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A New Subclass of Harmonic Univalent Functions Defined by *q*-Calculus

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ANEWSUBCLASSOFHARMONICUNIVALENTFUNCTIONSDEFINEDBYQCALCULUS

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A New Subclass of Harmonic Univalent Functions Defined by q-Calculus

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I. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain. We can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f.

A necessary and sufficient condition for f to be locally univalent and sence preserving in D is that $|h'(z)| > |g'(z)|,\ z \in D$ see Clunie and Shiel-small [7].

Let S_H denote the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as,

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_k| < 1. \tag{1.1}$$

Note that S_H reduces to class S of normalized analytic univalent functions if the co-analytic part of its member is zero.

After the appearance of the paper of Clune and Sheil-Small [10] several researchers for example (Silverman [6], Jahangiri [11], Dixit and Porwal [13], Dixit et al. [14], Frasin [4], Kumar et al. [21]) presented a systematic and unified study of various sub classes of harmonic univalent function.

Now, we recall the concept of q-calculus which may be found in [2], for $n \in N$, the q-number is defined as follows:

Ref

2. B.A. Frasin, Comprehensive family of harmonic univalent functions, SUT J. Math., 42(1) (2006), 145-155.

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$$[K]_q = \frac{1-q^k}{1-q}, \quad 0 < q < 1. \tag{1.2}$$

Hence, $[K]_q$ can be expressed as a geometric series $\sum_{i=0}^{k-1} q^i$, when $k \to \infty$ the series converges to $\frac{1}{1-q}$. As $q \to 1$, $[k]_q \to k$ and this is the bookmark of a q-analogus the limit as $q \to 1$ recovers the classical object.

The q-derivative of a function f is defined by

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z} \quad q \neq 1, \quad z \neq 0$$

and $D_q(f(0)) = f'(0)$ provided $f'(0)$ exists.

For a function $h(z) = z^k$ observe that

$$D_q(h(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}.$$

Then

$$\lim_{q \to 1} D_q(h(z)) = \text{limit}_{q \to 1} [k]_q z^{k-1} = k z^{k-1} - h'(z)$$

where h' is the ordinary derivative.

The q-Jackson definite integral of the function f is defined by

$$\int_0^z f(t) d_q t = (1-q) z \sum_{n=0}^{\infty} f(zq^n) q^n, \quad z \in C.$$

Now for $1 < \beta < \frac{4}{3}$, $0 \leq \lambda \leq 1$, $0 < q < 1$.

Suppose that $M_H[\lambda, q, \beta]$ denote the family of harmonic function of the form $f = h + \overline{g}$ (1.1).

Satisfying the condition

$$Re\left[\frac{z(zD_q h(z))' - \overline{z(zD_q g(z))'}}{\lambda[z(zD_q h(z))' - \overline{z(zD_q g(z))'}] + (1-\lambda)[h(z) + \overline{(g(z))}]}\right] < \beta. \tag{1.3}$$

Further let M_H the subclasses of S_H consisting of functions of the form,

$$f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k \tag{1.4}$$

Further, we define $M_H(\lambda, q, \beta) = N_H(\lambda, q, \beta) \cap M_H$.

Notes

In this paper, we obtain coefficient bound, extreme point, distortion bound, convolution, convex combination for the class $M_H(\lambda, q, \beta)$. We also discuss a class preserving integral operator.

II. Main Results

Theorem 2.1 Let the function $f = h + \overline{g}$ be given by (1.1). If

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\beta\lambda) - (1-\lambda)\beta}{\beta - 1}|a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\beta\lambda) - (1-\lambda)\beta}{\beta - 1}|b_k| \leq 1 \tag{2.1}$$

where $1 < \beta \leq \frac{4}{3}$, $0 \leq \lambda \leq 1$, then $f \in N_H(\lambda, q, \beta)$.

