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Smoothness for Some Selected Test Functions Relative to Shape Parameter via IMQ

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SMOOTHNESSFORSOMESELECTEDTESTFUNCTIONSRELATIVETOSHAPEPARAMETERVIAIMQ

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Ref

1. J. Biazar, M. Hosami, Selection of an interval for variable shape parameter in approximation by radial basis funtions, Adv. Numer. Analy. http://dx.doi.org/10.1155/2016/1397849

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Abstract- Radial basis function (RBF) approximation has the potential to provide accurate func-tion approximations for large data site given at scattered node locations which yields smooth solutions for a given number of node points especially when the basis functions are scaled to be nearly at and when the shape parameter is choose wisely. In this paper, we concentrate on the choice of shape parameter, which must be choose wisely and the simplest strategy we adopt is to perform a series of interpolation experiments by varying the interval of shape parameter, and then pick the \best" one. The \best" was pick by checking the errors for different data sites and the smoothness of the error graphs. The results shows that the choice of interval for the shape parameter give better accuracy and smoothness of the graphs.

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I. Introduction

Radial basis function (RBF) approximation [8, 10] is emerging as an important method for approximation, and interpolation of test functions for data given at scattered node locations, with computational domains in higher dimensions. And the application of radial functions to the solution of the scattered data interpolation problem benefits from the fact that the interpolation problem becomes insensitive to the dimension (d) of the space in which the data sites lie. In recent years, various RBF-based methods have gained fast growing attention from a broad range of scientific computing and engineering applications, such as multivariate scattered data processing [8,11], interpolation of functions [1,4,6,7,18] numerical solutions of partial differential equations (PDEs) [3,12–15], just to mention a few. The main advantages are spectral convergence rates that can be achieved using infinitely smooth basis functions, geometrical flexibility, and ease of implementation with shape parameter play important role in determine the accuracy of the interpolant [1, 19].

a) Basic idea of RBF

Define the RBF interpolant of a function $f(x)$ as

$$\Gamma_f(x) = \sum_{k=1}^{N} \lambda_k \varphi(r, \epsilon), \;\; r = \parallel x - x_k \parallel, \;\; x \in \mathbb{R}^d \tag{1.1}$$

where the value of the interpolant at any location x is obtained as weighted sum of r and the coefficients $\lambda_k, k = 1, \cdots, N$ depend on the right hand side $f(x_j)$ and ϵ is called the

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shape parameter which must be choosing wisely, for better accuracy. The coefficients λ_k are determined by enforcing the interpolation condition

$$\Gamma_f(x_j) = f(x_j), \quad j = 1, \dots, K \tag{1.2}$$

and when $N = K$, (1.2) becomes $N \times N$ linear system of the form

$$\theta_{NN}\sigma = F \tag{1.3}$$

where $F = \{f(x_j)\}_{j=1}^{N}$, $\sigma = \{\lambda_j\}_{j=1}^{N}$ and θ_{NN} is an $N \times N$ matrix whose entries are $\theta_{NN_{(j,k)}} = \varphi(\| x_j - x_k \|, \epsilon)$.

Therefore, RBF interpolant at M evaluation points on set $\Pi := \{x_i\}_{i=1}^{M}$ read

$$\Gamma_f(x_i) = \theta_{MN}\theta_{NN}^{-1}F \tag{1.4}$$

and some of the commonly used RBF are shown in the Table 1

Table 1: Some commonly used RBF

Type of RBF	$\varphi(r, \epsilon)$
Gaussian	$\exp(-\epsilon^2 r^2)$
Inverse Multiquadric (IMQ)	$(1 + \epsilon^2 r^2)^{-\frac{1}{2}}$
Multiquadric	$(1 + \epsilon^2 r^2)^{\frac{1}{2}}$
Thin Plate Spline (TPS)	$r^3 \log(r)$
Wendland	$(1 - \epsilon r)_+^4 (4r + 1)$

II. Role of Shape Parameter (ϵ)

The shape parameter has important consequences on the stability and accuracy of RBF interpolants, as the different choices of ϵ lead to different results in terms of computational error and the smaller the shape parameter, the flatter or wider the basis function (see figure 1).

