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# Mathematical Modeling of a Predator-Prey Model with Modified Leslie-Gower and Holling-type II Schemes

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MATHEMATICALMODELINGOFAPREDATORPREYMODELWITHMODIFIEDLESLIEGOWERANDHOLLINGTYPEIISCHEMES

Ref

1. Camara, B. I. and Aziz-Alaoui, M. A., Complexity in a prey predator Model, International Conference in Honor of Claude Lobry, (2007).

# Mathematical Modeling of a Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes

Ahmed Buseri Ashine α & Dawit Melese Gebru (Ph.D.) σ

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## Chapter One

## I. Introduction

### *a) Background*

A predator is an organism that eats another organism. The prey is the organism which the predator eats. Some examples of predator and prey are lion and zebra, fish and shark, shark and fish, fox and rabbit, and bat and moth. The words "predator" and "prey" are almost always used to mean only animals that eat animals, but the same concept also applies to plants: Bear and berry, rabbit and lettuce, grasshopper and leaf.

The dynamic relationships between species and their complex properties are at the heart of many ecological and biological processes [1]. One of such relationships is the dynamical relationship between a predator and their prey which has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [2].

Nature can provide some degree of protection to a given number of prey populations by providing refuges. Such refugia can help in prolonging prey- predator interactions by reducing the chance of extinction due to predation [3, 4] and damp prey predator oscillations [5, 6]. The effects of prey refuges on the population dynamics are very complex in nature, but for modeling purposes, it can be considered as constituted by two components: the first effects, which affect positively the growth of prey and negatively that of predators, comprise the reduction of prey mortality due to decrease in predation success. The second one may be the trade-offs and by-products of the

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hiding behavior of prey which could be advantageous or detrimental for all the interacting populations [7, 8].

The goal of this paper is to introduce and to give a first study of a two-dimensional system of autonomous differential equations modeling a predator-prey system. This model incorporates a modified version of the Leslie-Gower functional response as well as that of the Holling-type II functional response.

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [9-11]. These problems may appear to be simple mathematically at first sight, they are, in fact, often very challenging and complicated. Although the predator–prey theory has seen much progress in the last 40 years, many long standing mathematical and ecological problems remain open [12–17].

Species interaction in natural and wildlife environments is unavoidable. Different species occupying similar ecosystems and living amongst each other will regularly interact. These leads to the question, how does this primal concern on survival impact the way in which species interact? Mathematical population models have been used to study the dynamics of prey predator systems since Lotka (1925) and Volterra (1927) proposed the simple model of prey predator interactions now called the Lotka-Volterra model. Since then, many mathematical models, some reviewed in this study, have been constructed based on more realistic explicit and implicit biological assumptions.

Some of the aspects that need to be critically considered in a realistic and plausible mathematical model include; carrying capacity which is the maximum number of prey that the ecosystem can sustain in absence of predator, competition among prey and functional responses of the species.

The main aim of this paper is to study the effect of prey refuge on the stability property of the coexistence equilibrium point a prey predator system with Holling type II functional response and the modified Leslie-Gower type dynamics incorporating constant proportion of prey refuge.

### b) *Basic Definitions*

#### i. *Stability Analysis*

Stability refers to system's ability to resist small perturbations. Stability analysis is an acceptable tool for studying long time behavior of dynamical systems. Stability concept has been studied extensively for almost one hundred years [6, 14, 16]. There are many kinds of stability concepts which are frequently associated with a dynamical system: local stability, global stability and Lyapunov stability of equilibrium points. The local stability of non-linear system gives the behavior of the system in a small neighborhood of an equilibrium point. As such it does not predict the overall behavior of the system. However, global stability captures the behavior of the non-linear system far from an equilibrium point.

#### ii. *Local asymptotical Stability*

Consider the following system of differential equations for n-interacting species:

$$\frac{dU}{dT} = f(U,s) \tag{1.1}$$

where $U = (U_1, U_2, \ldots, U_n)^t$ is a state variable and s is the control parameter. The equilibrium point, say $U_e$, of the system (1.1) is the point of satisfying $f(U_e, s) = 0$.

Ref

7. M. E. Hochberg and R. D. Holt, "Refuge evolution and the population dynamics of coupled Host parasitoid associations," Evolutionary Ecology, vol. 9.

Notes

Linear stability is used to study the stability of the steady state $U_e$ in a small neighborhood. The system (1.1) can be linearized at $U_e$ as

$$\frac{dU}{dT} = JU \tag{1.2}$$

where the the matrix **J** is the Jacobian matrix of the system (1.2) with elements defined as $a_{ij} = \left(\frac{\partial f_i}{\partial U_j}\right)|_{U_e}$. The polynomial characteristic equation corresponding to the system (1.2) is obtained as

$$\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \ldots + b_n = 0 \tag{1.3}$$

The stability of the steady state $U_e$ depends on the sign of the real parts of the eigenvalues˘ $\lambda_i$ of **J**. If one or more of the real parts of the eigenvalues $\lambda_i$ are positive, the state is unstable. The stationary states can be classified further according to the imaginary parts and signs of the corresponding eigenvalues. For example, in a two-dimensional phase space there are five types of fixed points: stable and unstable foci (imaginary eigenvalues with non-zero real part), stable and unstable nodes (real eigenvalues) and saddle-nodes (real eigenvalues with different sign).

iii. *Routh-Hurwitz Stability Criterion*

The Routh-Hurwitz Stability Criterion provides the necessary and sufficient conditions under which all the roots of the polynomial equation (1.3) lie in the left half of the complex plane:

$$b_1 > 0,\ \begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix} > 0,\ \begin{vmatrix} b_1 & b_3 & b_5 \\ 1 & b_2 & b_4 \\ 0 & b_1 & b_3 \end{vmatrix} > 0, \ldots, \begin{vmatrix} b_1 & b_3 & b_5 & \ldots & 0 \\ 1 & b_2 & b_4 & \ldots & 0 \\ 0 & b_1 & b_3 & \ldots & 0 \\ & & & \ldots & \\ 0 & 0 & 0 & \ldots & b_n \end{vmatrix} > 0$$

In particular, the Routh-Hurwitz criterion for n = 2 is:

n=2; $\quad b_1 = -trace\, J > 0,\ b_2 = \det J > 0$

iv. *Globally Asymptotical Stability*

If a limit set contains more than one equilibrium, then it must also contain orbits joining these equilibria. In essence, we can say that a bounded solution tends either to an equilibrium or to limit cycle, overlooking such "unlikely coincidences" as the possibility of a running from a saddle point to itself. Thus, if we can show that, for a given system, all solutions are bounded but there are no asymptotically stable equilibrium points, we can deduce that there must be at least one periodic orbit.

