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Ricci Solitons on *CR*-Submanifolds of Maximal *CR* Dimension of a Complex Space Form

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Finally, we study Ricci soliton on *CR*-hypersurfaces M^n of a complex space form $M^{\frac{n+1}{2}}$ (4k) such that the shape operator A has exactly two distinct eigen-values and show that a Ricci soliton (M, g, V, λ) for $k < 0$ is shrinking and expanding and for $k > 0$ is shrinking.

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Ricci Solitons on *CR*-Submanifolds of Maximal *CR* Dimension of a Complex Space Form

Z. Nazari [α] & E. Abedi [σ]

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I. Introduction

A Ricci soliton is defined on a Riemannian manifold (M, g) by

$$\frac{1}{2}\mathcal{L}_V g + Ric - \lambda g = 0 \tag{1.1}$$

where $L_V g$ is the Lie-derivative of the metric tensor g with respect to V and λ is a constant on M. The Ricci soliton is a natural generalization of an Einstein metric. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Compact Ricci solitons are the fixed points of the Ricci flow:

$$\frac{\partial}{\partial t} g(t) = -2Ric(g(t)) \tag{1.2}$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds. We denote a Ricci soliton by $(M, g, V; \lambda)$ and call the vector field V the potential vector field of the Ricci soliton. A trivial Ricci soliton is one for which V is Killing or zero. If its potential field $V = \nabla f$ such that f is some smooth function on M then a Ricci soliton $(M, g, V; \lambda)$ is called a gradient Ricci soliton and the smooth function f is called the potential function. It was proved by Grigory Perelman in [13] that any compact Ricci soliton is the sum of a gradient of some smooth function f up to the addition of a Killing field. Thus compact Ricci solitons are gradient Ricci solitons. In particular, Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904.

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Hamilton[7] and Ivey [9] proved that a Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3, respectively. In [10], Ki proved that there are no real hypersurfaces with parallel Ricci tensor in a complex space form $\overline{M}^n(c)$ with $c \neq 0$ when $n \geq 3$. Kim [11] proved that when $n = 2$, this is also true. In particular, these results give that there is not any Einstein real hypersurfaces in a non-flat complex space form.

In [2], Chen studied important results on Ricci solitons which occur obviously on some Riemannian submanifolds. He presented several recent new criterions of trivial compact shrinking Ricci solitons.

Cho and Kimura [3] studied on Ricci solitons of real hypersurfaces in a non-flat complex space form and showed that a real hypersurface M in a non-flat complex space form $\overline{M}^n(c \neq 0)$ does not admit a Ricci soliton such that the Reeb vector field ξ is potential vector field. They defined so called η-Ricci soliton, such that satisfies

$$\frac{1}{2}\mathcal{L}_V g + Ric - \lambda g - \mu\eta \otimes \eta = 0 \tag{1.3}$$

where λ, μ are constants. They first proved that a real hypersurface M of a non-flat complex space form $\overline{M}^n(c)$ which accepts an η-Ricci soliton is a Hopf-hypersurface and classified that η-Ricci soliton real hypersurfaces in a non-flat complex space form.

We study Ricci solitons on CR-submanifolds of maximal CR dimension M^n of a complex space form $\mathbb{C}^{\frac{n+p}{2}}$ such that the shape operator A has only one eigenvalue. We prove that Ricci soliton on CR-submanifolds of maximal CR dimension M^n with eigenvalue zero is expanding and with eigenvalue nonzero is expanding and shrinking.

Finally, we study Ricci solitons on CR-hypersurfaces M^n with exactly two distinct eigenvalues of a complex space form $\overline{M}^{\frac{n+1}{2}}(4k)$ and show that a Ricci soliton (M, g, V, λ) for $k < 0$ is shrinking and expanding and for $k > 0$ is shrinking.

