Ricci Solitons on $CR$-Submanifolds of Maximal $CR$ Dimension of a Complex Space Form

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Abstract- We study Ricci solitons on CR-submanifolds of maximal CR dimension $M^n$ of a complex space form $\mathbb{C}^{n+p}$ such that the shape operator $A$ has only one eigenvalue. We prove that Ricci soliton on CR-submanifolds of maximal CR dimension $M^n$ with eigenvalue zero is expanding and with eigen-value nonzero is expanding and shrinking.

Finally, we study Ricci soliton on CR-hypersurfaces $M^n$ of a complex space form $M^{n+1}$ (4k) such that the shape operator $A$ has exactly two distinct eigen-values and show that a Ricci soliton $(M, g, V, \lambda)$ for $k < 0$ is shrinking and expanding and for $k > 0$ is shrinking.

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Finally, we study Ricci soliton on CR-hypersurfaces $M'$ of a complex space form $M' = \mathbb{C}_{\mathbb{P}}^{n+1}$ such that the shape operator $A$ has exactly two distinct eigenvalues and show that a Ricci soliton $(M, g, V, \lambda)$ for $k < 0$ is shrinking and expanding and for $k > 0$ is shrinking.

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1. Introduction

A Ricci soliton is defined on a Riemannian manifold $(M, g)$ by

$$\frac{1}{2} L_V g + \text{Ric} - \lambda g = 0$$

where $L_V g$ is the Lie-derivative of the metric tensor $g$ with respect to $V$ and $\lambda$ is a constant on $M$. The Ricci soliton is a natural generalization of an Einstein metric. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Compact Ricci solitons are the fixed points of the Ricci flow:

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t))$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds. We denote a Ricci soliton by $(M, g, V; \lambda)$ and call the vector field $V$ the potential vector field of the Ricci soliton. A trivial Ricci soliton is one for which $V$ is Killing or zero. If its potential field $V = \nabla f$ such that $f$ is some smooth function on $M$ then a Ricci soliton $(M, g, V; \lambda)$ is called a gradient Ricci soliton and the smooth function $f$ is called the potential function. It was proved by Grigory Perelman in [13] that any compact Ricci soliton is the sum of a gradient of some smooth function $f$ up to the addition of a Killing field. Thus compact Ricci solitons are gradient Ricci solitons. In particular, Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904.

References


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Hamilton[7] and Ivey [9] proved that a Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3, respectively. In [10], Ki proved that there are no real hypersurfaces with parallel Ricci tensor in a complex space form $\mathbb{M}^n(c)$ with $c \neq 0$ when $n \geq 3$. Kim [11] proved that when $n = 2$, this is also true. In particular, these results give that there is not any Einstein real hypersurfaces in a non-flat complex space form.

In [2], Chen studied important results on Ricci solitons which occur obviously on some Riemannian submanifolds. He presented several recent new criterions of trivial compact shrinking Ricci solitons.

Cho and Kimura [3] studied on Ricci solitons of real hypersurfaces in a non-flat complex space form $\mathbb{M}^n(c)$ of a complex space form and showed that a real hypersurface $\eta$-Ricci soliton real hypersurfaces in a non-flat complex space form.

We study Ricci solitons on $CR$-submanifolds of maximal $CR$ dimension $M^n$ of a complex space form $\mathbb{C}^{n+2}$ such that the shape operator $A$ has only one eigenvalue. We prove that Ricci soliton on $CR$-submanifolds of maximal $CR$ dimension $M^n$ with eigenvalue zero is expanding and with eigenvalue nonzero is expanding and shrinking.

Finally, we study Ricci solitons on $CR$-hypersurfaces $M^n$ with exactly two distinct eigenvalues of a complex space form $\mathbb{M}^{n+1}_c(4k)$ and show that a Ricci soliton $(M, g, V, \lambda)$ for $k < 0$ is shrinking and expanding and for $k > 0$ is shrinking.