If there is only one periodic orbit, then it must be globally asymptotically stable in the sense that every orbit tends to it. If there is more than one periodic orbit, each must be asymptotically stable from at least one side: orbits may spiral toward it from the inside, from the outside, or both.

v. *Hopf Bifurcation*

Hopf bifurcation is defined as the appearance or disappearance of a periodic orbit through a local change in the stability properties of a steady point. It is named in the memory of the mathematician Eberhard Hopf. In a dynamical system, Hopf bifurcation occurs when the system loses its stability in the form of complex conjugate eigenvalues of the linearization around the fixed point which crosses the imaginary axis of the complex plane.

Notes

### c) *Statement of the problem*

Many of the most interesting dynamics in the biological world have to do with interactions between species. In ecology, predation explains a biological interaction where a predator forages on its prey. The predator-prey relationship is substantial in maintaining the equilibrium between various animal species. The focus of the thesis is that in the realm of biological mathematics, it is possible to mathematically represent the population variations of a predation relationship to a certain extent of accuracy. This will be done using the modified Leslie-Gower dynamics and Holling type II functional response incorporating prey refuge.

### d) *Objectives*

i. *General Objectives*

The general objective of this study is to establish and study a predator prey model with modified Leslie Gower type dynamics and Holling type II functional response incorporating prey refuge.

ii. *Specific objectives*

The specific objectives of this study are:-

- To formulate a mathematical model of modified Leslie Gower and Holling type II with prey refuge.
- To investigate the effect of prey refuge on the predator-prey population in Leslie-Gower and Holling type II model.
- To establish several sufficient conditions on the stability of a positive equilibrium.

### e) *Significance of the study*

It is hoped that if the effect of parameters such as intrinsic growth rate, carrying capacity of the environment, prey refuge, etc, on the long term stable co-existence of the 2 species is known, this will enable the environment authorities to manage the population of the prey, the predator in the that area, especially taking well established measures to avoid extinction of any of the species.

### f) *Methodology*

The model in the thesis is nonlinear 1$^{st}$ order ordinary differential equation. First of all, we will convert our model into non-dimensional form. This will reduce the number of parameters. Linear stability analysis and bifurcation analysis will be carried out to see the dynamical behavior of the system under consideration. Finally, numerical simulations will be done by using MATLAB.

### g) *Structure and Presentation*

This paper is presented in four chapters. Chapter 1 gives the background to the study, statement of the problem, objectives of the study and significance of the study. Chapter 2 presents the literature review, focusing on Leslie Gower and Holing type II prey-predator systems and mathematical models. Chapter 3 deals with formulating and qualitative analysis of the model while, Chapter 4 includes discussion of results and conclusion.

## Chapter Two

## II. Review Literature

### a) Historical Background

One of the important interactions among species is the predator-prey relationship and it has been extensively studied because of its universal existence. There are many factors affecting the dynamics of predator-prey models. One of the familiar factors is the functional response, referring to the change in the density of prey attached per unit time per predator as the prey density changes. In the classical Lotka-Volterra model, the functional response is linear, which is valid first-order approximations of more general interaction.

In the literatures studies show that refuges have both stabilizing [15] and destabilizing effect [18, 19]. The traditional ways in which the effect of refuge used by the preys has been incorporated in predator- prey models is to consider either a constant number or a constant proportion of the prey population being protected from predation [15, 19, 22]. Hassel [15] notes that in reality refugia fall between these two extremes. It is pointed out that those protecting a proportion of the prey population appearing to be more common [6]. However, as pointed out by the authors [17, 21], the refuges, which protect a constant number of preys, have a stronger stabilizing effect on population dynamics than the refuges, which protect a constant proportion of prey. For more biological backgrounds and results on the effects of a prey refuge, one could refer to [4, 5, 6, 17] and the references therein.

Mathematical modeling and analysis of multiple species ecological problems was first done by Volterra (1927). Volterra had been introduced to an ecological problem that in the years after the first World War, the proportion of the predatory fishes caught in the Upper Adriatic Sea was found to be considerably higher than in the years before the war, whereas the proportion of prey fishes was down. In order to come out with an explanation to this ecological problem, Volterra formulated and analyzed a system of ordinary differential equations which is represented as below:

$$x^{\cdot} = x(a - by)$$

$$y^{\cdot} = y(-c + dx),$$

where *x* and *y* were the densities of the prey and predator respectively. This system of differential equations was also studied independently by Lotka (1925) in the context of chemical kinetics and is now known as the Lotka-Volterra model. Volterra's study showed that the steady state for the co-existence of the prey and predatory was periodic and that a pause of fishery would indeed lead to an increase of the predators and a decrease in the prey.

Recently, many mathematical models incorporating diverse areas of interest such as Holling Type functional responses, ratio-dependent functional responses, bio-economic exploitation or harvesting, delayed harvesting and age-structured models have been formulated and analyzed.

### b) Predator-prey Models

It has been observed that the inclusion of the logistic growth and Holling Type II functional response in the model, guarantees permanence. This showed the importance of incorporating logistic growth in prey-predator models.

In the Holling type II functional response, the rate of prey consumption by a predator rises as prey density increases, but eventually levels off at a plateau (or

Ref

18. McNair, J. N., The effects of refuges on predator-prey interactions: A reconsideration, Theor. Popul. Biol., 29(1986), 38-63.

Notes

asymptote) at which the rate of consumption remains constant regardless of increases in prey density. Holling's disk equation, named after experiments he performed in which a blindfolded assistant picked up sandpaper disks from a table, describes the type II functional response.

Holling (1959) studied predation of small mammals on pine sawflies, and he found that predation rates increased with increasing prey population density. This resulted from 2 effects: (1) each predator increased its consumption rate when exposed to a higher prey density, and (2) predator density increased with increasing prey density. Holling considered these effects as 2 kinds of responses of predator population to prey density: (1) the functional response and (2) the numerical response.

In the late 1980s, a credible, simple alternative to the Lotka-Volterra predator-prey model (and its common prey dependent generalizations) emerged, the ratio dependent or Arditi-Ginzburg model. The two are the extremes of the spectrum of predator interference models.

## Chapter Three

## II. Model Formulation and Analysis

In this chapter, we present model description, formulation and analysis. Let $X(t)$, and $Y(t)$ represent the population of the prey and predator species at any time $t$. The main feature of the model is that the interaction of species affects both populations. Terms representing logistic growth of the prey species in the absence of the predator are included in the prey equations. The Leslie-Gower formulation of the model is based on the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food. The model has two non-linear autonomous ordinary differential equations describing how the population densities of the two species would vary with time.

