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Unified Local Convergence for Some High order Methods with One Parameter

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Unified Local Convergence for Some High order Methods with One Parameter

Ioannis K. Argyros [α] & Santhosh George [σ]

Abstract- The aim of this paper is to extend the applicability of some Chebyshev-Halley-type method with one parameter for solving nonlinear equations under weaker than before hypotheses on the second derivative.

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I. Introduction

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces and Ω be an open and convex subset of $\mathcal{B}_1$. The problem of finding a solution x^* of equation

$$F(x) = 0. \tag{1.1}$$

where $F : \Omega \longrightarrow \mathcal{B}_2$ is differentiable in the sense of Fréchet is an important problem in applied mathematics due its wide applications. Higher order methods like [2–17] are considered for approximating the solution x^* of (1.1). The convergence analysis of higher order methods, requires assumptions on higher order derivatives.

A typical example of (1.1), in which the Lipschitz-type condition on derivatives of order greater than two does not hold is the mixed Hammerstein type equation defined on $X = Y = C[0,1]$ by

$$x(s) = \int_0^1 K(s,t)(\frac{1}{2}x(t)^{\frac{5}{2}} + \frac{x(t)^2}{8})dt, \tag{1.2}$$

where the kernel K is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$K(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases} \tag{1.3}$$

Define $F : C[0,1] \longrightarrow C[0,1]$ by

$$F(x)(s) := x(s) - \int_0^1 K(s,t)(\frac{1}{2}x(t)^{\frac{5}{2}} + \frac{x(t)^2}{8})dt \tag{1.4}$$

and consider

$$F(x)(s) = 0. \tag{1.5}$$

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Ref

2. I. K. Argyros D.Chen, Q. Quian, The Jarratt method in Banach space setting, J.Comput.Appl.Math. 51,(1994), 103-106.

Then, we have that

$$F'(x)\mu(s) = \mu(s) - \int_0^1 K(s,t)(\frac{5}{8}x(t)^{\frac{3}{2}} + \frac{x(t)}{4})\mu(t)dt.$$

Notice that $x^*(s) = 0$ is one of the solutions of (1.1). Using (1.3), we obtain

$$\| \int_0^1 K(s,t)dt\| \leq \frac{1}{8}. \tag{1.6}$$

Then, by (1.3)–(1.6), we have that

$$\|F'(x) - F'(y)\| \leq \frac{1}{8}(5\|x-y\|^{\frac{1}{4}} + \|x-y\|). \tag{1.7}$$

Note that , F'' is not Lipschitz. Hence the results in [1–17] cannot be used to solve (1.5).

In this paper we study the local convergence of the method defined for each $n = 0, 1, 2...$ [7] by

$$\begin{aligned} x_{n+1} &= x_n - [I + \frac{1}{2}L_n G_n]F'(x_n))^{-1}F(x_n), \qquad (1.8)\\ G_n &= I + \frac{\alpha}{2}L_n J_n \\ J_n &= (I - \frac{1}{2}L_n)^{-1} \\ L_n &= F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n), \end{aligned}$$

with $x_0 \in \Omega$ is an initial guess and $\alpha \in \mathbb{R}$. The semilocal convergence of method (1.8) was shown in [7] using hypotheses on F'' satisfying Lipschitz or Hölder or $\omega-$continuity conditions. Here we use $\omega-$type weaker hypotheses to study the local convergence not studied in [7].

The paper is structured as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies [2–17]. Special cases and numerical examples are presented in the concluding Section 3.

II. Local Convergence

The convergence shall be computed based on scalar parameters and functions. Let $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and nondecreasing with $w_0(0) = w(0) = 0$. Let ρ_0 stand for the smallest positive number satisfying

$$w_0(t) = 1. \tag{2.1}$$

Let also w, v, z be real continuous and nondecreasing function defined on the interval $[0, \rho_0)$ with $w(0) = 0$. Define functions q, h_q on interval $[0, \rho_0)$ by

$$q(t) = \frac{1}{2}\left(\frac{1}{1 - w_0(t)}\right)^2 \int_0^1 v(\theta t)d\theta z(t)t$$

and

$$h_q(t) = q(t) - 1.$$

We have $h_q(0) = -1 < 0$ and $h_q(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. The application of the intermediate value theorem on $[0, \rho_0)$ guarantees the existence of solutions

Ref

7. J. A. Ezquero and M. A. Hernandez, On a class of iteration containing the Chebyshev and the Halley method, Publ. Math. Debrecen, 54 (1999), 403–415.