II. Preliminaries

Let $\mathbb{M}^{n+2}_c$ be a complex Kähler manifold with the natural almost complex structure $J$. A Kähler manifold $\mathbb{M}^{n+2}_c$ is called a complex space form if it has constant holomorphic sectional curvature. The Riemannian curvature tensor $\mathcal{R}$ of a complex space form is given by

$$
\mathcal{R}(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, JY)JZ).
$$

A $CR$-submanifold is a submanifold $M^n$ tangent to $\xi$ that admits an invariant distribution $D$ whose orthogonal complementary distribution $D^\perp$ is anti-invariant, that is, $TM = D \perp D^\perp$ with condition $\varphi(D_p) \subset D_p$ for all $p \in M$ and $\varphi(D^\perp_p) \subset T^\perp_p M$ for all $p \in M$, where $D = \text{span}\{X_1, \ldots, X_m, \varphi X_1, \ldots, \varphi X_m\}$ and $D^\perp = \text{span}\{\xi\}$ such that $m = \frac{n-1}{2}$.

Therefore, there exists a vector subbundles anti-invariant $\nu$ and $J$-invariant $\nu^\perp$ of the normal bundle such that

$$
J\nu_p \subset T_p M, \quad J\nu^\perp_p \subset \nu^\perp_p, \quad (2.2)
$$

for $p \in M$, where $\nu^\perp = \text{span}\{N_1, \ldots, N_q, N_1^*, \ldots, N_q^* = JN_q\}, q = \frac{p-1}{2}$ and $\nu = \text{span}\{N\}$ and $T^\perp M = \nu \oplus \nu^\perp$. 

Ref

If $M^n$ is an CR-submanifolds of maximal CR dimension of $\overline{M}^{n+p}$, then at each point $p \in M$, the real dimension of $JT_p(M) \cap T_p(M) = n - 1$.

Let $\nabla$ and $\nabla$ are the Riemannian connections of $\overline{M}$ and $M$, respectively and $\nabla\perp$ is the normal connection induced from $\nabla$ in the normal bundle $T\perp(M)$.

Let $M^n$ be a CR-submanifolds of maximal CR dimension of a complex space form $\overline{M}^{n+p}$ with constant holomorphic sectional curvature $4k$ and the normal vector field $N$ be parallel with respect to normal connection $\nabla\perp$. We can write

$$\nabla\perp X N = \sum_{a=1}^{q} \{s_a(X)N_a + s_a\ast(X)N_a\ast\}$$  \hspace{1cm} (2.3)

by the relation 2.3, we have the following lemma

**Lemma 2.1.** [6] for a CR-submanifold of maximal CR dimension, the vector field $N$ is parallel with respect to the normal connection $\nabla\perp$, if and only if $s_a = s_a\ast = 0$ for $a = 1, \ldots, q$.

We define a metric $g$ on CR-submanifolds $M^n$ of maximal CR dimension by

$$g(X, Y) = \overline{g}(\iota X, \iota Y),$$

for any $X, Y \in TM$. The Riemannian metric $g$ is said the induced metric from $\overline{g}$ on $\overline{M}^{n+1}(4k)$ and the $\iota$ is called an isometric immersion.

For any vector field $X \in \chi(M)$ the decomposition holds:

$$JX = \varphi X + \eta(X)N$$  \hspace{1cm} (2.4)

where, $\varphi$ is an endomorphism acting on $T(M)$ and $\eta$ is one-form on $M$ and $N$ is a unit normal vector field on $M^n$ such that $JN = -\xi$. The structure $(\varphi, \eta, \xi, g)$ is an almost contact metric structure on $M^n$ such that

$$\varphi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0. \hspace{1cm} (2.5)$$

and

$$\overline{g}(\varphi X, \varphi X) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = \overline{g}(X, \xi). \hspace{1cm} (2.6)$$

Now, the Gauss formula are given by

$$\nabla_X Y = \nabla_X Y + h(X, Y), \hspace{1cm} (2.1)$$

for any $X, Y \in \chi(M)$. Where, the $h$ is the second fundamental form such that

$$h(X, Y) = g(AX, Y)N + \sum_{a=1}^{q} \{g(A_a X, Y)N_a + g(A_a\ast X, Y)N_a\ast\}.$$  \hspace{1cm} (2.8)