In this work, we are generally interested in the range of the shape parameter that will incurs numerical stability, the higher accuracy of RBF interpolant and the smoothness of the error graphs. Hence, basically, we would like to choose the best shape parameter ϵ, to achieve best shape of the basis functions with respect to RBF interpolation error.

Notes

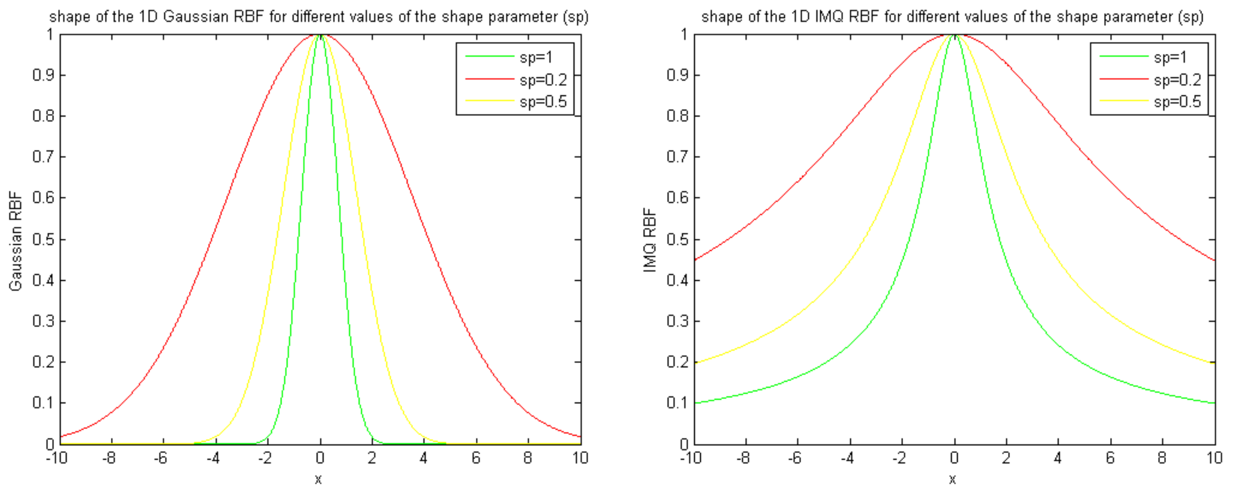

Figure 1: Examples of the shape of the 1D Gaussian and IMQ RBFs for di˙erent values of the shape parameter (sp)

III. Selection of Range of Shape Parameter (ϵ) for IMQ

To determine the best shape parameter ϵ^*, different strategies have been used by different authors, Xiang and Wang [22] applied trigonometric variable shape parameter (ϵ) to generalized Multiquadric RBF, Sarra and Sturgill [23] showed that the random variable shape parameter produces the most accurate results, if the centers are uniformly spaced, Biazar and Hosami [1] developed an algorithm for determining an interval for MQ shape parameter, In this work, we are focusing our attention on *Trial and Error* approach. The *Trial and Error* algorithm used in this work helps us to clarify quite well how the shape parameter acts on the basis functions (as shown in Figure 1) we choose for the interpolation, and so give us insight about the choice of a small range in which we will set ϵ. The algorithm is given below:

Trial and Error Algorithm

Data: f: function to be interpolated, Θ: set of data, B base of $\mathbf{S}$ interpolation subspace, Π: set of evaluation points

Result: ϵ^* optimal shape parameter

Define the data sites X from Θ

Define a large range for the choice of ϵ, in an interval $[a,\ b]$;

Define how many shape parameters are going to be tested, namely k;

for $i = 1, \ldots, k$

for **do**

$\epsilon_i = a + (i-1)\,\frac{b-a}{k}$;

Solve $\theta\,(X, B(X,\epsilon))\,\sigma = F$;

Evaluate $\Gamma_f(X,\epsilon;x)$;

Evaluate the error $\varepsilon_i = \mathrm{RMS} = \sqrt{\frac{\sum_{x\in\Pi}|f(x)-\Gamma_f(x)|^2}{\Pi}}$

end

Ref

22. S. Xiang, K. M.Wang, Y.-T. Ai, Y.-D. Sha, and H. Shi, Trigonometric variable shape parameter and exponent strategy for generalized multiquadric radial basis function approximation, Applied Mathematical Modelling, 36(5)(2012), 19311938.

