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A Note on the Representation and Definition of Dual Split Semi-Quaternions Algebra

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A Note on the Representation and Definition of Dual Split Semi-Quaternions Algebra

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Abstract- In this paper, dual split semi-quaternions algebra, $\tilde{H}_{ss}$, is defined for the first time, and some fundamental algebraic properties of its is studied. The set of all unit dual split semi-quaternions is a subgroup of $\tilde{H}_{ss}$. Fortheremore, by De-Moivre's formula, any powers of these quaternions are obtained.

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I. Introduction

The quaternion number system was discovered by Hamilton, who was looking for an extension of the complex number system to use in various areas of mathematics. The different type of quaternions are suitable algebraic instructure for expressing important space-time transformations as well as description of the classical and quantum filds.

Dual numbers and dual quaternions were introduced in the 19th century by W.K. Clifford, as a tool for his geometrical investigation.

In our previos work, we have studied the split semi-quaternions, and have presented some of their algebric properties. De Moivre's and Euler's formula for these quaternions are given (Jafari, 2015).We have shown that the set of all unit split semi-quaternions with the group operation of quaternion multiplication is a Lie group of 3-dimension and find its Lie algebra and Killing bilinear form (Jafari, 2016).

In this paper, we study the dual split semi-quaternions algebra and give some of their basic properties. We express De Moivre's and Euler's formulas for dual split semi-quaternions and find roots of a quaternion using these formulas. Finally, we give some examples for more clarification.We hope that these results will contribute to the study of physical science.

a) *Split Semi-quaternions Algebra*

A split semi-quaternion q has an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^{\,2} = 1,\ \vec{j}^{\,2} = \vec{k}^{2} = 0,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i},\ \vec{j}\vec{k} = 0 = -\vec{k}\vec{j},$$

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and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

The set of all split semi-quaternions is denoted by H_{ss}. For detailed information about this concept, we refer the reader to [3,4,6,9].

b) Dual Numbers Algebra

Let a and a^* be two real numbers, the combination

$$A = a + \varepsilon a^*,$$

is called a dual number. Here ε is the dual unit. Dual numbers are considered as polynomials in ε, subject to the rules

$$\varepsilon \neq 0,\ \varepsilon^2 = 0,\ \varepsilon.r = r.\varepsilon = \varepsilon, \text{ for all } r \in \mathbb{R}.$$

The set of dual numbers, D, forms a commutative ring having the εa^*(a^*real) as divisors of zero, not field. Some properties of dual numbers are

$$\sin(a + \varepsilon a^*) = \sin a + \varepsilon a^* \cos a,$$

$$\cos(a + \varepsilon a^*) = \cos a - \varepsilon a^* \sin a,$$

$$\sqrt{a + \varepsilon a^*} = \sqrt{a} + \varepsilon \frac{a^*}{2\sqrt{a}} \quad \text{for } a > 0.$$

For detailed information about dual numbers algebra, we refer the reader to (Keler, 2000).

c) Generalized Dual Quaternions Algebra

A generalized dual quaternion Q has an expression of form

$$Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

where A_0, A_1, A_2 and A_3 are dual numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units which satisfy the equalities

$$\vec{i}^2 = -\alpha,\quad \vec{j}^2 = -\beta,\quad \vec{k}^2 = -\alpha\beta,$$

$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i},\quad \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \alpha\vec{j} = -\vec{i}\vec{k},\quad \alpha, \beta \in \mathbb{R}.$$

The set of all generalized dual quaternions (abbreviated *GDQ*) are denoted by $\tilde{H}_{\alpha\beta}$. A generalized dual quaternion Q is a sum of a scalar and a vector, called scalar part, $S_Q = A_0$, and vector part $\vec{V}_Q = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ (Jafari,2015).

If $S_Q = 0$, then Q is called pure generalized dual quaternion, we may be called its generalized dual vector. The set of all generalized dual vectors denoted by $D^3_{\alpha\beta}$.

Special cases:

1. $\alpha = \beta = 1$, is considered, then $\tilde{H}_{\alpha\beta}$ is the algebra of dual quaternions.

Ref

3. Jafari M., *Some results on the matrices of Split Semi-quaternions*, 2016, submitted.

2. $\alpha=1,\ \beta=-1,$ is considered, then $\tilde{H}_{\alpha\beta}$ is the algebra of split dual quaternions.
3. $\alpha=1,\ \beta=0,$ is considered, then $\tilde{H}_{\alpha\beta}$ is the algebra of dual semi-quaternions.
4. $\alpha=-1,\ \beta=0,$ is considered, then $\tilde{H}_{\alpha\beta}$ is the algebra of dual split semi-quaternions.
5. $\alpha=0,\ \beta=0,$ is considered, then $\tilde{H}_{\alpha\beta}$ is the algebra of dual quasi-quaternions (Jafari, 2016).

