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Restricted Three-Body Problem with Albedo Effect when Smaller Primary is an Oblate Spheroid

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RESTRICTEDTHREEBODYPROBLEMWITHALBEDOEFFECTWHENSMALLERPRIMARYISANOBLATESPHEROID

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Restricted Three-Body Problem with Albedo Effect when Smaller Primary is an Oblate Spheroid

M. Javed Idrisi [α] & M. Shahbaz Ullah [σ]

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I. Introduction

The restricted three-body problem is one of well known problem in the field of celestial mechanics in which two finite bodies called primaries move around their center of mass in circular or elliptic orbits under the influence of their mutual gravitational attraction and a third body of infinitesimal mass is moving in the plane of the primaries which is attracted by the primaries and influenced by their motion but not influencing them. In classical case there exist five libration points out of which three are collinear and two are non-collinear. The collinear libration points L_1, L_2 and L_3 are unstable for $0 \leq \mu \leq \frac{1}{2}$ and the non collinear libration points $L_{4,5}$ are stable for a critical value of mass parameter $\mu < \mu_c = 0.03852...$, Szehebely (1967). Some studies related to the equilibrium points in R3BP or ER3BP, taken into account the oblateness and triaxiality of the primaries, Coriolis and Centrifugal forces, Yarkovsky effect, variation of the masses of the primaries and the infinitesimal mass etc. are discussed by Danby (1964); Vidyakin (1974); Sharma (1975); Choudhary R. K. (1977); Subbarao and Sharma (1975); Cid R. et. al. (1985); El-Shaboury (1991); Bhatnagar et al. (1994); Selaru D. et.al. (1995); Markellos et al. (1996); Subbarao and Sharma (1997); Khanna and Bhatnagar (1998, 1999); Roberts G.E. (2002); Oberti and Vienne (2003); Perdiou et. al. (2005); Sosnytskyi (2005); Ershkov (2012); Arredondo et.al. (2012); Idrisi and Taqvi (2013); Idrisi (2014); Idrisi and Amjad (2015). The photo-gravitational restricted three-body problem arises from the classical problem if one or both primaries is an intense emitter of radiation, formulated by Radzievskii (1950). He has considered only the central forces of gravitation and radiation pressure on the particle of infinitesimal mass without considering the other two components of light pressure field and studied this problem for three specific

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bodies; the Sun, a planet and a dust particle. The radiation repulsive force F_p exerted on a particle can be represented in terms of gravitational attraction F_g (Radzievskii, 1950) as $F_p = F_g(1-q)$, where $q = 1 - \frac{F_p}{F_g}$, a constant for a given particle, is a reduction factor expressed in terms of the particle radius a, density δ and radiation-pressure efficiency factor x (in c.g.s. system) as:

$$q = 1 - \frac{5.6*10^{-3}}{a\delta}x$$

The assumption that q is a constant implies that the fluctuations in the beam of solar radiation and the effect of planets shadow are neglected. Typical values for diameter of IDP (Interplanetary Dust Particles) are in the range of 50 - 500 μm and their densities range is $1-3g/cm^3$ with an average density of $2g/cm^3$. As the size of the particles increases, their density decreases (Grűn et.al. 2001). Some of the notable research in PRTBP are carried by Chernikov (1970); Bhatnagar and Chawla (1979); Schuerman D.W (1980); Simmons et. al. (1985); Kunitsyn and Tureshbaev (1985); Lukyanov (1988), Sharma (1987); Xuetang et.al. (1993); Ammar (2008); Singh and Leke (2010); Douskos (2010); Katour et.al. (2014) etc. In 2012, S. V. Ershkov studied the Yarkovsky effect in generalized photogravitational 3-body problem and proved the existence of maximally 256 different non-planar equilibrium points when second primary is non-oblate spheroid. The main contribution of the natural radiation pressure on the satellite is due to the direct solar radiation and the second main contribution of radiation forces is due to the Earth reflected radiation known as the Albedo studied by Anselmo et.al. (1983); Nuss (1998); McInnes (2000); Bhanderi (2005); Pontus (2005); MacDonald (2011) etc. Albedo effect is one of the most interesting non-gravitational force having significant effects on the motion of infinitesimal mass. Albedo is the fraction of solar energy reflected diffusely from the planet back into space (Harris and Lyle, 1969). It is the measure of the reflectivity of the planets surface. Therefore, the Albedo can be defined as the fraction of incident solar radiation returned to the space from the surface of the planet (Rocco, 2009) as

$$Albedo = \frac{radiation\ reflected\ back\ to\ space}{incident\ radiation}$$

In this paper the Albedo effect on the existence and stability of the libration points when smaller primary is a homogeneous ellipsoid has been studied. This paper is divided into five sections. In section-2, the equations of motion are derived. The existence of non-collinear and collinear libration points is shown in section-3. In section-4, the stability of non-collinear and collinear libration points is discussed. In section-5, a real application to Sun-Earth system has shown. In the last section, all the results are discussed.

II. Equations of Motion

Let m_2 be an oblate spheroid with axes a, b and c $(a = b > c)$ and m_1 a point mass and a source of radiation such that $m_1 > m_2$, are moving in the circular orbits around their center of mass O. An infinitesimal mass m_3 $<< 1$, is moving in the plane of motion of m_1 and m_2. The distances of m_3 from m_1, m_2 and O are r_1, r_2 and r respectively. F_1 and F_2 are the gravitational forces acting on m_3 due to m_1 and m_2 respectively, F_p is the solar radiation pressure on m_3 due to m_1 and F_A is the Albedo force (solar radiation reflected by m_2 in space) on m_3 due to m_2 (Fig. 1). Also, let us consider that the principal axes of spheroid remains parallel to the synodic axes $Oxyz$ throughout the motion and the equatorial plane of m_2 is coincide with the plane of motion of m_1 and m_2. Let the line joining m_1 and m_2 be taken as $X-axis$ and O their center of mass as origin. Let the line passing through O and perpendicular to OX and lying in the plane of motion m_1 and m_2 be the $Y-axis$. Let us consider a synodic system of co-ordinates $Oxyz$ initially coincide with the inertial system $OXYZ$, rotating with angular

Notes

Global Journal of Science Frontier Research (F) Volume XVII Issue V Version I Year 2017

Notes

velocity ω about $Z-axis$ (the $z-axis$ is coincide with $Z-axis$). We wish to find the equations of motion of m_3 using the terminology of Szebehely (1967) in the synodic co-ordinate system and dimensionless variables $i.e.$ the distance between the primaries is unity, the unit of time t is such that the gravitational constant $G=1$ and the sum of the masses of the primaries is unity $i.e.$ $m_1+m_2=1$.

