Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 17 Issue 6 Version 1.0 Year 2017
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A New Approximation to Standard Normal Distribution Function

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Abstract- This paper, presents three news-improved approximations to the Cumulative Distribution Function (C.D.F.). The first approximation improves the accuracy of approximation given by Polya (1945). In this first new approximation, we reduce the maximum absolute error (M.A.E.) from, 0.00314 to 0.00103. For this first new approximation, K. M. Aludaat and M. T. Alodat (2008) was reduce the (M.A.E.) from, 0.00314 to 0.001972. The second new approximation improve Tocher's approximation, we reduce the (M.A.E.) from, 0.166 to 0.00577. For the third new approximation, we combined the two previous approximations. Hence, this combined approximation is more accurate and its inverse is hard to calculate. This third approximation reduce the (M.A.E.) to be less than $2.232e-004$. The two improved previous approximations are less accurate, but his inverse is easy to calculate. Finally, we give an application to the third approximation for pricing a European Call using Black-Scholes Model.

Keywords: *cumulative distribution function, normal distribution, maximum absolute error.*

GJSFR-F Classification: *MSC 2010: 11N60*

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A New Approximation to Standard Normal Distribution Function

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Abstract- This paper, presents three news-improved approximations to the Cumulative Distribution Function (C.D.F.). The first approximation improves the accuracy of approximation given by Polya (1945). In this first new approximation, we reduce the maximum absolute error (M.A.E.) from, $\mathbf{0.00314}$ to $\mathbf{0.00103}$. For this first new approximation, K. M. Aludaat and M. T. Alodat (2008) was reduce the (M.A.E.) from, $\mathbf{0.00314}$ to $\mathbf{0.001972}$. The second new approximation improve Tocher's approximation, we reduce the (M.A.E.) from, $\mathbf{0.166}$to $\mathbf{0.00577}$. For the third new approximation, we combined the two previous approximations. Hence, this combined approximation is more accurate and its inverse is hard to calculate. This third approximation reduce the (M.A.E.) to be less than $\mathbf{2.232e-004}$. The two improved previous approximations are less accurate, but his inverse is easy to calculate. Finally, we give an application to the third approximation for pricing a European Call using Black-Scholes Model.

Keywords: cumulative distribution function, normal distribution, maximum absolute error.

I. Introduction

The cumulative distribution function (CDF) play an important role in financial mathematics and especially in pricing options with Black-Scholes Model. The European option pricing call given by Black-Scholes Model is

$$C(S,K,T,r,\sigma) = S\Phi(d) - Ke^{-rT}\Phi(d - \sigma\sqrt{T}) \tag{1}$$

Where
$$d = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{2}$$

S, the current price, K the exercise price, r interest rate, T time option and σ volatility. The cumulative distribution function (C.D.F.) is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{t^2}{2}}dt \tag{3}$$

The (C.D.F) have not a closed form. His evaluation is an expensive task. For evaluate the (CDF) at a point z we need compute the integral under the probability density function (PDF) given by $\varphi(t) = e^{-0.5t^2}/\sqrt{2\pi}$.

In much research, we find approximations, with closed forms, for the area under the standard normal curve. Otherwise, we need consulting Tables of cumulative standard normal probabilities. Hence, in the literature, we find several approximations to this function from polya (1945) to Yerukala (2015). For this raison; we use some approximations to this C.D.F. (Polya's approximation and Tocher's approximation).

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II. Improving Polya's Approximation

We consider the case of, $z \geq 0$. (For $z < 0, \Phi(z) = 1 - \Phi(-z)$).
The original Polya's approximation given by:

$$\Phi_{Polya}(z) \approx \frac{1}{2}\left\{1 + \sqrt{1 - e^{-az^2}}\right\}, \text{ where } a = \frac{2}{\pi}. \quad (2.1)$$

The Maximum Absolute Error (M.A.E.)

$$\text{M.A.E.}_{Polya} = \max_z\left|\Phi_{Polya}(z) - N(z)\right| = 0.003138181653387. \quad (2.2)$$

K.M.Aludaat and M.T.Alodat (2008) proposed the same formula with $a = \sqrt{\frac{\pi}{8}}$ instead of $a = \frac{2}{\pi}$. They have

$$\text{M.A.E.}_{Aludaat} = \max_z|\Phi_{Aludaat}(z) - \Phi(z)| = 0.001971820656170\,.$$

In this paper, we write the formula (2.1) and (2.2) as

$$\Phi_{Malki}(z) \approx a + b\sqrt{1 - e^{-cz^2}} \quad (2.3)$$

Hence, we search the parameters $a, b \text{ and } c$ that

$$\text{M.A.E.}_{Malki} = \max_z|\Phi_{Malki}(z) - \Phi(z)| \quad (2.4)$$

Was the smallest possible using the following algorithm?