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Theorem 2. Every unit generalized dual quaternion is a screw operator.

d) Dual split semi-quaternions Algebra

A dual split semi-quaternion Q is defined as

$$Q=A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$$

where A_0, A_1, A_2 and A_3 are dual numbers and $\vec{i},\vec{j},\vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^{\,2}=1,\quad \vec{j}^{\,2}=\vec{k}^{\,2}=0,$$
$$\vec{i}\vec{j}=\vec{k}=-\vec{j}\vec{i}\ ,\ \vec{j}\vec{k}=0=\vec{k}\vec{j}$$

and

$$\vec{k}\vec{i}=j=-\vec{i}\vec{k}.$$

In other words, this may also be given as $Q=q+\varepsilon q^*$, where q,q^* are split semi-quaternions. The set of all dual split semi-quaternions(abbreviated dual *SSQ*) is denoted by $\tilde{H}_{ss}$. We express the basic operations in terms of $\vec{i},\vec{j},\vec{k}$.

Given $Q=A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$, A_0 is called the *scalar part* of Q, denoted by

$$S_Q=A_0,$$

and $A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ is called the *vector part* of Q, denoted by

$$\vec{V}(Q)=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}.$$

If $S_Q=0$, then Q is called pure dual *SSQ*.

The addition becomes as

$$\begin{aligned}Q+P&=(A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k})+(B_0+B_1\vec{i}+B_2\vec{j}+B_3\vec{k})\\&=(A_0+B_0)+(A_1+B_1)\vec{i}+(A_2+B_2)\vec{j}+(A_3+B_3)\vec{k}\end{aligned}$$

This rule preserves the associativity and commutativity properties of addition.

The multiplication as

$$\begin{aligned}QP&=(A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k})(B_0+B_1\vec{i}+B_2\vec{j}+B_3\vec{k})\\&=(A_0B_0+A_1B_1)+(A_1B_0+A_0B_1)\vec{i}+\\&+(A_2B_0-A_3B_1+A_0B_2+A_1B_3)\vec{j}+(A_3B_0-A_2B_1+A_1B_2+A_0B_3)\vec{k}\end{aligned}$$

Also, this can be written as

$$QP = \begin{bmatrix} A_0 & A_1 & 0 & 0 \\ A_1 & A_0 & 0 & 0 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Obviously, the quaternion multiplication is associative and distributive with respect to addition and subtraction, but the commutativity law does not hold in general.

Corollary 1. $\tilde{\mathrm{H}}_{ss}$ with addition and multiplication has all the properties of a number field expect commutativity of the multiplication. It is therefore called the skew field of quaternions.

e) Some Properties of Dual Semi-Quaternions

1) The Hamilton *conjugate* of $Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ is

$$\overline{Q} = A_0 - A_1\vec{i} - A_2\vec{j} - A_3\vec{k}.$$

The dual *conjugate* of $Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ is

$$\begin{aligned} Q^* &= \overline{A}_0 + \overline{A}_1\vec{i} + \overline{A}_2\vec{j} + \overline{A}_3\vec{k} \\ &= (a_0 - \varepsilon a_0^*) + (a_1 - \varepsilon a_1^*)\vec{i} + (a_2 - \varepsilon a_2^*)\vec{j} + (a_3 - \varepsilon a_3^*)\vec{k}. \end{aligned}$$

The Hermitian *conjugate* of $Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ is

$$\begin{aligned} Q^\dagger &= \overline{A}_0 - \overline{A}_1\vec{i} - \overline{A}_2\vec{j} - \overline{A}_3\vec{k} \\ &= (a_0 - \varepsilon a_0^*) - (a_1 - \varepsilon a_1^*)\vec{i} - (a_2 - \varepsilon a_2^*)\vec{j} - (a_3 - \varepsilon a_3^*)\vec{k}. \end{aligned}$$

2) The *norm* of Q is

$$N_Q = Q\overline{Q} = \overline{Q}Q = A_0^2 - A_1^2$$

The norm Q canbe dual number, real number, or zero. If $N_Q = 1$, then Q is called a unit dual *SSQ*. We will use $\tilde{H}_{ss}^1$ to denote the set of all the unit dual *SSQ*.

If $N_Q = 0$, then Q is called a null dual *SSQ*. A dual split semi-quaternion Q for which $N_Q = 0$ has form $Q = A_2\vec{j} + A_3\vec{k}, (A_0 = A_1 = 0)$ and it is a zero divisor.

3) The *inverse* of Q with $N_Q \neq 0$, is

$$Q^{-1} = \frac{1}{N_Q}\overline{Q}.$$

Clearly $QQ^{-1} = 1 + 0\vec{i} + 0\vec{j} + 0\vec{k}$. Note also that $\overline{QP} = \overline{P}\,\overline{Q}$ and $(QP)^{-1} = P^{-1}Q^{-1}$.