Proof. Let

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\beta\lambda) - (1-\lambda)\beta}{\beta - 1}|a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\beta\lambda) + (1-\lambda)\beta}{\beta - 1}|b_k| \leq 1$$

It suffices to show that,

$$\left| \frac{\dfrac{z(zD_qh(z))' - \overline{z(zD_qg(z))'}}{\lambda[z(zD_qh(z))' - \overline{z(zD_qg(z))'}] + (1-\lambda)[h(z) + \overline{g(h)}]} - 1}{\dfrac{z(zD_qh(z))' - \overline{z(zD_qg(z))'}}{\lambda[z(zD_qh(z))' - \overline{z(zD_qg(z))'}] + (1-\lambda)[h(z) + \overline{g(h)}]} - (2\beta - 1)} \right| < 1$$

$$\leq \left| \frac{\dfrac{z + \sum_{k=2}^{\infty} k[k]_q a_k z^k - \overline{\sum_{k=1}^{\infty} k[k]_q b_k z^k}}{z + \sum_{k=2}^{\infty}(\lambda k[k]_q + 1 - \lambda)a_k z^k + \sum_{k=1}^{\infty}(\lambda k[k]_q - 1 + \lambda)\overline{b}_k \overline{z}^k} - 1}{\dfrac{z + \sum_{k=2}^{\infty} k[k]_q a_k z^k - \overline{\sum_{k=1}^{\infty} k[k]_q b_k z^k}}{z + \sum_{k=2}^{\infty}(\lambda k[k]_q + 1 - \lambda)a_k z^k + \sum_{k=1}^{\infty}(\lambda k[k]_q - 1 + \lambda)\overline{b}_k \overline{z}^k} - (2\beta - 1)} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty}[k[k]_q(1-\lambda) - (1-\lambda)]|a_k|\ |z|^{k-1} + \sum_{k=1}^{\infty}[k[k]_q(1-\lambda) + (1-\lambda)]|b_k|\ |z|^{k-1}}{\begin{array}{c} 2(\beta-1) - \sum_{k=2}^{\infty}[k[k]_q(1-\lambda(2\beta-1)) - (2\beta-1)(1-\lambda)]|a_k|\ |z|^{k-1} \\ - \sum_{k=1}^{\infty}[k[k]_q(1-\lambda(2\beta-1)) + (2\beta-1)(1-\lambda)]|b_k|\ |z|^{k-1} \end{array}}.$$

This last expression is bounded above by,

$$\sum_{k=2}^{\infty}[k[k]_q(1-\lambda)-(1-\lambda)]|a_k|+\sum_{k=1}^{\infty}[k[k]_q(1-\lambda)+(1-\lambda)]|b_k|$$

$$\leq 2(\beta-1)-\sum_{k=2}^{\infty}[k[k]_q(1-\lambda(2\beta-1))-(2\beta-1)(1-\lambda)]|a_k|$$

$$-\sum_{k=1}^{\infty}[k[k]_q(1-\lambda(2\beta-1))+(2\beta-1)-(1-\lambda)]|b_k|$$

$$\sum_{k=2}^{\infty}[k[k]_q(1-\lambda)-(1-\lambda)]|a_k|+\sum_{k=2}^{\infty}[k[k]_q(1-\lambda(2\beta-1))-(2\beta-1)(1-\lambda)]|a_k|$$

$$+\sum_{k=1}^{\infty}[k[k]_q(1-\lambda)+(1-\lambda)]|b_k|+\sum_{k=1}^{\infty}[k[k]_q(1-\lambda(2\beta-1))+(2\beta-1)(1-\lambda)]|b_k|\leq 2(\beta-1)$$

$$2\sum_{k=2}^{\infty}[k[k]_q(1-\lambda\beta)-(1-\lambda)\beta]|a_k|+2\sum_{k=1}^{\infty}[k[k]_q(1-\lambda\beta)+(1-\lambda)\beta]|b_k|\leq 2(\beta-1)$$

which is equivalent to

$$\sum_{k=2}^{\infty}\frac{[k[k]_q(1-\lambda\beta)-(1-\lambda)\beta]}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[k[k]_q(1-\lambda\beta)+(1-\lambda)\beta]}{\beta-1}|b_k|\leq 1.$$