1) $h = 0.00001; H = 20h;\ Er = 0.00314;$
2) $a_0 = 0.5, b_0 = 0.5, c_0 = \frac{2}{\pi},$
3) $for\ a = a_0 - H{:}\ h{:}\ a_0 + H\ for\ b = b_0 - H{:}\ h{:}\ b_0 + H$
4) $for\ c = c_0 - H{:}\ h{:}\ c_0 + H; M = a + b\sqrt{1 - e^{-cz^2}},$
5) $e = \max_z|\text{M} - N(z)|;\ if(Er > e)\ Er = e; A = a; B = b; C = c; end;$
6) $a_0 = A, b_0 = B, c_0 = C.$
7) Repeat 3) to 6) until convergence

Using our algorithm, we find the best parameters

$$a^* = 0.50103;\ b^* = 0.49794\,;\ c^* = 0.62632 \quad (2.4)$$

Hence the best formula is

$$\Phi_{Malki\,1}(z) = 0.50102976 + 0.49794047\sqrt{1 - e^{-0.626317743\ z^2}}, \quad (2.5)$$

Note that the absolute error as function of z variable noted by

$$\text{E(z)} = |\Phi_{\text{Malki 1}}(\text{z}) - \Phi(\text{z})| = 0.001029767666887 \quad (2.6)$$

Figure one, shows the graph of Absolute Error for Polya, Aludaat and Malki1 as function of $-5 \leq z \leq 5$.

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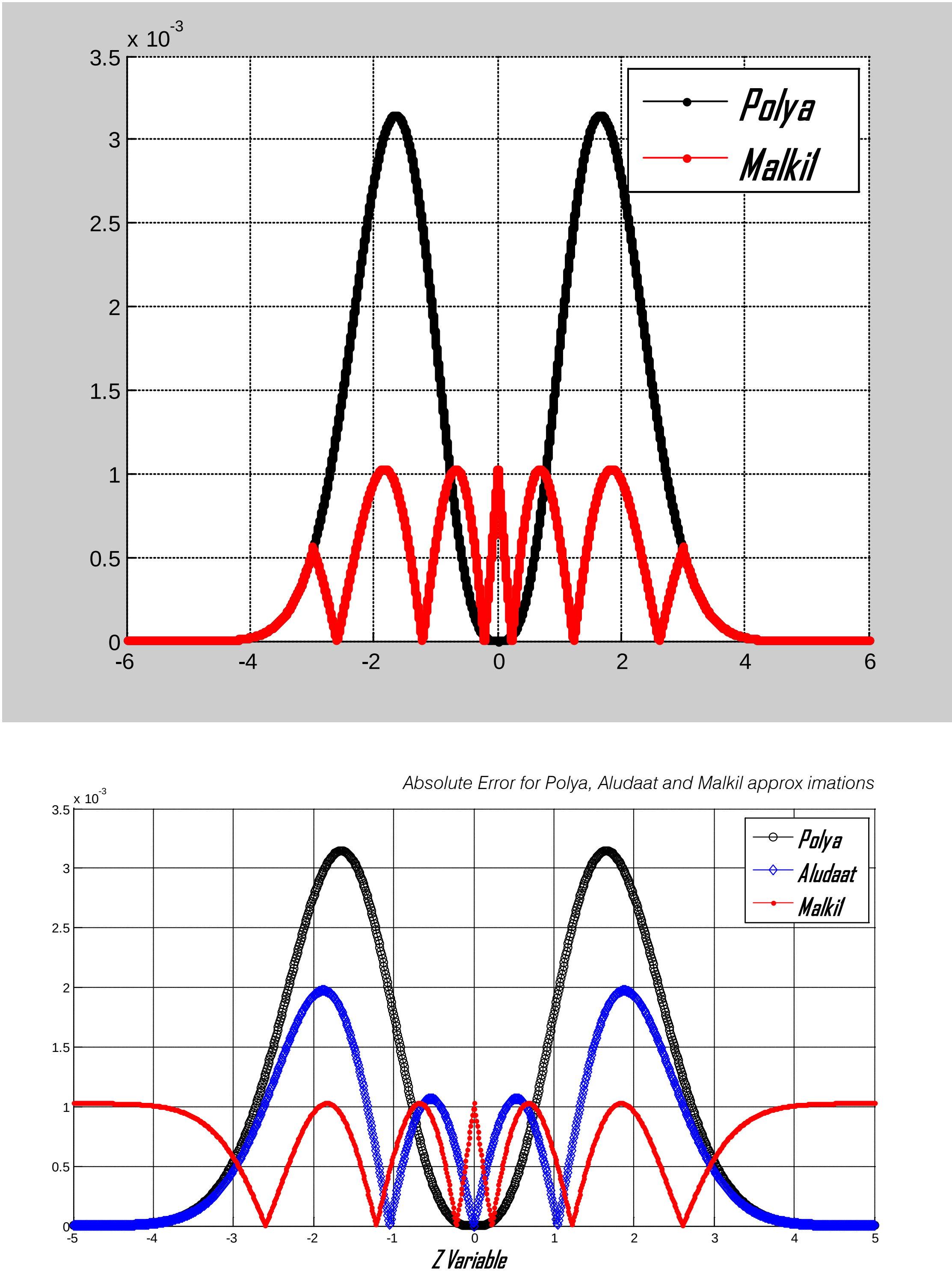