Theorem 1. The set $\tilde{H}_{ss}^1$ of unit dual *SSQ* is a subgroup of the group $\tilde{\mathrm{H}}_{ss}^0$ where $\tilde{\mathrm{H}}_{ss}^0$ is the set of all non-zero dual split semi-quaternions.

Proof: Let $Q, P \in \tilde{H}_{ss}^1$. We have $N_{QP} = 1$, *i.e.* $QP \in \tilde{H}_{ss}^1$ and thus the first subgroup requirement is satisfied. Also, by the property

$$N_Q = N_{\overline{Q}} = N_{Q^{-1}} = 1,$$

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the second subgroup requirement $Q^{-1} \in \tilde{H}_{ss}^{1}$.

Example 4. Consider the dual split semi-quaternions

$$Q_1 = 1 + (1+\varepsilon)\vec{i} - \varepsilon\vec{j} + (\varepsilon - 1)\vec{k}, \; Q_2 = 2\varepsilon + \vec{i} - (1-\varepsilon)\vec{j} + (-1+2\varepsilon)\vec{k},$$

and

$$Q_3 = (1-\varepsilon) + (1-\varepsilon)\vec{i} + (1-3\varepsilon)\vec{j} + \vec{k}, \; Q_4 = \sqrt{2} + 2\varepsilon\vec{i} + \vec{j} - \varepsilon\vec{k},$$

1. The vector parts of Q_1, Q_2 are

$$\vec{V}_{Q_1} = (1+\varepsilon)\vec{i} - \varepsilon\vec{j} + (-1+2\varepsilon)k, \; \vec{V}_{Q_2} = \vec{i} - (1-\varepsilon)\vec{j} + (-1+2\varepsilon)k.$$

2. The Hamilton conjugates of Q_2, Q_3 are

$$\overline{Q_2} = 2\varepsilon - \vec{i} + (1-\varepsilon)\vec{j} - (-1+2\varepsilon)\vec{k}, \; \overline{Q_3} = (1-\varepsilon) - (1-\varepsilon)\vec{i} - (1-3\varepsilon)\vec{j} - \vec{k},$$

3. The dual conjugates of Q_2, Q_3 are

$$Q_2^* = -2\varepsilon + \vec{i} + (1+\varepsilon)\vec{j} - (-1-2\varepsilon)\vec{k}, \; Q_3^* = (1+\varepsilon) + (1+\varepsilon)\vec{i} + (1+3\varepsilon)\vec{j} + \vec{k},$$

4. The Hermitian conjugate of Q_1, Q_4 are

$$Q_1^\dagger = 1 - (1-\varepsilon)\vec{i} - \varepsilon\vec{j} + (1+\varepsilon)\vec{k}, \; Q_4^\dagger = \sqrt{2} + 2\varepsilon\vec{i} - \vec{j} - \varepsilon\vec{k},$$

5. The norms are given by

$$N_{Q_1} = -2\varepsilon, \; N_{Q_2} = -1, \; N_{Q_3} = 0, N_{Q_4} = 2$$

6. The inverses of Q_1, Q_2 are

$$Q_1^{-1} = \frac{1}{2\varepsilon}[1 - (1+\varepsilon)\vec{i} + \varepsilon\vec{j} - (\varepsilon - 1)\vec{k}], \; Q_2^{-1} = -[2\varepsilon - \vec{i} + (1-\varepsilon)\vec{j} - (-1+2\varepsilon)\vec{k}],$$

and Q_3 not invertible.

7. One can realize the following operations

$$Q_1 + Q_2 = (1+2\varepsilon) + (2+\varepsilon)\vec{i} - (1+2\varepsilon)\vec{j} + (-2+3\varepsilon)\vec{k}$$

$$Q_2 - Q_3 = (-1+3\varepsilon) + \varepsilon\vec{i} + (-2+2\varepsilon)\vec{j} + (-2+2\varepsilon)k$$

$$Q_1 Q_4 = (\sqrt{2} + 2\varepsilon) + [\sqrt{2} + (2+\sqrt{2})\varepsilon]\vec{i} +$$
$$+ [1 + (1-\sqrt{2})\varepsilon]\vec{j} - [(1-\sqrt{2}) + \sqrt{2}\varepsilon]\vec{k}.$$

f) Trigonometric Form and De Moivre's Theorem

In this section, we express De-Moivre's formula for dual *SSQ*. For this, we can cosider two different cases:

Case 1. Let the norm of dual *SSQ* be positive.
The trigonometric (polar) form of a non-null dual *SSQ*

$$Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

is

$$Q = R(\cosh\phi + \vec{W}\sinh\phi)$$

where $R = \sqrt{N_Q}$, and

$$\cosh\phi = \frac{|A_0|}{R}\ ,\ \sinh\phi = \frac{\sqrt{A_1^2}}{R} = \frac{|A_1|}{\sqrt{A_1^2 - A_2^2}}.$$

$\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit dual vector $\vec{W}$ is given by

$$\vec{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{A_1^2}}[A_1\vec{i} + A_2\vec{j} + A_3\vec{k}\,] = \frac{1}{|A_1|}(A_1, A_2, A_3).$$

This is similar to polar coordinate expression of asplit quaternion [7], split semi-quaternion [3].

