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By Frederic Ayant

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On a General Class of Multiple Eulerian Integrals with Multivariable Aleph-Functions

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Abstract- Recently, Raina and Srivastava [5] and Srivastava and Hussain [12] have provided closed-form expressions for a number of a Eulerian integral involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerians integral involving the product of two multivariable Aleph-functions, a class of multivariable polynomials and the general sequence of functions. These integrals will serve as a capital formula from which one can deduce numerous integrals.

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I. Introduction and Prerequisites

The well-known Eulerian Beta integral [6]

$$\int_a^b (z-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(Re(\alpha)>0, Re(\beta)>0, b>a)$$

is a basic result of evaluation of numerous other potentially useful integrals involving various special functions and polynomials. The mathematicians Raina and Srivastava [7], Saigo and Saxena [8], Srivastava and Hussain [14], Srivastava and Garg [13] et cetera have established some Eulerian integrals involving a various general class of polynomials, Meijer's G-function and Fox's H-function of one and more variables with general arguments. Recently, several Author study some multiple a Eulerian integrals, see Bhargava [4], Goyal and Mathur [5], Ayant [3] and others. In this paper we obtain general multiple Eulerians integral of the product of two multivariable Aleph-functions, a general class of multivariable polynomials [12] and the general sequence of functions.

For this study, we need the following series formula for the general sequence of functions introduced by Agrawal and Chaubey [1] and was established by Salim [9].

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}] = \sum_{w,v,u,t',e,k_1,k_2} \psi(w,v,u,t',e,k_1,k_2)x^R$$

where

$$\psi(w,v,u,t',e,k_1,k_2) = \frac{(-)^{t'+w+k_2}(-v)_u(-t')_e(\alpha)_t l^n}{w!v!u!t'!e!l'_n k_1!k_2!}\frac{\mathfrak{s}^{w+k_1}F^{\gamma n-t'}}{(1-\alpha-t')_e}(-\alpha-\gamma n)_e(-\beta-\delta n)_v$$

$$g^{v+k_2}h^{\delta n-v-k_2}\,(v-\delta n)_{k_2}E^{t'}\left(\frac{pe+\tau w+\lambda+qu}{l}\right)_n \tag{1.3}$$

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Ref

8. M. Saigo, M. and R.K. Saxena, Unified fractional integral formulas forthe multivariable H-function. J. Fractional Calculus 15 (1999), 91-107.

and $$\sum_{w,v,u,t',e,k_1,k_2} = \sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t'=0}^{n}\sum_{e=0}^{t'}\sum_{k_1,k_2=0}^{\infty}$$

The infinite series on the right-hand side of (1.3) is convergent and $R = ln + qv + pt' + \tau w + \tau k_1 + k_2 q$

We shall note $R_n^{\alpha,\beta}[x; E, F, g, h; p, q; \gamma; \delta; e^{-sx^{\tau}}] = R_n^{\alpha,\beta}(x)$

The class of multivariable polynomials defined by Srivastava [29], is given in the following manner:

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1,K_1;\cdots;N_v,K_v]\, y_1^{K_1}\cdots y_v^{K_v} \quad (1.4)$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary real or complex constants.

We shall note $a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1,K_1;\cdots;N_v,K_v]$

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [11], itself is a generalization of G and H-functions of several variables studied by Srivastava et Panda [16,17]. The multiple Mellin-Barnes integrals occurring in this paper will refer to as the multivariable Aleph-function of r-variables throughout our present study and will be defined and represented as follows (see Ayant [2]).

We have

$$\aleph(z_1,\cdots,z_r) = \aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot & \cdot \\ \cdot & \\ \cdot & \cdots\cdots\cdots\cdots , \\ \mathrm{z}_r & \end{array}\right.$$

$$[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;$$
$$[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}] : [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;$$

$$\left.\begin{array}{c} [(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k) z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.5)$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}$$

Ref

11. C.K. Sharma and S.S. Ahmad, On the multivariable I-function. Acta ciencia Indica Math, 1994 vol 20,no2, 113-116.

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0 \quad (1.6)$$

With $k = 1, \cdots, r, i = 1, \cdots, R \;, i^{(k)} = 1, \cdots, R^{(k)}$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions concerning the multivariable Aleph-function.