The force acting on m_3 due to m_1 and m_2 is F_1 (1 - F_p/F_1)=$F_1(1-\alpha)$ and F_2 (1 - F_A/F_2) = F_2 ($1-\beta$) respectively, where $\alpha=F_p/F_1 << 1$ and $\beta=F_A/F_2 << 1$. Also, α and β can be expressed as:

$$\alpha=\frac{L_1}{2\pi G\ m_1 c\sigma};\ \beta=\frac{L_2}{2\pi G\ m_2 c\sigma}$$

where L_1 is the luminosity of the large primary m_1, L_2 is the luminosity of small primary m_2, G is the gravitational constant, c is the speed of velocity

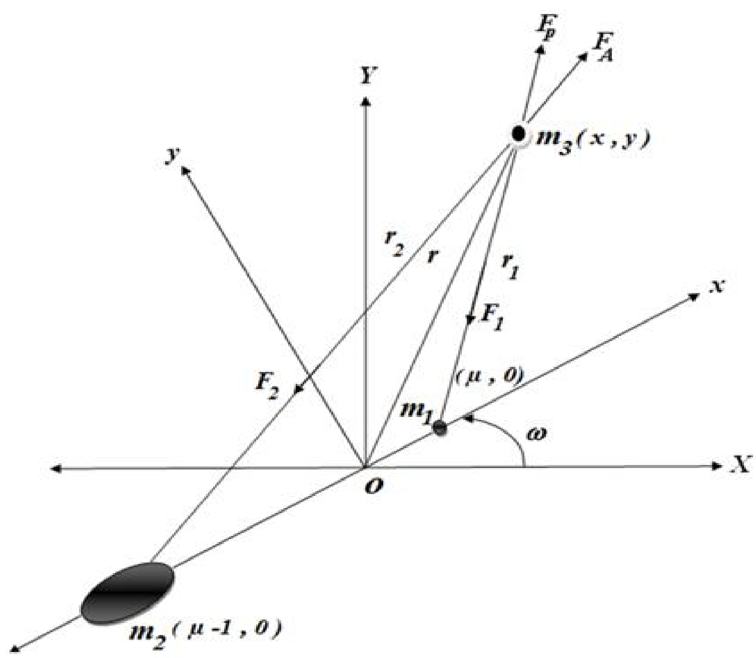

Fig. 1: Configuration of the R3BP under Albedo effect when m_2 is an oblate spheroid

and σ is mass per unit area. Now,

$$\frac{\beta}{\alpha}=\frac{L_2}{L_1}\frac{m_1}{m_2}\Rightarrow\beta=\alpha\left(\frac{1-\mu}{\mu}\right)k;k=\frac{L_2}{L_1}=constant. \tag{1}$$

The equations of motion of the infinitesimal mass m_3 in the synodic coordinate system and dimensionless variables are given by

$$\ddot{x}-2n\dot{y}=\Omega_x;\ddot{y}+2n\dot{x}=\Omega_y \tag{2}$$

where

$$\Omega=\frac{n^2}{2}\{(1-\mu)\ r_1^2\}+\frac{(1-\mu)(1-\alpha)}{r_1}+\frac{\mu(1-\beta)}{r_2}\left(1+\frac{\sigma}{2r_2^2}\right)$$

$$\Omega_x=\ n^2x-\frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3}-\frac{\mu(x+1-\mu)(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right)$$

$$\Omega_y=y\left\{n^2-\frac{(1-\mu)(1-\alpha)}{r_1^3}-\frac{\mu(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right)\right\}$$

Global Journal of Science Frontier Research (F) Volume XVII Issue V Version I Year 2017

$$n^2 = 1 + \frac{3\sigma}{2} \text{ is the mean motion of the primaries,} \quad (3)$$

$$\sigma = \frac{a^2 - c^2}{5} \text{ is the oblateness factor,}$$

$$r_1^2 = (x - \mu)^2 + y^2, \quad (4)$$

$$r_2^2 = (x + 1 - \mu)^2 + y^2, \quad (5)$$

$$0 < \mu = \frac{m_2}{m_1 + m_2} < \frac{1}{2} \Rightarrow m_1 = 1 - \mu;\ m_2 = \mu.$$

III. Libration Points

At the libration points all the derivatives are zero *i.e.*

$$\dot{x} = 0, \dot{y} = 0, \ddot{x} = 0, \ddot{y} = 0, \Omega_x = 0, \Omega_y = 0.$$

Therefore, the libration points are the solutions of the equations

$$\Omega_x = n^2 x - \frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3} - \frac{\mu(x+1-\mu)(1-\beta)}{r_2^3}\left(1 + \frac{3\sigma}{2r_2^2}\right) = 0$$

$$\Omega_y = y\left\{n^2 - \frac{(1-\mu)(1-\alpha)}{r_1^3} - \frac{\mu(1-\beta)}{r_2^3}\left(1 + \frac{3\sigma}{2r_2^2}\right)\right\} = 0$$

a) *Non-collinear Libration Points*

The non-collinear libration points are the solution of the Equations $\Omega_x = 0$ and $\Omega_y = 0,\ y \neq 0$ *i.e.*

$$n^2 x - \frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3} - \frac{\mu(x+1-\mu)(1-\beta)}{r_2^3}\left(1 + \frac{3\sigma}{2r_2^2}\right) = 0 \quad (6)$$

$$n^2 - \frac{(1-\mu)(1-\alpha)}{r_1^3} - \frac{\mu(1-\beta)}{r_2^3}\left(1 + \frac{3\sigma}{2r_2^2}\right) = 0 \quad (7)$$

On substituting $\sigma = 0$, $\alpha = 0$ and $\beta = 0$, the solution of Eqns. (6) and (7) is $r_1 = 1$, $r_2 = 1$ and from Eqn. (3), $n = 1$. Now we assume that the solution of Eqns. (6) and (7) for $\sigma \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$ as $r_1 = 1 + \xi_1$, $r_2 = 1 + \xi_2$, $\xi_1, \xi_2 << 1$. Substituting these values of r_1 and r_2 in the Eqns. (4) and (5), we get

$$x = \mu - \frac{1}{2} + \xi_2 - \xi_1;\ y = \pm\frac{\sqrt{3}}{2}\left\{1 + \frac{2}{3}(\xi_2 + \xi_1)\right\} \quad (8)$$

Table 1: Non-collinear Libration Points $L_{4,5}(x \pm y)$ for $\mu = 0.1$ and $\sigma = 10^{-3}$

