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On The Applications of Transmuted Inverted Weibull Distribution

By Ogunde Adebisi Ade, Fatoki Olayode & Oseghale Osezuwa Innocent
Joseph Ayo Babalola University

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ONTHEAPPLICATIONSOFTRANSMUTEDINVERTEDWEIBULLDISTRIBUTION

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Ref
1. Flair, H. Elsalloukh, E. Mendi and M. Milanova. The exponentiated inverted Weibull Distribution, Appl. Math. Inf. Sci. 6, No. 2, 167-171 (2012)

On the Applications of Transmuted Inverted Weibull Distribution

Ogunde Adebisi Ade [α], Fatoki Olayode [σ] & Oseghale Osezuwa Innocent [ρ]

Abstract- In this paper, we introduce a new distribution called transmuted inverted Weibull distribution (TIWD).The new distribution was used in analyzing bathtub failure rates lifetime data.We consider the standard transmuted inverted Weibull distribution (TIWD) that generalizes the standard inverted Weibull distribution (IWD), the new distribution has two shape parameters. The moments, median, survival function, hazard function, maximum likelihood estimators, least-squares estimators, fisher information matrix and asymptotic confidence intervals were obtained. A real data set is analyzed and it is observed that the (TIWD) distribution can provide a better fitting than (IWD) distribution.

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I. Introduction

The inverted Weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates. Ref. [11] explained the flexibility of the three parameters inverted Weibull distribution and its interested properties. Ref. [1]studied the properties of the inverted Weibull distribution and its application to failure data. Ref. [2] introduced the exponentiated Weibull distribution as generalization of the standard Weibull distribution and applied the new distribution as a suitable model to the bus-motor failure time data. Ref. [3] reviewed the exponentiated Weibull distribution with new measures. Ref. [4] studied the properties of transmuted Weibull distribution. Ref. [6] proposed andstudied the various structural properties of the transmuted Rayleigh distribution. Ref. [7] introduced the transmuted modified Weibull distribution. Transmuted Lomax distribution is presented by Ref. [8]. Ref. [9] introduces transmuted Pareto distribution. Transmuted Generalized Linear Exponential Distribution introduced by Ref. [10].

In this article we use transmutation map approach suggested by Shaw *et al.* [5] to define a new model which generalizes the inverted Weibull model. We will call the generalized distribution as the transmuted inverted Weibull distribution. According to the Quadratic Rank Transmutation Map (QRTM), approach the cumulative distribution function(cdf) satisfy the relationship

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2 |\lambda| \le 1 \qquad (1)$$

The probability density function (PDF) is given by

Author α: Department of Mathematics and Statistics, The Federal Polytechnic Ado-Ekiti, Nigeria. e-mail: debiz95@yahoo.com
Author σ: Department of Statistics, Ogun state Institute of Technology, Ogun State, Nigeria. e-mail: fatoki_olayode@yahoo.com
Author ρ: Joseph Ayo Babalola University, Ikeji Arakeji, Nigeria. e-mail: oseghaleinnocent@gmail.com

$$f(x) = \frac{dF}{dx} = [(1+\lambda) - 2\lambda G(x)]g(x) \qquad (2)$$

Where $g(x)$ is the *pdf* of the base line distribution.

II. Transmuted Inverted Weibull Distribution (TIWD)

We say that the random variable X has a standard inverted Weibull distribution (IWD) if its distribution function takes the following form:

$$G(x) = e^{-x^{-\beta}} \qquad x > 0,\ \beta > 0 \qquad (3)$$

The pdf of Inverted Weibull distribution is given by

$$g(x) = \beta x^{-\beta} e^{-x^{-\beta}} \qquad (4)$$

Now using (3) and (4) we have the cdf of a (TIWD) given by

$$F(x) = e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right] \qquad (5)$$

Where $|\lambda| \leq 1$ is simply the transmutation parameter of the distribution function of the standard inverted Weibull distribution (IWD). Here λ and β are the shape parameters. Therefore, the probability density function is:

$$f(x) = \beta x^{-(\beta+1)} e^{-x^{-\beta}}\left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right] \qquad (6)$$

Note that for $\lambda = 0$, we have the pdf of a standard inverted Weibull distribution. Figure 1 illustrates some of the possible shapes of the density function of (TIWD) for selected parameters.