If all the poles of (1.8) are simples, then the integral (1.6) can be evaluated with the help of the residue theorem to give

$$\aleph(z_1, \cdots, z_r) = \sum_{G_k=1}^{m_k} \sum_{g_k=0}^{\infty} \phi \frac{\prod_{k=1}^{r} \phi_k z_k^{\eta_{G_k,g_k}} (-)^{\sum_{k=1}^{r} g_k}}{\prod_{k=1}^{r} \delta_{G^{(k)}}^{(k)} \prod_{k=1}^{r} g_k!}$$

where

$$\phi = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(i)} S_k)}{\sum_{i=1}^{R} [\tau_i \prod_{j=\mathfrak{n}+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} S_k) \prod_{j=1}^{q_k} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} S_k)]}$$

$$\phi_k = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} S_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} S_k)}{\sum_{i^{(i)}=1}^{R^{(i)}} [\tau_{i^{(i)}} \prod_{j=m_i+1}^{q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} S_k) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} S_k)]}$$

and

$$S_k = \eta_{G_k,g_k} = \frac{d_{g_k}^{(k)} + G_k}{\delta_{g_k}^{(k)}} \text{ for } k = 1, \cdots, r$$

which is valid under the following conditions: $\epsilon_{M_k}^{(k)}[p_j^{(k)} + p'_k] \neq \epsilon_j^{(k)}[p_{M_k} + g_k]$

We shall note $\aleph(z_1, \cdots, z_r) = \aleph_1(z_1, \cdots, z_r)$ and

$$\aleph(z'_1, \cdots, z'_s) = \aleph^{0,n':m'_1,n'_1,\cdots,m'_s,n'_s}_{p'_i,q'_i,\iota_i;r':p'_{i(1)},q'_{i(1)},\iota_{i(1)};r^{(1)};\cdots;p'_{i(s)},q'_{i(s)};\iota_{i(s)};r^{(s)}} \left(\begin{array}{c|c} \mathrm{z}'_1 & [(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(s)})_{1,n'}], \\ \cdot & \cdot \\ \cdot & \\ \cdot & \cdots\cdots\cdots\cdots\cdots\cdots , \\ \mathrm{z}'_s & \end{array} \right.$$

$$[\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(s)})_{n'+1,p'_i}] : [(\mathrm{a}_j^{(1)};\alpha_j^{(1)})_{1,n'_1}], [\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n'_1+1,p_i'^{(1)}}]; \cdots ;$$

$$[\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(s)})_{m'+1,q'_i}] : [(\mathrm{b}_j^{(1)};\beta_j^{(1)})_{1,m'_1}], [\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m'_1+1,q_i'^{(1)}}]; \cdots ;$$

Ref

2. F.Y.Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31 No.3 (2016), 142-154.

$$\left.\begin{matrix} [(\mathrm{a}_j^{(s)};\alpha_j^{(s)})_{1,n_s'}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n_s'+1,p_i'^{(s)}}] \\ \cdot \\ [(\mathrm{b}_j^{(s)};\beta_j^{(s)})_{1,m_s'}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m_s'+1,q_i'^{(s)}}] \end{matrix}\right) = \frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k'^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \quad (1.8)$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n'}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=n'+1}^{p_i'}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{q_i'}\Gamma(1-\upsilon_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]}$$

and $$\phi_k(t_k)=\frac{\prod_{j=1}^{m_k'}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{n_k'}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=m_k'+1}^{q_{i^{(k)}}'}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=n_k'+1}^{p_{i^{(k)}}'}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]}$$

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.14) is obtained by extension of the corresponding conditions for multivariable H-function given by as:

$|argz_k'|<\frac{1}{2}B_i^{(k)}\pi$ Where

$$B_i^{(k)}=\sum_{j=1}^{n'}\mu_j^{(k)}-\iota_i\sum_{j=n'+1}^{p_i'}\mu_{ji}^{(k)}-\iota_i\sum_{j=1}^{q_i'}\upsilon_{ji}^{(k)}+\sum_{j=1}^{n_k'}\alpha_j^{(k)}-\iota_{i^{(k)}}\sum_{j=n_k'+1}^{p_{i^{(k)}}'}\alpha_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k'}\beta_j^{(k)}-\iota_{i^{(k)}}\sum_{j=m_k'+1}^{q_{i^{(k)}}'}\beta_{ji^{(k)}}^{(k)}>0 \quad (1.9)$$

with $k=1,\cdots,s;i=1,\cdots,r';i^{(k)}=1,\cdots,r^{(k)}$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form:

$$\aleph(z_1,\cdots,z_s)=0(\,|z_1|^{\alpha_1'},\cdots,|z_s|^{\alpha_s'}\,),max(\,|z_1|,\cdots,|z_s|\,)\to 0$$

$$\aleph(z_1,\cdots,z_s)\;=0(\,|z_1|^{\beta_1'},\cdots,|z_s|^{\beta_s'}\,),min(\,|z_1|,\cdots,|z_s|\,)\to\infty$$

Where $k=1,\cdots,z:\alpha_k'=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m_k'$ and

$$\beta_k'=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n_k'$$

We shall note $\aleph(z_1',\cdots,z_s')=\aleph_2(z_1',\cdots,z_s')$

II. Integral Representation of Generalized Hypergeometric Function

The following generalized hypergeometric function regarding multiple integrals contour is also required [15, page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\,[(A_P);(B_Q);-(x_1+\cdots+x_r)]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \qquad (2.1)$$

Ref

2. F.Y.Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31 No.3 (2016), 142-154.

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (2.1) is easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$
The equivalent form of Eulerian beta integral is given by (1.1):

III. Main Integral

Notes

We shall note:

$$V = m_1', n_1'; \cdots; m_s', n_s'; 1,0; \cdots; 1,0; 1,0; \cdots; 1,0$$

$$W = p_{i^{(1)}}', q_{i^{(1)}}', \iota_{i^{(1)}}; r^{(1)}; \cdots; p_{i^{(s)}}', q_{i^{(s)}}', \iota_{i^{(s)}}; r^{(s)};\ 0,1; \cdots; 0,1; 0,1; \cdots; 0,1$$

$$\mathbb{A} = [1+\sigma_i^{(1)} - \sum_{k'=1}^{v} K_{k'} \rho_i''^{(1,k')} - \sum_{k=1}^{r} \eta_{G_k,g_k} \rho_i^{(1,k)} - \theta_i^{(1)} R; \rho_i'^{(1,1)}, \cdots, \rho_i'^{(1,s)}, \tau_i^{(1,1)}, \cdots, \tau_i^{(1,l)}, 1, 0, \cdots, 0]_{1,s}, \cdots,$$

$$[1+\sigma_i^{(T)} - \sum_{k'=1}^{v} K_{k'} \rho_i''^{(T,k')} - \sum_{k=1}^{r} \eta_{G_k,g_k} \rho_i^{(T,k)} - \theta_i^{(T)} R; \rho_i'^{(T,1)}, \cdots, \rho_i'^{(T,s)}, \tau_i^{(T,1)}, \cdots, \tau_i^{(T,l)}, 1, 0, \cdots, 0]_{1,s},$$

$$[1-A_j; 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0]_{1,P},$$

$$[1-\alpha_i - \sum_{k'=1}^{v} K_{k'} \delta_i''^{(k')} - \sum_{k=1}^{r} \eta_{G_k,g_k} \delta_i^{(k)} - R\zeta_i; \delta_i'^{(1)}, \cdots, \delta_i'^{(s)}, \mu_i^{(1)}, \cdots, \mu_i^{(l)}, 1, \cdots, 1, 0, \cdots, 0]_{1,s},$$

