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Positive Definite and Related Functions in the Product of Hypercomplex Systems

By A. S. Okb El Bab, A. M. Zabel, Hossam A. Ghany & M. Zakarya
Helwan University

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POSITIVEDEFINITEANDRELATEDFUNCTIONSINTHEPRODUCTOFHYPERCOMPLEXSYSTEMS

Positive Definite and Related Functions in the Product of Hypercomplex Systems

A. S. Okb El Bab [α], A. M. Zabel [σ], Hossam A. Ghany [ρ] & M. Zakarya [ω]

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I. Introduction

Harmonic analysis in hypercomplex systems (HCSs) dates back to Delsartes and Levitans work during the 1930s and 1940s, but the substantial development had to wait till the 1990s when Berezansky and Kondratiev [1] put HCSs in the right setting for harmonic analysis. Recently, some authors as Zabel and Bin Dehaish [2,3], Bin Dehaish [4], Okb El Bab, Zabel and Ghany [5] and Okb El Bab, Ghany and Boshnaa [6], studied some important subjects related to harmonic analysis in HCSs. Furthermore, Okb El Bab, Ghany, Hyder and Zakarya [7,8], studied some important subjects related to a construction of non-Gaussian white noise analysis using the theory of HCSs.

Generalized translation operators (GTOs) were first introduced by Delsarte [9] as an object that generalizes the idea of translation on a group. Later, they were systematically studied by Levitan [10–14], for some classes of GTOs, he obtained generalizations of harmonic analysis, the Lie theory, the theory of almost periodic functions, the theory of group representations, etc. The fact that GTOs arise in various problems of analysis is explained by Vainerman and Litvinov in [15]. Transformations of Fourier type for which the Plancherel theorem and the inversion formula hold, as a rule, are closely related to families of GTOs. According to Section 1 in [1], each hypercomplex system (HCS) can be associated with a family of GTOs. So, we begin with recalling some necessary facts deal with theory of GTOs and reviewing the conditions that distinguish the class of HCSs from the class of GTOs [2,14].

Let $L_1(Q, m)$ be a HCS with basis Q and let Φ be a space of complex valued functions on Q. Assume that an operator valued function $Q \ni p \mapsto L_p : \Phi \to \Phi$ is given such that the function $g(p) = (L_p f)(q)$ belongs to Φ for any $f \in \Phi$ and any fixed $q \in Q$.

Ref

1. Yu. M. Berezansky and A. A. Kalyuzhnyi, Harmonic analysis in hypercomplex systems, Naukova Dumka, Kyiv: Ukrania, 1992.

Author ρ: Department of Mathematics, Helwan University, Cairo, Egypt. e-mail: h.abdelghany@yahoo.com
Author α σ: Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt.
Author ω: Department of Mathematics, Faculty of Science, Al-Azhar University, Egypt. e-mail: mohammed zakarya@rocketmail.com

Definition 1.1. The operators $L_p,\ p \in Q$ are called GTOs, provided that the following axioms are satisfied.

I. Associativity axiom. The equality

$$\big(L_p^q(L_q f)\big)(r) = \big(L_q^r(L_p f)\big)(r) \tag{1.1}$$

holds for any elements $p, q \in Q$.

II. There exists an element $e \in Q$ such that $L_e = I$, where I is the identity operator in Φ.

Definition 1.2. The GTOs are called commutative if for any $p, q \in Q$, we have $\big(L_p^r(L_q f)\big)(r) = \big(L_q^r(L_p f)\big)(r)$. For commutative GTOs L_p the following equality is satisfied.

$$\big(L_p f\big)(q) = \big(L_q f\big)(p), \quad p, q \in Q. \tag{1.2}$$

In this paper we are interest in the case where Q is locally compact space with regular Borel measure m positive on open sets and bounded GTOs L_p act in the space of functions $\Phi = L_2(Q, m)$.

Definition 1.3. Given an involutive homeomorphism $Q \ni p \mapsto p^* \in Q$. The GTOs L_p are involutive if the equalities

$$\big(L_p f\big)(q) = \overline{\big(L_{q^*} f^*\big)(p^*)}, \quad \big(f \in L_2(Q, m), f^*(p) = \overline{f(p^*)}\,\big), \tag{1.3}$$

and $e^* = e$ hold for almost all $p, q \in Q$.

Definition 1.4. The GTOs L_p preserve positivity if $\big(L_p f\big)(q) \geq 0$ almost everywhere in m whenever $f(q) \geq 0$.

Definition 1.5. The family of operators L_p is called weakly continuous if the operator-valued function $Q \ni p \mapsto L_p$ is weakly continuous.

Definition 1.6. Let L_p^* be the operator adjoint to L_p. The measure m is called strongly invariant if $L_p^* = L_{p^*}$ for all $p \in Q$. We say that the measure m unimodular if $m(A) = m(A^*)$ for all $A \in \mathcal{B}(Q)$.

Assume that the GTOs L_p satisfy the finiteness condition:

(F) For any $A, B \in \mathcal{B}_0(Q)$, there is a compact set F so large that $\big(L_p f\big)(q) = 0$ for almost all $p \in A$ and $q \in B$ provided that $\operatorname{supp} f \cap F = \emptyset$.

Lemma 1.1. [1] If weakly continuous GTOs L_p are commutative, then relation (1.2) holds for almost all p and q.

*Lemma 1.2. [1] Let m be a measure strongly invariant with respect to the GTOs L_p $(p \in Q)$ which preserve the unit element and satisfy the finiteness condition **(F)**, Then*

$$\int_Q \big(L_p f\big)(q)dq = \int_Q f(q)dq, \qquad (p \in Q \text{ and } f \in L_{2,0}(Q, m)), \tag{1.4}$$

where $L_{2,0}(Q, m)$ is the subspace of finite functions from $L_2(Q, m)$.

Theorem 1.3. [1] There exists a one-to-one correspondence between normal HCSs with basis unity e and weakly continuous families of bounded involutive GTOs L_p satisfying the finiteness condition, preserving positivity in the space $L_2(Q, m)$ with unimodular strongly invariant measure m,

Notes

and preserving the unit element. Convolution in the HCS $L_1(Q,m)$ and the corresponding family of GTOs L_p satisfy the relation

$$(f*g)(p)=\int_Q (L_pf)(q)\,g(q^*)dq=(L_pf,g^*),\ \ (f,g\in L_2(Q,m)). \tag{1.5}$$

Moreover, the HCS $L_1(Q,m)$ is commutative if and only if the GTOs L_p, $p\in Q$ are commutative.