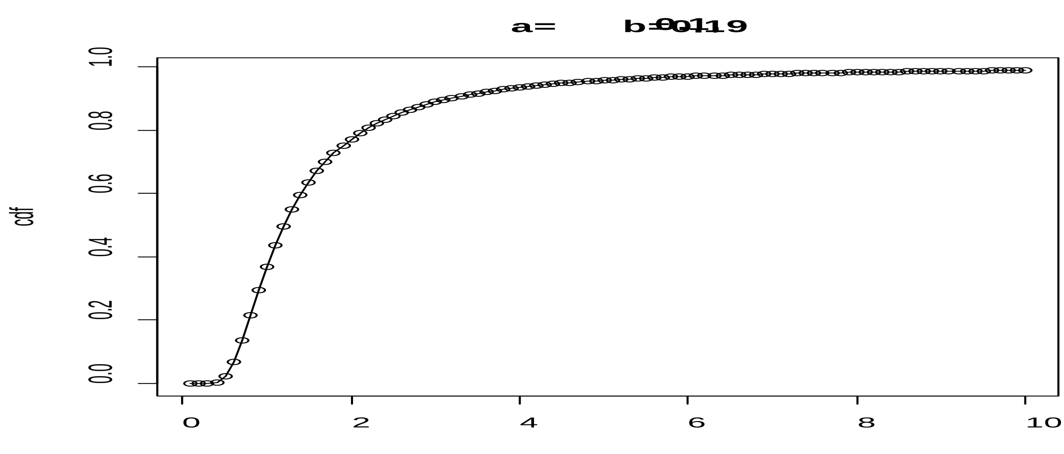

Fig. 1: The graph of the cdf of TIWD

Notes

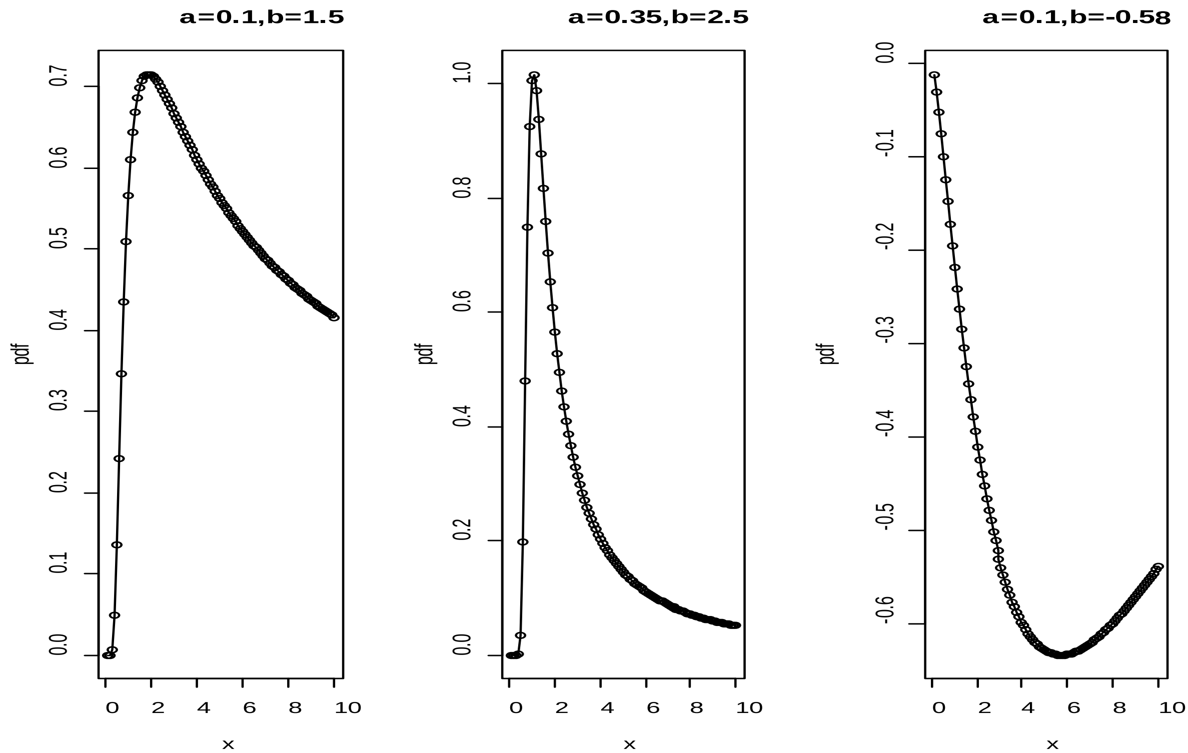

Fig. 2: The graph of the pdf of TIWD

III. Moments, Mean, Variance, Median and Quantile of (TIWD)

In this section we shall present the moments and qunatiles for the (TIWD). The k^{th}order moments, for $<\beta$, of (TIWD) can be obtained as follows for a random variable X,

$$E(X)^k = \int_{-\infty}^{\infty} x^k f(x) d \tag{7}$$

Putting eq. (6) in eq. (7), we have

$$E(X)^k = \beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-x^{-\beta}} \left[1+\lambda-2\lambda e^{-x^{-\beta}}\right] dx \tag{8}$$

Expanding eq. (8) and splitting it into three, we obtain

$$P_1 = \beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-x^{-\beta}} dx \tag{9}$$

$$P_2 = \lambda\beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-x^{-\beta}} dx \tag{10}$$

$$P_3 = -2\lambda\beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-2x^{-\beta}} dx \tag{11}$$