Moreover, the Weingarten formulae can be written as follows

$$\nabla_X N = -AX + \nabla\perp_X N$$

$$= -AX + \sum_{a=1}^{q} \{s_a(X)N_a + s_a\ast(X)N_a\ast\}, \hspace{1cm} (2.9)$$

$$\nabla_X N_a = -A_a X + \nabla\perp_X N_a$$

$$= -A_a X - s_a(X)N + \sum_{b=1}^{q} \{s_{ab}(X)N_b + s_{ab}\ast(X)N_b\ast\}, \hspace{1cm} (2.10)$$
\[ \nabla_X N_a^* = -A_a^* X + \nabla_X^i N_a^* \]

(2.11) \[ = -A_a^* X - s_a^* (X) N + \sum_{b=1}^{q} \{ s_a^* b (X) N_b + s_a^* b^* (X) N_b^* \}, \quad (2.11) \]

where \( A, A_a, A_a^* \) are the shape operators for the normals \( N, N_a, N_a^* \), respectively, and \( s^* \)'s are called the coefficients of the third fundamental form of \( M \).

Therefore, taking the covariant derivative of \( N_a^* = JN_a \) and using (2.4), (2.10), (2.11) and \( JN = -\xi \), we compute

\[ A_a^* X = \varphi A_a X - s_a (X) \xi, \quad (2.12) \]
\[ A_a X = -\varphi A_a^* X + s_a^* (X) \xi, \quad (2.13) \]
\[ s_a^* (X) = \eta (A_a X) = g (A_a \xi, X), \quad (2.14) \]
\[ s_a (X) = -\eta (A_a^* X) = -g (A_a^* \xi, X), \quad (2.15) \]
\[ s_a^* b = s_{ab}, \quad s_a^* b^* = -s_{ab^*}. \quad (2.16) \]

for all \( X, Y \in TM \) and \( a, b = 1, \ldots, q \). Further, since \( \varphi \) is skew-symmetric and \( A_a, A_a^*, a = 1, \ldots, q \) are symmetric, using relations (2.12) and (2.13), we compute

\[ trace A_a^* = \sum_{i=1}^{n} g (A_a^* e_i, e_i) = s_a (\xi), \]
\[ trace A_a = s_a^* (\xi). \quad (2.17) \]

By the note the vector field \( N \) is parallel with respect to the normal connection \( \nabla^i \), using Lemma 2.1 and relations (2.12)- (2.15), we conclude

\[ A_a \xi = 0, \quad A_a^* \xi = 0, \]
\[ A_a X = -\varphi A_a^* X, \quad A_a^* X = \varphi A_a X, \quad (2.18) \]

for all \( X \in TM \) and all \( a = 1, \ldots, q \). Further, we differentiate (2.4) and \( JN = -\xi \) covariantly and compare the tangential part and the normal part. Then we obtain

\[ \nabla_X \varphi Y = \eta (Y) AX - g (AY, X) \xi, \]
\[ \nabla_X \xi = \varphi AX. \quad (2.19) \]

Then from (2.4), The Gauss equation are written as follow:

\[ R (X, Y) Z = k \{ g (Y, Z) X - g (X, Z) Y \]
\[ + g (\varphi Y, Z) \varphi X - g (\varphi X, Z) \varphi Y - 2g (\varphi X, Y) \varphi Z \}
\[ + g (AY, Z) AX - g (AX, Z) AY \]
\[ + \sum_{a=1}^{q} \{ g (A_a Y, Z) A_a X - g (A_a X, Z) A_a Y \}
\[ + g (A_a^* Y, Z) A_a^* X - g (A_a^* X, Z) A_a^* Y \}, \quad (2.20) \]

by Lemma 2.1, the Codazzi equation become

\[ (\nabla_X A) Y - (\nabla_Y A) X = k \{ \eta (X) \varphi Y - \eta (Y) \varphi X - 2g (\varphi X, Y) \xi \}, \quad (2.21) \]
hence, by the relations (2.17), (2.18), Ricci tensor is obtained as
\[
Ric(X,Y) = k\{(n + 2)g(X,Y) - 3\eta(X)\eta(Y)\} + (\text{trace} A)g(AX,Y) - g(AX,AY) - 2\sum_{a=1}^{q}g(A_{a}X,A_{a}Y).
\]
for any tangent vector fields \(X,Y,Z\) on \(M\), where \(R\) and \(Ric\) are the curvature and Ricci tensors of \(M\), respectively.