Hence,

$$\left|\frac{\dfrac{z(zD_qh(z))'-\overline{z(zD_qg(z))'}}{\lambda[z(zD_qh(z))'-\overline{z(zD_qg(z))'}]+(1-\lambda)[h(z)+\overline{g(h)}]}-1}{\dfrac{z(zD_qh(z))'-\overline{z(zD_qg(z))'}}{\lambda[z(zD_qh(z))'-\overline{z(zD_qg(z))'}]+(1-\lambda)[h(z)+\overline{g(h)}]}-(2\beta-1)}\right|<1,$$

$z\in U$, and the theorem is proved. □

Theorem 2.2 A function of the form (1.4) is in $M_H(\lambda,q,\beta)$ if and only if,

$$\sum_{k=2}^{\infty}\frac{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}|a_k|+\sum_{k=2}^{\infty}\frac{k[k]_q(1-\beta\lambda)+(1-\lambda)\beta}{\beta-1}|b_k|\leq 1. \qquad (2.2)$$

Notes

Notes

Proof. Since $M_H(\lambda, q, \beta) \subset N_H(\lambda, q, \beta)$, we only need to prove the "only iff" Part of the theorem. For this we show that $f \in M_H(\lambda, q, \beta)$ if the above condition does not hold. Note that a necessary and sufficient condition for $f = h + \overline{g}$ given by (1.4) is in $M_H(\lambda, q, \beta)$

$$Re\left\{\frac{z(zD_qh(z))' - \overline{z(zD_qg(z))'}}{\lambda[zD_qh(z))' - \overline{z(zD_qg(z))'}] + (1-\lambda)[h(z) + \overline{g(z)}}\right\} < \beta,$$

is equivalent to

$$Re\left\{\frac{(\beta-1)z - \sum_{k=2}^{\infty}[k[k]_q(1-\lambda\beta) - (1-\lambda)\beta]|a_k|z^k - \sum_{k=1}^{\infty}[k[k]_q(1-\beta\lambda) + (1-\lambda)\beta]|b_k|\overline{z}^k}{z + \sum_{k=2}^{\infty}[\lambda k[k]_q + (1-\lambda)]|a_k|z^k + \sum_{k=1}^{\infty}[\lambda k[k]_q - (1-\lambda)]\,|b_k|\overline{z}^k}\right\} \geq 0$$

The above condition must hold for all values of z, $|z| = r < 1$, upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\left\{\frac{(\beta-1)z - \sum_{k=2}^{\infty} k[k]_q(1-\lambda\beta) - (1-\lambda)\beta|a_k|r^{k-1} - \sum_{k=1}^{\infty} k[k]_q(1-\lambda\beta) + (1-\lambda)\beta|b_k|r^{k-1}}{1 + \sum_{k=2}^{\infty} \lambda k[k]_q + (1-\lambda)|a_k|r^{k-1} - \sum_{k=1}^{\infty} \lambda k[k]_q - (1-\lambda)|b_k|r^{k-1}}\right\} \geq 0 \tag{2.3}$$

If the condition (2.2) does not hold then the numerator of (2.3) is negative for r sufficiently close to 1. Thus there exist a $z_0 = r_0$ in (0,1) for which the quotient in (2.3) is negative. This contradicts the required condition for $f \in M_H(\lambda, q, \beta)$ and so the proof is complete.