### *a) Assumptions*

The following assumptions are made in order to construct the model:

- The species live in an ecosystem where external factors such as droughts, fires, epidemics are stable or have a similar effect on the interacting species.
- The predator is completely dependent on the prey as the only favorite food source
- The prey species have an unlimited food supply.
- There is logistic growth of the prey in absence of the predator or human poaching of the prey. That is the population of the prey would increase (or decrease) exponentially until it reaches the maximum density of the living area, which is its carrying capacity.
- There is no threat to the prey besides the predator species being studied.
- Some of the preys were reserved that they are free from predator.

### *b) Model Variables and Parameters*

The following variables and parameters are used in the model:

i) X(t) - the population of the prey at time $t$.
ii) Y(t) - the population of the predator at time $t$.
iii) $r$ is per capita intrinsic growth rates for prey.
iv) $s$ is gives the maximal per-capita growth rate of predator.
v) $K$ is the carrying capacity of the environment.
vi) $k_1$ measures the extent of which environment provides protection to prey.

vii) $k_2$ measures the extent of which environment provides protection to predator.

viii) $c_1$ is the maximum value which per capita reduction rate of prey.

ix) $c_2$ is the crowding effect for the predator.

*c) The Mathematical Model*

Notes

Besides the model description, assumptions and definition of variables and parameters in section 3.1, it is assumed that a constant proportion $m \in [0,1)$ of the prey can take refuge to avoid predation, this leaves $(1-m)X$ of the prey available for predation.

Thus, the model under the above assumptions with Holling type II functional response and the modified Leslie-Gower type predator dynamics is given by:-

$$\begin{cases} \dfrac{dX}{dT} = r\left(1-\dfrac{X}{K}\right)X - \dfrac{c_1(1-m)XY}{k_1+(1-m)X} \\ \dfrac{dY}{dT} = Y\left(s - \dfrac{c_2 Y}{k_2+(1-m)X}\right) \end{cases} \tag{3.1}$$

where all the parameters in the model assumes positive values and with initial value $X(0) \geq 0$ and $Y(0) \geq 0$. The following non-dimensional state variables and parameters are chosen.

$$x = \frac{X}{K} \qquad y = \frac{Y}{K} \qquad t = rT$$

$$\alpha = \frac{c_1}{r} \qquad \beta = \frac{k_1}{K} \qquad \gamma = \frac{s}{r} \qquad \sigma = \frac{c_2}{r} \qquad \omega = \frac{k_2}{K}$$

The system (3.1) takes the following non-dimensional form

$$\begin{cases} \dfrac{dx}{dt} = (1-x)x - \left(\dfrac{\alpha(1-m)xy}{\beta+(1-m)x}\right) \equiv F(x,y) \\ \dfrac{dy}{dt} = y\left(\gamma - \dfrac{\sigma y}{\omega+(1-m)x}\right) \equiv G(x,y) \end{cases} \tag{3.2}$$

$x(0) = x_0 \geq 0$; $y(0) = y_0 \geq 0$.

*Theorem 3.1:* All the solutions $(x(t), y(t))$ of the system (3.2) are nonnegative. i.e $x(t) \geq 0$ $y(t) \geq 0$ for all $t \geq 0$.

*Proof:* Considering the biological significance, we investigate the dynamical system (3.2). From first equation of (3.2) it follows that x=0 is an invariant subset. i.e $x \equiv 0$ if and only if x =0 for some time t. This implies that $x(t) > 0 \quad \forall t$ if $x(0) > 0$. The same argument follows the second equation the system (3.2). i.e any trajectory in $\Re_+^2$, cannot cross the coordinate planes.

i. *Boundedness of the solution*

*Theorem 3.2:* All the solution $(x(t), y(t))$ of the system (3.2) are bounded.

*Proof:* The first equation of (3.2) gives us

$$\frac{dx}{dt} = x\left(1 - x - \frac{\alpha(1-m)y}{\beta + (1-m)x}\right)$$
$$< \quad x(1-x)$$

Therefore, $\limsup_{t\to\infty} x(t) < 1$. Hence, $x(t)$ is always bounded.

Similarly,

$$\frac{dy}{dt} = y\left(\gamma - \frac{\sigma y}{\omega + (1-m)x}\right)$$
$$\leq y\left(\gamma - \frac{\sigma y}{\omega + 1 - m}\right)$$
$$= \gamma y\left(1 - \frac{y}{\omega + 1 - m}\frac{\sigma}{\gamma}\right)$$
$$= \gamma y\left(1 - \frac{y}{\omega + 1 - m}\lambda\right)$$
$$= y\left(1 - \frac{y}{(\omega + 1 - m)/\lambda}\right)$$

Therefore, we have $y(t) \leq \max\left\{\frac{\omega + 1 - m}{\lambda}, y(0)\right\} \equiv M_2$, $\lambda = \frac{\sigma}{\gamma}$.

Hence, the solutions $(x(t), y(t))$ of the system (3.2) with the given initial conditions are bounded.

ii. *Equilibrium points and their stability*

One can see that the system (3.2) has three boundary equilibrium points, $E_1(0,0)$, $E_2\left(0, \frac{\omega\gamma}{\sigma}\right)$ and $E_3(1,0)$. Besides these three boundary equilibrium points the system (3.2) has one positive equilibrium points, say $E(x^*, y^*)$. $E(x^*, y^*)$ is obtained by solving the following simultaneous equation

$$\frac{\alpha(1-m)y^*}{\beta + (1-m)x^*} = 1 - x^*$$
$$y^* = \frac{\gamma(\omega + (1-m)x^*)}{\sigma} \tag{3.3}$$

One can easily see that $x^*$ satisfies the quadratic equation

$$Ax^{*2} + Bx^* + C = 0 \tag{3.4}$$

$$A = (1-m)\sigma, B = \alpha\gamma m^2 + (\sigma - 2\alpha\gamma)m + \alpha\gamma + \sigma\beta - \sigma,\ C = \alpha\gamma\omega - \alpha\gamma\omega m - \sigma\beta.$$

*Proposition 3.1:* The system (3.2) has a unique interior equilibrium $E^*\left(x^*, y^*\right)$ (i.e $x^* > 0$ and $y^* > 0$) provided

Notes

$$m > 1 - \frac{\sigma\beta}{\alpha\gamma\omega}$$

*Proof:* From the second equation of (3.3), we can see that $y^*$ is positive whenever $x^*$ is. $x^*$ is the unique positive root of the quadratic equation (3.4). Equation (3.4) will have a unique positive root when $C < 0$. Hence, $C < 0$ whenever $m > 1 - \dfrac{\sigma\beta}{\alpha\gamma\omega}$. Therefore, $E^*\left(x^*, y^*\right)$ exists uniquely whenever $m > 1 - \dfrac{\sigma\beta}{\alpha\gamma\omega}$.