If we let $u = x^{-\beta}$ in eq. (9) and eq. (10) and also let $u = 2x^{-\beta}$ in eq. (11), then we have

$$P_1 = -\Gamma(1-\frac{k}{\beta}) \tag{12}$$

$$P_2 = -\lambda\Gamma(1-\frac{k}{\beta}) \tag{13}$$

$$P_3 = \left(\frac{1}{2}\right)^{-\frac{k}{\beta}} \lambda\Gamma\left(1-\frac{k}{\beta}\right) \tag{14}$$

It should be noted that $\beta < k$

Then combining eq.(12), eq.(13) and eq.(14) we obtain the k^{th} moment of (TIWD), we have

$$\mu_k = E(X)^k = \left(1-\frac{k}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{k}{\beta}} - 1\right] - 1\right\} \tag{15}$$

Using eq. (15), we obtain the $1^{st}, 2^{nd}, 3^{rd}, and\ 4^{th}$ moment for $k = 1, 2, 3, 4$

$$\mu_1 = \Gamma\left(1-\frac{1}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{1}{\beta}} - 1\right] - 1\right\} \tag{16}$$

$$\mu_2 = \Gamma\left(1-\frac{2}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{2}{\beta}} - 1\right] - 1\right\} \tag{17}$$

$$\mu_3 = \Gamma\left(1-\frac{3}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{3}{\beta}} - 1\right] - 1\right\} \tag{18}$$

$$\mu_4 = \Gamma\left(1-\frac{4}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{4}{\beta}} - 1\right] - 1\right\} \tag{19}$$

The mean of (TIWD) is the first moment about the origin (μ_1) which corresponds to eq. (16)

And the variance of (TIWD) can be obtained using the relation

$$V(X) = \mu_2 - (\mu_1)^2 \tag{20}$$

Inserting eq. (16) and eq. (17) in eq. (20) we have

$$V(X) = \Gamma\left(1-\frac{2}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{2}{\beta}} - 1\right] - 1\right\} - \left[\Gamma\left(1-\frac{1}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{1}{\beta}} - 1\right] - 1\right\}\right]^2 \tag{21}$$

The quantile function for the (TIWD) is given as,

$$x_q = \left[-ln\frac{\lambda-\sqrt{\lambda(\lambda+4q)}}{2\lambda}\right]^{-\frac{1}{\beta}} \tag{22}$$

The lower quartile, median and the upper quartile of the (TIWD) can be obtained by letting $q = 0.25, q = 0.5$ and $q = 0.75$ respectively in eq. (22).

Table 1: Moments based values for TIWD

β	λ	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	$\hat{\mu}_4$
−1.0	−3.0	0.50	2.50	9.75	43.50
−1.0	−3.5	0.75	3.25	12.30	54.75
−1.0	−4.0	1.00	4.00	15.00	66.00
−1.0	−4.5	1.25	4.75	17.63	386.25

Notes

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Table 2: Moments based measures of the TIWD

β	λ	*Mean*	*Variance*	*CV*	*CS*	*CK*
−1.0	−3.0	0.50	2.25	3.17	2.47	6.96
−1.0	−3.5	0.75	2.69	2.19	2.10	5.18
−1.0	−4.0	1.00	3.00	1.73	1.89	4.12
−1.0	−4.5	1.25	3.19	1.43	1.70	17.12

Notes

IV. Reliability Analysis

The survival function, also known as the reliability function in engineering, is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some certain mechanical wearing system with time. It is the conditional probability that the system will survive beyond a specified time. The (TIWD) can be a useful model to characterize failure time of a given system because of the analytical structure. The reliability function $R(t)$, which is the probability of an item not failing prior to time t, is defined by $R(t) = 1-F(t)$. The reliability function of a (TIWD) is given by

The reliability function define as,$R(x) = 1 - F(x)$, for (TIWD) is given by:

$$R(x) = 1 - e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right] \tag{23}$$

The other characteristic of interest of a random variable is the hazard rate function also known as instantaneous failurerate defined by

$$h(x) = \frac{f(x)}{1-F(x)} \tag{24}$$

Is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time *t:* The hazard rate function for a (TIWD) is given by

$$h(x) = \frac{\beta x^{-(\beta+1)} e^{-x^{-\beta}}\left[1+\lambda-2\lambda e^{-x^{-\beta}}\right]}{1-e^{-x^{-\beta}}\left[(1+\lambda)-\lambda e^{-x^{-\beta}}\right]} \tag{25}$$

For $\lambda = 0$, it represents the hazard of the standard inverted Weibull distribution. The graph (fig.3) drawn below depict the graph of the hazard rate function for TIWD for various values of the parameters.

