5$, $y_3 = 3.5$ and $y_4 > 4$

$\Rightarrow \phi = (2-\epsilon)+i_1x_2+i_2x_3+i_1i_2x_4$ and

$\Psi = 2+2.5i_1+3.5i_2+(4+\delta)i_1i_2$ where $\epsilon, \delta > 0$

Consider an element B in $\mathbb{C}_2$ such that $B = 2+2.5i_1+3.5i_2+3i_1i_2$

$$\Rightarrow B \in (\phi,\Psi)_R \qquad \ldots(38)$$

Since ${}^1B = (5-i_1)$

$$\Rightarrow {}^1B < {}^1\alpha,\ {}^1B < {}^1\beta$$

$$\Rightarrow B <_{ID} \alpha,\ B <_{ID} \beta$$

Notes

$$\Rightarrow B \notin (\alpha, \beta)_{ID} \quad \ldots(39)$$

From (38), (39) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(xiv) If $x_1 = 2$, $x_2 < 2.5$, $y_1 = 2$, $y_2 = 2.5$, $y_3 = 3.5$ and $y_4 > 4$

$\Rightarrow \phi = 2+(2.5-\epsilon)i_1+i_2x_3+i_1i_2x_4$ and

$\Psi = 2+2.5i_1+3.5i_2+(4+\delta)i_1i_2$ where $\epsilon, \delta>0$

Notes

Also in this situation $\quad B \in (\phi, \Psi)_R \quad \ldots(40)$

Therefore from (39), (40) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(xv) If $x_1 = 2$, $x_2 = 2.5$, $x_3 < 3.5$, $y_1 = 2$, $y_2 = 2.5$, $y_3 = 3.5$ and $y_4 > 4$

$\Rightarrow \phi = 2+2.5i_1+(3.5-\epsilon)i_2+i_1i_2x_4$ and

$\Psi = 2+2.5i_1+3.5i_2+(4+\delta)i_1i_2$ where $\epsilon, \delta > 0$

$$\Rightarrow B \in (\phi, \Psi)_R \quad \ldots(41)$$

Therefore from (39), (41) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Case-(xvi) If $x_1 = y_1 = 2$, $x_2 = y_2 = 2.5$, $x_3 = y_3 = 3.5$ and $x_4 < 4 < y_4$

$\Rightarrow \phi = 2+2.5i_1+3.5i_2+(4-\epsilon)i_1i_2$ and

$\Psi = 2+2.5i_1+3.5i_2+(4+\delta)i_1i_2$ where $\epsilon, \delta > 0$

Consider an element D in $\mathbb{C}_2$ such that $D = 2+2.5i_1+3.5i_2+ai_1i_2$ where $(4-\epsilon) < a < 4$

$$\Rightarrow D \in (\phi, \Psi)_R \quad \ldots(42)$$

Since ${}^1D = \{(2 + a)\text{-}i_1\}$

Since $a < 4 \Rightarrow 2 + a < 6$

$$\Rightarrow {}^1D < {}^1\alpha \Rightarrow D <_{ID} \alpha$$

$$\Rightarrow D \notin (\alpha, \beta)_{ID} \quad \ldots(43)$$

Therefore from (42), (43) $(\phi,\Psi)_R \not\subseteq (\alpha,\beta)_{ID}$

Hence all open interval (w.r.t. real ordering) which contain the element 'p' they cannot be subset of the set 'B'.

Since B is open w.r.t. Idempotent order topology therefore it shows real order topology is not finer than Idempotent order topology.

Hence it proves that both topologies are not comparable.

Theorem 3.2.1, 2.3.10, 2.3.11and corollary 2.3.12, 2.3.14 submerge together to give a new corollary which is started below.

3.2.2 Corollary

The topology $\mathrm{T}_1^{\#}$ (and therefore $\mathrm{T}_2^{\#}$ and $\mathrm{T}_3^{\#}$) and the topology T_1 (and therefore T_2, T_3, T_1^{*}, T_2^{*} and T_3^{*}) on $\mathbb{C}_2$ are not comparable.

c) Comparison of the real order topology and norm topology on the bicomplex space

3.3.1 Lemma

The set $(\zeta, \Psi)_R$ is the proper subset of $B(\xi = x_1+i_1x_2+i_2x_3+i_1i_2x_4, r)$ where $\zeta = x_1+i_1x_2+i_2x_3+i_1i_2(x_4-\epsilon)$, $\Psi = x_1+i_1x_2+i_2x_3+i_1i_2(x_4+\epsilon)$ and either $\epsilon = \mathrm{Min}(d_1,d_2)$, $d_1^2+d_2^2 < 2r^2$ and $\epsilon > 0$ or $0 < \epsilon < r$

Global Journal of Science Frontier Research (F) Volume XVI Issue IV Version I Year 2016

Proof- Let suppose

$$\eta \in (\zeta, \Psi)_R \qquad \dots(44)$$

$$\Rightarrow \zeta <_R \eta <_R \Psi$$

$$\Rightarrow \eta = x_1+i_1x_2+i_2x_3+i_1i_2q \text{ where } (x_4-\epsilon) < q < (x_4+\epsilon)$$

Since $\eta = x_1+i_1x_2+i_2x_3+i_1i_2q$

$$\Rightarrow {}^1\eta = (x_1+q)+i_1(x_2-x_3) \text{ and } {}^2\eta = (x_1-q)+i_1(x_2+x_3)$$

Since $\xi = x_1+i_1x_2+i_2x_3+i_1i_2x_4$

$$\Rightarrow {}^1\xi = (x_1+x_4)+i_1(x_2-x_3) \text{ and } {}^2\xi = (x_1-x_4)+i_1(x_2+x_3)$$

Now $\left|{}^1\xi - {}^1\eta\right| = |x_4 - q| = \pm(x_4 - q)$

Since $(x_4 - \epsilon) < q < (x_4 + \epsilon)$ therefore $\pm(x_4 - q) < \epsilon$

$\Rightarrow \left|{}^1\xi - {}^1\eta\right| < \epsilon$

Similarly $\left|{}^2\xi - {}^2\eta\right| = \pm(x_4 - q) < \epsilon$

Since $\epsilon = \text{Min}(d_1, d_2)$ and $d_1^2 + d_2^2 < 2r^2$

Therefore $\left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2 < 2r^2$

$$\Rightarrow \sqrt{\frac{\left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2}{2}} < r$$

Therefore $\|\xi - \eta\| < r$

$$\Rightarrow \eta \in B(\xi = x_1+i_1x_2+i_2x_3+i_1i_2x_4\,,\, r) \qquad \dots(45)$$

Now consider an element P in $\mathbb{C}_2$ such that ${}^1P = {}^1\xi + r$ and ${}^2P = {}^2\xi$

Constitute $\sqrt{\frac{\left|{}^1\xi - {}^1P\right|^2 + \left|{}^2\xi - {}^2P\right|^2}{2}} = \frac{r}{\sqrt{2}} < r$

Therefore $\qquad P \in B(\xi, r) \qquad \dots(46)$

Since ${}^1\xi = (x_1+x_4)+i_1(x_2-x_3)$ and ${}^2\xi = (x_1-x_4)+i_1(x_2+x_3)$

$\therefore {}^1P = (x_1+x_4+r)+i_1(x_2-x_3)$ and ${}^2P = (x_1-x_4)+i_1(x_2+x_3)$

$\therefore P = (x_1+r/2)+i_1x_2+i_2x_3+i_1i_2\,(x_4+r/2)$

$$\Rightarrow P \notin (\zeta, \Psi)_R \qquad \dots(47)$$

From (44), (45), (46) and (47)

Hence the set $(\zeta, \Psi)_R$ is the proper subset of $B(\xi = x_1+i_1x_2+i_2x_3+i_1i_2x_4\,,\, r)$

3.3.2 Lemma

If $(\zeta, \Psi)_C$ is an open interval in $\mathbb{C}_2$ such that $u_1 \neq v_1$ or $u_1 = v_1$, $a_3 \neq b_3$ then there exist no open ball $B(\xi, r)$; $r < \infty$ which contain the set $(\zeta, \Psi)_C$.

Or

If $(\zeta, \Psi)_R$ is an open interval in $\mathbb{C}_2$ such that $a_1 \neq b_1$ or $a_1 = b_1$, $a_2 \neq b_2$ or $a_1 = b_1$, $a_2 = b_2$, $a_3 \neq b_3$ then there exist no open ball $B(\xi, r)$; $r < \infty$ which contain the set $(\zeta, \Psi)_R$.

Where $\zeta = u_1+i_2u_2 = (a_1+i_1a_2) + i_2(a_3+i_1a_4)$ and

$\Psi = v_1+i_2v_2 = (b_1+i_1b_2) + i_2(b_3+i_1b_4)$

Notes

Proof- Let $B(\xi, r)$ be an arbitrary open ball in $\mathbb{C}_2$ such that $r < \infty$

Let suppose $\xi = z_1+i_2z_2 = (x_1+i_1x_2) + i_2(x_3+i_1x_4)$ and

let $\eta = w_1+i_2w_2 = (y_1+i_1y_2) + i_2(y_3+i_1y_4)$ be the arbitrary element of $B(\xi, r)$.

$$\Rightarrow \eta \in B(\xi, r)$$

$$\Rightarrow \|\xi - \eta\| < r$$

$$\Rightarrow \sqrt{|z_1 - w_1|^2 + |z_2 - w_2|^2} < r$$

$$\text{Or } [(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2+(x_4-y_4)^2]^{1/2} < r$$

$$\Rightarrow |z_1 - w_1| < r \text{ and } |z_2 - w_2| < r$$

$$\text{Or } (x_i-y_i) < r;\ i = 1, 2, 3, 4$$

Let us consider an arbitrary open interval $(\zeta, \Psi)_C$ in $\mathbb{C}_2$ which contain the element η

$\Rightarrow \eta \in (\zeta, \Psi)_C$

$\Rightarrow \zeta <_C \eta <_C \Psi$

Since $\zeta <_C \eta$

Therefore either $u_1 < w_1$ or $u_1 = w_1$, $u_2 < w_2$

Since $\eta <_{ID} \Psi$

Therefore either $w_1 < v_1$ or $w_1 = v_1$, $w_2 < v_2$

Case-A If $u_1 \neq v_1$ then there will be three possibilities.