II. Preliminaries

Let $\overline{M}^{\frac{n+p}{2}}$ be a complex Kähler manifold with the natural almost complex structure J. A Kähler manifold $\overline{M}^{\frac{n+p}{2}}$ is called a complex space form if it has constant holomorphic sectional curvature. The Riemannian curvature tensor $\overline{R}$ of a complex space form is given by

$$\begin{aligned}\overline{R}(X,Y)Z &= k\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y \\ &+ \overline{g}(JY,Z)JX - \overline{g}(JX,Z)JY - 2\overline{g}(JX,Y)JZ\}.\end{aligned} \tag{2.1}$$

A CR-submanifold is a submanifold M^n tangent to ξ that admits an invariant distribution D whose orthogonal complementary distribution $D^\perp$ is anti-invariant, that is, $TM = D \oplus D^\perp$ with condition $\varphi(D_p) \subset D_p$ for all $p \in M$ and $\varphi(D_p^\perp) \subset T_p^\perp M$ for all $p \in M$, where $D = span\{X_1, ..., X_m, \varphi X_1, ..., \varphi X_m\}$ and $D^\perp = span\{\xi\}$ such that $m = \frac{n-1}{2}$.

Therefore, there exists a vector subbundles anti-invariant ν and J-invariant $\nu^\perp$ of the normal bundle such that

$$\begin{aligned} &J\nu_p \subset T_pM, \\ &J\nu_p^\perp \subset \nu_p^\perp, \end{aligned} \tag{2.2}$$

for $p \in M$, where $\nu^\perp = span\{N_1, ..., N_q, N_{1^*} = JN_1, ..., N_{q^*} = JN_q\}, q = \frac{p-1}{2}$ and $\nu = span\{N\}$ and $T^\perp M = \nu \oplus \nu^\perp$.

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11. U. K. Kim, *Nonexistence of Ricci-parallel real hypersurfaces in P_2C or H_2C*, Bull. Korean Math. Soc. 41 (2004), 699708.

If M^n is an CR-submanifolds of maximal CR dimension of $\overline{M}^{\frac{n+p}{2}}$, then at each point $p \in M$, the real dimension of $JT_p(M) \cap T_p(M) = n-1$.

Let $\overline{\nabla}$ and ∇ are the Riemannian connections of $\overline{M}$ and M, respectively and $\nabla^{\perp}$ is the normal connection induced from $\overline{\nabla}$ in the normal bundle $T^{\perp}(M)$.

Let M^n be a CR-submanifolds of maximal CR dimension of a complex space form $\overline{M}^{\frac{n+p}{2}}$ with constant holomorphic sectional curvature $4k$ and the normal vector field N be parallel with respect to normal connction $\nabla^{\perp}$. We can write

$$\nabla^{\perp}_X N = \sum_{a=1}^{q} \{s_a(X)N_a + s_{a^*}(X)N_{a^*}\} \tag{2.3}$$

by the relation 2.3, we have the following lemma

Lemma 2.1. [6] *for a CR- submanifold of maximal CR dimension, the vector field N is parallel with respect to the normal connection $\nabla^{\perp}$, if and only if $s_a = s_{a^*} = 0$ for $a = 1, ..., q$.*

We define a metric g on CR-submanifolds M^n of maximal CR dimension by

$$g(X, Y) = \overline{g}(\iota X, \iota Y),$$

for any $X, Y \in TM$. The Riemannian metric g is said the induced metric from $\overline{g}$ on $\overline{M}^{n+1}(4k)$ and the ι is called an isometric immersion.

For any vector field $X \in \chi(M)$ the decomposition holds:

$$JX = \varphi X + \eta(X)N \tag{2.4}$$

where, φ is an endomorphism acting on $T(M)$ and η is one-form on M and N is a unit normal vector field on M^n such that $JN = -\xi$. The structure (φ, η, ξ, g) is an almost contact metric structure on M^n such that

$$\varphi^2 = -Id + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0, \qquad \eta o \varphi = 0. \tag{2.5}$$

and

$$\overline{g}(\varphi X, \varphi X) = \overline{g}(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = \overline{g}(X, \xi). \tag{2.6}$$

Now, the Gauss formula are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

for any $X, Y \in \chi(M)$. Where, the h is the second fundamental form such that

$$\begin{aligned} h(X, Y) &= g(AX, Y)N \\ &+ \sum_{a=1}^{q} \{g(A_a X, Y)N_a + g(A_{a^*}X, Y)N_{a^*}\}. \end{aligned} \tag{2.8}$$