In this paper we can generalized the concept of HCS to the direct product of HCSs. This work can be immediately generalized to a direct product of any finite number of HCSs. While, the case of infinite number of HCSs is still open. Moreover, it is fairly easy to observe that all our results for direct product of HCSs can be easily investigated for direct products of semigroups and hypergroups (See [16, 17]).

This paper is organized as follows: In section 2, we give the basic definition of the direct product of HCSs and discuss its objects like convolution, characters, normality and commutativity preserving. In section 3, we give an example to improve the concept of direct product of HCSs. In section 4, we introduce and analyze the concept of positive definite functions on the direct product of commutative normal HCSs, and we present the integral representation of positive definite functions. Section 5 is employed for conclusion.

II. Direct Product of Hypercomplex Systems

Suppose that $L_{p_i}(i=1,2)$ be GTOs associated with normal HCSs $L_1(Q_1,m_1)$ and $L_1(Q_2,m_2)$ with basis unity e_1 and e_{2x} respectively. We denote, $\mathbf{H}_1=L_1(Q_1,m_1)$ and $\mathbf{H}_2=L_1(Q_2,m_2)$. The direct product of the GTOs L_{p_1} and L_{p_2} $(p_1\in Q_1,\ p_2\in Q_2)$ is defined as the operator-valued function

$$Q_1\times Q_2\ni(p_1,p_2)\mapsto L_{(p_1,p_2)}=L_{p_1}\otimes L_{p_2}:\mathbf{H}_1\otimes\mathbf{H}_2\to\mathbf{H}_1\otimes\mathbf{H}_2. \tag{2.1}$$

It is clear that the operators $L_{(p_1,p_2)}$ $((p_1,p_2)\in Q_1\times Q_2)$ form in $\mathbf{H}_1\otimes\mathbf{H}_2$ a family of GTOs satisfying the conditions of Theorem 1.3. The HCS $\mathbf{H}_1\otimes\mathbf{H}_2$ constructed from the GTOs $L_{(p_1,p_2)}$ is called the direct product of the HCSs $L_1(Q_1,m_1)$ and $L_1(Q_2,m_2)$. To denote the operation of taking the direct product, we write

$$\mathbf{H}_1\otimes\mathbf{H}_2=L_1(Q_1\times Q_2,m_1\otimes m_2)=L_1(Q_1,m_1)\otimes L_1(Q_2,m_2).$$

The following Lemma shows that the operation of taking the direct product preserves commutativity

Lemma 2.1. The direct product of two commutative HCSs is commutative.

Proof. Let $\mathbf{H}_1,\mathbf{H}_2$ be two commutative HCSs and L_p,L_q be the corresponding GTOs, respectively. According to the above Theorem, it is sufficient to prove that the GTOs $L_{(p_1,p_2)}=L_{p_1}\otimes L_{p_2}$, $L_{(q_1,q_2)}=L_{q_1}\otimes L_{q_2}$ $\big((p_1,p_2),(q_1,q_2)\in Q_1\times Q_2\big)$ are commutative. So from definition 1.2 and Eq.1.2 we have,

$$\big(L_{(p_1,p_2)}\big(L_{(q_1,q_2)}f\big)\big)(r_1,r_2)=\big(L_{(q_1,q_2)}\big(L_{(p_1,p_2)}f\big)\big)(r_1,r_2), \tag{2.2}$$

and hence, the Lemma is proved. ■

There are important concepts related to any HCS like structure measure, multiplicative measure, characters and normality. Now, we transfer these concepts to the direct product of two HCSs.

Let Q_1 and Q_2 be complete separable locally compact metric spaces. $\mathcal{B}(Q_1 \times Q_2)$ is the σ-algebra of Borel subsets from $Q_1 \times Q_2$, and $\mathcal{B}_0(Q_1 \times Q_2)$ be the subring of $\mathcal{B}(Q_1 \times Q_2)$ which consists of sets with compact closure. We will consider the Borel measures, that is, positive regular measures on $\mathcal{B}(Q_1 \times Q_2)$, finite on compact sets. The spaces of continuous functions, of finite continuous functions, of continuous functions vanishing at infinity and of bounded functions are denoted by $C(Q_1 \times Q_2)$, $C_0(Q_1 \times Q_2)$, $C_\infty(Q_1 \times Q_2)$ and $C_b(Q_1 \times Q_2)$, respectively.

Let $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}_0(Q_1 \times Q_2)$ and let $\mathcal{K}_{(A_1 \times A_2)}$ and $\mathcal{K}_{(B_1 \times B_2)}$ be the characteristic functions of $A_1 \times A_2, B_1 \times B_2$, respectively. By using Eq.(1.5), we can set up the structure measure of the HCS $\mathbf{H}_1 \otimes \mathbf{H}_2$ as follows

$$\begin{aligned} c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big) &= \mathcal{K}_{(A_1 \times A_2)} * \mathcal{K}_{(B_1 \times B_2)}(r_1, r_2) \\ &= \int\limits_{Q_1 \times Q_2} \big(L_{(r_1,r_2)} \mathcal{K}_{(A_1 \times A_2)}\big)(q_1, q_2) \mathcal{K}_{(B_1 \times B_2)}(q_1^*, q_2^*) d(q_1, q_2), \end{aligned} \tag{2.3}$$

where $(r_1, r_2) \in Q_1 \times Q_2$. This structure measure is said to be commutative whenever

$$c\Big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\Big) = c\Big(B_1 \times B_2, A_1 \times A_2, (r_1, r_2)\Big). \tag{2.4}$$

A regular Borel measure $m := m_1 \otimes m_2$ on $\mathcal{B}_0(Q_1 \times Q_2)$ is called multiplicative if

$$\int\limits_{Q_1 \times Q_2} c\Big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\Big) dm(r_1, r_2) = m(A_1 \times A_2) m(B_1 \times B_2). \tag{2.5}$$

By using Eq.(1.5), we can define the convolution in $\mathbf{H}_1 \otimes \mathbf{H}_2$ as follows

$$\begin{aligned} \big(f * g\big)(p_1, p_2) &= \int\limits_{Q_1 \times Q_2} \big(L_{(p_1,p_2)} f\big)(q_1, q_2) g(q_1^*, q_2^*) d(q_1, q_2) \\ &= \big(L_{(p_1,p_2)} f, g^*\big)_{(\mathbf{H}_1 \otimes \mathbf{H}_2)_2}, \end{aligned} \tag{2.6}$$

where $f, g \in L_2\big(Q_1 \times Q_2, m_1 \otimes m_2\big) := (\mathbf{H}_1 \otimes \mathbf{H}_2)_2$.