Possibility-1st: If $u_1 < w_1$ and $w_1 < v_1$

Consider an element $y = q_1+i_2q_2 \in \mathbb{C}_2$ such that $q_1 = w_1$ and $|z_2 - q_2| > r$

Since $q_1 = w_1$

$\Rightarrow u_1 < q_1$ and $q_1 < v_1$

$\Rightarrow \zeta <_C y$ and $y <_C \Psi$

Therefore $y \in (\zeta, \Psi)_C$

Since $|z_2 - q_2| > r$

$\Rightarrow y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta, \Psi)_C$ such that $y \notin B(\xi, r)$

$$\Rightarrow (\zeta, \Psi)_C \not\subset B(\xi, r)$$

Possibility-2nd: If $u_1 < w_1$, $w_1 = v_1$ and $w_2 < v_2$

Consider an element $y = q_1+i_2q_2 \in \mathbb{C}_2$ such that $u_1 < q_1 < w_1$ and $|z_2 - q_2| > r$

Since $u_1 < q_1 \Rightarrow \zeta <_C y$

Since $q_1 < w_1$ and $w_1 = v_1$

$\Rightarrow y <_C \Psi$

Therefore $y \in (\zeta , \Psi)_C$

Since $|z_2 - q_2| > r$

$\Rightarrow y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta , \Psi)_C$ such that $y \notin B(\xi, r)$

$$\Rightarrow (\zeta , \Psi)_C \not\subseteq B(\xi, r)$$

Possibility-3^rd: If $u_1 = w_1$, $u_2 < w_2$ and $w_1 < v_1$

Consider an element $y = q_1 + i_2 q_2 \in \mathbb{C}_2$ such that $w_1 < q_1 < v_1$ and $|z_2 - q_2| > r$

Since $u_1 = w_1$ and $w_1 < q_1 \Rightarrow \zeta <_C y$

Since $q_1 < v_1 \Rightarrow y <_C \Psi$

Therefore $y \in (\zeta , \Psi)_C$

Since $|z_2 - q_2| > r$

$\Rightarrow y \notin B(\xi, r)$

Therefore we have an element $y \in (\zeta , \Psi)_C$ such that $y \notin B(\xi, r)$

$\Rightarrow (\zeta , \Psi)_C \not\subseteq B(\xi, r)$

Case-B If $u_1 = v_1$ & $a_3 \neq b_3$

Since $a_3 \neq b_3 \Rightarrow a_3 < b_3$

Consider an element $s = a_1 + i_1 a_2 + i_2 c_3 + i_1 i_2 c_4 \in \mathbb{C}_2$ such that $a_3 < c_3 < b_3$ and $(x_4 - c_4) > r$

Therefore $s \in (\zeta , \Psi)_C$ and $s \notin B(\xi, r)$

Also in this situation $(\zeta , \Psi)_C \not\subseteq B(\xi, r)$

Finally the ball $B(\xi, r)$; $r < \infty$ cannot contain any open interval $(\zeta , \Psi)_C$ where $u_1 \neq v_1$ or $u_1 = v_1$, $a_3 \neq b_3$

Hence $(\zeta , \Psi)_C$ cannot be contained in any ball

$B(\xi, r)$; $r < \infty$

3.3.3 Theorem

The Real order topology is strictly finer than Norm topology.

Proof- Since the Real order topology and Idempotent order topology are not comparable.[By Theorem-3.2.1]

Therefore there exist a set $Q \subseteq \mathbb{C}_2$ which will be open w.r.t. Real order topology and will not be open w.r.t. Idempotent order topology.

Since Q is not open w.r.t. Idempotent order topology.

Therefore, from Theorem-3.1.3

Q will not be open w.r.t. Norm topology

Therefore we have a set Q which is open w.r.t. Real order topology and not open w.r.t. Norm topology.

Therefore Idempotent order topology and norm topology are not equivalent and Norm topology cannot be finer than Idempotent order topology

Now we want to show for all open ball $B(\xi, r)$ and for all $\eta \in B(\xi, r)$ then there exist $(\zeta , \Psi)_R$ such that $\eta \in (\zeta , \Psi)_R \subseteq B(\xi, r)$

Let us consider an arbitrary ball $B(\xi, r)$ and consider an arbitrary element η of $B(\xi, r) \Rightarrow \eta \in B(\xi, r)$

Let $\eta = y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4$

Notes

1st Method- Since $\eta \in B(\xi, r)$ then there exist a ball $B(\eta, s); s > 0$ such that $B(\eta, s) \subseteq B(\xi, r)$

Since $\eta = y_1+i_1y_2+i_2y_3+i_1i_2y_4$ then

$(\zeta, \Psi)_R$ will be the proper subset of $B(\eta=y_1+i_1y_2+i_2y_3+i_1i_2y_4, r)$

Where $\zeta = y_1+i_1y_2+i_2y_3+i_1i_2(y_4-\epsilon)$,

$\Psi = y_1+i_1y_2+i_2y_3+i_1i_2(y_4+\epsilon)$ and $\epsilon = \text{Min}(d_1, d_2)$ and $d_1^2+d_2^2 < 2r^2$

[By Lemma-3.3.1]

Therefore for $\eta \in B(\xi, r)$ we have a set $(\zeta, \Psi)_R$ such that $\eta \in (\zeta, \Psi)_R$ and $(\zeta, \Psi)_R \subseteq B(\xi, r)$.

2nd Method-Since $\eta \in B(\xi, r)$

$$\Rightarrow \sqrt{\frac{\left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2}{2}} < r$$

$$\Rightarrow \left|{}^1\xi - {}^1\eta\right|^2 + \left|{}^2\xi - {}^2\eta\right|^2 < 2r^2$$

Let $\left|{}^1\xi - {}^1\eta\right|^2 = d_1^2, \left|{}^2\xi - {}^2\eta\right| = d_2^2$

$\Rightarrow d_1^2+d_2^2 < 2r^2$

There exist $d_3, d_4 > 0$ such that $d_1 < d_3$, $d_2 < d_4$ and $d_3^2 + d_4^2 < 2r^2$

Consider a set $(\zeta, \Psi)_R$ such that

$\zeta = y_1+i_1y_2+i_2y_3+i_1i_2(y_4-\epsilon)$, $\Psi = y_1+i_1y_2+i_2y_3+i_1i_2(y_4+\epsilon)$ and $\epsilon = \text{Min}(d_3-d_1, d_4-d_2)$

Obviously $\eta \in (\zeta, \Psi)_R$

Let $$Y \in (\zeta, \Psi)_R \qquad \ldots(48)$$

$\Rightarrow \zeta <_R Y <_R \Psi$

$\Rightarrow Y = y_1+i_1y_2+i_2y_3+i_1i_2a$ where $(y_4-\epsilon) < a < (y_4+\epsilon)$

Therefore

${}^1Y = (y_1+a)+i_1(y_2-y_3)$ and ${}^2Y = (y_1-a)+i_1(y_2+y_3)$

Since $\eta = y_1+i_1y_2+i_2y_3+i_1i_2y_4$

Therefore ${}^1\eta = (y_1+y_4)+i_1(y_2-y_3)$ and ${}^2\eta = (y_1-y_4) + i_1(y_2+y_3)$

Now $\left|{}^1\xi - {}^1Y\right| \le \left|{}^1\xi - {}^1\eta\right| + \left|{}^1\eta - {}^1Y\right|$

$$\Rightarrow \left|{}^1\xi - {}^1Y\right| \le d_1 + |y_4 - a|$$

$$\Rightarrow \left|{}^1\xi - {}^1Y\right| \le d_1 \pm (y_4 - a)$$

Since $(y_4-\epsilon) < a < (y_4+\epsilon)$ therefore $\pm(y_4 - a) < \epsilon$

$$\Rightarrow \left|{}^1\xi - {}^1Y\right| < d_1+\epsilon$$

Since $\epsilon = \text{Min}(d_3-d_1, d_4-d_2)$

Therefore $\left|{}^1\xi - {}^1Y\right| < d_3$

Similarly $\left|{}^2\xi - {}^2Y\right| < d_4$

Since $d_3^2 + d_4^2 < 2r^2$

Therefore $\left|{}^1\xi - {}^1Y\right|^2 + \left|{}^2\xi - {}^2Y\right|^2 < 2r^2$

$$\Rightarrow \sqrt{\frac{\left|{}^1\xi - {}^1Y\right|^2 + \left|{}^2\xi - {}^2Y\right|^2}{2}} < r$$

Therefore $\|\xi - Y\| < r$

$$\Rightarrow Y \in B(\xi, r) \qquad \ldots(49)$$

From (48), (49)

$$(\zeta, \Psi)_R \subseteq B(\xi, r)$$

Therefore for $\eta \in B(\xi, r)$ we have a set $(\zeta, \Psi)_R$ such that $\eta \in (\zeta, \Psi)_R$ and $(\zeta, \Psi)_R \subseteq B(\xi, r)$. Hence it proves that real order topology is strictly finer than Norm topology.

Compiling Theorems 3.3.3, 2.3.1, 2.3.2 and corollary 2.3.3, 2.3.14 together, result to new corollary which states that

3.3.4 Corollary

The topology τ_1 (and therefore τ_2, τ_3, τ_1^*, τ_2^* and τ_3^*) on $\mathbb{C}_2$ is strictly finer than the topology τ_N (and therefore τ_I and τ_C) on $\mathbb{C}_2$.