III. Ricci Soliton on CR Hypersurfaces

Let \(M^{n}\) be a \(CR\)-submanifolds of maximal \(CR\) dimension of a complex space form \(\mathbb{M}^{n+2}\) with the vector field \(N\) be parallel with respect to normal connexion \(\nabla^{\perp}\) such that the shape operator \(A\) for unit normal vector field \(N\) has only one eigenvalue. Let \(\{e_{1},...,e_{n-1},\xi\}\) be a local orthonormal frame field such that \(D_{\xi} = \text{span}\{\xi\}\) and \(D = \text{span}\{e_{1},...,e_{m},e_{m+1}=\varphi e_{1},...,e_{2m}=\varphi e_{2m-1}\}\) such that \(m = \frac{n-1}{2}\).

In [6], proved that

**Theorem 3.1.** If the shape operator \(A\) with respect to unit normal vector field \(N\) of \(M^{n}\) has only one eigenvalue, then \(\mathbb{M}^{n+2}\) is a complex Euclidean space.

According to the assumption, it follows that \(A = 0\) or \(AX = \alpha X\) for all \(X \in T(M)\) such that \(\alpha \neq 0\).

Let \(AX = \alpha X\), therefore by the relation (2.22), we obtain
\[
Ric(e_{i},e_{j}) = \{(n - 1)\alpha^{2}\}\delta_{ij} - 2\sum_{a=1}^{q}g(A_{a}e_{i},A_{a}e_{j}),\quad i,j = 1,...,n-1,
\]
\[
Ric(\xi,\xi) = (n - 1)\alpha^{2},
\]
\[
Ric(e_{i},\xi) = 0, \quad i = 1,...,n-1.
\]

We consider \(CR\)-submanifolds of maximal \(CR\) dimension of a complex space form \(\mathbb{C}^{n+2}\) satisfying Ricci soliton equation
\[
\frac{1}{2}\mathcal{L}_{V}g + Ric - \lambda g = 0
\]
with respect to potential vector field \(V\) on \(M\) for constant \(\lambda\).

Putting
\[
V := f\xi, \quad (f : M \to \mathbb{R}, f \neq 0)
\]
Then definition of Lie derivative and second relation (2.19) imply
\[
(\mathcal{L}_{f\xi}g)(X,Y) = df(X)\eta(Y) + df(Y)\eta(X).
\]

We compute
\[
(\mathcal{L}_{f\xi}g)(\xi,\xi) = 2df(\xi),
\]
\[
(\mathcal{L}_{f\xi}g)(\xi,e_{i}) = df(e_{i}), \quad (i = 1,...,n-1),
\]
\[
(\mathcal{L}_{f\xi}g)(e_{i},e_{j}) = 0 \quad (i,j = 1,...,n-1).
\]
Using relations (3.1)-(3.3) and (3.7)-(3.9), Ricci soliton equation (3.4) is equivalent to
\[ df(\xi) = \lambda - (n - 1)\alpha^2, \quad (3.10) \]
\[ df(e_i) = 0, \quad (i = 1, ..., n - 1), \quad (3.11) \]
\[ ((n - 1)\alpha^2 - \lambda)\delta_{ij} - 2 \sum_{a=1}^{q} g(A_a e_i, A_a e_j) = 0, \quad (i, j = 1, ..., n - 1). \quad (3.12) \]

By the relation (3.12), for \( i = j \) we have \( \lambda = (n - 1)\alpha^2 - 2 \sum_{a=1}^{q} g(A_a e_i, A_a e_i) \) and thus the following theorem holds:

**Theorem 3.2.** Let \( M^n \) be a CR-submanifolds of maximal CR dimension of a complex space form \( \mathbb{C}^{\frac{n+p}{2}} \) with \( AX = \alpha X \). Then a Ricci soliton \((M, g, V, \lambda)\) with potential field \( V := f \xi \) is

(a) shrinking Ricci soliton if \((n - 1)\alpha^2 > 2 \sum_{a=1}^{q} g(A_a e_i, A_a e_i)\).