	$k=0$	$k=0$	$k=0.01$	$k=0.01$	$k=0.1$	$k=0.1$
α	x	$\pm y$	x	$\pm y$	x	$\pm y$
0.0	−0.399512	0.865737	−0.399512	0.865737	−0.399512	0.865737
0.1	−0.366167	0.846492	−0.369167	0.844761	−0.396167	0.829171
0.2	−0.332833	0.827247	−0.338833	0.823783	−0.392833	0.792606
0.3	−0.299512	0.808002	−0.308513	0.800806	−0.389512	0.756041
0.4	−0.266167	0.788757	−0.278167	0.781828	−0.386167	0.719475
0.5	−0.232833	0.769512	−0.247833	0.760851	−0.382833	0.682909
0.6	−0.199511	0.750267	−0.217502	0.739874	−0.379510	0.646344
0.7	−00.166167	0.731022	−0.187167	0.718897	−0.376167	0.609778
0.8	−0.132833	0.711777	−0.156833	0.697921	−0.372833	0.573213
0.9	−0.099512	0.692532	−0.126502	0.676943	−0.369514	0.536647

Notes

Global Journal of Science Frontier Research (F) Volume XVII Issue V Version I Year 2017

Notes

Now, substituting the values of $x,\ y$ from Eqns. (8), $r_1 = 1+\xi_1$ and $r_2 = 1+\xi_2$ in the Eqns. (6) and (7) and neglecting higher order terms, we obtain

$$\xi_1 = -\frac{\alpha}{3} + \frac{1}{2}\left(1 - \frac{\mu}{1-\mu}\right)\sigma;\ \xi_2 = -\frac{\beta}{3}$$

Thus, the coordinates of the non-collinear libration points $L_{4,5}$ are

$$x = \mu - \frac{1}{2} + \frac{1}{3}(\alpha - \beta) + \frac{\sigma}{2},$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}\left\{\frac{1}{3}(\alpha+\beta) + \frac{\sigma}{2}\right\}\right]$$

Using the relation (1) *i.e.* $\beta = \alpha(1-\mu)k/\mu$, we have

$$x = \mu - \frac{1}{2} + \frac{1}{3}\left\{1 - \frac{(1-\mu)k}{\mu}\right\}\alpha + \frac{\sigma}{2}, \tag{9}$$

$$y = \pm\frac{\sqrt{3}}{2}\left[1 - \frac{2}{3}\left\{\frac{\alpha}{3}\left(1 + \frac{(1-\mu)k}{\mu}\right)\alpha + \frac{\sigma}{2}\right\}\right] \tag{10}$$

Thus, we conclude that there exist two non-collinear libration points $L_{4,5}$ and these points are affected by oblateness as well as Albedo effect (Fig. 2), also these points form scalene triangle with the primaries as $r_1 \neq r_2$. The numerical location of $L_{4,5}$ is also calculated in Table 1 for $\mu = 0.1,\ \sigma = 0.001$ and different values of α and k and it is found that the abscissa and ordinate of non-collinear libration points are the decreasing functions of α and k *i.e.* as α and k increases, x and y decreases. For $\alpha = 0$, the results are in conformity with those of Bhatnagar and Hallan (1979). If $\alpha = 0$ and $\sigma = 0$, the classical case of the restricted three body problem is verified (Szebehely, 1967).

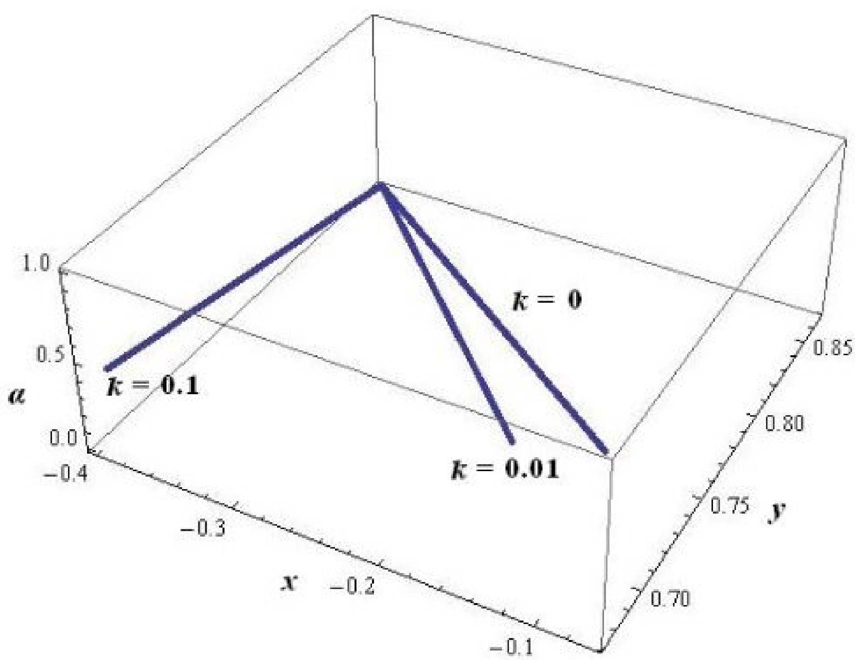

Fig. 2: L_4 verses α; $\mu = 0.1,\ \sigma = 10^{-3}$

b) Collinear Libration Points

The collinear libration points are the solution of the Equations $\Omega_x = 0$ and $y = 0$ *i.e.*

$$f(x) = n^2 x - \frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3} - \frac{\mu(x+1-\mu)(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right) = 0 \tag{11}$$

where $r_i = |x - x_i|, i = 1, 2$ is a seventh degree equation in x.

Since $f(x) > 0$ in each of the open interval $(-\infty, \mu-1)$, $(\mu-1, \mu)$ and (μ, ∞), the function $f(x)$ is strictly increasing in each of them. Also, $f(x) \to -\infty$ as $x \to -\infty$, $(\mu-1)+0$ or $(\mu+0)$ and $f(x) \to \infty$ as $x \to \infty$, $(\mu-1)-0$ or $(\mu-0)$. Therefore, there exists only one value of x in each of the open intervals $(-\infty, \mu-1)$, $(\mu-1, \mu)$ and (μ, ∞) such that $f(x) = 0$. Further, $f(\mu-2) < 0$, $f(0) \geq 0$ and $f(\mu+1) > 0$. Therefore, there are only three real roots lying in each of the intervals $(\mu-2, \mu-1)$, $(\mu-1, 0)$ and $(\mu, \mu+1)$. Thus there are only three collinear libration points.