References Références Referencias

1. Futagawa, M.: On the theory of functions of a quaternary Variable, Tohoku Math. J., 29 (1928), 175-222.
2. Futagawa, M.: On the theory of functions of a quaternary Variable, II, Tohoku Math. J., 35 (1932), 69-120.
3. Price, G.B.: An introduction to multicomplex spaces and Functions, Marcel Dekker Inc., New York, 1991.
4. Riley, J.D.: Contributions to the theory of function of a bicomplex variable, Tohoku Math. J., 2nd series 5 (1953), 132-165.
5. Ringleb, F.: Beitrage Zur Funktionen theorie in Hyperkomplexon Systemen, I, Rend. Circ. Mat. Palermo, 57 (1933), 311-340.
6. Segre, C.: Le Rappresentazioni Reali Delle Forme Complesse e Gli enti Iperalgebrici, Math. Ann. 40 (1892), 413-467.
7. Singh, Sukhdev: Certain order topologies on the bicomplex Space,M.Phil Dissertation, Dr. B. R. Ambedkar University, Agra, 2009.
8. Srivastava, Rajiv K.: Bicomplex Numbers: Analysis and applications, Math. Student, 72 (1-4) 2003, 69-87.
9. Srivastava, Rajiv K.: Certain topological aspects of the Bicomplex space, Bull. Pure & Appl. Maths., 2(2)(2008),222-234.

Notes

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Coordinate Systems, Numerical Objects and Algorithmic Operations of Computational Experiment in Fluid Mechanics

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Abstract- The paper deals with the computer implementation of direct computational experiments in fluid mechanics, constructed on the basis of the approach developed by the authors. The proposed approach allows the use of explicit numerical scheme, which is an important condition for increasing the efficiency of the algorithms developed by numerical procedures with natural parallelism. The paper examines the main objects and operations that let you manage computational experiments and monitor the status of the computation process. Special attention is given to a) realization of tensor representations of numerical schemes for direct simulation; b) realization of representation of large particles of a continuous medium motion in two coordinate systems (global and mobile); c) computing operations in the projections of coordinate systems, direct and inverse transformation in these systems. Particular attention is paid to the use of hardware and software of modern computer systems.

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Coordinate Systems, Numerical Objects and Algorithmic Operations of Computational Experiment in Fluid Mechanics

Alexander Degtyarev [α] & Vasily Khramushin [σ]

Abstract- The paper deals with the computer implementation of direct computational experiments in fluid mechanics, constructed on the basis of the approach developed by the authors. The proposed approach allows the use of explicit numerical scheme, which is an important condition for increasing the efficiency of the algorithms developed by numerical procedures with natural parallelism. The paper examines the main objects and operations that let you manage computational experiments and monitor the status of the computation process. Special attention is given to a) realization of tensor representations of numerical schemes for direct simulation; b) realization of representation of large particles of a continuous medium motion in two coordinate systems (global and mobile); c) computing operations in the projections of coordinate systems, direct and inverse transformation in these systems. Particular attention is paid to the use of hardware and software of modern computer systems.

I. Introduction

The problems of the Computational Fluid Dynamics (CFD) are among the most complex for implementation on modern computers both because of their strong connectivity and because of the complex nature of the problem. Not surprising that a large number of "grand challenges" to one degree or another gets reduced to CFD problems.

Traditionally, the calculation of flows is associated with the solution of Navier-Stokes equations. The problems in this area can be divided into three broad groups [1]:

1. Incorrectness of the Navier-Stokes equations underlying the simulation, since it is a certain idealization of real processes.
2. Idealization is also present in the geometry of these flows. This concerns sharp edges, corners, etc. in calculations. Such ideal forms were due to the method of approximation using large meshes. This causes a very big problem in the solution, because in reality most of these features are absent. Currently, the size of the mesh may be significantly reduced, which means that the methods of dealing with the problems that arise should be different.
3. Every time when the current problems are mapped on a new computer architecture, there arise porting problems of such applications. Since long ago it became clear that the appearance of new architectures is related with the appearance of new programming environments, which means the development of new libraries for currents will also be needed. Thus, we need to approach the problem of fluid dynamics programming in a different way.

A distinctive feature of the solution of the first problem is the traditional use of the implicit numerical schemes. Such a way reduces the solution to linear algebra problems that do not allow to

Ref

1. V. Bogdanov, A. B. Degtyarev and V. N. Khramushin, Computer Research and Modeling 7, 3, 429–438 (2015).

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efficiently control nonstationary processes and qualitative changes of the continuum for the considered complex problem. Moreover, the disadvantage of the implicit schemes as compared with the explicit ones is in the raise of complexity and parallelization problems. We know that it is a price to pay for the stability of the numerical scheme. However, the attractiveness of the explicit schemes has led to the development of an approach based on the description of the behavior of many large liquid particles [2]. Problems of numerical schemes instability in this case are avoided by providing a large particle with additional degrees of freedom. Geometrically this requires the introduction of a dual basis for describing the state of a large particle [3], [4].

The second group of problems is connected just with the geometric description of these flows. The efficiency of the calculations (especially in the case of direct numerical schemes) depends significantly on the description of the modeled objects, computational meshes and possibilities of their transformation and manipulation [5], [6]. This article is focused on the construction of such formal computational basis. Special attention is paid to programming procedures, which take into account the architecture of the computing systems.

II. The Subject of Application

The algorithmic realization of direct numerical simulations using explicit schemes has a beautiful historical analogy [2] in the form of calculus of fluxions by Isaac Newton. In the up-to-date algorithms, such implementation is represented as three-dimensional space numeric objects. Between them, connections are established for hydromechanics laws implementation.

Fluxions calculus[1] creates the basis of Newton classic mechanics for the motion in vector space and scalar time. Velocity $\vec{V}$ [m/s] becomes first fluxia in kinematics. It is differential (in accordance with Newton "moment") in multiplication with time step Δt [s] or just t. If we adapt such reasoning with reference to modern algorithms, then motion of control point in space is presented as follows[2]:

$$^{+}\vec{A} = \vec{R} + \vec{V} \cdot t + \overleftarrow{a} \cdot (\hat{\mathbf{r}} + \hat{\mathbf{v}} \cdot t) = {}^{0}\vec{A} + (\vec{V} + \overleftarrow{a} \cdot \hat{\mathbf{v}}) \cdot t, \qquad (1)$$

where t is the calculated time interval [s]; $\overleftarrow{a}$ is the vector count in the local basis (of the large particle) [m^{-2}]; $^{+}\vec{A}$ and $^{0}\vec{A}$ are new and initial positions of the control point in the global reference system [m]; $\vec{R}$ is the location of the local basis in the global reference system [m]; $\vec{V}$ is the velocity of the local basis translational displacement [m/s]; $\hat{\mathbf{v}}$ [m^3/s] is the tensor of rotation and deformation of the basis axes of the tensor form[3] $\hat{\mathbf{r}}$ [m^3].

This equation presents a point shift during one step of the direct computational experiment. As shown, it can be represented in the form of the recalculation from the local coordinates of the point into the global reference system in accordance with the time interval t.

Renewed spatial field of velocities and deformations are used at conjugate stage of computational experiment. They are necessary for the control of the dynamic state as well as for redefining rheological characteristics of fluid, which are presented in the form of scalar, vector and tensor objects. In up-to-date complex computational models similar hydrodynamic characteristics are associated with spatial distribution of polarized dipole cores, initiated sources of vortexes, etc.

Obviously, the direct computational experiment efficiency directly depends on the manipulation efficiency with reference systems, numerical objects and algorithmic operations on these objects.

III. Geometric Synthesis of Computational Objects and Related Algorithmic Operations

Let us consider the principles of construction of the computational objects in direct computational experiment. The described approach allows to partly automate the validation of code writing and to improve its computational efficiency.

---

[1] *fluens, fluentis* are functions x, y, z on the argument of time t; *fluxio* $\dot{x}, \dot{y}, \dot{z}$ are time derivatitives x, y, z.

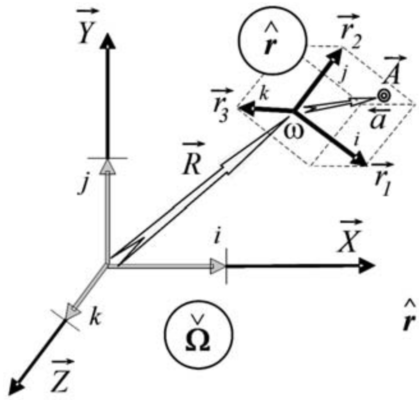
[2] In scalar notation:
$$^{+}A_x = {}^{0}A_x + (V_x + v_{xx}a^x + v_{xy}a^y + v_{xz}a^z) \cdot t;$$
$$^{+}A_y = {}^{0}A_y + (V_y + v_{yx}a^x + v_{yy}a^y + v_{yz}a^z) \cdot t;$$
$$^{+}A_z = {}^{0}A_z + (V_z + v_{zx}a^x + v_{zy}a^y + v_{zz}a^z) \cdot t.$$

[3] Tensor object without indices here and later are marked by bold.

Ref

2. A. B. Degtyarev and V. N. Khramushin, Mathematical Modeling 26, 11, 4–17 (2014).

The geometrical construction of spatial problems includes scalar, vector and tensor numerical objects. Algorithmic procedures and arithmetic-logic operations are defined in the dimensional physical form and associate numerical objects and interpolation basis in a tensor mesh space.



$$\vec{A} = \vec{R} + \overleftarrow{a}\cdot\hat{\mathbf{r}} \qquad \overleftarrow{a} = (\vec{A} - \vec{R})\cdot\check{\mathbf{r}}$$

$$\check{\mathbf{r}} = \overleftarrow{r}^{i} = \begin{pmatrix} \overleftarrow{r}^{x} \\ \overleftarrow{r}^{y} \\ \overleftarrow{r}^{z} \end{pmatrix} = r^{i,j} = \begin{pmatrix} r^{xx} & r^{xy} & r^{xz} \\ r^{yx} & r^{yy} & r^{yz} \\ r^{zx} & r^{zy} & r^{zz} \end{pmatrix}$$

$$\hat{\mathbf{r}} = \vec{r}_i = (\vec{r}_x \;\; \vec{r}_y \;\; \vec{r}_z) = r_{i,j} = \begin{pmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{pmatrix}$$

Figure 1 : Geometry of global space ($\check{\Omega}$) and local basis ($\hat{\mathbf{r}}$ (ω)); i, j, k) denote unitary vectors in the connected reference system

Elementary numerical objects are formed by non-coplanar basis vectors (Fig. 1). They serve to build indissoluble physical fields in the vicinity of adjacent mesh nodes (${}^{T}_{\Omega}\vec{R}$) and centers of mass (ω). Products of vector and tensor quantities are performed with convolution, i.e. by summation over a repeated index in the monomial product ($\overleftarrow{a} = \vec{a}\cdot\check{\mathbf{r}}$ or $a^j = a_i \cdot r^{ij}$), the transition to local basis and back ($\vec{a} = \overleftarrow{a}\cdot\hat{\mathbf{r}}$ or $a_k = a^j \cdot r^{jk}$). The latter occurs at the return to absolute coordinates.

