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A Litle Note on the Twisted/Warped Product Metrics and the Hopf Conjecture

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A Litle Note on the Twisted/Warped Product Metrics and the Hopf Conjecture

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Abstract- This work defines some notions on the twisted, warped product metric on the manifold $\mathbb{S}^2 \times \mathbb{S}^2$ before prouving that Hopf conjecture is true in the class of this metrics.

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I. Introduction

Let g_0 be the riemannian metric induced by $\mathbb{R}^6$ on $M = \mathbb{S}^2 \times \mathbb{S}^2$ and let T_pM denoted the tangent vector space of M at p. For any 2-plane $\sigma \subset T_pM$ spanned by two tangent vector fields, we define a function $K(\sigma)$ called the sectional curvature of σ. We will say that the riemannian manifold M has positive sectional curvature if for every point $p \in M$ the sectional curvature $K(\sigma)$ of every 2-plane $\sigma \subset T_pM$ is positive. An example of such manifolds are the n dimensional spheres $\mathbb{S}^n$ with the metric induced by $\mathbb{R}^{n+1}$. In this case its sectional curvature is equal to 1 for any 2-plane $\sigma \subset T_pM$. In general the question of deciding if a given manifold admits a riemannian metric with positive sectional curvature is a difficult one. One of the Hopf conjecture can be stated as follows: The product manifold $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit any riemannian metrics with strictly positive sectional curvature. For the relevance of this problem, we refer to the [4], [5] and [6].

A direct computation shows that when we endowed $\mathbb{S}^2 \times \mathbb{S}^2$ with the riemannian metric g_0, then for any $(a, b) \in \mathbb{S}^2 \times \mathbb{S}^2$ the sectional curvature of a plane spaned by a vector of the form $(v, 0) \in T_{(a,b)}(\mathbb{S}^2 \times \mathbb{S}^2)$ and a vector of the form $(0, w) \in T_{(a,b)}(\mathbb{S}^2 \times \mathbb{S}^2)$, with $v, w \in \mathbb{R}^3$, is zero, we also found out that the sectional curvature for the planes $T_{(a,b)}\mathbb{S}^2 \times \mathbb{S}^2 \cap \mathbb{R}^3 \times \{0\}$ and $T_{(a,b)}\mathbb{S}^2 \times \mathbb{S}^2 \cap \{0\} \times \mathbb{R}^3$ is 1, for any other plane the sectional curvature is a number between 0 and 1 ([4]).

The notion of warped product manifolds was introduced by Bishop and O'Neil [7] for contructing manifolds of negative curvature. Later this notion has been extented to be twisted product manifolds [[9], [8]].

More precisely, let (M_1, g_1) and (M_2, g_2) be two-riemannian manifolds of dimension m_1 and m_2 (respectively) and let $\pi_1 : M_1 \times M_2 \longmapsto M_1$ and $\pi_2 : M_1 \times M_2 \longmapsto M_2$ be the canonical projections. Also let $f : M_1 \times M_2 \longmapsto \mathbb{R}^+$ be a smooth function. Then the twisted product $M = M_1 \times_f M_2$ of the manifolds (M_1, g_1) and (M_2, g_2) with twisting

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function f, is defined to be the product manifold $M = M_1 \times M_2$ with the riemannian tensor $g = g_1 \oplus f^2 g_2$ given by $g = \pi_1^* g_1 + f^2 \pi_2^* g_2$. Moreover, if the function f only depends on the points of M_1, that is $f : M_1 \longmapsto \mathbb{R}^+$, then $M = M_1 \times_f M_2$ is called the warped product manifold of (M_1, g_1) and (M_2, g_2) with warping function f. In particular, if $f = 1$, then $M = M_1 \times_1 M_2$ is the product manifold of (M_1, g_1) and (M_2, g_2) with the product metric defined by $g = \pi_1^* g_1 + \pi_2^* g_2$.

Now, let $(\mathbb{S}^2 \times_f \mathbb{S}^2, g)$be a twisted product manifold with twisting function $f$and let $h = log(f)$. In this paper, we prove the following mains results.

Theorem 1.1. For any smooth function $h : \mathbb{S}^2 \times \mathbb{S}^2 \longmapsto \mathbb{R}$, the twisted product manifold $(\mathbb{S}^2 \times \mathbb{S}^2, g)$ has no positive sectional curvature.

Corollary 1.1. For any smooth function $h : \mathbb{S}^2 \longmapsto \mathbb{R}$, the warped product manifold $(\mathbb{S}^2 \times \mathbb{S}^2, g)$ has no positive sectional curvature.