From the Fig. 3, this is observed that the first collinear libration point L_1 always lie at the right of the primary m_2, the second libration point L_2 lies

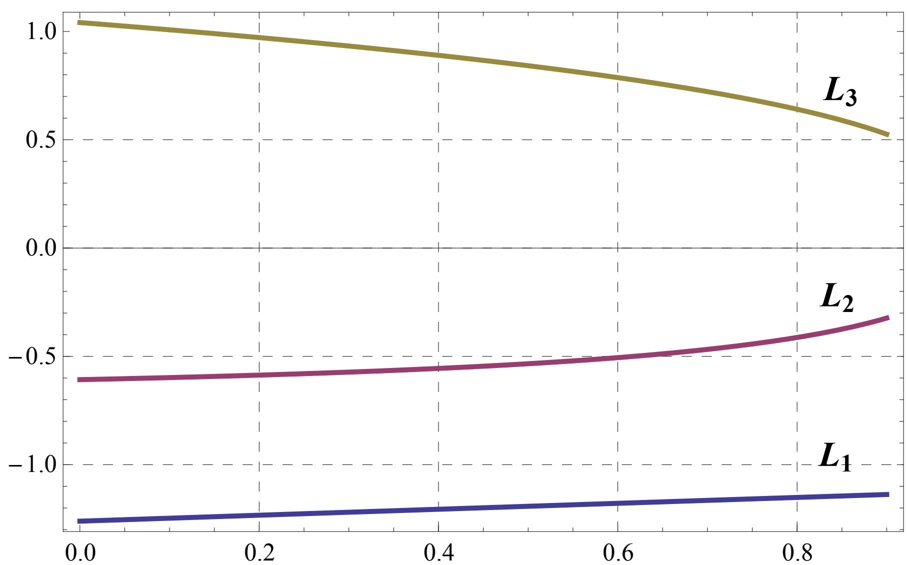

Fig. 3: α verses L_i, $i = 1, 2, 3$; $\mu = 0.1$, $\sigma = 10^{-3}$

between the center of mass of the primaries O and m_1 and the third libration point L_3 lies at the right of the primary m_1. This is also observed that the libration points L_1 move away from the center of mass as α increases while the second and third libration points L_2 and L_3 move toward the center of mass as α increases.

The Equation (11) determines the location of the collinear libration points $L_1(x_1, 0)$ $L_2(x_2, 0)$ and $L_3(x_3, 0)$ lie in the intervals $(-\infty, \mu-1)$, $(\mu-1, \mu)$ and (μ, ∞) respectively, where

$$x_1 = \mu - 1 - \xi_1,$$
$$x_2 = \mu - 1 + \xi_2,$$
$$x_3 = \mu + \xi_3. \tag{12}$$

Since the libration point L_1 lies in the interval $(-\infty, \mu-1)$ *i.e.* left to the smaller primary, we have $r_1 = \mu - x_1$ and $r_2 = \mu - 1 - x_1$ which when substituted in Equation (11), gives

Notes

Global Journal of Science Frontier Research (F) Volume XVII Issue V Version I Year 2017

$$n^2x+\frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3}+\frac{\mu(x+1-\mu)(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right)=0 \quad (13)$$

Similarly, for $L_2(x_2,0)$ and $L_3(x_3,0)$ the Equation (11) becomes

$$n^2x+\frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3}-\frac{\mu(x+1-\mu)(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right)=0 \quad (14)$$

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$$n^2x-\frac{(1-\mu)(x-\mu)(1-\alpha)}{r_1^3}-\frac{\mu(x+1-\mu)(1-\beta)}{r_2^3}\left(1+\frac{3\sigma}{2r_2^2}\right)=0 \quad (15)$$

Table 2: Collinear Libration Points $L_i(x_{i,0})(i=1,2,3)$ for $k=0$

	$\mu=.1,\sigma=10^{-3},k=0$	$\mu=.1,\sigma=10^{-3},k=0$	$\mu=.1,\sigma=10^{-3},k=0$
α	L_1	L_2	L_3
0.0	−1.26086	−0.607519	1.04112
0.1	−1.25296	−0.594138	1.00813
0.2	−1.24529	−0.578763	0.972618
0.3	−1.23783	−0.560863	0.934029
0.4	−1.23061	−0.539686	0.891595
0.5	−1.22361	−0.514114	0.844181
0.6	−1.21684	−0.482382	0.789997
0.7	−1.21029	−0.441432	0.725923
0.8	−1.20396	−0.385085	0.645599
0.9	−1.19786	−0.296465	0.531473

Table 3: Collinear Libration Points $L_i(x_{i,0})(i=1,2,3)$ for $k=0.01$

	$\mu=.1,\sigma=10^{-3},k=.01$	$\mu=.1,\sigma=10^{-3},k=.01$	$\mu=.1,\sigma=10^{-3},k=.01$
α	L_1	L_2	L_3
0.0	−1.26086	−0.607519	1.04112
0.1	−1.25182	−0.594889	1.00806
0.2	−1.24295	−0.580271	0.972458
0.3	−1.23431	−0.563125	0.933778
0.4	−1.22588	−0.542681	0.891246
0.5	−1.21768	−0.517794	0.843722
0.6	−1.20971	−0.486666	0.789412
0.7	−1.20197	−0.446175	0.725189
0.8	−1.19446	−0.390047	0.644678
0.9	−1.18717	−0.301225	0.530279

The solution of Equations (13), (14) and (15) is given in Table 2.

For $k=0$, the solutions obtained for the Equations (13), (14) and (15) are the libration points $L_i,(i=1,2,3)$ in the photogravitational restricted three-body problem when smaller primary is an oblate body but if $k\neq 0$ the libration points $L_i,(i=1,2)$ are affected by the Albedo and this effect displaced the libration points from its actual position as shown in Fig. 4 and 5 but L_3 not much affected by albedo, see Fig. 6.

IV. Stability of Libration Points

The variational equations are obtained by substituting $x=x_0+\xi$ and $y=y_0+\eta$ in the equations of motion (2), where (x_0,x_0) are the coordinates of L_4 or L_5 and $\xi,\eta<<1$ *i.e.*

$$\ddot{\xi}-2n\dot{\eta}=\xi\Omega^0_{xx}+\eta\Omega^0_{xy},$$

$$\ddot{\eta}+2n\dot{\xi}=\xi\Omega^0_{xy}+\eta\Omega^0_{yy}. \quad (16)$$

Here we have taken only linear terms in ξ and η. The subscript in Ω indicates the second partial derivative of Ω and superscript o indicates that the