Plot the (RMS) ϵ_i versus ε_i.
Find the index i^*=minimum ε_k
Set $\epsilon^* = \epsilon_{i^*}$
Compute the interpolant with ϵ^*

IV. Numerical Experiments

In this section, we implement the algorithm discussed in section 3 on three test functions in $\mathbb{R}^2$ (since RBF methods are dimension blind) using inverse multiquadrics (IMQ) RBF with different knots N and M evaluation points by setting $\Theta = [0,\ 1]^2$, $\mathbf{x} = (x, y)$, $\mathbf{x} \in \Theta \subseteq \mathbb{R}^2$, $f(\mathbf{x}) \in \mathbb{R}$. Firstly, the algorithm is applied to find the range of the shape parameter for each test function, then the interpolant is recomputed by using the range of the shape parameter strategy in the proposed interval (see Figure 2). The accuracy are computed by finding the Root Mean Square (RMS) error approach and then pick the best (see Figure 3). The Root Mean Square (RMS) errors were calculated by the formula

$$\varepsilon_i = \text{RMS} = \sqrt{\frac{\sum_{x\in\Pi} |f(\mathbf{x}) - \Gamma_f(\mathbf{x})|^2}{\Pi}}$$

Example 4.1: We consider 2D Franke function

$$f(\mathbf{x}) = \frac{3}{4}e^{-\left((9x-2)^2+\frac{(9y-2)^2}{4}\right)} + \frac{3}{4}e^{-\left(\frac{(9x+1)^2}{49}+\frac{(9y+1)^2}{10}\right)} + \frac{1}{2}e^{-\left((9x-7)^2+\frac{(9y-3)^2}{4}\right)} - \frac{1}{5}e^{-\left((9x-4)^2+(9y-7)^2\right)} \tag{4.1}$$

The RMS errors are tabulated in Table 2 with different ϵ, the corresponding graph is presented in Figure 2 while Figure 4 shows the interpolant at $N = 4225$ and its corresponding maximum error and the ϵ (with respect to the smoothness of the error graph) used for the computation of the interpolant is presented in Figure 3

Example 4.2: We consider 2D Ackley function

$$f(\mathbf{x}) = -20e^{\left(-\frac{1}{5}\sqrt{\frac{1}{2}(x^2+y^2)}\right.} - e^{\frac{1}{2}(\cos(2\pi x)+\cos(2\pi y))} + e^1 + 20 \tag{4.2}$$

The RMS errors are tabulated in Table 3 with different ϵ and the corresponding graph is presented in Figure 2 while Figure 5 shows the interpolant at $N = 4225$ and its corresponding maximum error and the ϵ (with respect to the smoothness of the error graph) used for the computation of the interpolant is presented in Figure 3

Example 4.3: We implement Trial and Error on 2D Beale function

$$f(\mathbf{x}) = \left(\frac{3}{2} - x + xy\right)^2 + \left(\frac{9}{4} - xy^2\right)^2 + \left(\frac{21}{8} - x + xy^3\right)^2 \tag{4.3}$$

Figure 6 display the interpolant that correspond to Figure 3 reference to the smoothness of the error graph while Table 4 is the RMS errors and its shape parameters.

V. Conclusion

In this paper, Trial and Error algorithm is proposed to determine the range of shape parameter for IMQ interpolation that produces the small error for different data sites (N) respect

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to the smooth error graph. The implementation of the scheme are illustrated on 3 selected test functions. Numerical results demonstrate that our scheme is an effective and reliable numerical technique for interpolating functions.