The proposed notation is similar to that in [7] and [8]. The symbol notations and the principles of their construction are summarized below.

– A Left upper index marks the current time, which may be indicated by a capital letter ${}^{T}\Omega$ in absolute terms or the calculated step in time ${}^{t}R$. In addition, badges ${}^{+}\omega$ and ${}^{-}\omega$ designate links to the next or previous time interval.

– A Left low index marks a location in the mesh space ${}_{\{X,Y,Z\}}\hat{\mathbf{r}}$, or links to adducent knots ${}_{\{+\}}\hat{\mathbf{r}}$, or centers of mass of liquid particles ${}_{\{-\}}\check{\mathbf{r}}$. It is performed on conjugate stages of the computational experiment.

Right indices connect vector and tensor components in absolute and local bases. They serve to a strict definition of the dynamics and deformation of numeric cells (particles of a continuous medium).

– Low right indices, tensor "box" and the right arrow show the belonging to an absolute coordinate system (Fig. 1). For example, the tensor $\hat{\mathbf{r}}$ [m^3] is a collection by columns of basis vectors $\vec{r}_i$ in matrices of geometric transformations (like $\vec{a} = \overleftarrow{a}\cdot\hat{\mathbf{r}}$ [m]).

– Upper right indices mark projections inside mobile and deformable mesh cells. The display of unitary vectors of absolute coordinates lies in row vectors in matrix of inverse coordinate transformations, $\check{\mathbf{r}} = \overleftarrow{r}^{j} = r^{jk} = \hat{\mathbf{r}}^{-1}$ [m^{-3}]. They are marked by tensor "tick" and vector left arrow $\overleftarrow{a} = \vec{a}\cdot\check{\mathbf{r}}$ [m^{-2}].

– Capital letters are used for big numerical values measured in scale of global space (Ω) and general absolute time (T);

– Lowercase letters are used for especially small quantities or finite differences which are commensurable with the physical dimensions of local bases of particle continuum ω, as well as in the range of the current time step t.

The absolute time T can contain the Julian date and time from the beginning of the day[4]: ${}^{k}T = T + k\cdot t$. Absolute values in space may also be presented[5]: $\vec{A} = \vec{R} + \overleftarrow{a}\cdot\hat{\mathbf{r}}$ [m] (geographical and

[4] The real time is set by the numeric structure Event with Julian data: D (from 4713 BC), and local time in hours from the day beginning: T.

[5] In software environment points in global coordinates (Point) are separated with free vectors in local bases (Vector). It unifies computing operations with tensor numerical objects Tensor and Basis.

Ref

7. G. Astarita and G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics* (Mc Graw-Hill, 1974)

other generalized coordinates). The need of involvement of absolute encoder in space ${}^{T}_{\Omega}\vec{R}$ and time T is eliminated in the balanced numerical schemes. In this case, the use of numerical values at nodes and centers of mass of conjugate mesh cells is sufficient at all stages of calculations $\hat{r}$ [m^3].

Kinematics of internal streams is defined by the speed difference tensor (Fig. 2). It is given on the large liquid particles basis vectors form shifted in time,

$$\hat{\mathbf{v}} \cdot t = \vec{v_i} \cdot t = {}^{\Delta}_{\Delta}\vec{r_i} = {}^{t}_{+}\vec{r_i} - {}^{0}_{\Omega}\vec{r_i}, \tag{2}$$

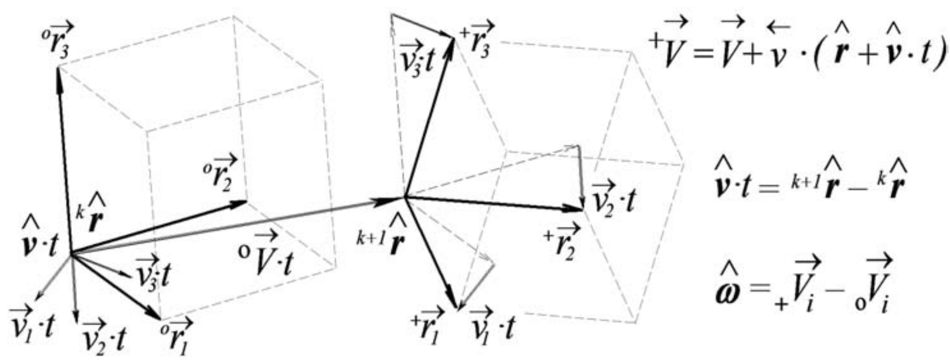


Figure 2 : Movement of basis vectors of calculation cell in space

The tensor $\hat{\mathbf{v}}$ [m^3/s] defines the current speed of the displacement of the liquid particle basis vectors on a local scale (lowercase letters) that is measured in the projection of the global coordinate system (lower indices). The independent convective rates tensor describing the local motion of the fluid is obtained after transformation of the velocity tensor reference frame[6] to the local basis of the large liquid particle (geometric normalization): $\overset{<}{\mathbf{v}} = \hat{\mathbf{v}} \cdot \check{\mathbf{r}}$ [1/s].

The tensor ($\overset{<}{\mathbf{v}}$ [1/s]) contains the extended set of kinematic elements of the differential equations with cross derivative components of deformable liquid particle motion:

$$\overset{<}{\mathbf{v}} = \hat{\mathbf{v}} \cdot \check{\mathbf{r}} = v_{ij} r^{jk} = \begin{pmatrix} v_{xx}r^{xx} + v_{xy}r^{yx} + v_{xz}r^{zx} & v_{xx}r^{xy} + v_{xy}r^{yy} + v_{xz}r^{zy} & v_{xx}r^{xz} + v_{xy}r^{yz} + v_{xz}r^{zz} \\ v_{yx}r^{xx} + v_{yy}r^{yx} + v_{yz}r^{zx} & v_{yx}r^{xy} + v_{yy}r^{yy} + v_{yz}r^{zy} & v_{yx}r^{xz} + v_{yy}r^{yz} + v_{yz}r^{zz} \\ v_{zx}r^{xx} + v_{zy}r^{yx} + v_{zz}r^{zx} & v_{zx}r^{xy} + v_{zy}r^{yy} + v_{zz}r^{zy} & v_{zx}r^{xz} + v_{zy}r^{yz} + v_{zz}r^{zz} \end{pmatrix}. \tag{3}$$

Alternatively, such products can be presented in the form of complete differentiation $\overset{<}{\mathbf{v}} = \hat{\mathbf{v}} \,/\, \hat{\mathbf{r}} = {}^{\Delta}_{\omega}\vec{v_i} / {}^{\Delta}_{\omega}\vec{r_i}$ executed without artificial exceptions of "small" or convective elements in substantial derivative approximations. Thus correct physical interpretation of rheological characteristics of liquid and living conditions of currents is remained.

IV. Construction of Tensor Numerical Objects and Modeling Algorithms

Computing space is constructed on fixed nonregularized nodes (`Point`) in indexed set of mesh cells. Three-dimensional interpolation is carried out using Euclidean bases (`Base`). Inside and in the vicinity, coordination of physical laws with the use of free vectors (`Vector`) is realized.

Object-oriented programming allows to move the control of the basic math operations correctness at the stage of the source program compilation, if they are applied to uniquely determined numerical objects (in accordance with physical characteristics).

The main computational objects (see, e.g., [9]) are defined by the requirement of fast and independent performance of computational operations. Particular attention is paid to the possibility of quick adjustment of the calculations for the application of hybrid algorithms depending upon conditions of the physical phenomena existing in a local subdomain. The following objects are introduced:

[6] Prohibition improving rank in product operation enables automatic permutation of factors in geometric transformations:

Ref

9. Code for numerical objects construction and functions of three-dimensional tensor mathematics for computational experiments in fluid mechanics (Tensor), Saint-Petersburg State University. FIPS N 2013619727

Notes

```
typedef double Real;   // scalar quantity in global space and time
typedef double real;   // local or differential counts in space and time
struct Tensor;         // tensor object without contextual links for quick calculation
struct Base;           // location coordinates and related Euclidean basis
struct Cell;           // numerical cell with contextual links to adjacent particles
struct Point;          // point in scale of absolute reference system
struct Vector;         // free difference vector in scale of local reference system
struct Space;          // space of nodal elements for net area in general
struct Volume;         // set of free/moving and deformable cells
```

The control of the correctness of the mathematical operations with respect to the listed objects can be illustrated as follows. The automatic conversion of a complex variable `Vector` or `Tensor` to a local value `real` determines the length of the vector and the determinant of the matrix. Any other transformation are marked as wrong by a translator. The difference between values of type `Point` can only give a value of `Vector`. Addition of objects of type `Vector` with `Vector` or `Point` values will result in the same type of data, and no other.

The development of modern computers goes rather towards adding more RAM, than towards improving the processing of large amounts of data. The basic data arrows `Space` and `Volume` have to be retained at current, previous and subsequent cuts in time for the technical support of the parallelization. This allows to synchronize the parallel processing in the case of complete separation of the stages of the experiment into independent physical processes, or to create any cycles for matching parameters of the computing environment in complex and ill-conditioned mathematical models, if necessary.

V. Conclusion

The solution of any computational problem consists of several stages:

– problem identification;
– formalization of the physical problem;
– construction of a computational procedure;
– creation of code for the chosen computer architecture.

At the transition between any successive stages, qualitative change in the essence of the described phenomenon can occur. Problems associated with incorrect mathematical models, numerical schemes to replace them, mapping problems on the architecture of a computer system, etc. are quite well known. For an adequate creation of a complete solution of the problem, it is necessary to develop consistent mapping of one stage onto another. The above approach is an attempt to adequately represent the final stage of solving the overall problem.