$$[1-\beta_i - \sum_{k'=1}^{v} K_{k'} \eta_i''^{(k')} - \sum_{k=1}^{r} \eta_{G_k,g_k} \eta_i^{(k)} - R\lambda_i; \eta_i'^{(1)}, \cdots, \eta_i'^{(s)}, \theta_i^{(1)}, \cdots, \theta_i^{(l)}, 1, \cdots, 1, 0, \cdots, 0]_{1,s}$$

$$\mathbf{A} = \{(u_j; \mu_j^{(1)}, \cdots, \mu_j^{(s)}, 0, \cdots, 0, 0 \cdots, 0)_{1,n'}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(s)}, 0, \cdots, 0, 0 \cdots, 0)_{n'+1,p_i'}\}:$$

$$;\{(a_j^{(1)}; \alpha_j^{(1)})_{1,n_1'}, \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{n_1'+1,p_{i^{(1)}}'}\}; \cdots; \{(a_j^{(s)}; \alpha_j^{(s)})_{1,n_s'}, \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{n_s'+1,p_{i^{(s)}}'}\};$$

$$(1,0), \cdots, (1,0); (1,0), \cdots, (1,0)$$

$$\mathbb{B} = [1+\sigma_i^{(1)} - \sum_{k'=1}^{v} K_{k'} \rho_i''^{(1,k')} - \sum_{k=1}^{r} \eta_{G_k,g_k} \rho_i^{(1,k)} - \theta_i^{(1)} R; \rho_i'^{(1,1)}, \cdots, \rho_i'^{(1,s)}, \tau_i^{(1,1)}, \cdots, \tau_i^{(1,l)}, 0, \cdots, 0]_{1,s}, \cdots,$$

$$[1+\sigma_i^{(T)} - \sum_{k'=1}^{v} K_{k'} \rho_i''^{(T,k')} - \sum_{k=1}^{r} \eta_{G_k,g_k} \rho_i^{(T,k)} - \theta_i^{(T)} R; \rho_i'^{(T,1)}, \cdots, \rho_i'^{(T,s)}, \tau_i^{(T,1)}, \cdots, \tau_i^{(T,l)}, 0, \cdots, 0]_{1,s},$$

$$[1-B_j; 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0]_{1,Q},$$

$$[1-\alpha_i - \beta_i - \sum_{k'=1}^{v} (\delta_i''^{(k')} + \eta_i''^{(k')}) K_{k'} - \sum_{k=1}^{r} (\delta_i^{(k)} + \eta_i^{(k)}) \eta_{G_k,g_k} - (\zeta_i + \lambda_i) R;$$

$$(\delta_i'^{(1)}+\eta_i'^{(1)}),\cdots,(\delta_i'^{(s)}+\eta_i'^{(s)}),(\mu_i^{(1)}+\theta_i^{(1)}),\cdots,(\mu_i^{(l)}+\theta_i^{(l)}),1,\cdots,1]_{1,s}$$

$$\mathbf{B}=\{\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(s)},0,\cdots,0,0,\cdots,0)_{m'+1,q_i'}\}:\{(b_j^{(1)};\beta_j^{(1)})_{1,m_1'},\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m_1'+1,q'_{i^{(1)}}}\};\cdots;$$

$$\{(b_j^{(s)};\beta_j^{(s)})_{1,m_s'},\iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m_s'+1,q'_{i^{(s)}}}\}\quad;(0,1),\cdots,(0,1);(0,1),\cdots,(0,1)$$

We have the following multiple Eulerian integrals, we obtain the Aleph-function of $(r+l+T)-$ variables.