Notes

The unique equilibrium density $x^*$ can be explicitly given as

$$x^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

The Jacobean matrix of the system (3.2) at the equilibrium points $E_i$ is given as

$$J(E_i) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F}{\partial x}(x_i, y_i) & \dfrac{\partial F}{\partial y}(x_i, y_i) \\ \dfrac{\partial G}{\partial x}(x_i, y_i) & \dfrac{\partial G}{\partial y}(x_i, y_i) \end{bmatrix};$$

where

$$\frac{\partial F}{\partial x}(x_i, y_i) = 1 - 2x - \frac{\alpha\beta(1-m)y}{(\beta + (1-m)x)^2}$$

$$\frac{\partial F}{\partial y}(x_i, y_i) = -\frac{\alpha(1-m)x}{\beta + (1-m)x}$$

$$\frac{\partial G}{\partial x}(x_i, y_i) = \frac{\sigma(1-m)y^2}{(\omega + (1-m)x)^2}$$

$$\frac{\partial G}{\partial y}(x_i, y_i) = \gamma - \frac{2\sigma y}{\omega + (1-m)x}$$

*Proposition 3.2:* The equilibrium point $\mathrm{E}_1(0,0)$ is unstable node.

*Proof:* At $\mathrm{E}_1(0,0)$, the Jacobean matrix becomes

$$J(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$$

The eigen values of this matrix are $\lambda_1 = 1$ and $\lambda_2 = \gamma$, which are both positive. Hence $\mathrm{E}_1(0,0)$ is an unstable node.

*Proposition 3.3:* The equilibrium point $E_2\left(0, \dfrac{\omega\gamma}{\sigma}\right)$ is locally asymptotically stable for

$$m < 1 - \frac{\sigma\beta}{\alpha\gamma\omega}.$$

*Proof:* At $E_2\left(0, \frac{\omega\gamma}{\sigma}\right)$, the Jacobean matrix is

$$J(E_2) = \begin{pmatrix} 1 - \frac{\alpha\gamma\omega(1-m)}{\sigma\beta} & 0 \\ \frac{\gamma^2}{\sigma} & -\gamma \end{pmatrix}$$

The eigenvalues of the matrix $J(E_2)$ are

$$\lambda_1 = 1 - \frac{\alpha\gamma\omega(1-m)}{\sigma\beta}, \qquad \lambda_2 = -\gamma < 0$$

For $E_2$ to be locally asymptotically stable, we should have $\lambda_1 < 0$, since $\lambda_2 < 0$. This implies that $m < 1 - \frac{\sigma\beta}{\alpha\gamma\omega}$.

*Proposition 3.4:* The equilibrium point $E_3(1,0)$ is unstable.
*Proof:* At $E_3(1,0)$, the Jacobean matrix becomes

$$J(E_3) = \begin{bmatrix} -1 & \frac{-\alpha(1-m)}{\beta + (1-m)} \\ 0 & \gamma \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -1 < 0$, $\lambda_2 = \gamma > 0$
Thus the equilibrium point $E_3(1,0)$ is a saddle point. i.e. Unstable.

*Proposition 3.5:* The coexistence equilibrium point $E^*(x^*, y^*)$ is locally asymptotically stable
provided

$$\gamma > \left(1 - 2x^* - \frac{\alpha\beta(1-m)y^*}{(\beta + (1-m)x^*)^2}\right). \tag{3.5}$$

*Proof:* At $E^*(x^*, y^*)$, the Jacobean matrix takes the form

$$J(E^*) = \begin{pmatrix} x^*\left(-1 + \frac{\alpha(1-m)^2 y^*}{(\beta + (1-m)x^*)^2}\right) & \frac{-\alpha(1-m)x^\circ}{\beta + (1-m)x^*} \\ \frac{(1-m)\gamma^2}{\sigma} & -\gamma \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$trace(J(E^*)) = a_{11} + a_{22} = x^*\left(-1 + \frac{\alpha(1-m)^2 y^*}{(\beta + (1-m)x^*)^2}\right) - \gamma$$

Thus, $trace(J(E^*) < 0$ if and only if $\gamma > x^*\left(-1 + \frac{\alpha(1-m)^2 y^*}{(\beta + (1-m)x^*)^2}\right)$.

Notes

$$\det(J(E^*)) = a_{11}a_{22} - a_{12}a_{21} = \gamma x^*\left(\frac{(\beta+(1-m)x^*)^2\sigma + \alpha\gamma(1-m)^2(\beta-\omega)}{(\beta+(1-m)x^*)^2\sigma}\right) > 0$$

Hence, the equilibrium point $E^*$ is locally asymptotically stable provided $\gamma > -x^* - \dfrac{\alpha(1-m)^2 x^* y^*}{(\beta+(1-m)x^*)^2}$.

Notes

iii. *Global Stability*

*Proposition 3.6.* The system (3.2) does not admit any periodic solution for $m > 1-\beta$.

*Proof:* Let $(x(t), y(t))$ be solutions of the system (3.2). Define Dulac function

$$H(x,y) = \frac{\beta+(1-m)x}{xy}$$

Then

$$Q = \frac{\partial(HF)}{\partial x} + \frac{\partial(HG)}{\partial y}$$

$$= -\left(\frac{(\beta-1+m)+2(1-m)x}{y} + \frac{(x(1-m)+\beta)\delta}{x((1-m)x+\omega)}\right)$$

It is observed that $Q<0$ for $m > 1-\beta$. Therefore, by Dulac criterion, the system (3.2) has no non-trivial periodic solutions.

*Theorem 3.3.* If $m > 1-\beta$ then the local asymptotical stability of the system (3.2) ensures its global asymptotical stability around the unique positive interior equilibrium point $E^*(x^*, y^*)$.