A non zero measurable and bounded almost everywhere function $Q_1 \times Q_2 \ni (r_1, r_2) \mapsto \chi(r_1, r_2) \in \mathbb{C}$ is said to be a character of HCS $\mathbf{H}_1 \otimes \mathbf{H}_2$, if the equality

$$\int\limits_{Q_1 \times Q_2} c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big) \chi(r_1, r_2) dm(r_1, r_2) = \chi(A_1 \times A_2) \chi(B_1 \times B_2) \tag{2.7}$$

holds for any $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}_0(Q_1 \times Q_2)$. Every direct product of HCSs has at least one character, namely, the function $\chi = 1$. A non zero measurable complex-valued function $\chi(r_1, r_2)$, $(r_1, r_2) \in Q_1 \times Q_2$ is called a generalized character of $\mathbf{H}_1 \otimes \mathbf{H}_2$, if the equality (2.7) holds.

The HCS $\mathbf{H}_1 \otimes \mathbf{H}_2$ is said to be normal, if there exists an involution homomorphism $Q_1 \times Q_2 \ni (r_1, r_2) \mapsto (r_1^*, r_2^*) \in Q_1 \times Q_2$, such that $m(E_1 \times E_2) = m(E_1^* \times E_2^*)$ $\big(E_1 \times E_2 \in \mathcal{B}(Q_1 \times Q_2)\big)$ and for all $A_1 \times A_2, B_1 \times B_2, C_1 \times C_2 \in \mathcal{B}_0(Q_1 \times Q_2)$, we have

$$\begin{aligned} c\big(A_1 \times A_2, B_1 \times B_2, C_1 \times C_2\big) &= c\big(C_1 \times C_2, B_1^* \times B_2^*, A_1 \times A_2\big), \\ &= c\big(A_1^* \times A_2^*, C_1 \times C_2, B_1 \times B_2\big), \end{aligned} \tag{2.8}$$

Notes

where

$$c\big(A_1 \times A_2, B_1 \times B_2, C_1 \times C_2\big) = \int\limits_{C_1 \times C_2} c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big) dm(r_1, r_2). \tag{2.9}$$

A normal HCS $\mathbf{H}_1 \otimes \mathbf{H}_2$ possesses a basis unity if there exists a point $(e_1, e_2) \in Q_1 \times Q_2$ such that $(e_1, e_2) = (e_1^*, e_2^*)$ and

$$c\big(A_1 \times A_2, B_1 \times B_2, (e_1 \times e_2)\big) = m\big((A_1^* \times A_2^*) \cap (B_1 \times B_2)\big), \tag{2.10}$$

Notes

where $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}(Q_1 \times Q_2)$.

A normal HCS $\mathbf{H}_1 \otimes \mathbf{H}_2$ is called Hermitian if $(r_1^*, r_2^*) = (r_1, r_2)$ fore all $(r_1, r_2) \in Q_1 \times Q_2$. Every Hermitian direct product of HCSs is commutative. We should remark that, for a normal HCS $\mathbf{H}_1 \otimes \mathbf{H}_2$, the mapping

$$\mathbf{H}_1 \otimes \mathbf{H}_2 \ni f(r_1, r_2) \mapsto f^*(r_1, r_2) \in \mathbf{H}_1 \otimes \mathbf{H}_2 \tag{2.11}$$

is an involution in the Banach algebra $\mathbf{H}_1 \otimes \mathbf{H}_2$. A character χ of a normal HCS $\mathbf{H}_1 \otimes \mathbf{H}_2$ is said to be Hermitian if

$$\chi(r_1^*, r_2^*) = \overline{\chi(r_1, r_2)}, \quad (r_1, r_2) \in Q_1 \times Q_2. \tag{2.12}$$

Denote the families of characters, of generalized characters and of bounded Hermitian characters by $\mathbf{X}$, $\mathbf{X}_g$ and $\mathbf{X}_h$, respectively.

The following result gives us the criterium of the generalized characters of a normal commutative direct product of HCSs.

Lemma 2.2. In order that a function $\chi(r_1, r_2) \in C(Q_1 \times Q_2)$ be a generalized character of the normal commutative direct product of HCS $\boldsymbol{H}_1 \otimes \boldsymbol{H}_2$ with basis unity (e_1, e_1) it is necessary and sufficient that the equality

$$\big(L_{(p_1,p_2)}\chi\big)(q_1, q_2) = \chi(p_1, p_2)\chi(q_1, q_2), \tag{2.13}$$

hold for almost all $(p_1, p_2), (q_1, q_2) \in (Q_1 \times Q_2)$.

Proof. Assume that a function $\chi \in X_g$. Then, we have

$$\begin{aligned}
\chi(A_1 \times A_2)\chi(B_1 \times B_2) &= \int\limits_{Q_1 \times Q_2} c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big)\chi(r_1, r_2) d(r_1, r_2) \\
&= \int\limits_{Q_1 \times Q_2} \int\limits_{B_1^* \times B_2^*} \big(L_{(r_1,r_2)}\mathcal{K}_{(A_1 \times A_2)}\big)(s_1, s_2) d(s_1, s_2)\chi(r_1, r_2) d(r_1, r_2) \\
&= \int\limits_{B_1 \times B_2} \int\limits_{Q_1 \times Q_2} \big(L_{(s_1^*,s_2^*)}\mathcal{K}_{(A_1 \times A_2)}\big)(r_1, r_2)\chi(r_1, r_2) d(r_1, r_2) d(s_1, s_2) \\
&= \int\limits_{B_1 \times B_2} \int\limits_{A_1 \times A_2} \big(L_{(s_1,s_2)}\chi\big)(r_1, r_2) d(r_1, r_2) d(s_1, s_2)
\end{aligned} \tag{2.14}$$

for any $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}_0(Q_1 \times Q_2)$, which yields 2.13. The converse statement can be proved by analogy. ∎

Practically, to illustrate the concept of direct product of HCSs, we give an example as follows:
Example 2.1. Let $Q_1 = G_1, Q_2 = G_2$ be commutative locally compact groups. It is easy to see that $Q_1 \times Q_2 = G_1 \times G_2$ is commutative locally compact group with unity (e_1, e_2), where e_1 and e_2 are the unities of G_1 and G_2, respectively. Consider its group algebra, i.e., a set $L_1(G_1 \times G_2, m)$ of functions defined on the group $G_1 \times G_2$ and summable with respect to the Haar measure $m := m_1 \otimes m_2$. So, we can define the involution

$$G_1 \times G_2 \ni (p_1, p_2) \longmapsto (p_1^*, p_2^*) \in G_1 \times G_2. \tag{2.15}$$