Next we determine the extreme points of the closed convex hulls of $M_H(\lambda, q, \beta)$ denoted by clco $M_H(\lambda, q, \beta)$ □

Theorem 2.3 If $f \in$ clco $M_H(\lambda, q, \beta)$, if and only if

$$f(z) = \sum_{k=1}^{\infty}\{x_k h_k(z) + y_k g_k(z)\}, \tag{2.4}$$

where

$$h_1(z) = z,\ \ h_k(z) = z + \frac{\beta - 1}{k[k]_q(1-\beta\lambda) - (1-\lambda)\beta} z^k, \quad k = (2, 3, ...)$$

and

$$g_k(z) = z - \frac{\beta - 1}{k[k]_q(1-\lambda\beta) + (1-\lambda)\beta} z^k, \quad k = (2, 3, ...),$$

$$\sum_{k=1}^{\infty}(x_k + y_k) = 1,\ \ x_k \geq 0,\ \ y_k \geq 0$$

In particular extreme points of $M_H(\lambda, q, \beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (1.4), we have,

$$f(z) = \sum_{k=2}^{\infty}[x_k h_k(z) + y_k g_k(z)]$$

$$= z+\sum_{k=2}^{\infty}\frac{\beta-1}{k[k]_q(1-\lambda\beta)-(1-\lambda)\beta}x_k z^k-\sum_{k=1}^{\infty}\frac{\beta-1}{k[k]_q(1-\lambda\beta)+(1-\lambda)\beta}y_k\overline{z}^k$$

Then by theorem (2.1)

$$\sum_{k=2}^{\infty}\frac{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}\left\{\frac{\beta-1}{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}x_k\right\}$$

$$+\sum_{k=1}^{\infty}\frac{k[k]_q(1-\beta\lambda)+(1-\lambda)\beta}{\beta-1}\left\{\frac{\beta-1}{k[k]_q(1-\beta\lambda)+(1-\lambda)\beta}y_k\right\}$$

$$= \sum_{k=2}^{\infty}x_k + \sum_{k=1}^{\infty}y_k$$

$$= 1 - x_1 \leq 1,$$

and so $f \in$ clco $M_H(\lambda, q, \beta)$ Set $x_k = \frac{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}|a_k|$, k=2,3,4,...

and $y_k = \frac{k[k]_q(1-\beta\lambda)+(1-\lambda)\beta}{\beta-1}|b_k|$, $k = 1, 2, 3, ...$

Then note that by Theorem 2.2, $0 \leq x_k \leq 1, (k = 1, 2, 3, ...)$.

We define $x_1 = 1 - \sum_{k=2}^{\infty}x_k - \sum_{k=1}^{\infty}y_k$ and by Theorem 2.2, $x_1 \geq 0$.

Consequently, we obtain $f(z) = \sum_{k=1}^{\infty}\{x_k h_k(z) + y_k g_k(z)\}$ as required. □

Theorem 2.4 Let $f \in M_H(\lambda, q, \beta)$. Then for $|z| = r < 1$, we have,

$$|f(z)| \leq (1+|b_1|)r+\left(\frac{\beta-1}{2[2]_q(1-\lambda\beta)-(1-\lambda)\beta}-\frac{\beta+1}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}|b_1|\right)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r-\left(\frac{\beta-1}{2[2]_q(1-\lambda\beta)-(1-\lambda)\beta}-\frac{\beta+1}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}|b_1|\right)r^2.$$

Proof. We only prove the right hand inequality. The proof for left hand inequality is similar and will be omitted. Let $f(z) \in M_H(\lambda, q, \beta)$, taking the absolute value of f, we have,

$$|f(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k$$

$$\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$

$$= (1+|b_1|)r + r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)$$

$$= (1+|b_1|)r + r^2\frac{\beta-1}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}\sum_{k=2}^{\infty}\frac{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}(|a_k|+|b_k|)$$

$$= (1+|b_1|)r + r^2\frac{\beta-1}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}\sum_{k=2}^{\infty}\frac{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}(|a_k|+|b_k|)$$

$$\leq (1+|b_1|)r + r^2\frac{\beta-1}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}$$

$$\sum_{k=2}^{\infty}\left(\frac{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}|a_k| + \frac{k[k]_q(1-\beta\lambda)+(1-\lambda)\beta}{\beta-1}|b_k|\right)$$

$$= (1+|b_1|)r + r^2\frac{\beta-1}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}\left(1-\frac{1+\beta-2\beta\lambda}{\beta-1}|b_1|\right)$$

$$= (1+|b_1|)r + r^2\left(\frac{\beta-1}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta} - \frac{1+\beta-2\beta\lambda}{2[2]_q(1-\beta\lambda)-(1-\lambda)\beta}|b_1|\right).$$