VI. Acknowledgements

The research was carried out using computational resources of Resource Center Computer Center of Saint-Petersburg State University and supported by Russian Foundation for Basic Research (project N 13-07-00747) and St. Petersburg State University (projects N 9.38.674.2013, 0.37.155.2014).

References Références Referencias

1. V. Bogdanov, A. B. Degtyarev and V. N. Khramushin, Computer Research and Modeling 7, 3, 429–438 (2015).
2. A. B. Degtyarev and V. N. Khramushin, Mathematical Modeling 26, 11, 4–17 (2014)
3. V. N. Khramushin, Vestnik of Saint-Petersburg University, Series 10 : Applied Mathematics. Computer Science. Control Processes, 2, 108–117 (2015).
4. V. N. Khramushin, *3D Tensor Mathematics for the Computational Fluid Mechanics Experience* (FEB RAS, Vladivostok, 2005).
5. I. D. Faux and M. J. Pratt, *Computational Geometry for Design and Manufacture (Mathematics and Its Applications)* (Ellis Horwood, 1981).

6. A. J. McConnell, *Applications of Tensor Analysis* (Dover Publications, 2011).
7. G. Astarita and G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics* (Mc Graw-Hill, 1974).
8. N. E. Kochin, *Vector calculus and beginnings of tensor calculus* (9th ed., Nauka, Moscow, 1965).
9. Code for numerical objects construction and functions of three-dimensional tensor mathematics for computational experiments in fluid mechanics (Tensor), Saint-Petersburg State University. FIPS N 2013619727.

Notes

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An a Priori Estimate for a Scalar Transmission Problem of the Laplacian in $\mathbb{R}^3$

By Ospino Portillo Jorge Eliécer
Fundación Universidad del Norte

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Keywords: scalar transmission problem, laplacian, a priori estimate.

GJSFR-F Classification : MSC 2000: 65N15, 35A35.

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Strictly as per the compliance and regulations of :

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1. A. Kufner, Weighted Sobolev spaces, John Wiley and Sons, (1985).

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Abstract- In this work, we present an a priori estimate for a scalar transmission problem of the Laplacian with parameter in R^3. The solution at infinity is described in the family of weighted Sobolev spaces.

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I. The Scalar Transmission Problem

Let Ω_- be a bounded region in $\mathbb{R}^3$ and $\Omega_+ = \mathbb{R}^3 \setminus \overline{\Omega_-}$. Let $\Sigma = \partial\Omega_- = \partial\Omega_+$ the interface is of class C^∞, see figure 1. Throughout this work, $\mathfrak{D}$ denote the space consisting of all C^∞-functions with compact support and $\mathfrak{D}'$ is the topological dual space of $\mathfrak{D}$ (space of distributions).

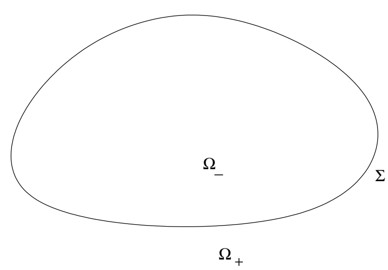

Figure 1 : Region of the problem

Consider the basic weight

$$\ell(r) = \sqrt{1+r^2}, \tag{1}$$

with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, for $\mathbf{x} = (x_1, x_2, x_3)$, is the distance of the origin. For any scalar function $u = u(x_1, x_2, x_3)$, we define the laplace and grad operator of u by

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2},$$

and

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right).$$

Due to the unboundedness of the exterior domain $A = \Omega_+$, the transmission problem is based on the weighted Sobolev spaces, also known as the Beppo-Levi spaces (see [**1**], [**2**]), these spaces were introduced and studied by Hanouzet in [**3**].
For any multi-index α in $\mathbb{N}^3$, we denote by ∂^α the differential operator of order α:

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$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}, \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

Then, for all m in $\mathbb{N}$ and all k in $\mathbb{Z}$, we define the weighted Sobolev space:

$$\mathbb{W}_k^m(\Omega^{is}) := \left\{ v \in \mathfrak{D}'(\Omega^{is}) \mid \forall \alpha \in \mathbb{N}^3,\ 0 \le |\alpha| \le m,\ \ell(r)^{|\alpha|-m+k}\partial^\alpha v \in L^2(\Omega^{is}) \right\}, \quad (2)$$

which is a Hilbert space for the norm:

$$\|v\|_{\mathbb{W}_k^m(\Omega^{is})} = \left\{ \sum_{|\alpha|=0}^{m} \|\ell(r)^{|\alpha|-m+k}\partial^\alpha v\|^2_{L^2(\Omega^{is})} \right\}^{\frac{1}{2}}.$$

And a wide range of basic elliptic problems were solved in these spaces by Giroire in [**4**],

$$\mathbb{W}_0^1(A) = \{ u \in \mathfrak{D}'(A) \mid (\ell(r))^{-1} u \in L^2(A), \nabla u \in \mathbf{L}^2(A) \} \quad (3)$$

and

$$\mathbb{W}_1^2(A) = \left\{ u \in \mathfrak{D}'(A) \,\middle|\, \frac{u}{\ell(r)} \in L^2(A), \nabla u \in \mathbf{L}^2(A), \ell(r)\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(A), 1 \le i,j \le 3 \right\} \quad (4)$$

They are reflexive Banach spaces equipped, respectively, with natural norms:

$$\|u\|_{\mathbb{W}_0^1(A)} = \left(\|(\ell(r))^{-1}u\|^2_{L^2(A)} + \|\nabla u\|^2_{\mathbf{L}^2(A)} \right)^{\frac{1}{2}}, \quad (5)$$

and

$$\|u\|_{\mathbb{W}_1^2(A)} = \left(\left\| \frac{u}{\ell(r)} \right\|^2_{L^2(A)} + \|\nabla u\|^2_{\mathbf{L}^2(A)} + \sum_{1 \le i,j \le 3} \left\| \ell(r)\frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(A)} \right)^{\frac{1}{2}}$$

We also define semi-norms

$$|u|_{\mathbb{W}_0^1(A)} = \|\nabla u\|_{\mathbf{L}^2(A)},$$

and

$$|u|_{\mathbb{W}_1^2(A)} = \left(\sum_{1 \le i,j \le 3} \left\| \ell(r)\frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(A)} \right)^{\frac{1}{2}}.$$

Here $\mathbf{L}^2(A) = (L^2(A))^3$, and also we define for all m in $\mathbb{N} \cup \{0\}$ and all k in $\mathbb{Z}$

$$L^2_{m,k}(\mathbb{R}^3) := \left\{ u \in \mathbb{R} \mid \forall \alpha \in \mathbb{N}^3,\ 0 \le |\alpha| \le m,\ \ell(r)^{|\alpha|-m+k} u \in L^2(\mathbb{R}^3) \right\},$$

with the norm

$$\|u\|_{L^2_{m,k}(\mathbb{R}^3)} = \left\{ \sum_{|\alpha|=0}^{m} \|\ell(r)^{|\alpha|-m+k} u\|_{L^2(\mathbb{R}^3)} \right\}^{\frac{1}{2}}.$$

Hence

$$\mathbb{W}_0^0(\Omega_+) = L^2(\Omega_+) \quad \text{and} \quad \mathbb{W}_{-1}^0(\mathbb{R}^3) = L^2_{0,-1}(\mathbb{R}^3).$$

We set the following spaces:

$$\mathring{\mathbb{W}}_0^1(A) = \overline{\mathfrak{D}(A)}^{\|\cdot\|_{\mathbb{W}_0^1(A)}} \quad \text{and} \quad \mathring{\mathbb{W}}_1^2(A) = \overline{\mathfrak{D}(A)}^{\|\cdot\|_{\mathbb{W}_1^2(A)}}.$$

We denote by $\mathbb{W}_0^{-1}(A)$ (respectively $\mathbb{W}_1^0(A)$) the dual space of $\mathring{\mathbb{W}}_0^1(A)$ (respectively of $\mathring{\mathbb{W}}_1^2(A)$). They are spaces of distributions.
With $a(\mathbf{x}) = a_- \in \Omega_-$, $a(\mathbf{x}) = a_+ \in \Omega_+$ for constants $a\pm$, its jump $[a]_\Sigma = a_+ - a_-$, across Σ and the restriction $\varphi^+(\varphi^-)$ of a function φ to $\Omega_+(\Omega_-)$ we consider the problem:
For given

$$f \in L^2(\Omega_-) \cup \mathbb{W}_1^0(\Omega_+) \quad \text{and} \quad g \in H^{\frac{1}{2}}(\Sigma), \quad (6)$$

Ref

4. J. Giroire, Etude de quelques probl'emes aux limites extjerieurs et rjesolution par jequations intjegrales, PhD thesis, UPMC, Paris, France, (1987).

find $\varphi \in \mathcal{V}$, such that

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla \quad ^+} dx + a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla \quad ^-} dx = -\int_{\Omega_+ \cup \Omega_-} f \cdot \overline{\psi} dx + [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds, \quad \forall \quad \in \mathcal{V} \quad (7)$$

with

$$\varphi \in \mathcal{V} = H_0^1(\Omega_-) \cup \mathbb{W}_0^1(\Omega_+), \;\; H_0^1(\Omega_-) = \left\{ \varphi \in H^1(\Omega_-) \;\middle|\; \int_{\Omega_-} \varphi dx = 0 \right\}, \quad (8)$$

and φ satisfies the decay condition at infinity

$$\varphi = O\left(\frac{1}{|\mathbf{x}|}\right), \;\; \partial_{\mathbf{n}}\varphi = o\left(\frac{1}{|\mathbf{x}|^2}\right) \quad \text{as} \quad |\mathbf{x}| \longrightarrow \infty, \quad (9)$$

The transmission problem (6)-(7) is elliptic. By elliptical regularity, φ has more regularity on sub-domains when the data are more regular.
We introduce

$$PH^2(\mathbb{R}^3) = \{\varphi = (\varphi^+, \varphi^-) \;\mid\; \varphi^+ \in \mathbb{W}_1^2(\Omega_+) \;\text{ and }\; \varphi^- \in H^2(\Omega_-)\},$$

with norm

$$\|\varphi\|^2_{PH^2(\mathbb{R}^3)} = \|\varphi^-\|^2_{H^2(\Omega_-)} + \|\varphi^+\|^2_{\mathbb{W}_1^2(\Omega_+)}. \quad (10)$$

The following result is an extension of Peron's results [**5**] (for a bounded exterior domain) to an unbounded exterior domain Ω_+.