Recently, it is observed that the reversed hazard function plays an important role in the reliability analysis; see Gupta and Gupta (2007). The reversed hazard function of the (TIWD) is given as

$$R(x) = \beta x^{-(\beta+1)}\left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right] \tag{26}$$

Notes

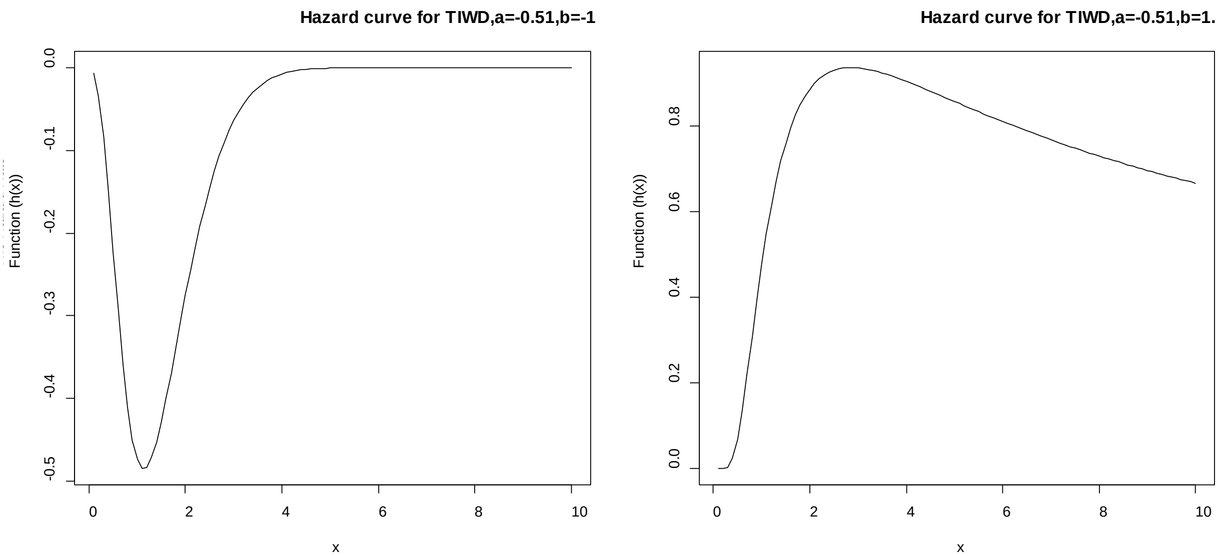

Fig. 3: The graph of hazard rate function for TIWD

V. Moment Generating Function of (TIWD)

The moment generating function of a random variable x is defined by

$$M_t(x) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{27}$$

The above expression can further be simplify as

$$M_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k f(x)dx \tag{28}$$

Since,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^k x^k}{k!} \tag{29}$$

Inserting eq. (15) in eq. (28) we have

$$M_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(1 - \frac{k}{\beta}\right) \left\{ \lambda \left[\left(\frac{1}{2}\right)^{-\frac{k}{\beta}} - 1 \right] - 1 \right\} \tag{30}$$

The above expression is the moment generating function of (TIWD).

VI. Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with cdf $F_X(X)$ *and pdf* $f_X(x)$.

Then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1 - F_X(x)]^{n-j} \tag{31}$$

For $j = 1, 2, 3, \ldots, n$

We have from (5) and (6) the pdf of the j^{th}order inverted Weibull random variable X_j given by

$$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!}\beta x^{-\beta} e^{-x^{-\beta}} \left[e^{-x^{-\beta}}\right]^{j-1} \left[1-e^{-x^{-\beta}}\right]^{n-j} \tag{32}$$

If we consider the series expansion,

$$(1-z)^m = \sum_{j=0}^{\infty}(-1)^i \binom{m}{i} z^i \tag{33}$$

Applying eq. (33) in eq. (31) will yield

$$f_{X_{(j)}} = \frac{n!}{(j-1)!\,(n-j)!}\beta x^{-\beta} \sum_{i=0}^{\infty}(-1)^i \binom{n-j}{i} \left[e^{-x^{-\beta}}\right]^{i+j} \tag{34}$$

Therefore, the pdf of the n^{th}order inverted Weibull random variable X_n is given by

$$f_{X_{(n)}} = n!\,\beta x^{-\beta} \left[e^{-x^{-\beta}}\right]^{n} \tag{35}$$

Now we provide the distribution of the order statistics for transmuted inverted Weibull random variable. The pdf of the j^{th} order statistic for the transmuted inverted Weibull random variable is given by

$$f_{X_{(j)}} = \frac{n!}{(j-1)!\,(n-j)!}\beta x^{-(\beta+1)} e^{-x^{-\beta}} \Big[1+\lambda - 2\lambda e^{-x^{-\beta}}\Big] \left[e^{-x^{-\beta}}\left[(1+\lambda)-\lambda e^{-x^{-\beta}}\right]\right]^{j-1} \left[1-e^{-x^{-\beta}}\left[(1+\lambda)-\lambda e^{-x^{-\beta}}\right]\right]^{n-j}$$