(b) expanding Ricci soliton if \((n - 1)\alpha^2 < 2 \sum_{a=1}^{q} g(A_a e_i, A_a e_i)\).

Now, let \( A = 0 \), using relation (2.22), it follows that

\[ \text{Ric}(e_i, e_j) = -2 \sum_{a=1}^{q} g(A_a e_i, A_a e_j), \quad i, j = 1, ..., n - 1, \quad (3.13) \]
\[ \text{Ric}(\xi, \xi) = 0, \quad (3.14) \]
\[ \text{Ric}(e_i, \xi) = 0, \quad i = 1, ..., n - 1. \quad (3.15) \]

CR-submanifolds of maximal CR dimension \( M^n \) \((n \geq 3)\) is considered in a complex space form \( \mathbb{C}^{\frac{n+p}{2}} \) satisfying Ricci soliton equation with potential vector field \( f \xi \). From relations (3.13)-(3.15) and (3.7)-(3.9), Ricci soliton equation (3.4) is equivalent to

\[ df(\xi) = \lambda, \quad (3.16) \]
\[ df(e_i) = 0, \quad (i = 1, ..., n - 1), \quad (3.17) \]
\[ (-\lambda)\delta_{ij} - 2 \sum_{a=1}^{q} g(A_a e_i, A_a e_j) = 0, \quad (i, j = 1, ..., n - 1). \quad (3.18) \]

Using the relation (3.18), it follows \( \lambda = -2 \sum_{a=1}^{q} g(A_a e_i, A_a e_i) \) and hence

**Theorem 3.3.** Let \( M^n \) be a CR-submanifolds of maximal CR dimension of complex space form \( \mathbb{C}^{\frac{n+p}{2}} \) with \( A = 0 \). Then a Ricci soliton \((M, g, V, \lambda)\) with potential field \( V := f \xi \) is expanding Ricci soliton.

Let \( M^n \) \((n \geq 3)\) is a CR-hypersurface in a complex space form \( \overline{M}^{\frac{n+1}{2}} \). We assume that the shape operator \( A \) with respect to \( N \) has exactly two distinct eigenvalues \( \alpha \) and \( \beta \). The following lemma holds[6]

**Lemma 3.4.** Let \( \overline{M}^{\frac{n+1}{2}} \) be a Kähler manifold of constant holomorphic sectional curvature \( 4k \), with \( k \neq 0 \). If the shape operator \( A \) has exactly two distinct eigenvalues, then \( \xi \) ia an eigenvector of \( A \).
By the lemma above, let $A\xi = \alpha \xi$. Differentiating $A\xi = \alpha \xi$ covariantly and the second relation (2.19) imply

$$(\nabla_X A)\xi = \alpha \varphi AX - A\varphi AX + (X\alpha)\xi$$

The Codazzi equation is obtained as

$$(\nabla_\xi A)X = k\varphi X + \alpha \varphi AX - A\varphi AX + (X\alpha)\xi$$

Since $\nabla_\xi A$ is self-adjoint, we conclude the relation:

$$0 = -2g(A\varphi AX, Y) + 2k\eta(\varphi X, Y) + \alpha g((A\varphi + \varphi A)X, Y) + (X\alpha)\eta(Y)$$

Substituting $Y$ for $\xi$ in (3.19) and using the fact that $\alpha$ is an eigenvalue of $A$, we conclude the relation:

$$\gamma = \frac{(\alpha^2 + 2k)}{(2\beta - \alpha)}$$

As $A$ has exactly two distinct eigenvalues, we have the following three cases:

If $\alpha = \beta$, we conclude that $\gamma = \frac{(\alpha^2 + 2k)}{(\alpha)}$ and $\text{trace} A = \frac{n\alpha^2 + k(n-1)}{\alpha}$.