Table 2: Shape parameters (ϵ) and its corresponding RMS errors for Franke function (Example 4.1)

data sites (N)	$\epsilon = 6.0$	$\epsilon = 6.2$	$\epsilon = 6.4$	$\epsilon = 6.6$	$\epsilon = 6.8$	$\epsilon = 7.0$
9	1.921311e-001	1.940839e-001	1.960197e-001	1.979357e-001	1.998295e-001	2.016994e-001
25	3.080634e-002	3.158354e-002	3.240742e-002	3.327625e-002	3.418810e-002	3.514088e-002
36	1.928667e-002	1.972654e-002	2.019069e-002	2.067860e-002	2.118985e-002	2.172405e-002
289	2.500556e-004	2.751982e-004	3.009742e-004	3.272891e-004	3.540543e-004	3.811861e-004
1089	3.543262e-006	4.311747e-006	5.184915e-006	6.167679e-006	7.264324e-006	8.478486e-006
4225	6.912828e-009	7.890668e-009	9.963406e-009	1.362776e-008	1.814020e-008	2.402932e-008

Table 3: Shape parameters (ϵ) and its corresponding RMS errors for Ackley function (Example 4.2)

data sites (N)	$\epsilon = 7.0$	$\epsilon = 7.2$	$\epsilon = 7.4$	$\epsilon = 7.6$	$\epsilon = 7.8$	$\epsilon = 8.0$
9	4.372483e-001	4.638774e-001	4.903400e-001	5.166114e-001	5.426708e-001	5.685013e-001
25	1.191346e-001	1.188800e-001	1.187683e-001	1.188320e-001	1.191021e-001	1.196076e-001
36	8.979158e-002	8.958214e-002	8.934477e-002	8.909150e-002	8.883506e-002	8.858881e-002
289	4.959255e-003	5.231761e-003	5.504290e-003	5.776340e-003	6.047447e-003	6.317180e-003
1089	1.357913e-004	1.505501e-004	1.665895e-004	1.838989e-004	2.024630e-004	2.222622e-004
4225	6.598121e-006	6.093603e-006	6.372372e-006	6.208202e-006	6.156283e-006	6.123180e-006

Table 4: Shape parameters (ϵ) and its corresponding RMS errors for Beale function (Example 4.3)

data sites (N)	$\epsilon = 6.0$	$\epsilon = 6.2$	$\epsilon = 6.4$	$\epsilon = 6.6$	$\epsilon = 6.8$	$\epsilon = 7.0$
9	4.687441e-001	5.164511e-001	5.695731e-001	6.267950e-001	6.870994e-001	7.497095e-001
25	2.558036e-001	2.492158e-001	2.418695e-001	2.338618e-001	2.252999e-001	2.163039e-001
36	2.088475e-001	2.086083e-001	2.077394e-001	2.062678e-001	2.042220e-001	2.016324e-001
289	1.056356e-002	1.152670e-002	1.251113e-002	1.351372e-002	1.453141e-002	1.556133e-002
1089	1.656199e-004	1.978607e-004	2.340463e-004	2.743377e-004	3.188737e-004	3.677704e-004
4225	3.468167e-007	3.752169e-007	4.869402e-007	6.527897e-007	8.634783e-007	1.130024e-006