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for the equation $h_q(t) = 0$ in $(0, \rho_0)$. Let r_q stand for the smallest such solution. Moreover, define functions f and h_f on the interval $[0, r_q)$ by

$$f(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)} + \frac{q(t)(1 + |\alpha - 1|q(t))\int_0^1 v(\theta t)d\theta}{(1 - w_0(t))(1 - q(t))}$$

and

$$h_f(t) = f(t) - 1.$$

We also get $h_f(0) = -1 < 0$ and $h_f(t) \longrightarrow +\infty$ as $t \longrightarrow r_q^-$. Denote by r the smallest solution of equation $h_f(t) = 0$. Then, we have that for each $t \in [0, r)$

$$0 \leq q(t) < 1 \tag{2.2}$$

and

$$0 \leq f(t) < 1. \tag{2.3}$$

The set $B(u, \lambda) = \{x \in \mathcal{B}_1 : \|x - x_0\| < \lambda\}$ is called the closed ball in $\mathcal{B}_1$ with center $u \in \mathcal{B}_1$ and radius $\lambda > 0$, whereas $\bar{B}(u, \lambda)$ is its closure.

The local convergence analysis is based on the notation introduced previously.

Theorem 2.1 *Let $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchet differentiable operator. Suppose: there exist $x^* \in \Omega$, function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ continuous and nondecreasing with $w_0(0) = 0$ such that*

$$F(x^*) = 0, \ F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{2.4}$$

and for all $x \in D$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|). \tag{2.5}$$

Let $\Omega_0 := \Omega \cap B(x^*, \rho_0)$. There exist functions $w, v, z : [0, \rho_0) \longrightarrow [0, +\infty)$ continuous, nondecreasing with $w(0) = 0$ such that

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|), \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|) \tag{2.7}$$

$$\|F'(x^*)^{-1}F''(x)\| \leq z(\|x - x^*\|) \tag{2.8}$$

and

$$\bar{B}(x^*, r) \subseteq \Omega, \tag{2.9}$$

where the radius r is defined previously. Then, iteration $\{x_n\}$produced for $x_0 \in B(x^*, r) - \{x^*\}$ by method (1.8) exists, lies in $B(x^*, r)$ and converges to x^*, so that

$$\|x_{n+1} - x^*\| \leq f(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.10}$$

where the functions f is defined previously. Moreover, if there exists $r^* \geq r$ such that

$$\int_0^1 w_0(\theta r^*)d\theta < 1,$$

the x^* is the only solution of equation $F(x) = 0$ in $\Omega_1 = \Omega \cap \bar{B}(x^*, r^*)$.

Proof. We shall base the proof on mathematical induction. By hypothesis $x_0 \in B(x^*, r) - \{x^*\}$, (2.5) and the definition of r, we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq w_0(\|x_0 - x^*\|) < w_0(r) < 1. \tag{2.11}$$

The Banach perturbation lemma [1] in combination with (2.11) assert the existence of $F'(x_0)^{-1}$ and the estimate

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.12}$$

Next, we show the existence of x_1 as follows: By (2.4) we can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.13}$$

The point $x^* + \theta(x_0 - x^*) \in B(x^*, r)$, since $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| \leq \|x_0 - x^*\| < r$. Then, by (2.7) and (2.13) we get that

$$\|F'(x^*))^{-1}F(x_0)\| \leq \int_0^1 v(\theta\|x_0 - x^*\|)\|x_0 - x^*\|d\theta. \tag{2.14}$$

We need an upper bound on $\frac{1}{2}L_0$. Using (2.2), (2.8), (2.12) and (2.14), we obtain in turn that

$$\begin{aligned}
\frac{1}{2}\|L_0\| &\leq \frac{1}{2}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F''(x_0)\| \\
&\quad \times\|F'(x_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\leq \frac{1}{2}\left(\frac{1}{1 - w_0(\|x_0 - x^*\|}\right)^2 \int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|z(\|x_0 - x^*\|) \\
&= q(\|x_0 - x^*\|) \leq q(r) < 1,
\end{aligned} \tag{2.15}$$

so J_0^{-1} exists and

$$\||J_0^{-1}\| \leq \frac{1}{1 - q(\|x_0 - x^*\|)}. \tag{2.16}$$

Hence, x_1 is well defined by the first substep of method (1.8) for $n = 0$. By the definition of G_0 we can write

$$G_0 = I + \frac{\alpha}{2}L_0J_0 = [I + \frac{\alpha - 1}{2}L_0]J_0$$

so

$$\begin{aligned}
\|G_0\| &\leq \|I + \frac{\alpha - 1}{2}L_0\|\|J_0\| \\
&\leq \frac{1 + |\alpha - 1|q(\|x_0 - x^*\|)}{1 - q(\|x_0 - x^*\|)}.
\end{aligned} \tag{2.17}$$

Ref

1. I.K. Argyros, Computational theory of iterative methods. Series: Studies in Computational Mathematics, 15, Editors: C.K.Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