Figure 1: Comparison of absolute error for Polya, Aludaat and Malki1

III. Improving Tocher's Approximation

The Original Tocher's approximation is $\Phi_{Tocher}(z) = 1/(1 + e^{-\sqrt{\frac{2}{\pi}}z})$ with

$$\max_z |\Phi_{Tocher}(z) - \Phi(z)| = 0.165811983691380 \approx 0.166. \qquad (3.1)$$

This approximation have the form:

$$\Phi_{Malki\,2}(z) = \frac{a}{b+e^{-cz}} \tag{3.2}$$

Hence, we search the parameters a, b and c that

$$\text{M.A.E.}_{Malki\,2} = \max_z |N_{Malki\,2}(z) - N(z)| \tag{3.3}$$

Was the smallest possible using the following algorithm?

1) $h = 0.00001; H = 20h;\ Er = 0.166;$
2) $a_0 = 1, b_0 = 1, c_0 = \sqrt{\frac{2}{\pi}},$
3) $for\ a = a_0 - H{:}\,h{:}\,a_0 + H\ for\ b = b_0 - H{:}\,h{:}\,b_0 + H$
4) $for\ c = c_0 - H{:}\,h{:}\,c_0 + H;\ M = \frac{a}{b+e^{-cz}},$
5) $e = \max_z |M - \Phi(z)|;\ if(Er > e)\ Er = e; A = a; B = b; C = c; end;$
6) $a_0 = A, b_0 = B, c_0 = C.$
7) Repeat 3) to 6) until convergence

Using our algorithm, we find the best parameters

$$a^* = 0.97186;\ b^* = 0.96628;\ c^* = 1.69075 \tag{3.4}$$

Hence the best-improved formula for Tocher's approximation is

$$\Phi_{Malki\,2}(z) = \frac{0.97186}{0.96628 + e^{-1.69075\,z}} \tag{3.5}$$

$$\max_z |\Phi_{Malki\,2}(z) - N(z)| = 0.005774676414954 \tag{3.6}$$

Figure 2 gives the curves of original absolute error and the new absolute error

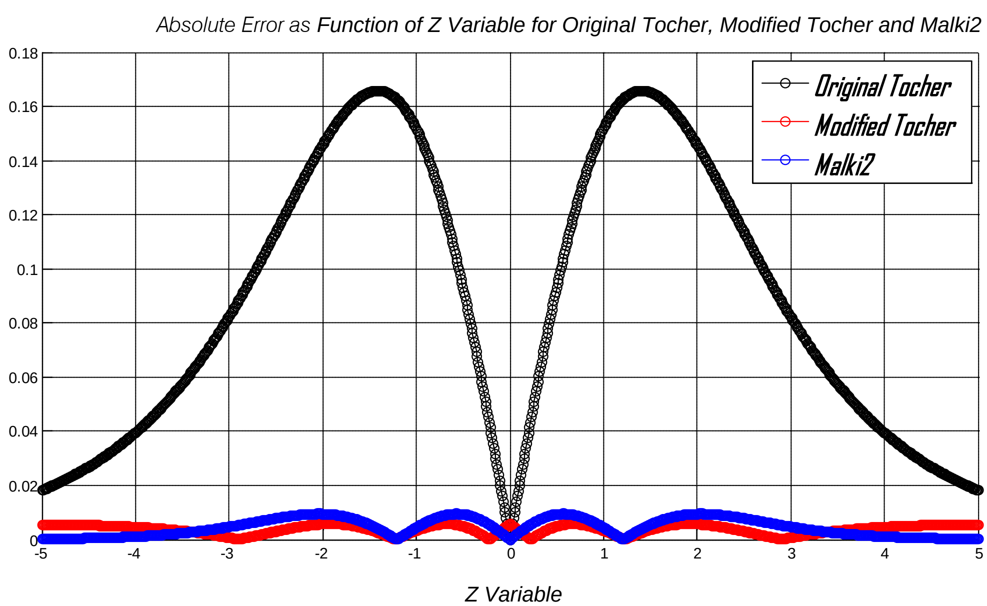

Figure 2: Comparison of Absolute Error for Original Tocher, Modified Tocher and Malki2 as function of z variable $(-5 \le z \le 5)$