In this case, where

$$\big(L_{(p_1,p_2)} f\big)(q_1, q_2) = f\big((q_1, q_2)(p_1, p_2)\big), \quad (p_1, p_2), (q_1, q_2) \in G_1 \times G_2, \tag{2.16}$$

we have the convolution

$$\big(f * g\big)(p_1, p_2) = \int\limits_{Q_1 \times Q_2} f\big((q_1, q_2)(p_1, p_2)\big) g(q_1^*, q_2^*) d(q_1, q_1) \tag{2.17}$$

Also, the structure measure has the form,

$$c\big(A_1 \times A_2, B_1 \times B_2, (r_1, r_2)\big) = m\big((A_1^{-1} \times A_2^{-1})(r_1, r_2) \cap (B_1 \times B_2)\big), \tag{2.18}$$

where $A_1 \times A_2, B_1 \times B_2 \in \mathcal{B}(G_1 \times G_2), (r_1, r_2) \in G_1 \times G_2$. Thus, we obtain the direct product of the commutative HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$. This direct product is also commutative and with basis unity (e_1, e_2). In particular, if $G_1 \times G_2 = \mathbb{R} \times \mathbb{R}$ is an additive groups of all real numbers. For such HCSs it is possible to introduce generalized translation $L_{(p_1,p_2)}$:

$$\mathbb{R} \times \mathbb{R} \ni (p_1, p_2) \longmapsto \big(L_{(p_1,p_2)} f\big)(q_1, q_2) \in \mathbb{C}, \quad f \in C(\mathbb{R} \times \mathbb{R}),$$

where $\big(L_{(p_1,p_2)} f\big)(q_1, q_2) = f\big((q_1, q_2) + (p_1, p_2)\big)$. By using the operators $L_{(p_1,p_2)}$, one can rewrite the involution and convolution as follows respectively:

$$\mathbb{R} \times \mathbb{R} \ni (p_1, p_2) \longmapsto (p_1^*, p_2^*) := (p_1^{-1}, p_2^{-1}) \in \mathbb{R} \times \mathbb{R}, \tag{2.19}$$

$$\begin{aligned} \big(f * g\big)(p_1, p_2) &= \int\limits_{\mathbb{R} \times \mathbb{R}} f(q_1, q_2) \big(L_{(q_1^*, q_2^*)} g\big)(p_1, p_2) d(q_1, q_1) \\ &= \int\limits_{\mathbb{R} \times \mathbb{R}} f(q_1, q_2) g\big((p_1, p_2) - (q_1, q_2)\big) d(q_1, q_1), \end{aligned} \tag{2.20}$$

where $(q_1^*, q_2^*) = (-q_1, -q_2)$ in additive groups $\mathbb{R} \times \mathbb{R}$, $f, g \in \mathbf{H}_1 \otimes \mathbf{H}_2$ and the functions $\chi(t_1, t_2) = e^{i(t_1,t_2)(s_1,s_2)}$, $\big((s_1, s_2) \in \mathbb{R} \times \mathbb{R}\big)$ are characters.

Actually, there are many examples can be modified to the case of direct product of HCSs. For more details see [1, 19].

III. Positive Definite Functions on Direct Product of HCSs

In this section, we present a concept of positive definite functions on a commutative normal direct product of HCSs with basis unity. So, we give the following definitions and the important concepts of positive definite functions.

Notes

Notes

Definition 3.1. An essentially bounded function $\Theta(p_1,p_2)$ $\big((p_1,p_2)\in Q_1\times Q_2\big)$ is called positive definite if

$$\int\limits_{Q_1\times Q_2}\Theta(p_1,p_2)\Big(x^**x\Big)(p_1,p_2)d(p_1,p_2)\geq 0 \tag{3.1}$$

for all $x\in \mathbf{H}_1\otimes\mathbf{H}_2$. We also, present another definition of positive definiteness as the following.

Definition 3.2. A continuous bounded function $\Theta(p_1,p_2)$ $\big((p_1,p_2)\in Q_1\times Q_2\big)$ is called positive definite if the inequality

$$\sum_{i,j=1}^{n}\lambda_i\overline{\lambda_j}\big(L_{\left((p_1)^*,(p_2)^*\right)_i}\Theta\big)(p_1,p_2)_j\geq 0 \tag{3.2}$$

holds for all $(p_i,p_j),...,(p_n,p_n)\in Q_1\times Q_2$, $(p_1,p_2)_i^*:=\big((p_1)_i^*,(p_2)_i^*\big),(p_1,p_2)_j:=\big((p_1)_j,(p_2)_j\big)$, $\big(i,j=1,...,n\ (n\in\mathbb{N})\big)$ and $\lambda_1,...,\lambda_n\in\mathbb{C}$.

Lemma 3.1. If the GTOs $L_{(t_1,t_2)}$ extended to $L_\infty:C_b(Q_1\times Q_2)\to C_b\Big((Q_1\times Q_2)\times(Q_1\times Q_2)\Big)$. Then the definitions 3.1 and 3.2 are equivalent for the functions $\phi\in C_b(Q_1\times Q_2)$.

Proof. From definition 3.1, we have

$$\begin{aligned}
&\int\limits_{Q_1\times Q_2}\phi(r_1,r_2)\Big(x^**x\Big)(t_1,t_2)d(t_1,t_2)\\
&=\int\limits_{Q_1\times Q_2}\phi(t_1,t_2)\int\limits_{Q_1\times Q_2}\big(L_{(s_1,s_2)}x\big)(t_1,t_2)\overline{x(s_1,s_2)}d(s_1,s_2)d(t_1,t_2)\\
&=\int\limits_{Q_1\times Q_2}\int\limits_{Q_1\times Q_2}\big(L_{(s_1^*,s_2^*)}\phi\big)(t_1,t_2)\overline{x(s_1,s_2)}d(s_1,s_2)x(t_1,t_2)d(t_1,t_2)\\
&=\int\limits_{Q_1\times Q_2}\int\limits_{Q_1\times Q_2}\big(L_{(t_1,t_2)}\phi\big)(s_1^*,s_2^*)x(t_1,t_2)\overline{x(s_1,s_2)}d(t_1,t_2)d(s_1,s_2)\\
&\geq\ 0,
\end{aligned}\tag{3.3}$$