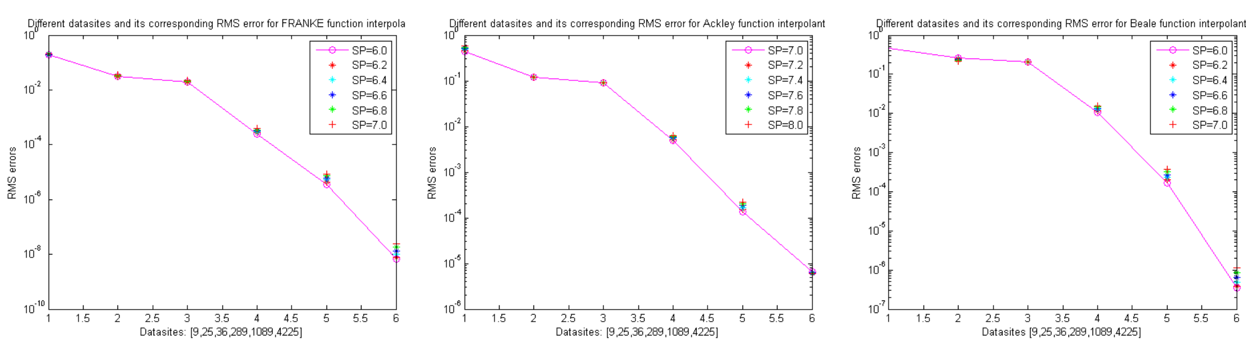

Figure 2: Test functions with different shape parameters (sp) and their corresponding RMS errors

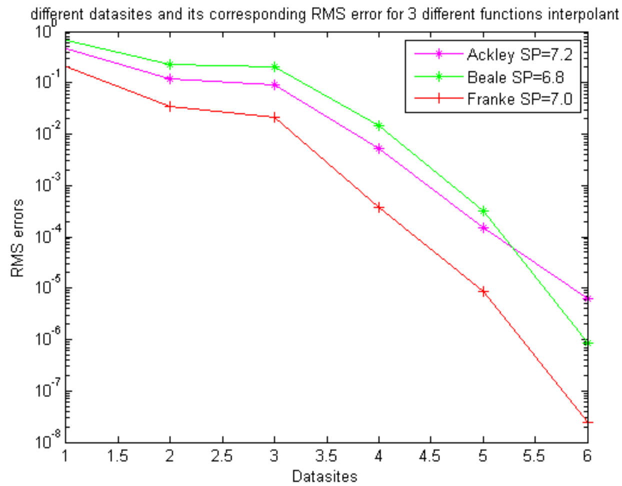

Figure 3: RMS error for the 3 Test functions with best shape parameter (sp)