Using the relation in eq. (32) the above expression can be simplified as

$$f_{X_{(j)}} = \frac{n!}{(j-1)!\,(n-j)!}\beta x^{-(\beta+1)} e^{-x^{-\beta}} \Big[1+\lambda - 2\lambda e^{-x^{-\beta}}\Big] \sum_{j=0}^{\infty}(-1)^i \binom{n-j}{i} \left[e^{-x^{-\beta}}\left[(1+\lambda)-\lambda e^{-x^{-\beta}}\right]\right]^{i+j-1} \tag{36}$$

Therefore the pdf of the largest order statistic $X_{(n)}$ for the transmuted inverted Weibull random variable is given by

$$f_{X_{(n)}} = n!\,\beta x^{-(\beta+1)} e^{-x^{-\beta}} \left[1+\lambda-2\lambda e^{-x^{-\beta}}\right] \left[e^{-x^{-\beta}}\left[(1+\lambda)-\lambda e^{-x^{-\beta}}\right]\right]^{n-1} \tag{37}$$

And for the first order we have

$$f_{X_{(1)}} = \beta x^{-(\beta+1)} e^{-x^{-\beta}} \left[1+\lambda-2\lambda e^{-x^{-\beta}}\right] \tag{38}$$

VII. Entropy

The entropy of a random variable X with probability density $f(x)$ is a measure ofvariation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Renyi, A. (1961) introduced the Renyi entropy defined as

Notes

$$I_R(\delta) = \frac{1}{1-\delta} log\left\{\int_0^\infty f(x)^\delta dx\right\} \tag{39}$$

Where $\delta > 0$ and $\delta \neq 1$. the integral in $I_R(\delta)$ of the TIWD$(x;\, \lambda, \beta)$ can be define as

$$\int_0^\infty f(x)^\delta dx = \int_0^\infty \left\{x^{-(\beta+1)} e^{-x^{-\beta}} \left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right]\right\}^\delta dx \tag{40}$$

This can be simplify as

$$\beta^\delta \int_0^\infty x^{-\delta(\beta+1)} e^{-\delta x^{-\beta}} \left[1 - \lambda(2e^{-x^{-\beta}} - 1\right]^\delta dx$$

$$= \beta^\delta \sum_{k=1}^\infty (-1)^k \binom{\delta}{k} \lambda^k \int_0^\infty x^{-\delta(\beta+1)} e^{-\delta x^{-\beta}} \left(2e^{-x^{-\beta}} - 1\right)^k dx \tag{41}$$

But $\left(2e^{-x^{-\beta}} - 1\right)^k = (-1)^k \left(1 - 2e^{-x^{-\beta}}\right)^k$, therefore the RHS equation (41) can be simplify as

$$\beta^\delta \sum_{k=1}^\infty \sum_{l=1}^k (-1)^{2k} (-1)^l \binom{\delta}{k} \lambda^k \binom{k}{l} 2^l \int_0^\infty x^{-\delta(\beta+1)} e^{-(\delta+l)x^{-\beta}} dx \tag{42}$$

Letting $P = x^{-\beta(\delta+l)}$, then $x = P^{\frac{-1}{\beta(\delta+l)}}$ and $dx = -\frac{P^{\frac{-1}{\beta(\delta+l)}-1}}{\beta(\delta+l)} dP$

Then substituting in the above integral function we have

$$-\frac{1}{\beta(\delta+l)} \int_0^\infty P^{\frac{\delta-1-\beta l}{\beta(\delta+l)}} e^{-P} dP = -\frac{1}{\beta(\delta+l)} \Gamma\left(\frac{\delta-1-\beta l}{\beta(\delta+l)} + 1\right) \tag{43}$$

Then substituting equation (43), (42) in (39), we have the entropy of TIWD as

$$I_R(\delta) = \frac{-1}{1-\delta} log\left\{\beta^\delta \sum_{k=1}^\infty \sum_{l=1}^k (-1)^{2k} (-1)^l \binom{\delta}{k} \lambda^k \binom{k}{l} 2^l \frac{1}{\beta(\delta+l)} \Gamma\left(\frac{\delta - 1 - \beta l}{\beta(\delta+l)} + 1\right)\right\} \tag{44}$$

VIII. Parameters Estimation

In this section, we discuss the maximum likelihood estimators of the two-parameter transmuted inverted Weibull distribution and their fisher information matrix as well as asymptotic confidence intervals.

a) Maximum likelihood estimators and fisher information matrix

If $x_1, x_2, . . ., x_n$ is a random sample from transmuted inverted Weibull distribution given by (6), then the Likelihood function (L) becomes:

$$L = \prod_{i=1}^n f(x_i, \lambda, \beta) \tag{45}$$

By substituting from equation (6) into Equation (39), we get

Notes

$$L = \prod_{i=1}^{n} \beta x^{-(\beta+1)} e^{-x^{-\beta}} \left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right] \tag{46}$$