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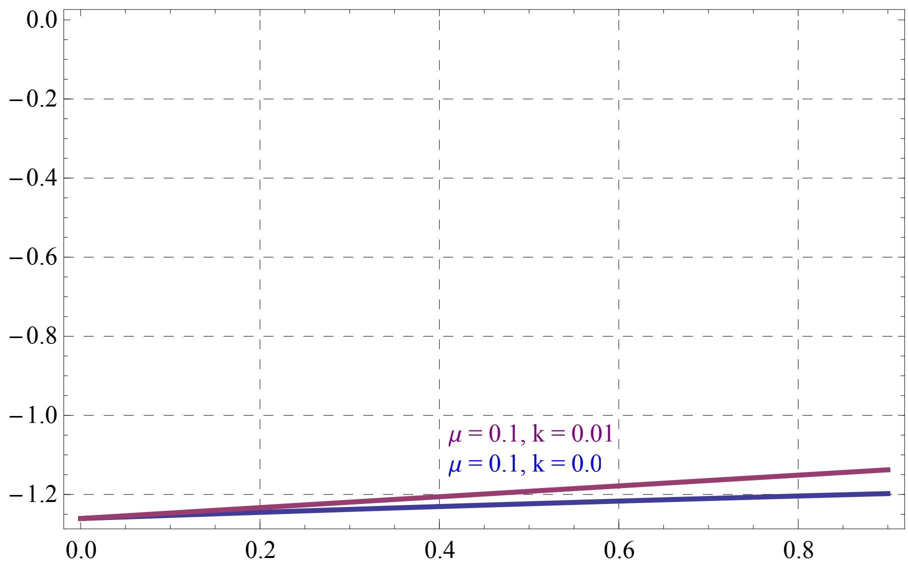

Fig. 4: α versus L_1; $\sigma = 10^{-3}$

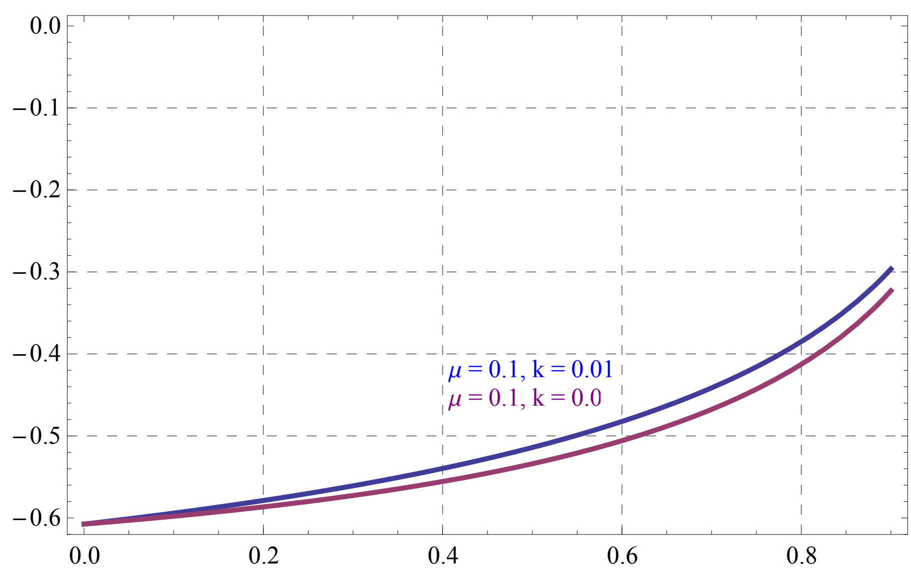

Fig. 5: α versus L_2; $\sigma = 10^{-3}$

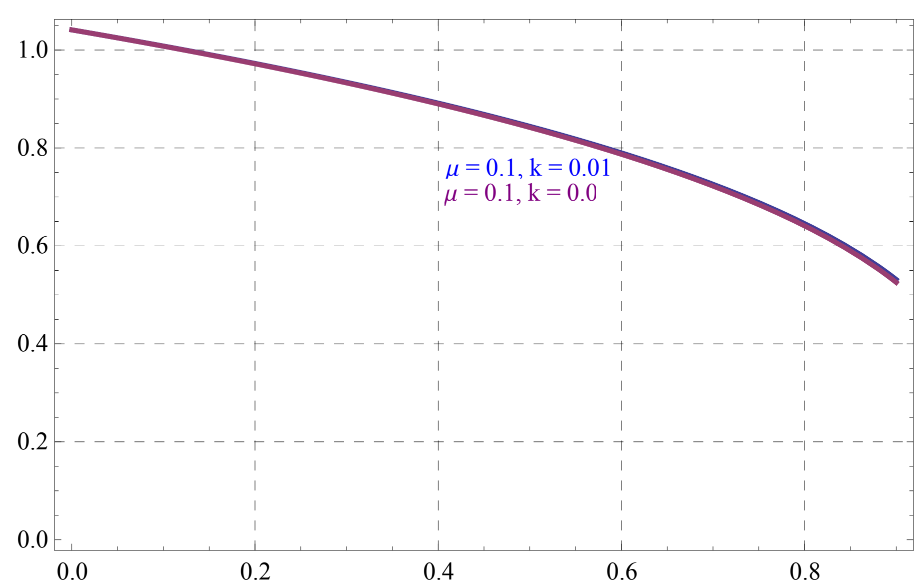

Fig. 6: α versus L_3; $\sigma = 10^{-3}$

derivative is to be evaluated at the libration point (x_0, y_0). The characteristic equation corresponding to Eqn. (16) is

$$\lambda^4 + \left(4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0\right)\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{xy}^0)^2 = 0 \quad (17)$$

where

$$\Omega_{xx}^0 = \frac{3}{4}\left\{1 - \frac{2}{3}(1-3\mu)\alpha + \frac{2}{3}(2-3\mu)\beta + \left(\frac{13}{2} - \frac{23\mu}{4} - \frac{2\mu^2}{1-\mu}\right)\sigma_1\right\}$$

$$\frac{3}{4}\left\{\left(-6 + \frac{39}{4}\mu + \frac{2\mu^2}{1-\mu}\right)\sigma_2\right\}$$

$$\Omega_{xy}^0 = \frac{3\sqrt{3}}{2}\left\{\frac{1}{9}(1+\mu)\alpha + \left(-\frac{25}{12} + \frac{85}{24}\mu - \frac{\mu}{6(1-\mu)} - \frac{\mu^2}{6(1-\mu)}\right)\sigma_1\right\}$$

$$+\frac{3\sqrt{3}}{2}\left\{\left(\mu - \frac{1}{2}\right) - \frac{1}{9}(2-\mu)\beta + \left(\frac{3}{2} - \frac{11}{8}\mu + \frac{\mu}{6(1-\mu)} + \frac{\mu^2}{6(1-\mu)}\right)\sigma_2\right\}$$

$$\Omega_{yy}^0 = \frac{9}{4} + \frac{1}{2}(1-3\mu)\alpha + \left(\frac{33}{8} + \frac{135\mu}{16} - \frac{33\mu}{8(1-\mu)} + \frac{45\mu^2}{8(1-\mu)}\right)\sigma_1$$

$$-\frac{1}{2}(2-3\mu)\beta + \left(\frac{135\mu}{16} + \frac{33\mu}{8(1-\mu)} - \frac{45\mu^2}{8(1-\mu)}\right)\sigma_2.$$

a) *Stability of Non-collinear Libration points*

Let $\lambda^2 = \Pi$, therefore Equation (17) becomess

$$\Pi^2 + q_1\Pi + q_2 = 0 \quad (18)$$

which is a quadratic equation in Π and its roots are given by

$$\Pi_{1,2} = \frac{1}{2}(-q_1 \pm \sqrt{D}) \quad (19)$$

where $q_1 = 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0; q_2 = \Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{xy}^0)^2;\ D = q_1^2 - 4q_2$.
The motion near the Libration point (x_0, y_0) is said to be bounded if $D \geq 0$ *i.e.*

$$1 - 27\mu + 27\mu^2 - 6(1-\mu)\{\mu + (1-\mu)k\}\alpha + \tfrac{3}{4}\left(8 - 237\mu + 267\mu^2\right)\sigma_1$$
$$+\tfrac{3}{4}(-4 + 37\mu - 111\mu^2)\sigma_2 \geq 0 \quad (20)$$