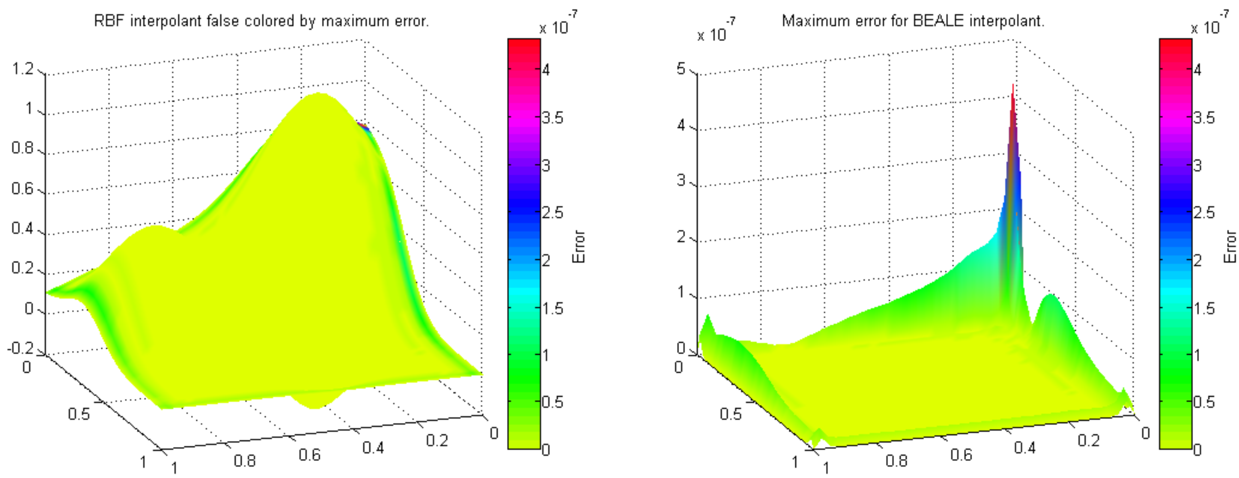

Figure 4: Franke function interpolant and its maximum error

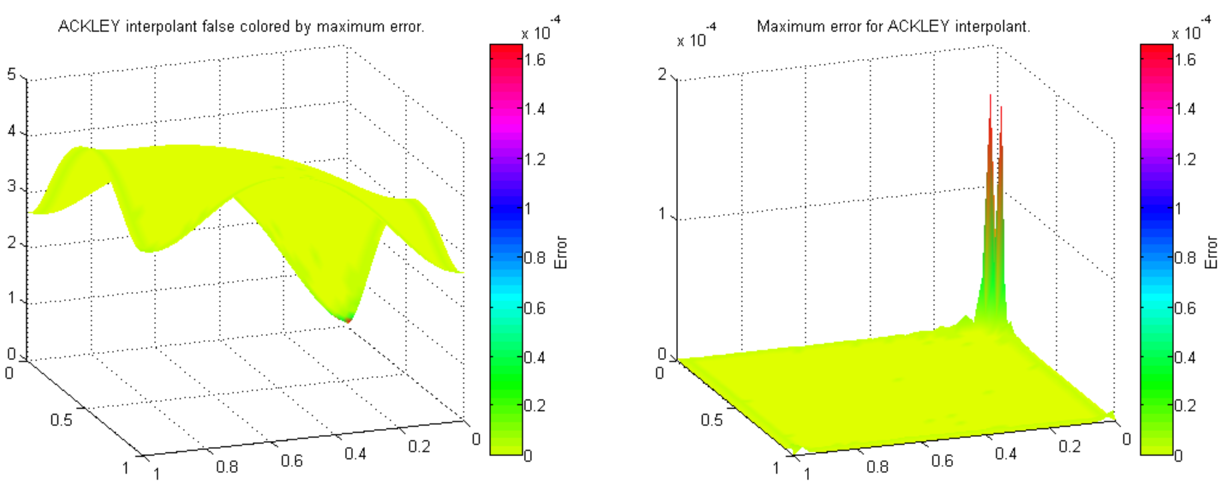

Figure 5: Ackley function interpolant and its maximum error

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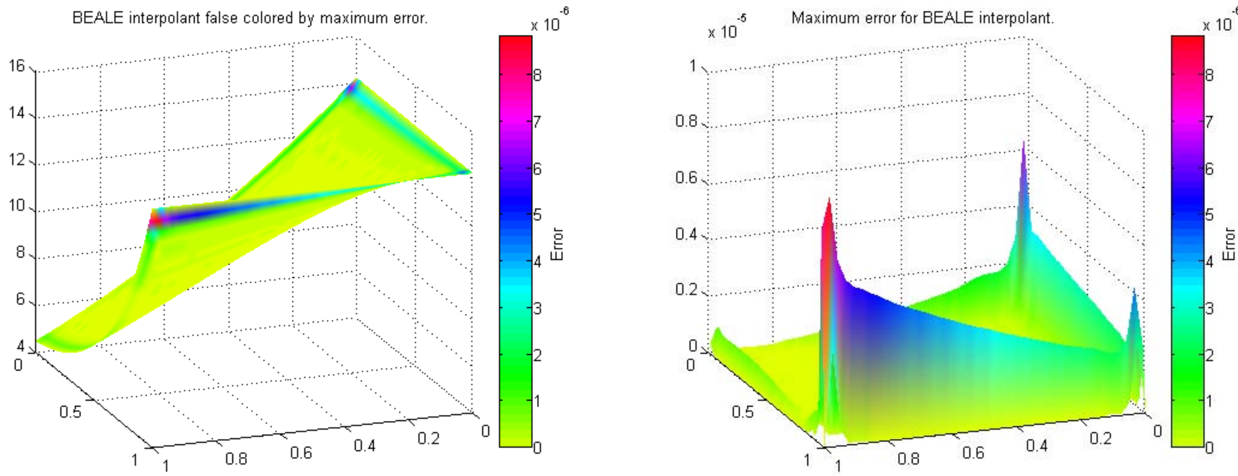

Figure 6: Beale function interpolant and its maximum error

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