where $x\in\mathbf{H}_1\otimes\mathbf{H}_2$. By the condition, we have $\big(L_{(t_1,t_2)}\phi\big)(s_1^*,s_2^*)\in C_b\big((Q_1\times Q_2)\times(Q_1\times Q_2)\big)$, then the last inequality clearly implies (3.2). Let us prove the converse assertion. Let $Q_n\times Q_n$ be an increasing sequence of compact sets covering the entire $Q_1\times Q_2$. We consider a function $\Omega(r_1,r_2)\in C_0(Q_1\times Q_2)$ and set $\lambda_i=\Omega(r_1,r_2)_i$ in (3.2) This yields

$$\sum_{i,j=1}^{n}\Big(L_{\left(r_1^*,r_2^*\right)_i}\phi\Big)(r_1,r_2)_j\,\Omega(r_1,r_2)_i\,\overline{\Omega(r_1,r_2)_j}\geq 0. \tag{3.4}$$

By integrating this inequality with respect to each $(r_i,r_j),...,(r_n,r_n)$, over the set $Q_k\times Q_k$ $(k\in\mathbb{N})$ and collecting similar terms, we conclude that

$$\begin{aligned}
&nm(Q_k\times Q_k)\int\limits_{Q_k\times Q_k}\big(L_{(r_1^*,r_2^*)}\phi\big)(r_1,r_2)\,|\Omega(r_1,r_2)|^2d(r_1,r_2)\\
&+\ n(n-1)\int\limits_{Q_k\times Q_k}\int\limits_{Q_k\times Q_k}\big(L_{(r_1^*,r_2^*)}\phi\big)(s_1,s_2)\,\Omega(r_1,r_2)\,\overline{\Omega(s_1,s_2)}d(r_1,r_2)d(s_1,s_2)\\
&\geq\ 0
\end{aligned}\tag{3.5}$$

Further, we divide this inequality by n^2 and pass to the limit as $n \to \infty$. We get

$$\int\limits_{Q_k \times Q_k} \int\limits_{Q_k \times Q_k} \big(L_{(r_1^*, r_2^*)}\phi\big)(s_1, s_2)\, \Omega(r_1, r_2)\, \overline{\Omega(s_1, s_2)} d(r_1, r_2) d(s_1, s_2) \geq 0 \tag{3.6}$$

for each $k \in \mathbb{N}$. By passing to the limit as $k \to \infty$. and applying Lebesgue theorem, we see that (3.1) holds for all functions from $C_0(Q_1 \times Q_2)$. Approximating an arbitrary function from $\mathbf{H}_1 \otimes \mathbf{H}_2$ by finite continuous functions, we arrive at (3.1) ■

By $\mathcal{P}(Q_1 \times Q_2)$ we denote the set of all positive definite functions.

Lemma 3.2. If x belongs to $(\boldsymbol{H}_1 \otimes \boldsymbol{H}_2)_2$, then $\big(x^* * x\big) \in \mathcal{P}(Q_1 \times Q_2)$.

Proof. The proof is an immediately consequence of Lemma 3.3 in [1]. ■

Definition 3.3. For any function $x \in \mathbf{H}_1 \otimes \mathbf{H}_2$ and any character $\chi \in \mathbf{X}$, we set

$$\widehat{x}(\chi) = \int\limits_{Q_1 \times Q_2} x(r_1, r_2) \overline{\chi(r_1, r_2)} d(r_1, r_2). \tag{3.7}$$

This integral exists and $\widehat{x}$ is a continuous function on $\mathbf{X}$. It is called a Fourier transform of the function $x \in \mathbf{H}_1 \otimes \mathbf{H}_2$.

It well known that, every positive definite function in a HCS has a unique integral representation with respect to a nonnegative finite regular measure defined on the family of Hermitian characters (see Theorem 3.1 in [1]). Theorem 3.3 below gives a similar representation, but for positive definite functions in $\mathbf{H}_1 \otimes \mathbf{H}_2$.

Theorem 3.3. Every function $\Theta \in \mathcal{P}(Q_1 \times Q_2)$ admits a unique representation in the form of an integral

$$\Theta(r_1, r_2) = \int\limits_{\boldsymbol{X}_h} \chi(r_1, r_2) d\mu(\chi), \qquad (r_1, r_2) \in Q_1 \times Q_2, \tag{3.8}$$

where μ is a nonnegative finite regular measure on the space $\boldsymbol{X}_h$. Conversely, each function of the form (3.8) belongs to $\mathcal{P}(Q_1 \times Q_2)$.

Proof. Let $\Theta \in \mathcal{P}(Q_1 \times Q_2)$. Consider a continuous functional Φ in $\mathbf{H}_1 \otimes \mathbf{H}_2$ defined as follows

$$\Phi(x) = \int\limits_{Q_1 \times Q_2} \Theta(r_1, r_2) x(r_1, r_2) d(r_1, r_2), \quad (x \in \mathbf{H}_1 \otimes \mathbf{H}_2). \tag{3.9}$$

It is clear that this functional is positive. The functional Φ can be extended to a positive functional $\widetilde{\Phi}$ in a commutative normal direct product of HCSs with basis unity $\mathbf{H}_1 \otimes \mathbf{H}_2$. To do this, it suffices to show that

- The functional Φ is real $\big($i.e., $\Phi(x^*) = \overline{\Phi(x)}$ for all $x \in \mathbf{H}_1 \otimes \mathbf{H}_2\big)$,
- The inequality $\big|\Phi(x)\big|^2 \leq C\Phi(x^*)$ holds, where C is a constant.