Theorem

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$\aleph_1\begin{pmatrix} \mathrm{z}_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{pmatrix}\aleph_2\begin{pmatrix} \mathrm{z}_1'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(s)}}(v_i-x_i)^{\eta_i'^{(s)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,s)}}}\right]\end{pmatrix}$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \mathrm{z}_1''\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_v''\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right]\end{pmatrix}R_n^{\alpha,\beta}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi'(w,v,u,t',e,k_1,k_2)$$

Notes

Notes

$$\aleph^{0,sT+P+2s+n':V}_{U_{sT+P+2s;sT+Q+s}:W}\left(\begin{array}{c|c} z'_1w_1 & \mathbb{A},\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z'_sw_s & \cdot \\ g_1W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ g_lW_l & \cdot \\ G_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ G_T & \mathbb{B},\ \mathbf{B} \end{array}\right) \tag{3.1}$$

where $U_{st+P+2s;sT+Q+s} = sT+P+2s+p'_i, sT+Q+s+q'_i, \iota_i; r'$

$$\psi'(w,v,u,t',e,k_1,k_2) = \frac{\psi(w,v,u,t,e,k_1,k_2)}{\prod_{i=1}^t\prod_{j=1}^W (u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^v \rho_i''^{(j,k')}K_{k'}+\sum_{k=1}^r \rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}R}}$$

$$\times\frac{\prod_{i=1}^t (v_i-u_i)^{\sum_{k'=1}^v(\delta_i''^{(k')}+\eta_i''^{(k')})K_{k'}+\sum_{k=1}^r(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}+(\zeta_i+\lambda_i)R}}{\prod_{i=1}^t\prod_{j=W+1}^T (u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^v \rho_i''^{(j,k')}K_{k'}+\sum_{k=1}^r \rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}R}}$$

$$w_m = \prod_{i=1}^t\left[(v_i-u_i)^{\delta_i'^{(m)}+\eta_i'^{(m)}}\prod_{j=1}^W\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i'^{(j,m)}}\prod_{j=W+1}^T\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\rho_i'^{(j,m)}}\right], m=1,\cdots,s$$

$$W_k = \prod_{i=1}^t\left[(v_i-u_i)^{\mu_i^{(k)}+\theta_i^{(k)}}\prod_{j=1}^W\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\prod_{j=W+1}^T\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\right], k=1,\cdots,l$$

$$G_j = \prod_{i=1}^t\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=1,\cdots,W$$

$$G_j = -\prod_{i=1}^t\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=W+1,\cdots,T$$

$$\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty} = \sum_{G_1,\cdots,G_r=1}^{m_1,\cdots,m_r}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

Provided that:

(A) $W\in[0,T]; u_i,v_i\in\mathbb{R}; i=1,\cdots,t$

(B) $min\{\delta_i^{(g)},\eta_i^{(g)},\delta_i'^{(h)},\eta_i'^{(h)},\delta_i''^{(k)},\eta_i''^{(k)},\zeta_i,\eta_i\}\geqslant 0; g=1,\cdots,r; i=1,\cdots,t; h=1,\cdots,s; k=1,\cdots,v$

$min\{\rho_i^{(j,g)},\rho_i'^{(j,h)},\rho_i''^{(j,k')},\theta_i^{(j)},\tau_i^{(j,k)}\}\geqslant 0; j=1,\cdots,T; i=1,\cdots,t; g=1,\cdots,r; h=1,\cdots,s; k'=1,\cdots,v, k=1,\cdots,l$

(C) $\sigma_i^{(j)} \in \mathbb{R}, U_i^{(j)}, V_i^{(j)} \in \mathbb{C}, z_{i'}, z_{j'}, z''_{k'}, g_k, G_j \in \mathbb{C}; i=1,\cdots,t; j=1,\cdots,T; i'=1,\cdots,r;$

$j'=1,\cdots,s; k'=1,\cdots,v; k=1,\cdots,l$

(D) $max\left[\left|\dfrac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right] < 1, i=1,\cdots,s; j=1,\cdots,W$ and

$$max\left[\left|\dfrac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right] < 1, i=1,\cdots,s; j=W+1,\cdots,T$$