Since the shape operator $A$ is self-adjoint, for any $X, Y \in D$

$$\alpha g(\varphi X, Y) = g(AX, \varphi Y) = g(X, A\varphi Y) = \gamma g(X, \varphi Y)$$

therefore, $\alpha = \beta = \gamma$, which is a contradiction since the shape operator $A$ with respect to $N$ has exactly two distinct eigenvalues.

Now, if $\gamma = \alpha$, we conclude that $\alpha \beta - \alpha^2 = 2k$ and $\text{trace} A = \frac{n-1}{2}\beta + \frac{n+1}{2}\alpha$.

By the note the shape operator $A$ is self-adjoint, we have

$$\alpha g(X, \varphi Y) = g(X, AX) = g(AX, \varphi Y) = \beta g(X, \varphi Y)$$

therefore, $\gamma = \alpha = \beta$, which is a contradiction since the shape operator $A$ with respect to $N$ has exactly two distinct eigenvalues. Thus the multiplicity of the eigenvalue $\alpha$ corresponding to the eigenvector $\xi$ is one.

Therefore, we suppose that the shape operator $A$ has exactly two distinct eigenvalues, $\alpha, \beta = \gamma$. Then it follows that $\beta^2 - \alpha \beta = k$ and $A\varphi = \varphi A$ and $\text{trace} A = \alpha + (n-1)\beta$.

Hence, by the relation (2.22), Ricci tensor related to a CR-hypersurface $(M^n, g)$ is written as

$$\text{Ric}(e_i, e_j) = (2kn + (n-1)\beta^2)\delta_{ij}, \quad (i, j = 1, ..., n-1),$$

$$\text{Ric}(\xi, \xi) = (n-1)(k + \beta^2),$$

$$\text{Ric}(e_i, \xi) = 0, \quad (i = 1, ..., n-1),$$

Notes
We consider a CR-hypersurface $M^n$ ($n \geq 3$) in complex space form $\mathbb{M}^{n+1}_2(4k)$ that satisfies Ricci soliton

$$\frac{1}{2} \mathcal{L}_V g + \text{Ric} - \lambda g = 0 \quad (3.28)$$

with respect to potential vector field $V$ on $M$ for constant $\lambda$. We put

$$V := f \xi, \quad (f : M \to \mathbb{R}, f \neq 0) \quad (3.29)$$

Definition of Lie derivative and the second relation (2.19) imply

$$(\mathcal{L}_{f \xi} g)(X,Y) = df(X)\eta(Y) + df(Y)\eta(X). \quad (3.30)$$

We obtain

$$(\mathcal{L}_{f \xi} g)(\xi,\xi) = 2df(\xi), \quad (3.31)$$

$$(\mathcal{L}_{f \xi} g)(\xi,e_i) = df(e_i), \quad (i = 1,\ldots,n-1), \quad (3.32)$$

$$(\mathcal{L}_{f \xi} g)(e_i,e_j) = 0, \quad (i,j = 1,\ldots,n-1). \quad (3.33)$$

Using relations (3.25)-(3.27) and (3.31)-(3.33), Ricci soliton equation (3.28) follows

$$\lambda = 2kn + (n-1)\beta^2, \quad (3.34)$$

$$df(\xi) = k(n+1), \quad (3.35)$$

$$df(e_i) = 0, \quad (i = 1,\ldots,n-1), \quad (3.36)$$

**Theorem 3.5.** Let $M$ be a CR-hypersurface of complex space form $\mathbb{M}^{n+1}_2(4k)$. If $k > 0$, then a Ricci soliton $(M,g,V,\lambda)$ with potential field $V := f \xi$ is shrinking Ricci soliton.

**Theorem 3.6.** Let $M$ be a CR-hypersurface of complex space form $\mathbb{M}^{n+1}_2(4k)$ with $k < 0$.

a) If $|k| > \frac{(n-1)\beta^2}{2n}$, Then a Ricci soliton $(M,g,V,\lambda)$ with potential field $V := f \xi$ is expanding Ricci soliton.

b) If $|k| < \frac{(n-1)\beta^2}{2n}$. Then a Ricci soliton $(M,g,V,\lambda)$ with potential field $V := f \xi$ is shrinking Ricci soliton.

**References**