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IV. Combined Formula

As the third new approximation formula, we consider the two previous formula

$$\Phi_{Malki\,1}(z) = 0.50103 + 0.49794\sqrt{1 - e^{-0.62632\,z^2}},$$

and,
$$\Phi_{Malki\,2}(z) = \frac{0.97186}{0.96628 + e^{-1.69075\,z}}.$$

Hence, we consider the third new formula as

$$\Phi_{Malki\,3}(z) = \omega\Phi_{Malki\,1}(z) + (1-\omega)\Phi_{Malki\,2}(z),\ \text{for},\ (0 \le \omega \le 1) \tag{4.1}$$

We search the optimum parameter ω that the

$$\text{M.A.E.}_{Malki\,3} = \max_z |\Phi_{Malki\,3}(z) - \Phi(z)|$$

Was the smallest possible. We find optimum parameter $\omega^* = 0.16$

The new third approximation is

$$\Phi_{Malki\,3}(z) = 0.16\Phi_{Malki\,1}(z) + 0.84\Phi_{Malki\,2}(z) \tag{4.2}$$

The adjusted formula is

$$\Phi_{\text{Malki 3}}(z) = \frac{0.1544976}{0.96568 + e^{-1.68975\,z}} + 0.4212652 + 0.4189696\sqrt{1 - e^{-0.62642\,z^2}} \tag{4.3}$$

For, this approximation we have:

$$\max_z |\Phi_{\text{Malki 3}}(z) - \Phi(z)| = 2.231943559627414\text{e} - 004 \tag{4.4}$$

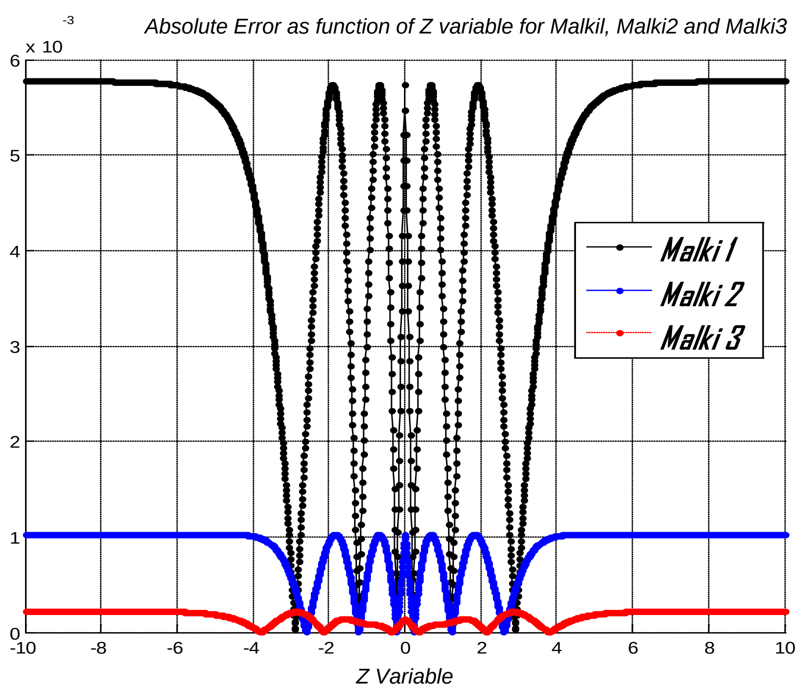

Figure 3

V. Application with Black-Scholes Model

For

$$S = 35; K = 30; r = 0.065; T = 1.2;\ \sigma = 0.35; \tag{5.1}$$

To calculate a Call European option we compute

$$d = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 0.797198914562755 \quad (5.2)$$

And
$$d - \sigma\sqrt{T} = 0.413793124309139 \quad (5.3)$$

Hence

$$C = S\Phi(d) - Ke^{-rT}\Phi\left(d-\sigma\sqrt{T}\right) = 9.228813813962439 \quad (5.4)$$

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Using, $\Phi_{Malki\,3}$ we have

$$C_3 = S\Phi_{Malki\,3}(d) - Ke^{-rT}\Phi_{Malki\,3}\left(d-\sigma\sqrt{T}\right) = 9.231095739041432 \quad (5.5)$$

The absolute error is$|C - C_3| \leq\ 0.0023$ (5.6).

VI. Conclusion

We have proposed three approximations to the cumulative distribution function of the standard normal distribution. The first approximation improve the Polya's formula in accuracy. The second new approximation improve the accuracy of Tocher's formula. The third formula is a combination of the two previous formula. The M.A.E. for the first approximation is 0.00103. The M.A.E. for the second approximation is 0.00577. For the third approximation the M.A.E. is less than$2.232\mathrm{e}-004$. Finally, we insert an application to option pricing of a Call European option based on Black-Scholes formula.

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