Proposition 1. Let φ be a solution of the problem (7). For f and g satisfying (6) we have

$$\varphi \in PH^2(\mathbb{R}^3), \quad (11)$$

φ solves

$$\begin{aligned} & a_+\Delta\varphi^+ = f^+ \;\; in \;\; \Omega_+, \\ & a_-\Delta\varphi^- = f^- \;\; in \;\; \Omega_-, \\ & \varphi^+ = \varphi^-, \\ & a_+\partial_{\boldsymbol{n}}\varphi^+ - a_-\partial_{\boldsymbol{n}}\varphi^- = [a]_\Sigma \cdot g \;\; on \;\; \Sigma, \\ & \varphi = O\left(\frac{1}{|\boldsymbol{x}|}\right), \;\; \partial_{\boldsymbol{n}}\varphi = o\left(\frac{1}{|\boldsymbol{x}|^2}\right) \;\; as \;\; |\boldsymbol{x}| \longrightarrow \infty, \end{aligned} \quad (12)$$

where $\partial_{\boldsymbol{n}}$ denote the normal derivative.

Proof. We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_-. Let $\Omega_+ = \lim_{R\to\infty} \Omega_R$ and $\Omega_R = B_R \cap \Omega_+$, with $\partial\Omega_R = \partial B_R \cup \Sigma$, see figure 2.

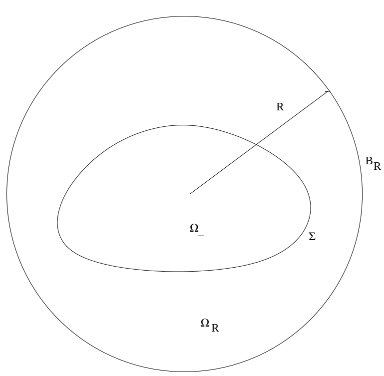

Figure 2 : The domain $\Omega_R = B_R \cap \Omega_+$

Ref

5. V. Peron, Modjelisation mathjematique de phjenom'enes jelectromagnjetiques dans des matjeriaux 'a fort contraste, PhD thesis, Universitje de Rennes I, Rennes, France, (2009).

Notes

Then, the first term in (7) is

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = \lim_{R\to\infty} a_+ \int_{\Omega_R} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx,$$

and by integration by parts in Ω_R

$$a_+ \int_{\Omega_R} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = a_+ \int_{\Omega_R} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\partial\Omega_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds$$

$$= a_+ \int_{\Omega_R} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds + a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds,$$

then, when $R \to \infty$, comes

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx = a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx + a_+ \int_{\Sigma} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds + \lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds.$$

The second term in (7) by integration by parts, yields

$$a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla\psi} dx = a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx - a_- \int_{\Sigma} \partial_{\mathbf{n}}\varphi^- \cdot \overline{\psi} ds$$

then

$$a_+ \int_{\Omega_+} \nabla\varphi^+ \cdot \overline{\nabla\psi} dx + a_- \int_{\Omega_-} \nabla\varphi^- \cdot \overline{\nabla\psi} dx =$$

$$= a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx + a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx +$$

$$+ \int_{\Sigma} (a_+ \partial_{\mathbf{n}}\varphi^+ - a_- \partial_{\mathbf{n}}\varphi^-) \cdot \overline{\psi} ds + \lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds.$$

The right part in (7) is

$$-\int_{\Omega_+\cup\Omega_-} f \cdot \overline{\psi} dx + [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds = -\int_{\Omega_+} f^+ \cdot \overline{\psi} dx - \int_{\Omega_-} f^- \cdot \overline{\psi} dx + [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds,$$

then, we have

$$a_+ \int_{\Omega_+} \Delta\varphi^+ \cdot \overline{\psi} dx = -\int_{\Omega_+} f^+ \cdot \overline{\psi} dx,$$

$$a_- \int_{\Omega_-} \Delta\varphi^- \cdot \overline{\psi} dx = -\int_{\Omega_-} f^- \cdot \overline{\psi} dx,$$

$$\int_{\Sigma} (a_+ \partial_{\mathbf{n}}\varphi^+ - a_- \partial_{\mathbf{n}}\varphi^-) \cdot \overline{\psi} ds = [a]_\Sigma \int_\Sigma g \cdot \overline{\psi} ds,$$

and

$$\lim_{R\to\infty} a_+ \int_{\partial B_R} \partial_{\mathbf{n}}\varphi^+ \cdot \overline{\psi} ds. = 0.$$

This implies (12), because φ satisfies (9). q.e.d.

Next we set $a_+ = 1$, $a_- = \rho \in \mathbb{C}$, and consider:
Find $\varphi_\rho \in \mathcal{V}$, such that, for all $\ \ \in \mathcal{V}$,

$$\int_{\Omega_+} \nabla\varphi_\rho^+ \cdot \overline{\nabla \ \ ^+} dx + \rho \int_{\Omega_-} \nabla\varphi_\rho^- \cdot \overline{\nabla \ \ ^-} dx = -\int_{\Omega_+\cup\Omega_-} f \cdot \overline{\psi} dx + (1-\rho) \int_\Sigma g \cdot \overline{\psi} ds, \quad (\mathbf{P}_\rho)$$

with f and g satisfying (6) independent of ρ and φ satisfying (9).
We construct a mapping $\rho \longmapsto \varphi_\rho$ where φ_ρ solves $(\mathbf{P}_\rho)$ and consider its behavior when $|\rho| \to \infty$.
We assume

$$\int_{\Omega_+\cup\Omega_-} f dx = 0 \quad \text{and} \quad \int_\Sigma g ds = 0. \tag{13}$$

and show an a priori estimate for φ_ρ uniformly in ρ.
We show now that $\varphi_\rho \in \mathcal{V}$. By construction, φ_ρ is a solution of problem (12), with $a_- = \rho$, $a_+ = 1$. Especially $\varphi_\rho \in H^1(\Omega_-) \cup \mathbb{W}_0^1(\Omega_+)$. Finally $\int_{\Omega_-} \varphi_\rho^- dx = 0$, because φ_ρ^- has integral mean zero.

To complete the proof of Proposition 1 let as now to prove the following a priori estimate. Its application gives the assertion of Proposition 1.

Ref

7. G. Caloz and M. Dauge and V. Peron, Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism, Journal of Mathematical Analysis and applications. 370 (2) (2010), 555–572.

II. A Priori Estimate

The main result for this work is to show a priori estimate in PH^2 uniformly in ρ for a solution $\varphi_\rho \in \mathcal{V}$ of $(\mathbf{P}_\rho)$; that is the following theorem ([**6**, Teorema 3]).

Theorem 1. Assuming (6) and (13), there exists a constant $\rho_0 > 0$ such that for all $\rho \in \{\vec{z} \in \mathbb{C} | |\vec{z}| \geq \rho_0\}$, problem $(\boldsymbol{P}_\rho)$ has a solution $\varphi_\rho \in PH^2(\mathbb{R}^3)$ with

$$\|\varphi_\rho\|_{PH^2(\mathbb{R}^3)} \leq C_{\rho_0}(\|f^-\|_{L^2(\Omega_-)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}), \tag{14}$$

where $C_{\rho_0} > 0$ is independent of ρ, f and g.

The proof of Theorem 1 follows the same steps as the approach in [**5**, **7**] and is given via the following steps.

First we expand φ_ρ in a power series in ρ^{-1}.

$$\varphi_\rho = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \varphi_n^+ \rho^{-n}, & \text{in} \quad \Omega_+, \\ \displaystyle\sum_{n=0}^{\infty} \varphi_n^- \rho^{-n}, & \text{in} \quad \Omega_-. \end{cases} \tag{15}$$

We show that these series converge in the norm in the space PH^2 to a solution of problem $(\mathbf{P}_\rho)$.

Inserting (15) in (12) and identifying terms of like powers of ρ^{-1} we obtain a family of problems independent of ρ, coupled by their conditions on Σ, and the decay condition at infinity. Then by simple calculation we obtain:

$$\begin{aligned} \Delta\varphi_0^- &= 0, \quad \text{in} \quad \Omega_-, \\ \partial_{\mathbf{n}}\varphi_0^- &= g, \quad \text{on} \quad \Sigma, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \Delta\varphi_0^+ &= f^+, \quad \text{in} \quad \Omega_+, \\ \varphi_0^+ &= \varphi_0^-, \quad \text{on} \quad \Sigma, \end{aligned} \tag{17}$$

and for $k \in \mathbb{N}$ with the Kronecker symbol $\delta_{k,1}$

$$\begin{aligned} \Delta\varphi_k^- &= \delta_{k,1} f^-, \quad \text{in} \quad \Omega_-, \\ \partial_{\mathbf{n}}\varphi_k^- &= -\delta_{k,1} g + \partial_{\mathbf{n}}\varphi_{k-1}^+, \quad \text{on} \quad \Sigma, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \Delta\varphi_k^+ &= 0, \quad \text{in} \quad \Omega_+, \\ \varphi_k^+ &= \varphi_k^-, \quad \text{on} \quad \Sigma, \end{aligned} \tag{19}$$

and the condition at infinity

$$\varphi_\rho = O\left(\frac{1}{|\mathbf{x}|}\right), \quad \partial_{\mathbf{n}}\varphi_\rho = o\left(\frac{1}{|\mathbf{x}|^2}\right) \quad \text{as} \quad |\mathbf{x}| \longrightarrow \infty, \tag{20}$$

We construct every term successively φ_n^- and φ_n^+, by beginning in φ_0^- and φ_0^+.
Let us assume that $\{\varphi_k^-\}_{k=0}^{n-1}$ and $\{\varphi_k^+\}_{k=0}^{n-1}$ are known. Then, problem (18) defines a unique φ_n^-. Its trace on Σ is inserted in (19) as Dirichlet data to determine the external part φ_n^+.