II. Preliminaries

Let M_1 and M_1 be two riemannian manifolds, and let $f > 0$ be smooth fonction on M_1, the warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ furnished with the riemannian metric

$$g = \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2, \tag{1}$$

where π_1 and π_2 are canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. This notion has been extented to several forms. Let (M, g_1) and (M_2, g_2) be riemannian manifolds, $h : M_1 \times M_2 \longmapsto \mathbb{R}$ a positive and smooth function, g a riemannian metric on the manifold $M = M_1 \times M_2$ and assume that the canonical foliation L_1 and L_2 intersects perpendicularly everywhere. Then g is the metric tensor of

(1) a twisted product $M_1 \times_h M_2$ if only if L_1 is a totally geodesic foliation and L_2 is a totally umbilic foliation,

(2) a warped product $M_1 \times_h M_2$ if only if L_1 is a totally geodesic foliation and L_2 is a spheric foliation,

(3) a usual product of riemannian manifolds if only if L_1 and L_2 are totally geodesic foliations.

let (M, g_1) and (M_2, g_2) be riemannian manifolds with the Levi-Civita connections ∇^1 and ∇^2, respectively, and let ∇ denote the Levi-Civita connection of the twisted product manifold $M_1 \times_f M_2$ of (M_1, g_1) and (M_2, g_2) with twisting function f. Also, let $h = log(f)$ and let $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$ be the sets of lifts of vector fields on M_1 and M_2 to $M_1 \times M_2$, respectively.
We have the following proposition for a twisted product manifold.

Proposition 2.1. ([7], [9], [8]) Let $M = M_1 \times_f M_2$ be a twisted product manifold with the metric $g = g_1 \oplus f^2 g_2$ and let $X, Y \in \mathcal{L}(M_1)$ and $V, W \in \mathcal{L}(M_2)$. Then
1) $\nabla_X Y = \nabla^1_X Y$,
2) $\nabla_X V = \nabla_V X = X(h)V$,
3) $\nabla_V W = \nabla^2_V W + V(h)W + W(h)V - g(V, W)\nabla h$,
where $h = log(f)$ and ∇h is the gradient of the function h.

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We define the hessian $Hess_1h(X,Y) = XY(h) - (\nabla^1_X Y)(h)$, for $X, Y \in \mathcal{L}(M_1)$. If $X, Y \in \mathcal{L}(M_1)$ and $V \in \mathcal{L}(M_2)$, we have $XV(h) = VX(h)$ and the hessian $hessh$ of h on the manifold $M = M_1 \times_f M_2$ satisfies

$$Hessh(X,V) = XV(h) - X(h)V(h), \tag{2}$$

$$Hessh(X,Y) = Hess_1h(X,Y). \tag{3}$$

Notes

We have the following proposition for the riemannian curvature tensor of a twisted product manifold.

Proposition 2.2. *([7], [9], [8]) Let $M = M_1 \times_f M_2$ be a twisted product manifold and let $X, Y, Z \in \mathcal{L}(M_1)$, $V, W, U \in \mathcal{L}(M_2)$. Then, we have*

1) $R(X,Y)Z = R^1(X,Y)Z$,
2) $R(X,Y)V = 0$,
3) $R(V,W)U = R^2(V,W)U - |\nabla h|^2[g_2(V,U)W - g_2(W,U)V]$,
4) $R(X,W)V = [X(h)V(h) + Hessh(X,Y)]W - g(W,V)[X(h)\nabla h + H^h(X)]$,

where H^h is the hessian tensor of h on $M_1 \times_f M_2$, that is $Hessh(X,Y) = g(H^h(X),Y)$.

Lemma 2.1. *Let $M = M_1 \times_f M_2$ be a twisted product manifold. Then, for orthonormal vectors X and V, where $X \in \mathcal{L}(M_1)$, $V \in \mathcal{L}(M_2)$. The mixed sectional curvature is given by*

$$K(X,V) = g(R(X,V)V,X) = -[g(\nabla h, X)^2 + Hessh(X,X)], \tag{4}$$

where $h = log(f)$ and f positive smooth function.

Proof of the Lemma. Using the formula in 4) of the Proposition 2.2 with the definition of the sectional curvature in orthonormal mixed coordinate systems $\{X,V\}$, we obtain the result (4) of the lemma. □

III. Proof of the Theorem and the Corollary

In this section we prove the theorem and the corollary. The proof of the corollay is strongly related to that of the theorem, it is the same proof.

Proof of the Theorem. we will assume that the sectional curvature of the manifold (M,g) is everywhere positive.
We consider the case where M_1 and M_2 are the standard unit sphere $\mathbb{S}^2$, and the twisted product manifold $\mathbb{S}^2 \times_f \mathbb{S}^2$ with the twisting function f and $h = log(f)$. Let X and Y be orthonormal vectors, tangents to the left sphere $\mathbb{S}^2$, and V and W orthonormal vectors, tangents to the right sphere $\mathbb{S}^2$. Then, it readily follows from 1) and 3) of the Proposition 2.2 that the sectional curvatures $K(X,Y) = g(R(X,Y)Y,X)$ and $K(V,W) = g(R(V,W)W,V)$ are strictly positive, or still if for every unit tangent vector X one have

$$g(\nabla h, X)^2 + hessh(X,X) < 0. \tag{5}$$

Finally, Since M is a compact manifold, the smooth function $h : \mathbb{S}^2 \times \mathbb{S}^2 \longmapsto \mathbb{R}$ has a minimum $m = (m_1, m_2) \in \mathbb{S}^2 \times \mathbb{S}^2$. It is sufficient to show that at critical point m of h, we have $g(\nabla h, X) = 0$ and the mixed sectional curvature in (4) becomes

$$K(X, V) = -XXh. \tag{6}$$

Since the critical point m of h is a minimum then, at m we have that $XXh \geq 0$. And in this case, the sectional curvature $K(X, V)$ is not positive. This leads to a contradiction.

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