Let $e_n \in \mathbf{H}_1 \otimes \mathbf{H}_2$ be an approximative unit, that is, $e_n(r_1, r_2) \geq 0$, $e_n(r_1, r_2) = e_n(r_1^*, r_2^*)$ $(r_1, r_2) \in Q_1 \times Q_2$, $\|e_n\|_{\mathbf{H}_1 \otimes \mathbf{H}_2} = 1$ and for all $x \in \mathbf{H}_1 \otimes \mathbf{H}_2$, $\lim\limits_{n \to \infty} e_n * x = x$ weakly in $\mathbf{H}_1 \otimes \mathbf{H}_2$. Since Φ is positive, we have

$$\Phi(x^*) = \lim_{n \to \infty} \Phi(e_n^*(r_1, r_2) * x^*) = \lim_{n \to \infty} \overline{\Phi(x * e_n(r_1, r_2))} = \overline{\Phi(x)} \tag{3.10}$$

Ref

1. Yu. M. Berezansky and A. A. Kalyuzhnyi, Harmonic analysis in hypercomplex systems, Naukova Dumka, Kyiv: Ukrania, 1992.

for all $x \in \mathbf{H}_1 \otimes \mathbf{H}_2$. Further, by using Lemma 1.3 in [1], we obtain

$$\begin{aligned} \left|\Phi(x)\right|^2 &= \lim_{n\to\infty} \left|\Phi\left(e_n(r_1,r_2) * x\right)\right|^2 \\ &\le \lim_{n\to\infty} \Phi\left(e_n^*(r_1,r_2) * e_n(r_1,r_2)\right)\Phi\left(x^* * x\right) \\ &\le \|\Phi\|\Phi\left(x^* * x\right). \end{aligned} \tag{3.11}$$

Consequently, it is possible to extend Φ to a positive functional $\widetilde{\Phi}$ on $\mathbf{H}_1 \otimes \mathbf{H}_2$. By virtue of the theorem on representations of positive functionals on commutative Banach $*$-algebras with identity element, the functional $\widetilde{\Phi}$ (and, hence, Φ) can be uniquely represented in the form

$$\Phi(x) = \int\limits_{\mathbf{X}_h} \int\limits_{Q_1\times Q_2} x(r_1,r_2)\chi(r_1,r_2)d(r_1,r_2)d\mu(\chi), \tag{3.12}$$

where μ is a finite regular Borel measure on $\mathcal{B}_0(\mathbf{X}_h)$. From Eqs.(3.9) and (3.12), we obtain the following relation

$$\Theta(r_1,r_2) = \int\limits_{\mathbf{X}_h} \chi(r_1,r_2)d\mu(\chi),$$

almost everywhere on $Q_1 \times Q_2$. Since the characters of $\mathbf{H}_1 \otimes \mathbf{H}_2$ are continuous, both functions in this equality are also continuous. This yields Eq.(3.8). The second part of the theorem follows from the relation

$$\begin{aligned} &\int\limits_{Q_1\times Q_2} \int\limits_{\mathbf{X}_h} \chi(r_1,r_2)d\mu(\chi)\left(x^* * x\right)(r_1,r_2)d(r_1,r_2) \\ &= \int\limits_{\mathbf{X}_h} \int\limits_{Q_1\times Q_2} \left(x^* * x\right)(r_1,r_2)\chi(r_1,r_2)d(r_1,r_2)d\mu(\chi) \\ &= \int\limits_{\mathbf{X}_h} \left|\widehat{x}(\chi)\right|^2 d\mu(\chi) \ge 0, \end{aligned} \tag{3.13}$$

where $\widehat{x}(\chi)$ is the Fourier transform of the functions $x \in \mathbf{H}_1 \otimes \mathbf{H}_2$. For all $\chi \in \mathbf{X}_h$, we have $\widehat{(x^*)}(\chi) = \overline{(\widehat{x})(\chi)}$, In particular, $\left(\widehat{x^* * x}\right)(\chi) = \left|\widehat{x}(\chi)\right|^2$. See [1] and the Lebesgue theorem on the limit transition. ■

Corollary 3.4. If the product of any two Hermitian characters is positive definite in $\mathbf{H}_1 \otimes \mathbf{H}_2$, then the product of any two continuous positive definite functions in $\mathbf{H}_1 \otimes \mathbf{H}_2$ is also positive definite.

Proof. Let χ and   are two Hermitian characters and positive definite in $\mathbf{H}_1 \otimes \mathbf{H}_2$, by virtue of Theorem 3.3, we have

$$\begin{aligned} &\int\limits_{Q_1\times Q_2} f(r_1,r_2)g(r_1,r_2)\left(x^* * x\right)(r_1,r_2)d(r_1,r_2) \\ &= \int\limits_{Q_1\times Q_2} \int\limits_{\mathbf{X}_h} \chi(r_1,r_2)d\mu(\chi) \int\limits_{\mathbf{X}_h} \quad (r_1,r_2)d\nu(\quad)\left(x^* * x\right)(r_1,r_2)d(r_1,r_2) \\ &= \int\limits_{\mathbf{X}_h}\int\limits_{\mathbf{X}_h}\int\limits_{Q_1\times Q_2} \chi(r_1,r_2) \quad (r_1,r_2)\left(x^* * x\right)(r_1,r_2)d(r_1,r_2)d\mu(\chi)d\nu(\quad) \ge 0 \end{aligned} \tag{3.14}$$

for all $f, g \in \mathcal{P}(Q_1 \times Q_2)$, $x \in \mathbf{H}_1 \otimes \mathbf{H}_2$. ■

Ref

1. Yu. M. Berezansky and A. A. Kalyuzhnyi, Harmonic analysis in hypercomplex systems, Naukova Dumka, Kyiv: Ukrania, 1992.

Corollary 3.5. Assume that $\mathbf{H}_1 \otimes \mathbf{H}_2$ *is a commutative direct product of HCSs with basis unity, then a continuous bounded function* $\varphi(r_1, r_1)$ *is positive definite in the sense of (3.1) if and only if it is positive definite in the sense of (3.2). Moreover, it has the following properties.*

(i) $\varphi(e_1, e_2) \geq 0$,

(ii) $\varphi(r_1^*, r_2^*) = \overline{\varphi(r_1, r_2)}$,

(iii) $|\,\varphi(r_1, r_2)\,| \leq \varphi(e_1, e_2)$,

(iv) $|\,\big(L_{(s_1,s_2)}\varphi\big)(t_1, t_2)\,|^2 \leq \big(L_{(s_1^*,s_2^*)}\varphi\big)(s_1, s_2)\big(L_{(t_1^*,t_2^*)}\varphi\big)(t_1, t_2)$,

(v) $|\,\varphi(s_1, s_2) - \varphi(t_1, t_2)\,|^2 \leq 2\varphi(e_1, e_2)\big[\varphi(e_1, e_2) - Re\big(L_{(s_1,s_2)}\varphi\big)(t_1^*, t_2^*)\big]$.