Then the log – likelihood function becomes

$$l = \ln(\beta) - (\beta + 1) \sum_{i=1}^{n} ln\,(x_i) - \sum_{i=1}^{n} x_i^{-\beta} + \sum_{i=1}^{n} \ln(1 + \lambda - 2\lambda e^{-x_i^{-\beta}}) \tag{47}$$

Notes

And the score vector is given as

$$\frac{dl}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} ln\,(x_i) + \sum_{i=1}^{n} x_i^{-\beta} \ln(x_i) + 2\lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} e^{-x_i^{-\beta}}}{(1 + \lambda - 2\lambda e^{-x_i^{-\beta}})} \tag{48}$$

$$\frac{dl}{d\lambda} = \sum_{i=1}^{n} \frac{\left(1 - 2e^{-x_i^{-\beta}}\right)}{(1 + \lambda - 2\lambda e^{-x^{-\beta}})} \tag{49}$$

Therefore, the MLEs of λ and β which maximize equation (47) must satisfy the nonlinear normal equations given by:

$$\frac{dl}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} ln\,(x_i) + \sum_{i=1}^{n} x_i^{-\beta} \ln(x_i) + 2\lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} e^{-x_i^{-\beta}}}{(1 + \lambda - 2\lambda e^{-x_i^{-\beta}})} = 0 \tag{50}$$

$$\frac{dl}{d\lambda} = \sum_{i=1}^{n} \frac{\left(1 - 2e^{-x_i^{-\beta}}\right)}{(1 + \lambda - 2\lambda e^{-x^{-\beta}})} = 0 \tag{51}$$

The maximum likelihood estimator $\hat{\Theta} = \left(\hat{\lambda}, \hat{\beta}\right)'$ of $\Theta = (\lambda, \beta)'$ is obtained by setting the score vector to zero and solving the nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function given in (47).

b) Least square Estimates

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the ordered sample of size n from TIWD. Then the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right] = \frac{i}{n+1};\; i = 1,2,3,..n \tag{52}$$

Then the least square estimates of $\hat{\lambda}_{LS}$ and $\hat{\beta}_{LS}$ of λ and β are obtained by minimizing $\mathcal{Z}$ giving as

$$\mathcal{Z}(\lambda, \beta) = \sum_{i=1}^{n} \left(e^{-x^{-\beta}} \left[(1 + \lambda) - \lambda e^{-x^{-\beta}}\right] - \frac{i}{n + 1}\right)^2 \tag{53}$$

Therefore the $\hat{\lambda}_{LS}$ and $\hat{\beta}_{LS}$ of λ and β can be obtained as the solution of the following equations

$$\frac{\delta \mathcal{Z}(\lambda, \beta)}{\delta \lambda} = \sum_{i-1}^{n} -\left\{e^{-x^{-\beta}}\right\}^2 \left(e^{-x^{-\beta}} \left[(1 + \lambda) - \lambda e^{-x^{-\beta}}\right] - \frac{i}{n + 1}\right) = 0 \tag{54}$$

$$\frac{\delta \mathcal{Z}(\lambda, \beta)}{\delta \beta} = \sum_{i-1}^{n} \left\{\beta x^{-`(\beta+1)} e^{-x^{-\beta}} (1 + \lambda)\right\} \left(e^{-x^{-\beta}} \left[(1 + \lambda) - \lambda e^{-x^{-\beta}}\right] - \frac{i}{n + 1}\right) = 0 \tag{55}$$

IX. Asymptotic Confidence Bounds

In this section, we derive the asymptotic confidence intervals of these parameters when$\lambda > 0$ and $\beta > 0$ and as the MLEs of the unknown parameters λ and β cannot be obtained in closed forms, by using variance covariance matrix I_0^{-1} see Lawless(2003), where I_0^{-1} is the inverse of the observed information matrix

$$I_0^{-1} = \begin{bmatrix} \frac{\partial^2 l}{\partial \lambda^2} & \frac{\partial^2 l}{\partial \lambda \partial \beta} \\ \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \beta^2} \end{bmatrix} \tag{56}$$

Thus

$$I_0^{-1} = \begin{bmatrix} var\hat{\lambda} & cov(\hat{\lambda}, \hat{\beta}) \\ cov(\hat{\lambda}, \hat{\beta}) & var\hat{\beta} \end{bmatrix} \tag{57}$$