*Proof:* The unique equilibrium point $E^*(x^*, y^*)$ is the only stable point in the $xy$ plane. The boundedness of the solutions of the system together with the non-existence of periodic solutions establishes the global asymptotical stability.

d) *Mathematical Model without prey refuge*

If it is assumed that all the prey are accessible to the predator species, our mathematical model becomes,

$$\begin{cases} \dfrac{dX}{dT} = r\left(1-\dfrac{X}{K}\right)X - \dfrac{c_1 XY}{k_1+X} \\ \dfrac{dY}{dT} = Y\left(s - \dfrac{c_2 Y}{k_2+X}\right) \end{cases} \tag{3.6}$$

where all the parameters in the model are positive.

The following non-dimensional state variables and parameters are chosen.

$$x = \frac{X}{k} \qquad y = \frac{Y}{k} \qquad t = rT$$

$$\alpha = \frac{c_1}{r} \qquad \beta = \frac{k_1}{K} \qquad \gamma = \frac{s}{r} \qquad \sigma = \frac{c_2}{r} \qquad \omega = \frac{k_2}{K}$$

The system (3.6) takes the following non-dimensional form

$$\begin{cases} \dfrac{dx}{dt} = \left(1 - x - \dfrac{\alpha y}{\beta + x}\right)x \equiv F(x,y) \\ \dfrac{dy}{dt} = y\left(\gamma - \dfrac{\sigma y}{\omega + x}\right) \equiv G(x,y) \end{cases} \tag{3.7}$$

$$x(0) = x_0 \geq 0\,;\; y(0) = y_0 \geq 0$$

Notes

i. *Equilibrium Points*

We now study the existence of equilibria of system (3.7). All possible equilibria are

(i) The trivial equilibrium $E_0(0,0)$

(ii) Equilibrium in the absence of predator (y = 0) $E_1(1,0)$

(iii) Equilibrium in the absence of prey (x= 0) $E_2\left(0, \dfrac{\omega\gamma}{\sigma}\right)$

(iv) The interior (positive) equilibrium $E_3(x^*, y^*)$ where $x^*$ is the unique positive root of the quadratic equation

$$\sigma x^{*2} + (\alpha\gamma + \sigma\beta - \sigma)x^* + \alpha\gamma\omega - \sigma\beta = 0\,;$$

$$x^* = \frac{-B + \sqrt{B^2 - 4\sigma C}}{2\sigma}\;,\; y^* = \frac{\gamma(\omega + x^*)}{\sigma}$$

where $B = \alpha\gamma + \sigma\beta - \sigma$, $C = \alpha\gamma\omega - \sigma\beta$

ii. *Local stability of the equilibrium points*

The local asymptotic stability of each equilibrium point is studied by computing the Jacobean matrix and finding the eigenvalues evaluated at each equilibrium point. For stability of the equilibrium points, the real parts of the eigenvalues of the Jacobean matrix must be negative. From equations (3.7), the Jacobean matrix of the system is given by

$$J(E_i) = \begin{bmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{bmatrix}$$

which gives

$$J(E_i) = \begin{bmatrix} 1 - 2x - \dfrac{\alpha\beta y}{(\beta + x)^2} & -\dfrac{\alpha x}{\beta + x} \\ \dfrac{\sigma y^2}{(\omega + x)^2} & \gamma - \dfrac{2\sigma y}{\omega + x} \end{bmatrix}$$

The local asymptotic stability for each equilibrium point is analyzed as below:

i. $E_0(0,0)$ is unstable point.

The Jacobean matrix evaluated at $E_0$ gives,

$$J_0 = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$$

The eigen values of $J(E_0)$ are 1 and $\gamma$. $E_0(0, 0)$ is an unstable node, since 1 and $\gamma$ are always positive.

Notes

ii. $E_1(1,0)$ is unstable point.

The Jacobean matrix evaluated at $E_1$ gives;

$$J_1 = \begin{bmatrix} -1 & \dfrac{-\alpha}{\beta+1} \\ 0 & \gamma \end{bmatrix}$$

The eigenvalues of matrix $J(E_1)$ are $-1$ and $\gamma$. The eigenvalues above are one negative and the other positive. Hence, the equilibrium point $E_1(1,0)$ is unstable, saddle.

iii. $E_2\left(0, \dfrac{\omega\gamma}{\sigma}\right)$ is locally asymptotically stable for $\gamma > \dfrac{\sigma\beta}{\alpha\omega}$

The Jacobean matrix evaluated at $E_2$ gives,

$$J_2 = \begin{pmatrix} 1 - \dfrac{\alpha\gamma\omega}{\sigma\beta} & 0 \\ \dfrac{\gamma^2}{\sigma} & -\gamma \end{pmatrix}$$

The eigen values of the matrix $J(E_2)$ are, $1 - \dfrac{\alpha\gamma\omega}{\sigma\beta}$ and $-\gamma$. Both the eigenvalues are negative if

$$\gamma > \frac{\sigma\beta}{\alpha\omega}.$$

For a positive equilibrium $E_3$ $(x^*, y^*)$, $J(E_3)$ can be simplified to

$$J(E_3) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} = 1 - 2x^* - \frac{\alpha\gamma b x^*}{\beta + x^*} \qquad a_{12} = \frac{-\alpha x^*}{\beta + x^*}, \quad a_{21} = \frac{\gamma^2}{\sigma}, \quad a_{22} = -\gamma$$

*Proposition 3.7:* The unique positive equilibrium point $E_3(x^*, y^*)$ is locally asymptotically stable provided $\gamma > a_{11}$.

*Proof:* The characteristic equation is $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0$.

The equilibrium point $E_3(x^*, y^*)$ is stable when $trace(J(E^*) = a_{11} + a_{22} < 0$ and $\det(J(E^*) = a_{11}a_{22} - a_{12}a_{21} > 0$

Now

$$trace(J(E^*) = a_{11} + a_{22}$$
$$= a_{11} - \gamma < 0$$
$$\Rightarrow \gamma > a_{11}$$

$$\det(J(E^*) = a_{11}a_{22} - a_{12}a_{21}$$

$$= \frac{2\,\delta x^3 + (\gamma\,\alpha - \delta + 4\,\beta\,\delta)x^2 + 2\beta\,(\,\alpha\gamma - \delta + \beta\delta)x + \beta(\alpha\gamma\omega - \beta\delta)}{(x+\beta)^2\,\delta} > 0$$

Hence, the unique positive equilibrium point $E_3(x^*, y^*)$ is locally asymptotically stable provided $\gamma > a_{11}$.

iii. *Global Stability*

*Proposition 3.7:* The system (3.7) does not admit any periodic solution for $\beta > 1$.