Proof. The first part of this Corollary we can found it from Lemma 3.1, from Theorem 3.3, we can proof the second part from **(i)** to **(v)** as following

$$\varphi(e_1, e_2) = \int_{\mathbf{X}_h} \chi(e_1, e_2)d\mu(\chi) = \mu(\mathbf{X}_h) \geq 0, \tag{3.15}$$

$$\varphi(r_1^*, r_2^*) = \int_{\mathbf{X}_h} \chi(r_1^*, r_2^*)d\mu(\chi) = \int_{\mathbf{X}_h} \overline{\chi(r_1, r_2)}d\mu(\chi) = \overline{\varphi(r_1, r_2)}, \tag{3.16}$$

$$\Big|\varphi(r_1, r_2)\Big| \leq \int_{\mathbf{X}_h} \Big|\chi(r_1, r_2)\Big|d\mu(\chi) \leq \mu(\mathbf{X}) = \varphi(e_1, e_2), \tag{3.17}$$

$$\begin{aligned}
\big|\big(L_{(s_1,s_2)}\varphi\big)(t_1, t_2)\big|^2 &= \Big|\int_{\mathbf{X}_h} \chi(s_1, s_2)\chi(t_1, t_2)d\mu(\chi)\Big|^2 \\
&\leq \int_{\mathbf{X}_h} \Big|\chi(s_1, s_2)\Big|^2 d\mu(\chi) \int_{\mathbf{X}_h} \Big|\chi(t_1, t_2)\Big|^2 d\mu(\chi) \\
&= \int_{\mathbf{X}_h} \chi(s_1, s_2)\chi(s_1^*, s_2^*)d\mu(\chi) \int_{\mathbf{X}_h} \chi(t_1, t_2)\chi(t_1^*, t_2^*)d\mu(\chi) \\
&= \big(L_{(s_1^*,s_2^*)}\varphi\big)(s_1, s_2)\big(L_{(t_1^*,t_2^*)}\varphi\big)(t_1, t_2),
\end{aligned} \tag{3.18}$$

Finally,

$$\begin{aligned}
\big|\varphi(s_1, s_2) - \varphi(t_1, t_2)\big|^2 &= \Big|\int_{\mathbf{X}_h} \chi(s_1, s_2)d\mu(\chi) - \int_{\mathbf{X}_h} \chi(t_1, t_2)d\mu(\chi)\Big|^2 \\
&\leq \Big|\int_{\mathbf{X}_h} \big(\chi(s_1, s_2) - \chi(t_1, t_2)d\mu(\chi)\Big|^2 \\
&\leq \mu(\mathbf{X}_h) \int_{\mathbf{X}_h} \Big|\big(\chi(s_1, s_2) - \chi(t_1, t_2)\Big|^2 d\mu(\chi) \\
&= \varphi(e_1, e_2) \int_{\mathbf{X}_h} \Big(\big|\chi(s_1, s_2)\big|^2 + \big|\chi(t_1, t_2)\big|^2 - 2Re\,\chi(s_1, s_2)\overline{\chi(t_1, t_2)}\Big)d\mu(\chi) \\
&\leq \varphi(e_1, e_2) \int_{\mathbf{X}_h} 2\Big(1 - Re\,\big(L_{(s_1,s_2)}\chi\big)(t_1^*, t_2^*)\Big)d\mu(\chi) \\
&= 2\varphi(e_1, e_2)\Big[\varphi(e_1, e_2) - Re\,\big(L_{(s_1,s_2)}\varphi\big)(t_1^*, t_2^*)\Big].
\end{aligned} \tag{3.19}$$

Notes

Hence, the Corollary is proved. ∎

In the remaining part of this section, we present the necessary and sufficient conditions guarantees that the property of positive definiteness on the direct product of HCSs is preserved under the usual function product.

Let $\mathbf{H}_1 \otimes \mathbf{H}_2$ be a commutative direct product of HCSs. The following two lemmas are in fact, an adaption of whatever done for semigroups in Berg et al. [18]. We will not repeat the proof, wherever the proof for semigroups can be applied to the HCSs [5]. In our work, we can applay it to the direct product of HCSs $\mathbf{H}_1 \otimes \mathbf{H}_2$ with necessary modification.

Lemma 3.6. *(i) The sum and the point-wise limit of positive definite functions in $\mathbf{H}_1 \otimes \mathbf{H}_2$ are also positive definite.*

(ii) Let ϕ be a continuous positive definite function on $Q \times Q$ and define $\Phi : \mathbf{H}_1 \otimes \mathbf{H}_2 \longrightarrow \mathbb{C}$ by $\Phi(x) := \int_{Q_1 \times Q_2} \phi(s_1, s_2) dm(s_1, s_2)$, $x \in \mathbf{H}_1 \otimes \mathbf{H}_2$. Then Φ is positive definite in $\mathbf{H}_1 \otimes \mathbf{H}_2$.

Proof. The proof is as the case of semigroups and HCSs [5, 18]. ∎

Lemma 3.7. *A bounded measurable function $\phi \in C_c(Q_1 \times Q_2)$ is positive definite if and only if there exists a in $(\mathbf{H}_1 \otimes \mathbf{H}_2)_2$ such that $\phi = \bullet \tilde{}$, where*

$$f \bullet \tilde{g}(r_1, r_2) = \int_{Q_1 \times Q_2} f\big((r_1, r_2) * (s_1, s_2)\big)\overline{g(s_1, s_2)} dm(s_1, s_2), \tag{3.20}$$

for all f, $g \in (\mathbf{H}_1 \otimes \mathbf{H}_2)_2$.

Proof. The proof is as Lemma 7.2.4 in Pederson [20]. ∎

Theorem 3.8. *Let ϕ_1 and ϕ_2 belongs to $C_c(Q_1 \times Q_2)$, then the product $\phi_1.\phi_2$ is positive definite on $Q_1 \times Q_2$ if and only if ϕ_1 and ϕ_2 are positive definite on $Q_1 \times Q_2$.*

Proof. From Lemma 3.7, there exist $f, g \in (\mathbf{H}_1 \otimes \mathbf{H}_2)_2$ such that $\phi_1 = f \bullet \tilde{f}$, $\phi_2 = g \bullet \tilde{g}$. So, we have

$$\begin{aligned}
\phi_1 . \phi_2(r_1, r_2) &= \Big(f \bullet \tilde{f}(r_1, r_2)\Big).\Big(g \bullet \tilde{g}(r_1, r_2)\Big) \\
&= \int_{Q_1 \times Q_2} f\Big((r_1, r_2) * (s_1, s_2)\Big)\overline{f(s_1, s_2)} dm(s_1, s_2) \\
&\times \int_{Q_1 \times Q_2} g\Big((r_1, r_2) * (t_1, t_2)\Big)\overline{g(t_1, t_2)} dm(t_1, t_2) \\
&= \int_{Q_1 \times Q_2} \int_{Q_1 \times Q_2} f\Big((r_1, r_2) * (s_1, s_2)\Big) g\Big((r_1, r_2) * (t_1, t_2)\Big) \\
&\times \overline{f(s_1, s_2) g(t_1, t_2)}\, dm(s_1, s_2)\, dm(t_1, t_2) \\
&= \int_{Q_1 \times Q_2} \int_{Q_1 \times Q_2} f . g\Big((r_1, r_2) * (s_1, s_2), (r_1, r_2) * (t_1, t_2)\Big) \\
&\times \overline{f.g\Big((s_1, s_2), (t_1, t_2)\Big)}\, dm(s_1, s_2)\, dm(t_1, t_2) \\
&= \int \int f . g\Big((r_1, r_2) * \big((s_1, s_2), (t_1, t_2)\big)\Big)
\end{aligned}$$