Example 1.5. The trigonometric forms of the dual split semi-quaternions

$$Q_1 = \sqrt{2} + \vec{i} + (1+\varepsilon)\vec{j} + 2\varepsilon\vec{k}, \text{is } Q_1 = \cosh\phi_1 + \vec{W}_1\sinh\phi_1,$$

$$Q_2 = (1+\varepsilon) + \vec{i} + (1-\varepsilon)\vec{j} - \vec{k}, \text{is } Q_2 = \sqrt{2\varepsilon}\ [\cosh\phi_2 + \vec{W}_2\sinh\phi_2]$$

where

$$\cosh\phi_1 = \sqrt{2},\ \sin\phi_1 = 1,\ \vec{W}_1 = (1, 1+\varepsilon, 2\varepsilon)$$

$$\cosh\phi_2 = \frac{1+\varepsilon}{\sqrt{2\varepsilon}},\ \sinh\phi_2 = \frac{1}{\sqrt{2\varepsilon}},\ \vec{W}_2 = (1, 1-\varepsilon, -1),$$

and $N_{\vec{W}_1} = N_{\vec{W}_2} = -1.$

Theorem 1.5. (De Moivre's Theorem) If $Q = R(\cosh\phi + \vec{W}\sinh\phi)$ be a dual *SSQ* and *n* is any positive integer, then

$$Q^n = R^n(\cosh n\phi + \vec{W}\sinh n\phi)$$

Proof: The proof is easily followed by induction on *n*.■

The Theorem holds for all integers *n*, since

$$Q^{-1} = R^{-1}(\cosh\phi - \vec{W}\sinh\phi),$$

$$Q^{-n} = R^{-n}[\cosh(-n\phi) + \vec{W}\sinh(-n\phi)]$$

$$= R^{-n}[\cosh n\phi - \vec{W}\sinh n\phi]$$

Example 2.5. Let $Q = 2 + \vec{i} + (1+\varepsilon)\vec{j} - 2\varepsilon\vec{k}$. Find Q^{10} and Q^{-45}.

Solution: First write Q in trigonometric form.

$$Q = \sqrt{3}(\cosh\phi + \vec{W}\sinh\phi),$$

where $\phi = \ln\sqrt{3}$, $\vec{W} = (1, 1+\varepsilon, -2\varepsilon)$

Applying de Moivre's Theorem gives:

Ref

7. Rosenfeld B., *Geometry of Lie groups*, Kluwer Academic Publishers, Netherlands, (1997).

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$$Q^{10} = 3^5(\cosh 10\phi + \vec{W}\sinh 10\phi) = 3^5(\frac{3^5+3^{-5}}{2} + \vec{W}\frac{3^5-3^{-5}}{2})$$

$$Q^{-45} = (\sqrt{3})^{-45}(\cosh 45\phi - \vec{W}\sinh 45\phi)$$

Corollary 1.5. The equation $Q^n = 1$, does not have solution for a unit dual split semi-quaternion.

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Example 3.5. Let $Q = 1 + 3\varepsilon i + (2+\varepsilon)j + \varepsilon k$, be a dual split semi-quaternion. There is no *n* (*n*¿ 0) such that $Q^n = 1$.

Case 2. Let the norm of dual *SSQ* be negative, *i.e.* $N_Q = A_0^2 - A_1^2 < 0$.

The polar form of a non-null dual *SSQ*

$$Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

is

$$Q = R(\sinh\psi + \vec{W}\cosh\psi)$$

where $R = \sqrt{|N_Q|}$, and

$$\sinh\psi = \frac{|A_0|}{R}\ ,\ \cosh\phi = \frac{\sqrt{A_1^2}}{R} = \frac{|A_1|}{\sqrt{A_1^2 - A_2^2}}.$$

$\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit dual vector $\vec{W}$ is given by

$$\vec{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{A_1^2}}[A_1\vec{i} + A_2\vec{j} + A_3\vec{k}\ ] = \frac{1}{|A_1|}(A_1, A_2, A_3).$$

Futher Work

By the Hamilton operators, dual split semi-quaternions have been expressed in terms of 4×4 matrices. With the aid of the De-Moivre's formula, we will obtain any power of these matrices.

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