The Equation (20) is quadratic in μ, on solving it we have

$$\mu_{1,2} = \frac{-p_2 \pm \sqrt{p_2^2 - 4p_1p_3}}{2p_1} \quad (21)$$

where

$$p_1 = 108 + 24(1-k)\alpha + 468\sigma,$$

$$p_2 = -108 - 24(1-2k)\alpha - 492\sigma,$$

$$p_3 = 1 - 6k\alpha + 3\sigma.$$

From the Fig. 7, $\mu_1 > 1/2$ and $\mu_2 < 1/2$ for all values of α. Thus the critical value of mass parameter μ_c for which the non-collinear libration points $L_{4,5}$ are stable is

$$\mu_c = \frac{-p_2 - \sqrt{p_2^2 - 4p_1p_3}}{2p_1} < 1/2 \tag{22}$$

If $\alpha = 0, \sigma = 0$, $\mu_c = 0.038520896504551...$ which is the critical value of mass parameter for classical case. Also, if $\alpha = 0, \sigma = 0$ then $\mu = \mu_0$ is the solution of the Equation (21) where $\mu_0 = 0.0385208965...$ (Szebehely, 1967). When $\alpha \neq 0, \sigma \neq 0$, we suppose that $\mu_c = \mu_0 + \xi_1\alpha + \xi_2\sigma_1 + \xi_3\sigma_2$ as the root of the equation (20), where ξ_1, ξ_2, ξ_3 are to be determined in a manner such that $D = 0$. Therefore we have

$$\xi_1 = -\frac{2\left(k+\mu_0-2k\mu_0-\mu_0^2+k\mu_0^2\right)}{9(1-2\mu_0)},$$

$$\xi_2 = \frac{(8-237\mu_0+267\mu_0^2)}{36(1-2\mu_0)},$$

$$\xi_3 = \frac{(-4+73\mu_0-111\mu_0^2)}{36(1-2\mu_0)}.$$

Therefore

$$\mu_c = 0.0385208965... - (0.00891747 + 0.222579k)\alpha - 0.0627796\sigma \tag{23}$$

Fig. 7: α versus $\mu_i (i = 1, 2)$; $k = 0.01, \sigma = 10^{-3}$

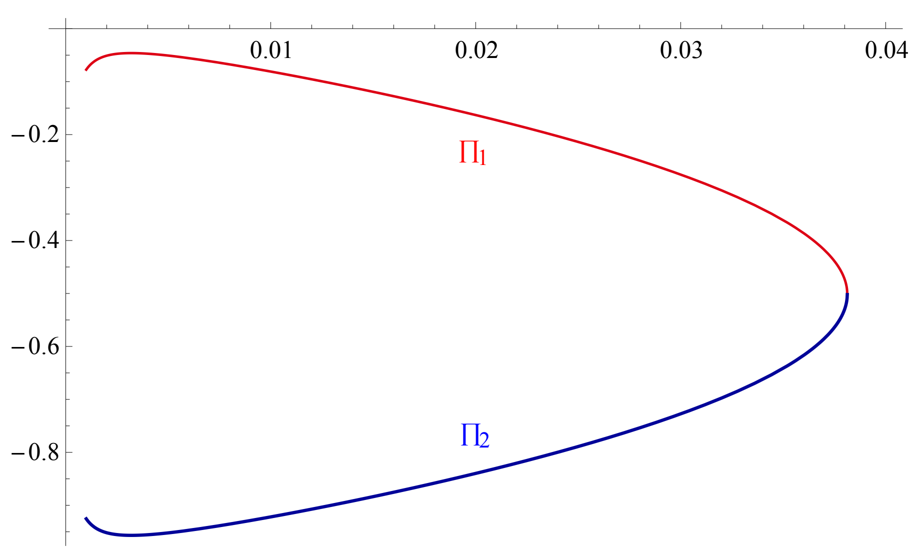

Fig. 8: μ versus Π; $\alpha = 0.01$, $k = 0.01$, $\sigma_1 = 10^{-3}$, $\sigma_2 = 10^{-4}$

Notes

Thus, the non-collinear libration points $L_{4,5}$ are stable for the critical mass parameter $0 < \mu \leq \mu_c$, where μ_c is defined in Equation (23).

As shown in the Fig. 8, $\Pi_{1,2} < 0$ for $\mu \leq \mu_c$. Thus the eigen-values of characteristic Equation (17) are given by $\lambda_{1,2} = \pm i\sqrt{\Pi_1}$, $\lambda_{3,4} = \pm i\sqrt{\Pi_2}$, therefore, the non-collinear libration points $L_{4,5}$ are periodic and bounded and hence stable for the critical mass parameter $\mu \leq \mu_c$, where μ_c is defined in Equation (23).

Notes

b) Stability of Collinear Libration points

First we consider the point lying in the interval $(\mu - 2, \mu - 1)$. For this point, $r_2 < 1$, $r_1 > 1$ and

$$\Omega^0_{xx} = n^2 + \frac{2(1-\mu)(1-\alpha)}{r_1^3} + \frac{2\mu(1-\beta)}{r_2^3} + \frac{6\mu\sigma}{r_2^5} > 0, \Omega^0_{xx} = 0$$

$$\Omega^0_{yy} = \mu\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\left(r_2 - \frac{1}{r_2^2}\right) + \frac{\mu}{r_2}\left(\frac{1}{r_2} - \frac{1}{r_1}\right)\beta + \frac{3\mu\sigma}{r_1} + \frac{3\mu\sigma}{2r_1r_2^4} - \frac{3\mu\sigma}{2r_2^5} < 0$$