The derivatives in $I_0^{-1}$are given as follows:

$$\frac{d^2 l}{d\lambda^2} = -\sum_{i=1}^{n} \left(\frac{1 - 2e^{-x_i{}^{-\beta}}}{1 + \lambda - 2\lambda e^{-x^{-\beta}}} \right)^2 \tag{58}$$

$$\frac{d^2 l}{d\beta^2} = -\frac{n}{\beta} - \sum_{i=1}^{n} x_i^{-\beta} (lnx)^2 + \sum_{i=1}^{n} \frac{x_i^{-\beta} lnx e^{-x_i{}^{-\beta}} \left[\left(x_i^{-\beta} - 1 \right) - 2x_i^{-\beta} e^{-x_i{}^{-\beta}} \right]}{\left(1 + \lambda - 2\lambda e^{-x_i{}^{-\beta}} \right)^2} \tag{59}$$

$$\frac{d^2 l}{d\lambda d\beta} = \sum_{i=1}^{n} \frac{2x_i^{-\beta} lnx e^{-x_i{}^{-\beta}} \left[\left(1 + \lambda - 2\lambda e^{-x^{-\beta}} \right) - \lambda \left(1 - 2e^{-x_i{}^{-\beta}} \right) \right]}{\left(1 + \lambda - 2\lambda e^{-x_i{}^{-\beta}} \right)^2} \tag{60}$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters λ and β by using variance covariance matrix as in the following forms

$$\hat{\lambda} \pm Z_{\frac{\delta}{2}} \sqrt{var\hat{\lambda}} \ , \ \ \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{var\hat{\beta}}$$

Where $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right)^{th}$ percentile of the standard normal distribution.

X. Applications

In this section, we use two real data sets to show that the TIWD can be a better model than one based on the IWD and the exponentiated inverted Weibull distribution (EIWD). In the first application, we consider a data set of the tensile strength of 100 observation of carbon fibers, the data was obtained from see Ref.[12]. These data are: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88,

Ref

12. Andrews, D. F. & Herzberg, A. M. (1985), Data: A Collection of Problems from Many Fields for the Student and Research Worker, Springer Series in Statistics, New York.

2.82, 2.05, 3.65. Table 3.0and Table 6.0 gives the exploratory data analysis of the data considered, Table 4.0 and Table 7.0 gives the maximum likelihood estimates of the parameters with their standard error in parenthesis and Table 5.0 and Table 8.0 gives the criteria for comparison. Fig. 4 and fig. 5 represents the empirical density and the cumulative density of the data considered.

Notes

Table 3: Descriptive Statistics on Breaking stress of Carbon fibres

Min	Lower quartile	Median	Upper quartile	Mean	Max.	Skewness	Kurtosis	Range
0.390	1.840	2.700	3.220	2.640	5.560	0.37378	0.17287	5.17

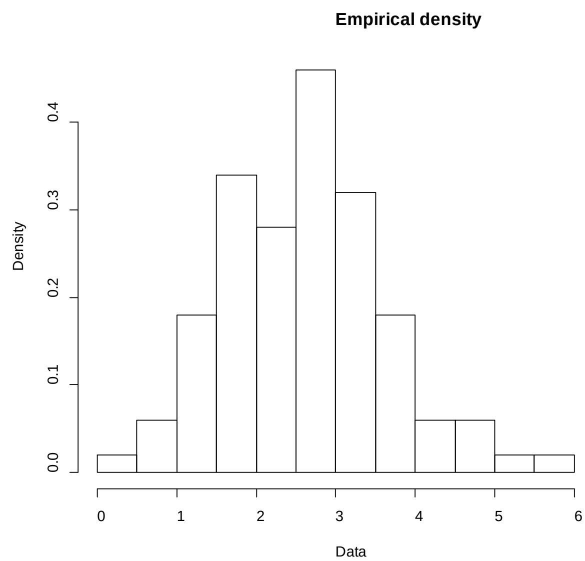

Fig. 4: The graph of the Empirical density and the cumulative distribution function of the carbon data

Table 4: Estimated parameters of the TIWD, EIWD and IWD

Mode I	Estimates		$l(\hat{\theta})$
$TIWD$ (λ, β)	-0.8965 (0.0908)	3.1175 (0.2668)	-44.571
$EIWD$ (θ, β)	0.2485 (1.9685)	0.2344 (2.8876)	-46.853
IWD (β)	- $(-)$	2.806 (2.723)	-58.482

Table 5: Measures of Goodness of Fit

Mode I	$K-S$	AD	W	AIC	BIC	HQIC	CAIC
$TIWD$	0.4591	6.3175	1.1915	93.142	97.428	94.827	93.827
$EIWD$	0.5173	5.7414	1.0727	97.907	101.993	99.392	97.907
IWD	0.4359	6.0615	1.1388	118.963	121.107	119.806	119.029

Also, we consider a data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. This data are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.For previous studies with these data sets see Andrews and Herzberg [12]

Table 6: Summary of data on fatigue fracture of Kevlar 373/epoxy at 90 % stress level

Min	Lower quartile	Median	Upper quartile	Mean	Max.	Variance	Skewness	Kurtosis	Range
0.0251	0.09048	1.7361	2.2960	1.9590	9.0960	2.4774	1.9406	8.1608	9.0709