Moreover, the Weingarten formulae can be written as follows

$$\begin{aligned} \overline{\nabla}_X N &= -AX + \nabla^{\perp}_X N \\ &= -AX + \sum_{a=1}^{q} \{s_a(X)N_a + s_{a^*}(X)N_{a^*}\}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \overline{\nabla}_X N_a &= -A_a X + \nabla^{\perp}_X N_a \\ &= -A_a X - s_a(X)N + \sum_{b=1}^{q} \{s_{ab}(X)N_b + s_{ab^*}(X)N_{b^*}\}, \end{aligned} \tag{2.10}$$

$$\begin{aligned}\overline{\nabla}_X N_{a^*} &= -A_{a^*}X + \nabla^{\perp}_X N_{a^*} \\ &= -A_{a^*}X - s_{a^*}(X)N + \sum_{b=1}^{q}\{s_{a^*b}(X)N_b + s_{a^*b^*}(X)N_{b^*}\}, \end{aligned} \quad (2.11)$$

where A, A_a, A_{a^*} are the shape operators for the normals N, N_a, N_{a^*}, respectively, and s's are called the coefficients of the third fundamental form of M in $\overline{M}$.

Therefore, taking the covariant derivative of $N_{a^*} = JN_a$ and using (2.4), (2.10), (2.11) and $JN = -\xi$, we compute

$$A_{a^*}X = \varphi A_a X - s_a(X)\xi, \quad (2.12)$$

$$A_a X = -\varphi A_{a^*}X + s_{a^*}(X)\xi, \quad (2.13)$$

$$s_{a^*}(X) = \eta(A_a X) = g(A_a\xi, X), \quad (2.14)$$

$$s_a(X) = -\eta(A_{a^*}X) = -g(A_{a^*}\xi, X), \quad (2.15)$$

$$s_{a^*b^*} = s_{ab}, \qquad s_{a^*b} = -s_{ab^*}. \quad (2.16)$$

for all $X, Y \in TM$ and $a, b = 1, ..., q$. Further, since φ is skew-symmetric and $A_a, A_{a^*}, a = 1, ..., q$ are symmetric, using relations (2.12) and (2.13), we compute

$$traceA_{a^*} = \sum_{i=1}^{n} g(A_{a^*}e_i, e_i) = s_a(\xi),$$

$$traceA_a = s_{a^*}(\xi). \quad (2.17)$$

By the note the vector field N is parallel with respect to the normal connection $\nabla^{\perp}$, using Lemma 2.1 and relations (2.12)- (2.15), we conclude

$$A_a\xi = 0, \qquad A_{a^*}\xi = 0,$$

$$A_a X = -\varphi A_{a^*}X, \qquad A_{a^*}X = \varphi A_a X, \quad (2.18)$$

for all $X \in TM$ and all $a = 1, ..., q$. Further, we differentiate (2.4) and $JN = -\xi$ covariantly and compare the tangential part and the normal part. Then we obtain

$$(\nabla_X\varphi)Y = \eta(Y)AX - g(AY, X)\xi,$$

$$\nabla_X\xi = \varphi AX. \quad (2.19)$$

Then from (2.4), The Gauss equation are written as follow:

$$\begin{aligned} R(X,Y)Z &= k\{g(Y,Z)X - g(X,Z)Y \\ &+ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY \\ &+ \sum_{a=1}^{q}\{g(A_a Y, Z)A_a X - g(A_a X, Z)A_a Y \\ &+ g(A_{a^*}Y, Z)A_{a^*}X - g(A_{a^*}X, Z)A_{a^*}Y\}, \end{aligned} \quad (2.20)$$

by Lemma 2.1, the Codazzi equation become

$$(\nabla_X A)Y - (\nabla_Y A)X = k\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\}, \quad (2.21)$$

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hence, by the relations (2.17), (2.18), Ricci tensor is obtained as

$$\begin{aligned} Ric(X,Y) &= k\{(n+2)g(X,Y)-3\eta(X)\eta(Y)\} \\ &+ (traceA)g(AX,Y)-g(AX,AY) \\ &- 2\sum_{a=1}^{q} g(A_aX,A_aY). \end{aligned} \tag{2.22}$$

for any tangent vector fields X, Y, Z on M, where R and Ric are the curvature and Ricci tensors of M, respectively.