The Neumann problem (16) has a unique solution $\varphi_0^- \in H^1(\Omega_-)$ if $\int_{\Omega_-} \varphi_0^- dx = 0$. We remember that we have the compatibility condition $\int_\Sigma g ds = 0$. Also, by elliptic regularity, $\varphi_0^- \in H^2(\Omega_-)$ and there is a constant $C_N > 0$, independent of ρ, such that (see [**8**, Theorem 2.5.2])

$$\|\varphi_0^-\|_{H^2(\Omega_-)} \leq C_N \|g\|_{H^{\frac{1}{2}}(\Sigma)}. \tag{21}$$

We are interested in φ_0^+ in (17). Problem (17) has a unique solution (see [**4**, Chapter 2]), $\varphi_0^+ \in \mathbb{W}_0^1(\Omega_+)$. Also, by elliptic regularity and since $\varphi_0^- \in H^2(\Omega_-)$, $\varphi_0^+ \in \mathbb{W}_1^2(\Omega_+)$ and there is a constant $C_{DN} > 0$ independent of ρ, such that (see [**2**, Theorem 6])

$$\|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}(\|\varphi_0^-\|_{H^2(\Omega_-)} + \|f^+\|_{\mathbb{W}_1^0(\Omega_+)}). \tag{22}$$

Now that (20) guaranties that $\varphi_0^+ \in \mathbb{W}_0^1(\Omega_+)$ and not only in $\mathbb{W}_0^1(\Omega_+)\diagdown\mathbb{R}$. Similarly we can deal with (18) and (19). Since φ_ρ satisfies the decay condition at infinity, φ_ρ can not behave like a constant. Therefore the constraints (23) are not necessary.
Next we show that the Neumann problem (18) is compatible.
For $k = 1$, is necessary to prove that

$$\int_{\Omega_-} f^- dx + \int_\Sigma (-g + \partial_{\mathbf{n}}\varphi_0^+) ds = 0. \tag{23}$$

According to (17) and (20)

$$\begin{aligned} \Delta\varphi_0^+ &= f^+, && \text{in} \quad \Omega_+, \\ \varphi_0^+ &= \varphi_0^-, && \text{on} \quad \Sigma, \\ \partial_{\mathbf{n}}\varphi_0^+ &= o\left(\frac{1}{|\mathbf{x}|^2}\right), && \text{as} \quad |\mathbf{x}| \longrightarrow \infty. \end{aligned} \tag{24}$$

We choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_- (see figure 2). Then for the bounded domain $\Omega_+ \cap B_R$, integrating by part in $(24)_1$ gives

$$\int_{\Omega_+ \cap B_R} f^+ \overline{\quad^+} dx = \int_{\Omega_+ \cap B_R} \Delta\varphi_0^+ \overline{\quad^+} dx$$

$$= \int_{\Omega_+ \cap B_R} \nabla\varphi_0^+ \cdot \overline{\nabla \quad^+} dx + \int_{\partial(\Omega_+ \cap B_R)} \overline{\quad^+} \cdot \partial_{\mathbf{n}}\varphi_0^+ ds,$$

for $\quad \equiv 1$ yields

$$\int_{\Omega_+ \cap B_R} f^+ dx = \int_{\partial(\Omega_+ \cap B_R)} \partial_{\mathbf{n}}\varphi_0^+ ds$$

and $\partial(\Omega_+ \cap B_R) = \partial B_R \cup \Sigma$, then

$$\int_{\Omega_+ \cap B_R} f^+ dx = \int_{\partial B_R} \partial_{\mathbf{n}}\varphi_0^+ ds + \int_\Sigma \partial_{\mathbf{n}}\varphi_0^+ ds$$

$$= \int_{\partial B_R} o\left(\frac{1}{R^2}\right) ds + \int_\Sigma \partial_{\mathbf{n}}\varphi_0^+ ds$$

Ref

4. J. Giroire, Etude de quelques probl'emes aux limites extjerieurs et rjesolution par jequations intjegrales, PhD thesis, UPMC, Paris, France, (1987).

$$= o\left(\frac{1}{R^2}\right) R^2 + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds,$$

then

$$\int_{\Omega_+} f^+ dx = o(1) + \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds, \quad \text{as} \quad R \longrightarrow \infty,$$

then

$$\int_{\Omega_+} f^+ dx = \int_\Sigma \partial_{\mathbf{n}} \varphi_0^+ ds.$$

Notes

Under the hypothesis (13)

$$\int_\Sigma g ds = 0, \quad \text{and} \quad \int_{\mathbb{R}^3} f dx = 0,$$

then

$$\int_{\Omega_+} f^+ dx = -\int_{\Omega_-} f^- dx,$$

the compatibility condition (23) is deducted.
For $k \geq 2$, we assume that the term φ_{k-1}^+ is constructed. It is necessary to show that

$$\int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds = 0. \tag{25}$$

According to (19) and (20)

$$\begin{aligned} &\Delta \varphi_{k-1}^+ = 0, && \text{in} \quad \Omega_+, \\ &\varphi_{k-1}^+ = \varphi_{k-1}^-, && \text{on} \quad \Sigma, \\ &\partial_{\mathbf{n}} \varphi_{k-1}^+ = o\left(\frac{1}{|\mathbf{x}|^2}\right), && \text{as} \quad |\mathbf{x}| \longrightarrow \infty. \end{aligned} \tag{26}$$

Again we choose a ball B_R with radius $R > 0$ and boundary ∂B_R containing Ω_-. Then for the bounded domain $\Omega_+ \cap B_R$, integrating by part in $(26)_1$ gives

$$0 = \int_{\Omega_+ \cap B_R} \Delta \varphi_{k-1}^+ \overline{\quad}^{+} dx = \int_{\Omega_+ \cap B_R} \nabla \varphi_{k-1}^+ \cdot \overline{\nabla \quad}^{+} dx + \int_{\partial(\Omega_+ \cap B_R)} \overline{\quad}^{+} \cdot \partial_{\mathbf{n}} \varphi_{k-1}^+ ds,$$

for $\quad \equiv 1$ yields

$$0 = \int_{\partial(\Omega_+ \cap B_R)} \partial_{\mathbf{n}} \varphi_{k-1}^+ ds$$

and $\partial(\Omega_+ \cap B_R) = \partial B_R \cup \Sigma$, then

$$0 = \int_{\partial B_R} \partial_{\mathbf{n}} \varphi_{k-1}^+ ds + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds$$

$$= \int_{\partial B_R} o\left(\frac{1}{R^2}\right) ds + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds$$

$$= o\left(\frac{1}{R^2}\right) R^2 + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds,$$

then

$$0 = o(1) + \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds, \quad \text{as} \quad R \longrightarrow \infty,$$

then

$$0 = \int_\Sigma \partial_{\mathbf{n}} \varphi_{k-1}^+ ds,$$

then (25) is deducted.

Consequently, the Neumann problem (18) admits a solution $\varphi_k^- \in H^1(\Omega_-)$, which is unique under condition $\int_{\Omega_-} \varphi_k^- dx = 0$ (see [**8**, Theorem 2.5.10]). Also, $\varphi_k^- \in H^2(\Omega_-)$ and (see [**5**, **7**])

$$\|\varphi_k^-\|_{H^2(\Omega_-)} \leq C_N[\delta_k^1(\|f^-\|_{L^2(\Omega_-)} + \|g\|_{H^{\frac{1}{2}}(\Sigma)}) + \|\partial_{\mathbf{n}}\varphi_{k-1}^+\|_{H^{\frac{1}{2}}(\Sigma)}]. \tag{27}$$

Finally, problem (19) has a unique solution $\varphi_k^+ \in \mathbb{W}_0^1(\Omega_+)$ (see [**4**, Chapter 2] and the estimate (see [**8**, Theorem 2.5.14])

$$\|\varphi_k^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_k^-\|_{H^2(\Omega_-)}. \tag{28}$$

Next, we demonstrate the convergence in $PH^2(\mathbb{R}^3)$ of the series (15) for large $|\rho|$.

For the Neumann trace (see [**5**, **7**])

$$\begin{array}{rccc} \gamma_{1,\Sigma}: & \mathbb{W}_1^2(\Omega_+) & \longrightarrow & H^{\frac{1}{2}}(\Sigma), \\ & \varphi & \longmapsto & \partial_{\mathbf{n}}\varphi \end{array}$$

we have with a constant $C_1 > 0$,

$$\|\gamma_{1,\Sigma}(\varphi)\|_{H^{\frac{1}{2}}(\Sigma)} \leq C_1\|\varphi\|_{\mathbb{W}_1^2(\Omega_+)}. \tag{29}$$

We pose $\alpha = C_N C_1 C_{DN}$, where C_N and C_{DN} are the respective constants of estimates (21) and (22). With (27), (28) and (29) we show by induction

$$\begin{aligned} \|\varphi_n^-\|_{H^2(\Omega_-)} &\leq \alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)}, \\ \|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN}\cdot\alpha^{n-1}\|\varphi_1^-\|_{H^2(\Omega_-)}. \end{aligned} \tag{30}$$

$(30)_1$ can be see as follows: For $n = 1$,

$$\|\varphi_1^-\|_{H^2(\Omega_-)} = \alpha^0\|\varphi_1^-\|_{H^2(\Omega_-)}.$$

With (27) we have for $k = 2$

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_1^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and with (29)

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)};$$

hence by (28) we have for $k = 1$

$$\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

and therefore

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \quad \leq C_N C_1 C_{DN}\|\varphi_1^-\|_{H^2(\Omega_-)} = \alpha\|\varphi_1^-\|_{H^2(\Omega_-)}.$$

We assume that $(30)_1$ is true for $k = n-1$, this is

$$\|\varphi_{n-1}^-\|_{H^2(\Omega_-)} \leq \alpha^{n-2}\|\varphi_1^-\|_{H^2(\Omega_-)},$$

then, according to (27), for $k = n$

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N\|\partial_{\mathbf{n}}\varphi_{n-1}^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (29)

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N C_1\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)};$$

according to (28) for $k = n-1$

$$\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN}\|\varphi_{n-1}^-\|_{H^2(\Omega_-)},$$

then

Ref

8. J. C. Nedelec, Acoustic and electromagnetic equations. Integral representations for harmonic problems, Springer-Verlag, (2001).

$$\begin{aligned}\|\varphi_n^-\|_{H^2(\Omega_-)} &\leq C_N C_1 C_{DN} \|\varphi_{n-1}^-\|_{H^2(\Omega_-)} \\ &\leq \alpha \cdot \alpha^{n-2} \|\varphi_1^-\|_{H^2(\Omega_-)} \\ &= \alpha^{n-1} \|\varphi_1^-\|_{H^2(\Omega_-)},\end{aligned}$$

then $(30)_1$ is true for all n.