Ref

18. C. Berg, J. P. R. Christensen, P. Ressel, Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions, Springer-Verlag: Berlin, Heidelberg, New York, 1984.

$$\times \quad \overline{\overset{Q_1\times Q_2\; Q_1\times Q_2}{f\,.\,g\big((s_1,s_2),(t_1,t_2)\big)}}\, dm(s_1,s_2)\, dm(t_1,t_2). \tag{3.21}$$

Applying Fubini's theorem to the right hand side, we get

$$\phi_1.\phi_2(r_1,r_2) \;= \int f\,.\,g\Big((r_1,r_2) * \big((s_1,s_2),(t_1,t_2)\big)\Big)$$

$$\times \quad \overline{\overset{(Q_1\times Q_2)\times(Q_1\times Q_2)}{f\,.\,g\big((s_1,s_2),(t_1,t_2)\big)}}\, dn\big((s_1,s_2),(t_1,t_2)\big). \tag{3.22}$$

This implies that $\phi_1.\phi_2(r_1,r_2) = f\,.\,g \bullet \widetilde{f\,.\,g}(r_1,r_2)$. ■

IV. Conclusion

A direct product of two HCSs is precisely defined via the theory of GTOs. We showed that, under some conditions, the properties of commutativity, normality are preserved under the operation of taking the direct product. Some examples were given to improve the concept of direct product of HCSs. Also, we transferred the objects of harmonic analysis, namely, the criteria of positive definite, the integral representation of positive definite functions, the positive definiteness of the product of two HCSs.

This work can be immediately generalized to a direct product of any finite number of HCSs. While, the case of infinite number of HCSs is still open. Moreover, it is fairly easy to observe that all our results for direct product of HCSs can be easily investigated for direct products of semigroups and hypergroups (See [16–18]).

References Références Referencias

1. Yu. M. Berezansky and A. A. Kalyuzhnyi, Harmonic analysis in hypercomplex systems, Naukova Dumka, Kyiv: Ukrania, 1992.
2. A. M. Zabel and B. A. Bin Dehaish, Negative definite functions on hypercomplex systems, KYUNGPOOK Math. Journal, 46(2006), 285-295.
3. A. M. Zabel and B. A. Bin Dehaish, Ljevy Khinchin formula on commutative hypercomplex system, KYUNGPOOK Math. Journal, 48 (2008), 559-575.
4. B. A. Bin Dehaish, Exponentially convex functions on hypercomplex systems, International Journal of Mathematics and Mathematical Sciences, 1 (2011), 1-11.
5. A. S. Okb El Bab, A. M. Zabel, H. A. Ghany, Harmonic analysis in hypercomplex systems, International Journal of Pure and Applied Mathematics, 80 (2012), 739-750.
6. A. S. Okb El Bab, H. A. Ghany and R. M. Boshnaa, The moment problem in hypercomplex systems, Journal of Advances in Mathematics 10 (2015), 3654-3663.
7. A. S. Okb El Bab, H. A. Ghany and M. Zakarya, A construction of Non-Gaussian White Noise Analysis using the Theory of Hypercomplex Systems, Global Journal of Science Frontier Research: F Mathematics and Decision Sciences, 16 (2016) 11-25.
8. A. Hyder and M. Zakarya, Non-Gaussian Wick calculus based on hypercomplex systems, International Journal of Pure and Applied Mathematics, (2016) accepted.
9. J. Delsarte, Sur certaines transformation fonctionelles relative aux equations linearies aux derivees partiels duseconde ordre, C. R. Acad. Sci., Ser. A., 17 (1938), 178-182.
10. B. M. Levitan, Generalization of translation operation and infinite-dimensional hypercomplex systems, Mat. Sb. I, 16(3) (1945), 259-280; II 17(1) (1945), 9-44; III 17(2) (1945), 163-192. (in Russian).

Ref

16. C. Berg, J. P. R. Christensen and P. Ressel, Harmonic Analysis on Semigroups, Springer-Verlage: Berlin, Heidelberg, New York, 1980.

Notes

11. B. M. Levitan, Application of generalized translation operators to linear differential equations of the second order, Usp. Mat. Nauk, 4 (1949), 3-112.(in Russian).
12. B. M. Levitan, Lie theorems for generalized translation operators, Usp. Mat. Nauk, 16 (1961), 3-30. (in Russian).
13. B. M. Levitan, Generalized translation operators and some their applications, Nauka: Moscow, 1962; English translation: Israel Program for Scientific Translations, 1964.
14. B. M. Levitan, Theory of generalized translation operators, Nauka: Moscow, 1973 (in Russian).
15. G. L. Litvinov, On generalized translation operators and their representations, Tr. Sem. Vekt. Anal. MGU, 18 (1978), 345-371. (in Russian).
16. C. Berg, J. P. R. Christensen and P. Ressel, Harmonic Analysis on Semigroups, Springer-Verlage: Berlin, Heidelberg, New York, 1980.
17. W. R. Bloom and P. Ressel, Positive definite and related functions on hypergroups, Canad. J. Math., 43 (1991), 242-254.
18. C. Berg, J. P. R. Christensen, P. Ressel, Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions, Springer-Verlag: Berlin, Heidelberg, New York, 1984.
19. Yu. M. Berezansky and S. G. Krein, Hypercomplex systems with a compact basis, Ukrainski Matematicheski Zhurnal, 3 (1951), 184-204.
20. G. K. Pederson, C^{*}-algebra and their automorphism groups, London Mathematical Society Monograph, Academic Press: London, New York, 1979.
21. C. Berg and G. Forst, Potential theory on locally compact abelian groups, Springer-Verlage: Berlin, Heidelberg, 1975.

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