*Proof:* Let $(x(t), y(t))$ be solutions of the system (3.7). Define Dulac function

$$H(x,y) = \frac{\beta + x}{xy}$$

Then

$$Q = \frac{\partial(HF)}{\partial x} + \frac{\partial(HG)}{\partial y}$$

$$= -\left(\frac{(\beta-1)+2x}{y} + \frac{(x+\beta)\delta}{x(x+\omega)}\right)$$

It is observed that $Q < 0$ for $\beta > 1$. Therefore, by Dulac criterion, the system (3.7) has no non-trivial periodic solutions.

*Theorem 3.4:* If $\beta > 1$ then the local asymptotical stability of the system (3.7) ensures its global asymptotical stability around the unique positive interior equilibrium point $E^*(x^*, y^*)$.

*Proof:* The unique equilibrium point $E^*(x^*, y^*)$ is the only stable point in the $xy$ plane. The boundedness of the solutions of the system together with the non-existence of periodic solutions establishes the global asymptotical stability.

## Chapter Four

## VI. Numerical Simulation

Many areas of science and engineering relays on quantitative analysis, as more complex mathematical models of the real world phenomena become available. Since most of these models don't have a closed form exact solution, numerical approximations are the only tools available for analyzing them. In this chapter we will solve the system equation (3.2) and (3.7) by using the in-built ordinary differential equation solver MatLab function ode45.

Notes

Notes

For solving system (3.2), we took the following parametric values. $\alpha = 1,\ \gamma = 0.2,\ \omega = 0.2,\ \sigma = 0.1,\ \beta = 0.2$ in appropriate units. For these values of parameter, we simplify the existence and stability properties of the equilibrium for the system.

For the given parametric values, it is found that the coexistence equilibrium point exists for $m > 0.5$. Hence, in our simulation we took the values of $m$ in the range $0.5 < m < 1$.

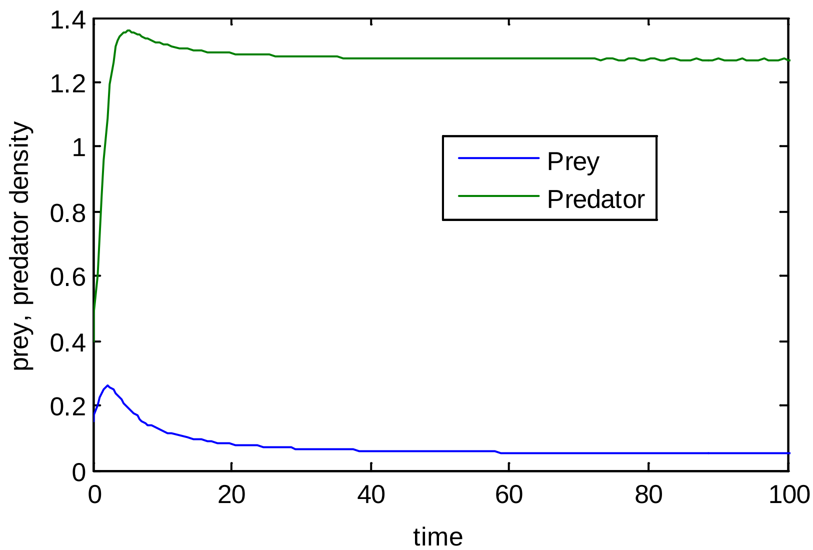


*Figure 1:* Times series plot of prey and predator at m=0.55

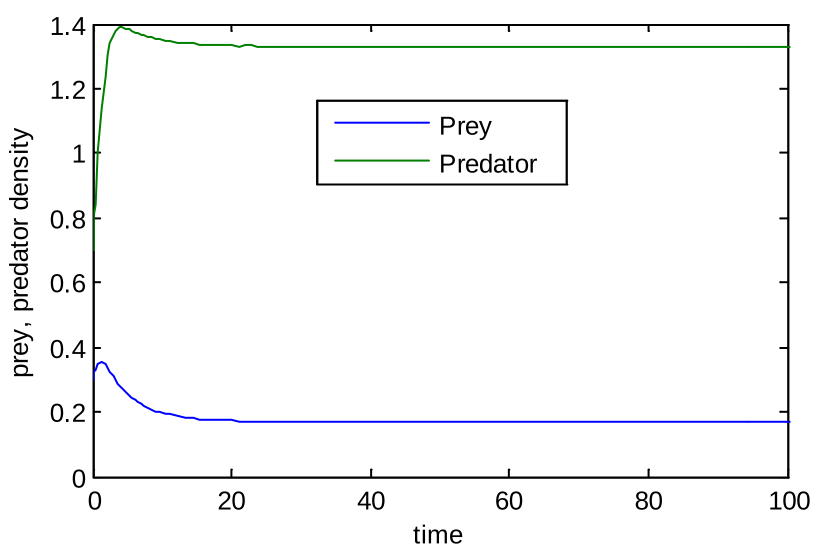


*Figure 2:* Time series plot of prey and predator at m=0.6.

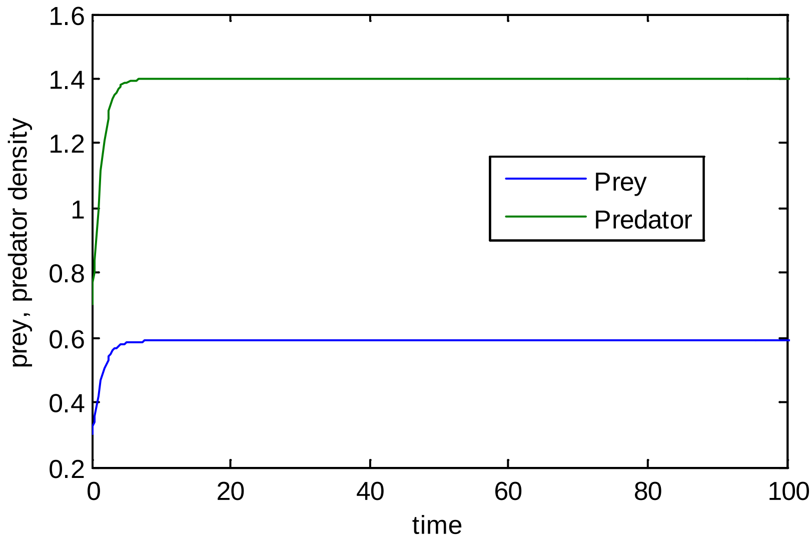


*Figure 3:* Time series plot of prey and predator at m=0.8.