$(30)_2$ can be see as follows: According to (28) for $k=1$

Notes

$$\|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \|\varphi_1^-\|_{H^2(\Omega_-)},$$

and for $k=2$

$$\|\varphi_2^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \|\varphi_2^-\|_{H^2(\Omega_-)}.$$

According to (27) for $k=2$

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N \|\partial_{\mathbf{n}} \varphi_1^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (29)

$$\|\varphi_2^-\|_{H^2(\Omega_-)} \leq C_N C_1 \|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)},$$

then

$$\begin{aligned}\|\varphi_2^+\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN} C_N C_1 \|\varphi_1^+\|_{\mathbb{W}_1^2(\Omega_+)} \\ &\leq C_{DN} \cdot \alpha \|\varphi_1^-\|_{H^2(\Omega_-)}.\end{aligned}$$

We assume that $(30)_2$ is true for $k=n-1$, this is

$$\|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \cdot \alpha^{n-2} \|\varphi_1^-\|_{H^2(\Omega_-)}$$

then, according to (28), for $k=n$

$$\|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \|\varphi_n^-\|_{H^2(\Omega_-)},$$

and according to (27) for $k=n$

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N \|\partial_{\mathbf{n}} \varphi_{n-1}^+\|_{H^{\frac{1}{2}}(\Sigma)},$$

and for (29)

$$\|\varphi_n^-\|_{H^2(\Omega_-)} \leq C_N C_1 \|\varphi_{n-1}^+\|_{\mathbb{W}_1^2(\Omega_+)},$$

then

$$\begin{aligned}\|\varphi_n^-\|_{H^2(\Omega_-)} &\leq C_N C_1 C_{DN} \cdot \alpha^{n-2} \|\varphi_1^-\|_{H^2(\Omega_-)} \\ &= \alpha^{n-1} \|\varphi_1^-\|_{H^2(\Omega_-)},\end{aligned}$$

then

$$\|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq C_{DN} \cdot \alpha^{n-1} \|\varphi_1^-\|_{H^2(\Omega_-)},$$

then $(30)_2$ is true for all n.

Hence for all $\rho \in \mathbb{C}$, with $|\rho|^{-1}\alpha < 1$, the series (15) converges in $\mathbb{W}_1^2(\Omega_+)$ and $H^2(\Omega_-)$, respectively.

Now we are in the position to prove Theorem 1.

We show first the estimate (14) for φ_ρ in (15). Let $\rho_0 > 0$, such that $\rho_0^{-1}\alpha < 1$, where $\alpha = C_N C_1 C_{DN}$.

Let $\rho \in \{z \in \mathbb{C} | |z| \geq \rho_0\}$. According to (30) φ_ρ converges geometrically in $PH^2(\mathbb{R}^3)$ with convergence ratio $|\rho^{-1}|\alpha$, bounded by $\rho_0^{-1}\alpha$. Hence,

$$\begin{aligned} \|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} &\leq C_{DN}\frac{1}{1-\rho_0^{-1}\alpha}\rho_0^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)}+\|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}, \\ \|\varphi_\rho^-\|_{H^2(\Omega_-)} &\leq \rho_0^{-1}\frac{1}{1-\rho_0^{-1}\alpha}\|\varphi_1^-\|_{H^2(\Omega_-)}+\|\varphi_0^-\|_{H^2(\Omega_-)} \end{aligned} \tag{31}$$

From $(15)_1$, $(30)_2$ and the triangular inequality, we have

$$\begin{aligned} \|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} &= \|\textstyle\sum_{n=0}^\infty \varphi_n^+\rho^{-n}\|_{\mathbb{W}_1^2(\Omega_+)} \\ &\leq \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}+\textstyle\sum_{n=1}^\infty \|\varphi_n^+\|_{\mathbb{W}_1^2(\Omega_+)}|\rho^{-n}| \\ &\leq \|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}+C_{DN}\cdot\alpha^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)}\textstyle\sum_{n=1}^\infty |\rho^{-n}|\alpha^n, \end{aligned}$$

and

$$\sum_{n=1}^\infty |\rho^{-n}|\alpha^n=\sum_{n=1}^\infty(\rho^{-1}\alpha)^n=\frac{1}{1-\rho^{-1}\alpha}\leq\frac{1}{1-\rho_0^{-1}\alpha}, \tag{32}$$

then

$$\|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)}\leq C_{DN}\frac{1}{1-\rho_0^{-1}\alpha}\rho_0^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)}+\|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}.$$

Using the triangle inequality, $(15)_2$ and $(30)_1$, we have

$$\begin{aligned} \|\varphi_\rho^-\|_{H^2(\Omega_-)} &= \|\textstyle\sum_{n=0}^\infty \varphi_n^-\rho^{-n}\|_{H^2(\Omega_-)} \\ &\leq \|\varphi_0^-\|_{H^2(\Omega_-)}+\textstyle\sum_{n=1}^\infty \|\varphi_n^-\|_{H^2(\Omega_-)}|\rho^{-n}| \\ &\leq \|\varphi_0^-\|_{H^2(\Omega_-)}+\alpha^{-1}\|\varphi_1^-\|_{H^2(\Omega_-)}\textstyle\sum_{n=1}^\infty |\rho^{-n}|\alpha^n, \end{aligned}$$

this and (32) implies $(31)_2$.
Now, from (27), for $k=1$

$$\|\varphi_1^-\|_{H^2(\Omega_-)}\leq C_N[\|f^-\|_{L^2(\Omega_-)}+\|g\|_{H^{\frac{1}{2}}(\Sigma)}+\|\partial_{\mathbf{n}}\varphi_0^+\|_{H^{\frac{1}{2}}(\Sigma)}], \tag{33}$$

according to (33) and (29), get

$$\|\varphi_1^-\|_{H^2(\Omega_-)}\leq C_N[\|f^-\|_{L^2(\Omega_-)}+\|g\|_{H^{\frac{1}{2}}(\Sigma)}+C_1\|\varphi_0^+\|_{\mathbb{W}_1^2(\Omega_+)}]. \tag{34}$$

From (34), (31), (21) and (22), we have

$$\begin{aligned} \|\varphi_\rho^+\|_{\mathbb{W}_1^2(\Omega_+)} \leq{}& C_{DN}\frac{1}{1-\rho_0^{-1}\alpha}\rho_0^{-1}C_N[\|f^-\|_{L^2(\Omega_-)}+\|g\|_{H^{\frac{1}{2}}(\Sigma)} \\ &+C_1C_{DN}(C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)}+\|f^+\|_{\mathbb{W}_1^0(\Omega_+)})]+C_{DN}(C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)}+\|f^+\|_{\mathbb{W}_1^0(\Omega_+)}), \end{aligned}$$

and

$$\begin{aligned} \|\varphi_\rho^-\|_{H^2(\Omega_-)} \leq{}& \rho_0^{-1}\frac{1}{1-\rho_0^{-1}\alpha}C_N[\|f^-\|_{L^2(\Omega_-)}+\|g\|_{H^{\frac{1}{2}}(\Sigma)} \\ &+C_1C_{DN}(C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)}+\|f^+\|_{\mathbb{W}_1^0(\Omega_+)})]+C_N\|g\|_{H^{\frac{1}{2}}(\Sigma)}. \end{aligned}$$

This yields the estimate (14).

Notes

III. Acknowledgements

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References Références Referencias

Notes

1. A. Kufner, Weighted Sobolev spaces, John Wiley and Sons, (1985).
2. T. Z. Boulmezaoud, On the Laplace operator and on the vector potential problems in the half-space: an approach using weighted spaces, Math. Meth. in the Appl. Scienc., 23, (2003), 633–669.
3. B. Hanouzet, Espaces de Sobolev avec poids. Application au probl'eme de Dirichlet dans un demi-espace, Rend. Sem. Mat. Univ. Padova XLVI, (1971), 227–272.
4. J. Giroire, Etude de quelques probl'emes aux limites extȷerieurs et rȷesolution par ȷequations intȷegrales, PhD thesis, UPMC, Paris, France, (1987).
5. V. Peron, Modȷelisation mathȷematique de phȷenom'enes ȷelectromagnȷetiques dans des matȷeriaux 'a fort contraste, PhD thesis, Universitȷe de Rennes I, Rennes, France, (2009).
6. J. E. Ospino P., Finite elements/boundary elements for electromagnetic interface problems, especially the skin effect, PhD thesis, Institut of Applied Mathematics, Hannover University, Germany, (2011)
7. G. Caloz and M. Dauge and V. Peron, Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism, Journal of Mathematical Analysis and applications. 370 (2) (2010), 555–572.
8. J. C. Nedelec, Acoustic and electromagnetic equations. Integral representations for harmonic problems, Springer-Verlag, (2001).

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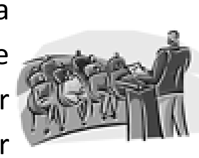

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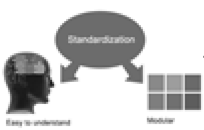

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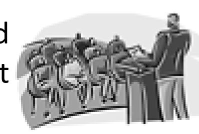

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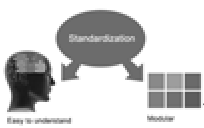

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2. Ethical Guidelines,

3. Submission of Manuscripts,

4. Manuscript's Category,

5. Structure and Format of Manuscript,

6. After Acceptance.

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The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
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- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
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- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

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- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
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- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
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Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

Index

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