Thus the proof of Theorem 2.4 is established. □

Theorem 2.5 For $1<\alpha\leq\beta\leq\frac{4}{3}, 0\leq\lambda\leq 1$, let $f\in M_H(\lambda,q,\alpha)$, and $F\in M_H(\lambda,q,\beta)$ then $f*F\in M_H(\lambda,q,\alpha)\subseteq M_H(\lambda,q,\beta)$.

Proof. Let $f(z) = z+\sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k$ be in $M_H(\lambda,q,\alpha)$ and

$$F(z) = z+\sum_{k=2}^{\infty}|A_k|z^k - \sum_{k=1}^{\infty}|B_k|\overline{z}^k \text{ be in } M_H(\lambda,q,\beta).$$

Then the convolution $f*F$ is given by

$$(f*F)(z) = f(z)*F(z)$$

$$= z+\sum_{k=2}^{\infty}|a_kA_k|z^k - \sum_{k=1}^{\infty}|b_kB_k|\overline{z}^k.$$

We wish to show that the coefficient of $f*F$ satisfy the required condition in Theorem 2.2 for $F(z)\in M_H(\lambda,q,\beta)$ we note that $|A_k|<1$ and $|B_K|<1$. Now for the convolution function $f*F$, we obtain,

Notes

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}|a_kA_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta)-(1-\lambda)\beta}{\beta-1}|b_kB_k|$$

$$\leq \frac{k[k]_q(1-\beta\lambda)-(1-\lambda)\beta}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta)-(1-\lambda)\beta}{\beta-1}|b_k|$$

≤ 1 Since $f(z) \in M_H(\lambda, q, \beta)$.

Therefore, $f * F \in M_H(\lambda, q, \alpha) \subseteq M_H(\lambda, q, \beta)$.

Thus the proof of the Theorem 2.5 is established. □

A family of class Preserving Integral Operator

Let $f(x) = h(x) + \overline{g(x)}$ be defined by (1.1). Let us defined $F(z)$ by the relation,

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)dt}, \quad (c > -1). \tag{2.5}$$

Theorem 2.6 Let $f(z) = h(z) + \overline{g(z)} \in S_H$ be given by (1.4) and $f \in M_H(\lambda, q, \beta)$ where $1 < \beta \leq \frac{4}{3}$, $0 < \lambda \leq 1$. Then $F(z)$ defined by (2.5) is also in the class $M_H(\lambda, q, \beta)$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ be in $M_H(\lambda, q, \beta)$ then by Theorem 2.2, we have

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta)-(1-\lambda)\beta}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta)+(1-\lambda)\beta}{\beta-1}|b_k| \leq 1.$$

From the representation (2.5) of $F(z)$, it follows that:

$$F(z) = z + \sum_{k=2}^{\infty} \frac{c+1}{c+k}|a_k|z^k - \sum_{k=1}^{\infty} \frac{c+1}{c+k}|b_k|\overline{z}^k.$$

Now,

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta)-(1-\lambda)\beta}{\beta-1}\left(\frac{c+1}{c+k}\right)|a_k|$$

$$+ \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta)+(1-\lambda)\beta}{\beta-1}\left(\frac{c+1}{c+k}\right)|b_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta)-(1-\lambda)\beta}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta)+(1-\lambda)\beta}{\beta-1}|b_k|$$

$\leq 1.$

Thus $F(z) \in M_H(\lambda, q, \beta)$.