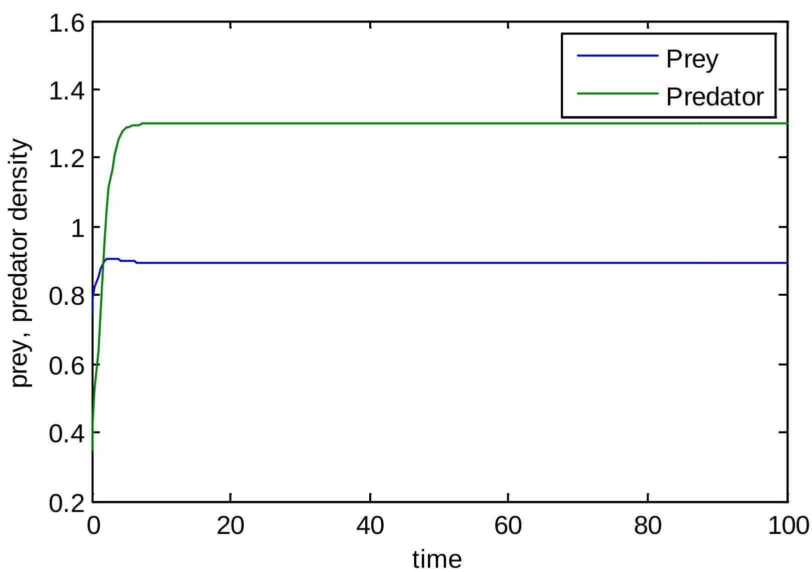


*Figure 4:* Time series plot of prey and predator at m=0.95.

For the system equation (3.7), that is the system in the absence of prey refuge, we have used the following parametric values as fixed and the parameter $\gamma$ as a control parameter. These values are $\alpha = 1$, $\omega = 0.2$, $\sigma = 0.1$, $\beta = 0.2$.

For these set of parametric values the coexistence equilibrium point exists whenever $\gamma < 0.1$.

The coexistence equilibrium point is locally asymptotically stable for $\gamma < 0.651234$ and hence unstable otherwise.

Figures 5-7 shows the stability of the coexistence equilibrium point. i.e. the solution, trajectory, of the prey and predator species approaches to the coexistence equilibrium point.