Similarly, for the points lying in the interval $(\mu-1, 0)$ and $(\mu, \mu+1)$, $\Omega^0_{xx} > 0$, $\Omega^0_{xy} = 0$, $\Omega^0_{yy} < 0$ Since the discriminant of Equation (19) is positive and the four roots of the characteristic equation (17) can be written as $\lambda_{1,2} = \pm s$ and $\lambda_{3,4} = \pm t$ (s and t are real). Hence the motion around the collinear libration points is unbounded and consequently the collinear libration points are unstable.

c) Application to Real System

Let us consider an example of the Sun-Earth system in the restricted three-body problem in which the smaller primary m_2 (ellipsoid) is taken as the Earth and the bigger one m_1 as Sun. From the astrophysical data we have: Mass of the Sun (m_1) $= 1.9891 \times 10^{30} kg$; Mass of the Earth (m_2) $= 5.9742 \times 10^{24} kg$; Axes of the Earth: $a = 6378.140 km$, $c = 6356.755 km$; Mean distance of Earth from the Sun $= 1AU = 1.5 \times 10^{11} m$; Luminosity of Sun $= 3.9 \times 10^{26} W$; Flux received on Earth by the Sun $= 1379 W/m^2$; Albedo of Earth $= 0.3$ *i.e.* 30 percentage of energy reflected back to space by the Earth, therefore the luminosity of Earth $= 5.2 \times 10^{16} W$.

In dimensionless system

$\mu = 0.00000300346$, $a = 0.0000426352$, $c = 0.0000424923$, $k = 1.3 \times 10^{-10}$. Therefore $\beta = 0.0000443931\alpha$, $\sigma_1 = 2.43294 \times 10^{-12}$, $\sigma_2 = 1.2793 \times 10^{-12}$, $n = 1.0000000000018248$. From the Equations (9), (10), (13), (14) and (15), the libration points obtained in Sun-Earth system are the given in Table 3.

Table 4: Libration Points in Sun-Earth System

α	L_1	L_2	L_3	$L_{4,5}(x, \pm y)$
0.0	−1.01003	−0.990027	1.000001	(−0.499997, ±0.866025)
0.1	−1.00512	−0.964684	0.965491	(−0.466665, ±0.846781)
0.2	−1.00378	−0.928121	0.928319	(−0.433333, ±0.827534)
0.3	−1.00312	−0.887822	0.887905	(−0.400001, ±0.808288)
0.4	−1.00272	−0.843389	0.843434	(−0.366671, ±0.789042)
0.5	−1.00244	−0.793674	0.793699	(−0.333338, ±0.769796)
0.6	−1.00223	−0.736789	0.736808	(−0.300006, ±0.750551)
0.7	−1.00206	−0.669421	0.669432	(−0.266674, ±0.731304)
0.8	−1.00193	−0.584795	0.584805	(−0.233342, ±0.712058)
0.9	−1.00182	−0.464153	0.464161	(−0.200011, ±0.692813)

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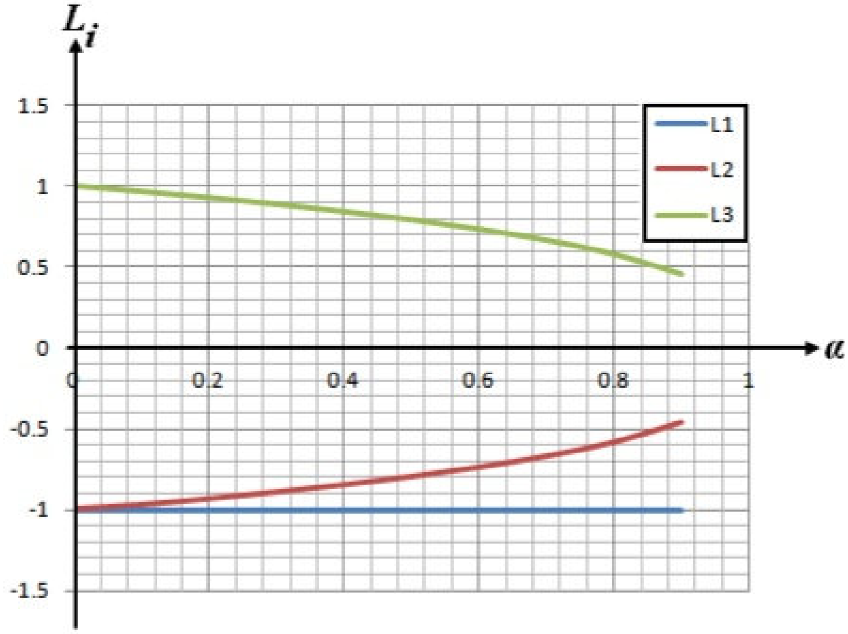

Fig. 9: α versus Collinear Libration Points $L_i(i = 1, 2, 3.)$ in Sun-Earth system

Since all the libration points in Sun-Earth system are the functions of α, so as α increases in the interval $0 \leq \alpha < 1$, the first collinear libration point L_1 slightly displaced while second and third libration points L_2 and L_3 move toward center of mass (Fig. 9). The abscissa and ordinate of non-collinear libration points also decreases as α increases and hence the shape of the scalene triangle formed by $L_{4,5}$ reduces (Fig. 10).

Since $\Pi_1 < 0$ in the interval $0 \leq \alpha \leq 0.00785$ and $\Pi_2 < 0$ in $0 \leq \alpha < 1$, the roots of the characteristic equation (17) are pure imaginary in the interval

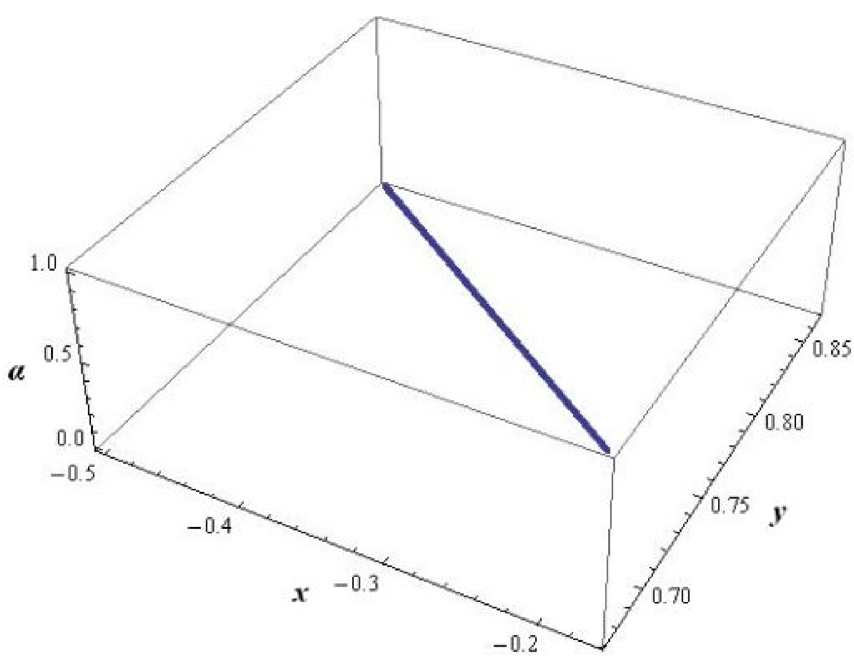

Fig. 10: α versus Non-collinear Libration Points $L_j(i = 1, 2.)$ in Sun-Earth system