The proof of following Theorem 2.6 is complete. □

Notes

Notes

Definition 2.1 Let $f = h + \overline{g}$ be defined, by (1.1); then the q-integral operator $F_q : H \to H$ is defined by the relation,

$$F_q(z) = \frac{[c]_q}{z^{c+1}} \int_0^z t^c h(t) d_q t + \frac{[c]_q}{z^{c+1}} \int_0^z t^c g(t) d_q t, \tag{2.6}$$

where $[a]_q$ is the q-number defiend by (1.2) and H is the class of functions of the form (1.1) which are harmonic in U.

Theorem 2.7 Let $f(z) = h(z) + \overline{g(z)}$ be given by (1.3) and $f \in M_H(\lambda, q, \beta)$ where $1 < \beta \leq \frac{4}{3}$, $0 < q < 1, 0 \leq \lambda \leq 1$. Then $F_q(z)$ defined by (2.6) is also in the class $M_H(\lambda, q, \beta)$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k$ be in $M_H(\lambda, q, \beta)$ then by Theorem 2.2. We have,

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta) - (1-\lambda)\beta}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta) + (1-\lambda)\beta}{\beta - 1} |b_k| \leq 1.$$

From the representation (2.6) of $F_q(z)$, it follows that

$$F_q(z) = z + \sum_{k=2}^{\infty} \frac{[c]_q}{[k+c+1]_q} |a_k| z^k - \sum_{k=1}^{\infty} \frac{[c]_q}{[k+c+1]_q} |b_k| \overline{z}^k.$$

Since

$$[k+c+1]_q - [c]_q$$

$$= \sum_{i=0}^{k+c} q^i - \sum_{i=0}^{c-1} q^i = \sum_{i=c}^{k+c} q^i > 0$$

$$[k+c+1]_q > [c]_q,$$

or

$$\frac{[c]_q}{[k+c+1]_q} < 1.$$

Now

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta) - (1-\lambda)\beta}{\beta - 1} \frac{[c]_q}{[k+c+1]_q} |a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta) + (1-\lambda)\beta}{\beta - 1} \frac{[c]_q}{[k+c+1]_q} |b_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta) - (1-\lambda)\beta}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta) + (1-\lambda)\beta}{\beta - 1} |b_k|$$

$$\leq 1.$$

Thus the proof of the Theorem 2.7 is established. □

Theorem 2.8. The class $M_H(\lambda, q, \beta)$ is closed under convex function.

Proof. For $i = \{1, 2, 3, ...\}$, let $f_i(z) \in M_H(\lambda, q, \beta)$ where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{k=2}^{\infty} |a_{k_i}| z^k - \sum_{k=1}^{\infty} |b_{k_i}| \overline{z}^k.$$

Then by Theorem 2.2,

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta) - (1-\lambda)\beta}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta) + (1-\lambda)\beta}{\beta - 1} |b_k| \leq 1. \tag{2.7}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$ the convex combination of f_i may be written as,

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}| \right) z^k - \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \overline{z}^k.$$

Then by (2.2), we have,

$$\sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta) - (1-\lambda)\beta}{\beta - 1} \left(\sum_{i=1}^{\infty} t_i |a_{k_i}| \right) + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta) + (1-\lambda)\beta}{\beta - 1} \left(\sum_{i=1}^{\infty} t_i |b_{k_i}| \right)$$

$$= \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{k[k]_q(1-\lambda\beta) - (1-\lambda)\beta}{\beta - 1} |a_{k_i}| + \sum_{k=1}^{\infty} \frac{k[k]_q(1-\lambda\beta) + (1-\lambda)\beta}{\beta - 1} |b_{k_i}| \right)$$

$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

This is the condition required by Theorem 2.8 and so $\sum_{i=1}^{\infty} t_i f_i(z) \in M_H(\lambda, q, \beta)$. The proof of the following Theorem 2.8 is complete. □

Notes

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