Notes

Notes

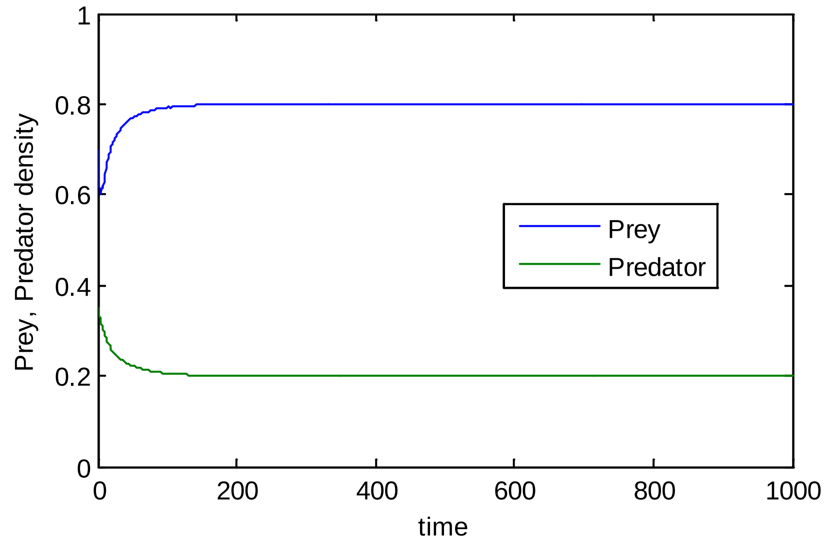


*Figure 5:* Time series plot of the prey and predator at $\gamma = 0.02$

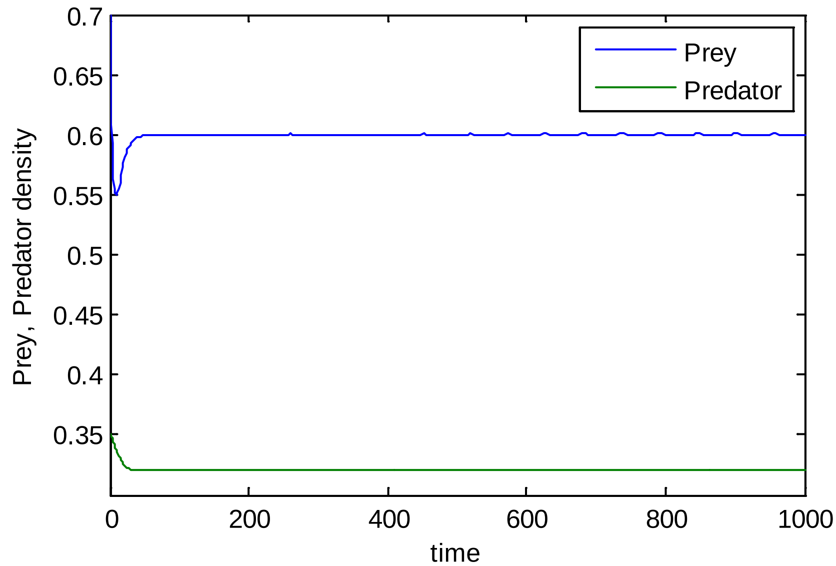


*Figure 6:* Time series plot of prey and predator at $\gamma = 0.04$

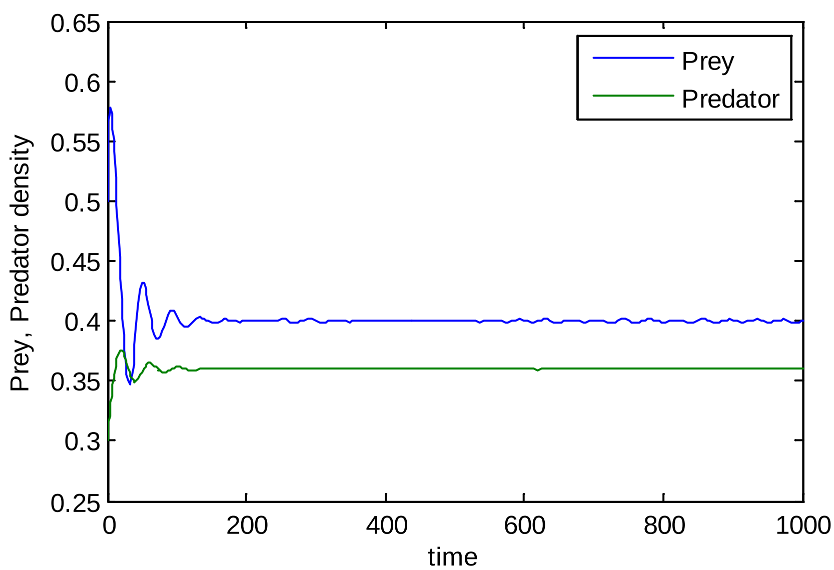


*Figure 7:* Time series plot of prey and predator at $\gamma = 0.06$

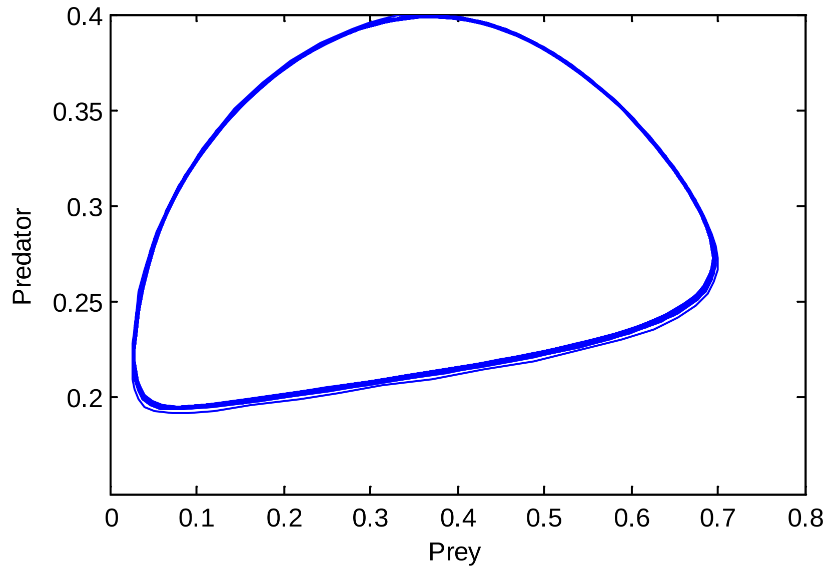


*Figure 8:* Phase portrait of prey and predator at $\gamma = 0.07$.

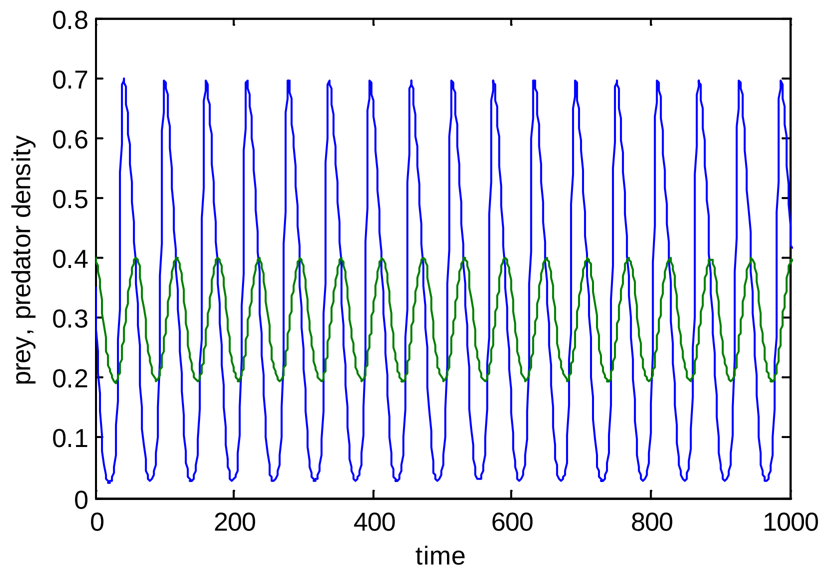


*Figure 9:* Time series plot of prey and predator at $\gamma = 0.07$.

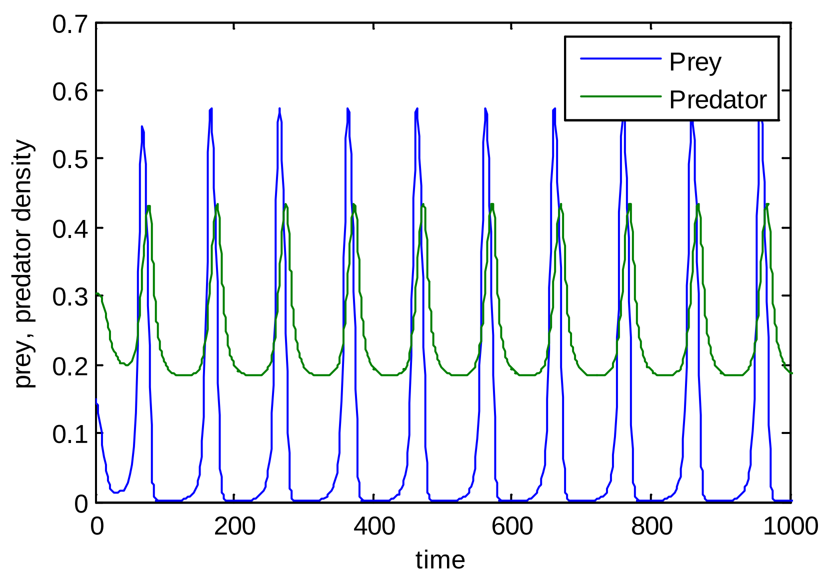


*Figure 10:* Time series plot of prey and predator at $\gamma = 0.09$.

Notes

Figures 8 shows the existence of a limit cycle, periodic solution. Figure 9 also shows the oscillatory nature of the predator prey system. Figure 10 represents the instability of the coexistence equilibrium point.

## Chapter Five

## V. Conclusion

Notes

This paper presents a prey-predator model with Holling type II functional response and modified Leslie Gower incorporating a constant proportion of prey refuge. Incorporating a refuge into system (3.2) provides a more realistic model. Refugi, therefore, can be considered as, areas in which the predator is not successfully controlling the prey and important for the biological control of a predator. The main focus of this paper was to introduce new mathematical models of biological systems and techniques for their analysis. Local asymptotic stability of the positive equilibrium implies its global asymptotic stability. Moreover, we established some new results such as the existence of stable or unstable equilibrium points under suitable values of parameters in the models. Two species can coexist in the case of stable condition; otherwise they might be extinct in the case of unstable condition.

*Declaration*

I, undersigned declare that this research entitled with "Mathematical modeling of a prey-predator Model with Modified Leslie-Gower and Holling -Type II incorporating prey refuge" is original and it has not been submitted to any institution elsewhere for the award of any academic degree or like, where other sources of information have been used, they have been acknowledged.

*Acknowledgement*

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