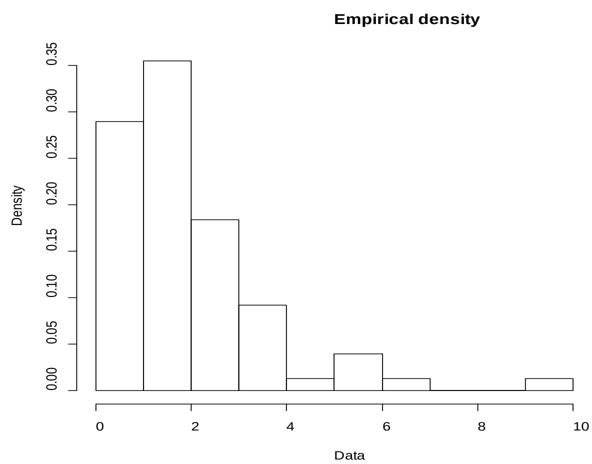

Fig. 7: The graph of the Empirical density and the cumulative distribution function of the Kevlar 373/epoxy data

Table 7: Estimated parameters of the TIWD, EIWD and IWD

Mode I	Estimates		$l(\hat{\theta})$
$TIWD$ (λ, β)	0.7074 (03994)	0.6903 (0.0575)	-152.483
$EIWD$ (θ, β)	0.8608 (0.1088)	0.7588 (0.0541)	-153.539
IWD (β)	- (–)	0.7322 (0.0474)	-154.278

Table 8: Measures of Goodness of Fit

Mode I	$K-S$	AD	W	AIC	BIC	HQIC	CAIC
$TIWD$	0.2440	4.9831	0.8506	308.967	313.128	310.829	308.967
$EIWD$	0.2290	5.9661	1.0351	311.078	315.740	312.941	311.243
IWD	0.2291	5.2691	0.9036	310.556	312.887	311.487	310.610

We employ the statistical tools for model comparison such as Kolmogorov-Smirnov (K-S) statistics, Anderson Darling statistic (AD), crammer von misses statistic

Ref

12. Andrews, D. F. & Herzberg, A. M. (1985), Data: A Collection of Problems from Many Fields for the Student and Research Worker, Springer Series in Statistics, New York.

(W), Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Hannan Quinine information criterion (HQIC) and Bayesian information criterion o choose the best possible model for the data sets among the competitive models.The selection criterion is that the lowest AIC, CAIC, BIC and HQIC correspond to the best fit model.

XI. Conclusion

Notes

Among the models considered the best model is the transmuted inverted Weibull distribution for the two data sets.

References Références Referencias

1. Flair, H. Elsalloukh, E. Mendi and M. Milanova. The exponentiated inverted Weibull Distribution, Appl. Math. Inf. Sci. 6, No. 2, 167-171 (2012)
2. G. S. Mudholka, D. K. Srivastava and M. Freimer, The exponentiated Weibull family: a real nalysis of the bus motorfailure data.
3. G. S. Mudholkar and A. D. Hutson, Exponentiated Weibull family: some properties and flood data application, Commun. Statistical Theory and Method. 25, 3050-3083 (1996).
4. G.R. Aryal, and Ch. P. Tsokos, (2011) Transmuted Weibull Distribution: A Generalization of the Weibull probability distribution, European journal of pure and applied mathematics Vol. 4, No.2, 89- 102.
5. W. T. Shaw, and I. R. Buckley, (2009) The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, arXiv preprint arXiv:0901.0434.
6. F. Merovci, (2013). Transmuted Rayleigh distribution. Austrian Journal OF Statistics, Vol. 42, 1, 21–31.
7. M. S. Khan, and R. King, (2013). Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution. European Journal of Pure and Applied Mathematics, 6, 66-88.
8. S. K. Ashour, and M. A. Eltehiwy, (2013) Transmuted Lomax Distribution, American Journal of Applied Mathematics and Statistics, Vol. 1, No. 6, 121-127.
9. F. Merovci, and L. Puka, (2014).Transmuted Pareto distribution. Prob Stat Forum, 7, 1-11.
10. I. Elbatal, L.S. Diab and N. A. Abdul-Alim, (2013) Transmuted Generalized Linear Exponential Distribution International Journal of Computer Applications, Vol. 83, No. 17, 29-37.
11. M. S. Khan, G. R. Pasha and A. H. Pasha, Theoretical analysis of inverse Weibull distribution, WSEAS Transactions on Mathematics, 7, 2 (2008).
12. Andrews, D. F. & Herzberg, A. M. (1985), Data: A Collection of Problems from Many Fields for the Student and Research Worker, Springer Series in Statistics, New York.
13. R. E. Barlow, R. H. Toland, and T. Freeman, (1984), A Bayesian analysis of stress rupture life of kevlar 49/epoxy
14. R.C. Gupta and R.D. Gupta (2007), "Proportional reversed hazard model and its applications", Journal of Statistical Planning and Inference, vol. 137, no. 11, 3525 - 3536.
15. Lawless, J. F. (2003). "Statistical Models and Methods for Lifetime Data". *John Wiley and Sons*, New York, 20, 1108-1113.

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