(E) $\left|arg\left(z_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i^{(j,k)}}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} - \delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^{T}\rho_i^{(j,k)} > 0$$

$$\left|arg\left(z'_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i'^{(j,k)}}\right)\right| < \frac{1}{2}B_i^{(k)}\pi, \text{ where}$$

$$B_i^{(k)} = \sum_{j=1}^{n'}\mu_j^{(k)} - \iota_i\sum_{j=n'+1}^{p'_i}\mu_{ji}^{(k)} - \iota_i\sum_{j=1}^{q'_i}\upsilon_{ji}^{(k)} + \sum_{j=1}^{n'_k}\alpha_j^{(k)} - \iota_{i^{(k)}}\sum_{j=n'_k+1}^{p'_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m'_k}\beta_j^{(k)} - \iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}$$

$$-\delta_i'^{(k)} - \eta_i'^{(k)} - \sum_{j=1}^{T}\rho_i'^{(j,k)} > 0$$

(F) $Re\left(\alpha_i+\zeta_iR+\sum_{j=1}^{r}\delta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\delta_i'^{(k)}\min_{1\leqslant j\leqslant m'_k} R\left(\dfrac{b_j^{(k)}}{\beta_j^{(k)}}\right) > 0$ and

$$Re\left(\beta_i+\lambda_iR+\sum_{j=1}^{r}\eta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\eta_i'^{(k)}\min_{1\leqslant j\leqslant m'_k} Re\left(\dfrac{b_j^{(k)}}{\beta_j^{(k)}}\right) > 0 \text{ for } i=1,\cdots,t$$

(G) $P \leqslant Q+1$. The equality holds, also,

either $P>Q$ and $\sum_{k=1}^{l}\left|g_k\left(\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\tau_i^{(j,k)}}\right)\right|^{\frac{1}{Q-P}} < 1 \qquad (u_i \leqslant x_i \leqslant v_i; i=1,\cdots,t)$

Notes

or $\quad P \leqslant Q$ and $\max_{1\leqslant k\leqslant l} \left[\left| \left(g_k \prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{-\tau_i^{(j,k)}} \right) \right| \right] < 1 \quad (u_i \leqslant x_i \leqslant v_i; i = 1, \cdots, t)$

Proof

To establish the formula (3.1), we first express the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ in series with the help of (1.4), the multivariable Aleph-function $\aleph_1(z_1,\cdots,z_r)$ in serie with the help of (1.7), the sequence of functions in series with the help of (1.4), use integral contour representation with the help of (1.8) for the multivariable Alephfunction $\aleph_2(z'_1,\cdots,z'_s)$ occurring in its left-hand side and use the integral contour representation with the help of (2.1) for the Generalized hypergeometric function ${}_PF_Q(.)$.

Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now we write:

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} \prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}$$

where $K_i^{(j)} = \upsilon_i^{(j)} - \theta_i^{(j)}R - \sum_{l=1}^{r}\rho_i^{(j,l)}\eta_{G_l,g_l} - \sum_{l=1}^{s}\rho_i'^{(j,l)}\psi_l - \sum_{l=1}^{v}\rho_i''^{(j,v)}K_l$ where $i = 1,\cdots,t; j = 1,\cdots,T$

and express the factors occurring in R.H.S. of (3.1) regarding the following Mellin-Barnes integrals contour, we obtain:

$$\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^W}\int_{L'_1}\cdots\int_{L'_W}\prod_{j=1}^{W}\Big[\Gamma(-\zeta'_j)\Gamma(-K_i^{(j)}+\zeta'_j)$$

$$\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i-u_i)}{(u_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta'_j}\Bigg]\,\mathrm{d}\zeta'_1\cdots\mathrm{d}\zeta'_W \tag{3.3}$$

and

$$\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} = \prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^{T-W}}\int_{L'_{W+1}}\cdots\int_{L'_T}\prod_{j=W+1}^{T}\Big[\Gamma(-\zeta'_j)\Gamma(-K_i^{(j)}+\zeta'_j)$$

$$\prod_{j=W+1}^{T}\left[-\frac{(U_i^{(j)}(v_i-x_i)}{(v_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta'_j}\Bigg]\,\mathrm{d}\zeta'_{W+1}\cdots\mathrm{d}\zeta'_T \tag{3.4}$$

We apply the Fubini's theorem for multiple integrals. Finally evaluating the innermost **x**-integral with the help of (1.1) and reinterpreting the multiple Mellin-Barnes integrals contour in terms of multivariable Aleph-function of $(r+l+T)$–variables, we obtain the formula (3.1).