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Furthermore, using method (1.8) for $n = 0$, (2.3), (2.6), (2.12), (2.14), (2.16) and (2.17) we have in turn from the identity

$$\begin{aligned} x_1 - x^* &= x_0 - F'(x_0)^{-1}F(x_0) - x^* \\ & -\frac{1}{2}L_0G_0F'(x_0)^{-1}F(x_0) \end{aligned} \quad (2.18)$$

that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|F'(x_0)^{-1}F(x^*)\| \\ &\times \|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta\| \\ &\frac{1}{2}\|L_0\|\|G_0\|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\leq \left[\frac{\int_0^1 w(\theta\|x_0 - x^*\|)d\theta}{1 - w_0(\|x_0 - x^*\|)}\right. \\ &+ \left.\frac{q(\|x_0 - x^*\|)(1 + |\alpha - 1|q(\|x_0 - x^*\|)\int_0^1 v(\theta\|x_0 - x^*\|)d\theta}{(1 - w_0(\|x_0 - x^*\|))(1 - q(\|x_0 - x^*\|))}\right]\|x_0 - x^*\| \\ &= f(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \quad (2.19)$$

which shows (2.10) for $n = 0$ and $x_1 \in B(x^*, r)$. Simply replacing x_0, x_1 by x_k, x_{k+1} in the preceding estimates, we arrive at (2.10). In view of the estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r, \; c = f(\|x_0 - x^*\|) \in [0, 1),$$

we deduce that $\lim x_k = x^*$ and $x_{k+1} \in B(x^*, r)$. Finally, to show the uniqueness part, let $F(y^*) = 0$ with $y^* \in \Omega_1$. Define $Q = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Using (2.5), we get that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \leq \int_0^1 w_0(\theta r^*)d\theta < 1. \quad (2.20)$$

That is $Q^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Using the identity

$$0 = F(y^*) - F(x^*) = Q(y^* - x^*),$$

we conclude that $x^* = y^*$.

Remark 2.2 *1. In view of (2.6) and the estimate*

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|) \end{aligned}$$

condition (2.8) can be dropped and v can be replaced by

$$v(t) = 1 + w_0(t).$$

2. *Let $w_0(t) = L_0 t$, $w(t) = Lt$, $v(t) = M$ for some $L_0 > 0, L > 0$ and $M \geq 1$. In this special case, the results obtained here can be used for operators F satisfying autonomous differential equations [1, 3] of the form*

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. *The radius $\rho = \frac{2}{2L_0+L}$, was shown by us to be the convergence radius of Newton's method [1, 3]*

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.21}$$

under the conditions (2.5)–(2.7). It follows from the definition of r that the convergence radius r of the method (1.8) cannot be larger than the convergence radius ρ of the second order Newton's method (2.21). As already noted in [1, 3] ρ is at least as large as the convergence ball given by Rheinboldt [14]

$$r_R = \frac{2}{3L}. \tag{2.22}$$

In particular, for $L_0 < L$ we have that

$$r_R < \rho$$

and

$$\frac{r_R}{\rho} \to \frac{1}{3} \text{ as } \frac{L_0}{L} \to 0.$$

That is our convergence ball ρ is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [15].

4. *It is worth noticing that method (1.8) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [3–17]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

III. Numerical Examples

The numerical examples are presented in this section.

Example 3.1 *Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, \Omega = \bar{B}(0, 1), x^* = (0, 0, 0)^T$. Define function F on Ω for $w = (x, y, z)^T$ by*

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14. W.C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, In: Mathematical models and numerical methods (A.N.Tikhonov et al. eds.) pub.3, (1977), 129-142 Banach Center, Warsaw Poland.

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.6)-(2.8) conditions, we get $L_0 = e - 1, L = e^{\frac{1}{L_0}} = M$, *so* $w_0(t) = L_0 t = (e-1)t$, $w(t) = Lt = e^{\frac{1}{L_0}} t$ *and* $z(t) = v(t) = M = e^{\frac{1}{L_0}}$. *Then the parameters are*

$$r_q = 0.8770, \; r = 0.1496.$$

Example 3.2 *Let* $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, *the space of continuous functions defined on* $[0,1]$ *and be equipped with the max norm. Let* $\Omega = \overline{B}(0,1)$. *Define function* F *on* Ω *by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$, $M = 2$ *so* $w_0(t) = 7.5t, w(t) = 15t$, $v(t) = 2$ *and* $z(t) = 30$. *Then the parameters are*

$$r_q = 0.8000, \; r = 0.0108.$$

Example 3.3 *Returning back to the motivational example at the introduction of this study, we have* $w_0(t) = w(t) = \frac{1}{32}(5t^{3/2} + t)$ $v(t) = 1 + w_0(t)$ *and* $z(t) = \frac{3}{16}$. *Then the parameters are*

$$r_q = 2.5084, \; r = 1.3679.$$

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