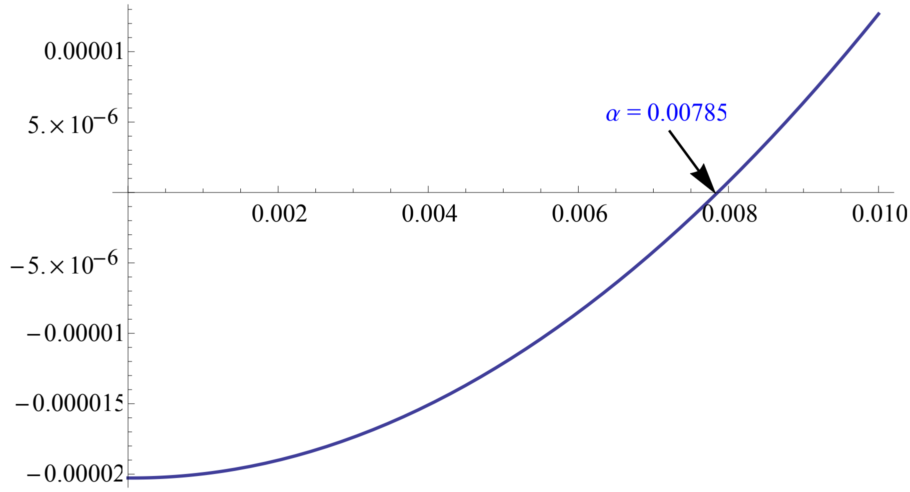

Fig. 11: α versus Π_1

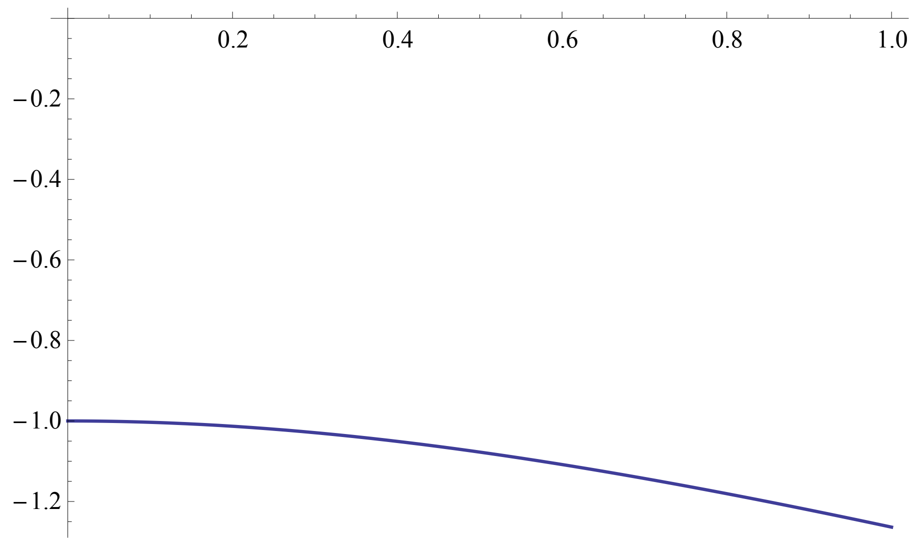

Fig. 12: α versus Π_2

$0 \leq \alpha \leq 0.00785$. Thus the non-collinear libration points $L_{4,5}$ in Sun-Earth system are stable if $0 \leq \alpha \leq 0.00785$.
For the collinear libration points $L_i (i = 1, 2, 3)$ in Sun-Earth system, $\Omega^0_{xx} > 0$, $\Omega^0_{xy} = 0$, $\Omega^0_{yy} < 0$, in $0 \leq \alpha < 1$. Since the discriminant of Equation (19) is positive and the four roots of the characteristic equation (17) an be written as $\lambda_{1,2} = \pm s$ and $\lambda_{3,4} = \pm t$ (s and t are real). Hence the motion around the collinear libration points is unbounded and consequently the collinear libration points are unstable.

V. Conclusion

In the present paper, the existence and stability of libration points in circular restricted three-body problem has been studied under Albedo effect when smaller primary is an oblate spheroid. The equations of motion in case of Albedo effect are derived, Eqn. (2). For For $\beta = 0$, the problem reduces to photogravitational restricted three-body problem when smaller primary is an oblate spheroid. It is found that there exist five libration points, three collinear and two non-collinear. The first collinear libration point L_1 lie at the right of the primary m_2, the second libration point L_2 lies between the center of

Notes

mass of the primaries O and m_1 and the third libration point L_3 lies at the right of the primary m_1. The libration points L_i (i = 1, 2, 3) are affected by the Albedo effect and this effect displaced the libration points from its actual position as shown in Figs. 4, 5 and 6. Also, there exist two non-collinear libration points $L_{4,5}$ and these points are affected by triaxiality as well as Albedo effect Figs. 3 and 4, these points form scalene triangle with the primaries as $r_1 \neq r_2$. The numerical location of $L_{4,5}$ is also calculated in Table 1 for $\mu = 0.1$ and different values of α and k and it is found that the abscissa and ordinate of non-collinear libration points are the decreasing functions of α and k *i.e.* as α and k increases, x and y decreases. For $\alpha = 0$, the results are in conformity with those of Bhatnagar and Hallan (1979). If $\alpha = 0$ and $\sigma = 0$, the classical case of the restricted three body problem is verified (Szebehely, 1967). The non-collinear libration points are stable for a critical value of mass parameter $\mu \leq \mu_c$ where $\mu_c = 0.0385208965 \ldots$ - (0.00891747 + 0.222579k) α - 0.0627796 σ but collinear libration points are still unstable. Also, an example of Sun-Earth system is taken in Section-5 as a real application and this is found that all the libration points in Sun-Earth system are the functions of α, so as α increases in the interval $0 \leq \alpha < 1$, the first collinear libration point L_1 slightly displaced while second and third libration points L_2 and L_3 move toward center of mass (Fig. 9). The abscissa and ordinate of non-collinear libration points also decreases as α increases and hence the shape of the scalene triangle formed by $L_{4,5}$ reduces (Fig. 10). The non-collinear libration points $L_{4,5}$ in Sun-Earth system are stable for $0 \leq \alpha < 0.00785$ but collinear libration points are unstable.

Notes

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