Notes

IV. Particular Cases

a) Multivariable H functions

Here, the multivariable Aleph-functions $\aleph_1$ and $\aleph_2$ reduce to multivariable H-functions defined by Srivastava and Panda [16,17], we have the following Eulerian integral:

Corollary 1

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$H_1\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{array}\right)H_2\left(\begin{array}{c} \mathrm{z}'_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_s\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(s)}}(v_i-x_i)^{\eta_i'^{(s)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,s)}}}\right]\end{array}\right)$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right]\end{array}\right)R_n^{\alpha,\beta}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi'(w,v,u,t',e,k_1,k_2)$$

Ref

17. H.M. Srivastava and R.Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. (1976), 265-274.

$$H^{0,sT+P+2s+n':V}_{sT+P+2s+p',sT+Q+s+q':W'}\left(\begin{array}{c|c} \mathrm{z}'_1 w_1 & \mathbb{A},\ \mathbf{A}' \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_s w_s & \cdot \\ \mathrm{g}_1 W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{g}_l W_l & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{G}_T & \mathbb{B},\ \mathbf{B}' \end{array}\right) \tag{4.1}$$

where $W = p'_1, q'_1; \cdots ; p'\ \ q'; 0\ 1; \cdots ; 0\ 1; 0\ 1; \cdots ; 0\ 1$ and the validity conditions are the same that (3.1) for $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}}, \iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \rightarrow 1$ and $R = R^{(1)} = \cdots = R^{(r)} = r' = r^{(1)} = \cdots = r^{(s)}. = 1$ (condition 1)
The quantities $\mathbf{A}', \mathbf{B}'$ are equal respectively to A, B with the conditions 1.

b) Aleph-functions of two variables

If $r = s = 2$, the multivariable Aleph-functions reduce to Aleph-functions of two variables defined by Sharma [10].

Corollary 2

$$\int_{u_1}^{v_1} \cdots \int_{u_t}^{v_t} \prod_{i=1}^{t} \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\aleph_1 \left(\begin{array}{c} \mathrm{z}_1 \prod_{i=1}^t \left[\frac{(x_i - u_i)^{\delta_i^{(1)}} (v_i - x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_2 \prod_{i=1}^t \left[\frac{(x_i - u_i)^{\delta_i^{(2)}} (v_i - x_i)^{\eta_i^{(2)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i^{(j,2)}}} \right] \end{array} \right) \aleph_2 \left(\begin{array}{c} \mathrm{z}'_1 \prod_{i=1}^t \left[\frac{(x_i - u_i)^{\delta_i'^{(1)}} (v_i - x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i'^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'_2 \prod_{i=1}^t \left[\frac{(x_i - u_i)^{\delta_i'^{(2)}} (v_i - x_i)^{\eta_i'^{(2)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i'^{(j,2)}}} \right] \end{array} \right)$$

$$S^{\mathfrak{M}_1, \cdots, \mathfrak{M}_v}_{N_1, \cdots, N_v} \left(\begin{array}{c} \mathrm{z}''_1 \prod_{i=1}^t \left[\frac{(x_i - u_i)^{\delta_i''^{(1)}} (v_i - x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i''^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}''_v \prod_{i=1}^t \left[\frac{(x_i - u_i)^{\delta_i''^{(v)}} (v_i - x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i''^{(j,v)}}} \right] \end{array} \right) R_n^{\alpha,\beta} \left[\prod_{j=1}^{t} \left[\frac{(x_i - u_i)^{\zeta_i} (v_i - x_i)^{\lambda_i}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\theta_i^{(j)}}} \right] \right]$$

Ref

10. K. Sharma, On the integral representation and applications of the generalized function of two variables, International Journal of Mathematical Engineering and Sciences, 3(1) (2014), 1-13.

$$ {}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t $$

$$ =\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right] $$

$$ \sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{G_2=1}^{m_2}\sum_{g_1,g_2=0}^{\infty}\phi_2\frac{\prod_{k=1}^{2}\phi_{2k}z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{2}g_k}}{\prod_{k=1}^{2}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{2}g_k!}\,{}_vz_1^{\prime\prime K_1}\cdots z_v^{\prime\prime K_v}\psi'(w,v,u,t',e,k_1,k_2) $$

$$ \aleph_{U_{sT+P+2s;sT+Q+s}:W_2}^{0,n'+sT+P+2s:V_2}\left(\begin{array}{c|c} \mathrm{z}'_1w_1 & \mathbb{A}_2,\ \mathbf{A}_2\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_2w_2 & \cdot\\ \mathrm{g}_1W_1 & ;\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{g}_lW_l & \cdot\\ \mathrm{G}_1 & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{G}_T & \mathbb{B}_2,\ \mathbf{B}_2 \end{array}\right) \tag{4.2} $$

The validity conditions are the same that (3.1) with $r=s=2$. The quantities $\phi_2,\phi_{2k},V_2,W_2,\mathbb{A}_2,\mathbb{B}_2,\mathbf{A}_2,\mathbf{B}_2$ are equal to $\phi,\phi_k,V,W,\mathbb{A},\mathbb{B},\mathbf{A},\mathbf{B}$ respectively for $r=s=2$.

c) Aleph-function of one variable

Corollary 3

If $r=s=1$, the multivariable Aleph-functions reduce to Aleph-functions of one variable defined by Sudland [18].

$$ \int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right] $$

$$ \aleph_1\left(\mathrm{z}\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\right)\aleph_2\left(\mathrm{z}'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\right) $$

Ref

18. N. Südland, B. Baumann and T.F. Nonnenmacher T, Open problem: who knows about the Aleph-functions? Fract. Calc. Appl. Anal, 1(4) (1998): 401-402.

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} z''_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ z''_v\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right] \end{pmatrix} R_n^{\alpha,\beta}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{g_1=0}^{\infty}\phi_1\frac{z_k^{\eta_{G_1,g_1}}(-)^{g_1}}{\delta_{G^{(1)}}^{(1)}g_1!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi'(w,v,u,t',e,k_1,k_2)$$

$$\aleph_{U_{sT+P+2s;sT+Q+s}:W_1}^{0,n'+sT+P+2s:V_1}\left(\begin{array}{c|c} \mathrm{z'w}_1 & \mathbb{A}_1,\ \mathbf{A}_1 \\ \mathrm{g}_1W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{g}_lW_l & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{G}_T & \mathbb{B}_1,\ \mathbf{B}_1 \end{array}\right) \tag{4.3}$$

The validity conditions are the same that (3.1) with $r=s=1$. The quantities $\phi_1, V_1, W_1, \mathbb{A}_1, \mathbb{B}_1, \mathbf{A}_1, \mathbf{B}_1$ are equal to $\phi_k, V, W, \mathbb{A}, \mathbb{B}, \mathbf{A}, \mathbf{B}$ respectively for $r=s=1$ and

$$U_{st+P+2s;sT+Q+s}=sT+P+2s+p'_{i^{(1)}},sT+Q+s+q'_{i^{(1)}},\iota_{i^{(1)}};r^{(1)}.$$

Remark: By the similar procedure, the results of this document can be extended to the product of any finite number of multivariable Aleph-functions and class of multivariable polynomials defined by Srivastava [12].

V. Conclusion

Our main integral formula is unified in nature and possesses manifold generality. It acts a fundamental expression and using various particular cases of the multivariable Aleph-function, the class of multivariable polynomials and a general sequence of functions, we can obtain a large number of other integrals involving simpler special functions and polynomials of one and several variables.

Ref

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Notes