III. Ricci Soliton on CR Hypersurfaces

Let M^n be a CR-submanifolds of maximal CR dimension of a complex space form $\overline{M}^{\frac{n+p}{2}}$ with the vector field N be parallel with respect to normal connction $\nabla^{\perp}$ such that the shape operator A for unit normal vector field N has only one eigenvalue. Let $\{e_1,...,e_{n-1},\xi\}$ be a local orthonormal fram field such that $D^{\perp} = span\{\xi\}$ and $D = span\{e_1,...,e_m,e_{m+1} = \varphi e_1,...,e_{2m=n-1} = \varphi e_m\}$ such that $m = \frac{n-1}{2}$.
In [6], proved that

Theorem 3.1. If the shape operator A with respect to unit normal vector field N of M^n has only one eigenvalue, then $\overline{M}^{\frac{n+p}{2}}$ is a complex Euclidean space.

According to the assumption, it follows that $A = 0$ or $AX = \alpha X$ for all $X \in T(M)$ such that $\alpha \neq 0$.
Let $AX = \alpha X$, therefore by the relation (2.22), we obtain

$$Ric(e_i,e_j) = \{(n-1)\alpha^2\}\delta_{ij} - 2\sum_{a=1}^{q} g(A_ae_i,A_ae_j), \qquad i,j = 1,...,n-1, \tag{3.1}$$

$$Ric(\xi,\xi) = (n-1)\alpha^2, \tag{3.2}$$

$$Ric(e_i,\xi) = 0, \qquad i = 1,...,n-1. \tag{3.3}$$

We consider CR-submanifolds of maximal CR dimension of a complex space form $\mathbb{C}^{\frac{n+p}{2}}$ satisfying Ricci soliton equation

$$\frac{1}{2}\mathcal{L}_V g + Ric - \lambda g = 0 \tag{3.4}$$

with respect to potential vector field V on M for constant λ.
Putting

$$V := f\xi, \qquad (f : M \to \mathbb{R}, f \neq 0) \tag{3.5}$$

Then definition of Lie derivative and second relation (2.19) imply

$$(\mathcal{L}_{f\xi}g)(X,Y) = df(X)\eta(Y) + df(Y)\eta(X). \tag{3.6}$$

We compute

$$(\mathcal{L}_{f\xi}g)(\xi,\xi) = 2df(\xi), \tag{3.7}$$

$$(\mathcal{L}_{f\xi}g)(\xi,e_i) = df(e_i), \qquad (i = 1,...,n-1), \tag{3.8}$$

$$(\mathcal{L}_{f\xi}g)(e_i,e_j) = 0 \qquad (i,j = 1,...,n-1). \tag{3.9}$$

Using relations (3.1)-(3.3) and (3.7)-(3.9), Ricci soliton equation (3.4) is equivalent to

$$df(\xi) = \lambda - (n-1)\alpha^2, \tag{3.10}$$

$$df(e_i) = 0, \qquad (i = 1, ..., n-1), \tag{3.11}$$

$$\{(n-1)\alpha^2 - \lambda\}\delta_{ij} - 2\sum_{a=1}^{q} g(A_a e_i, A_a e_j) = 0, \ \ (i, j = 1, ..., n-1). \tag{3.12}$$

By the relation (3.12), for $i = j$ we have $\lambda = (n-1)\alpha^2 - 2\sum_{a=1}^{q} g(A_a e_i, A_a e_i)$ and thus the following theorem holds:

Theorem 3.2. Let M^n be a CR-submanifolds of maximal CR dimension of a complex space form $\mathbb{C}^{\frac{n+p}{2}}$ with $AX = \alpha X$. Then a Ricci soliton (M, g, V, λ) with potential field $V := f\xi$ is

(a) shrinking Ricci soliton if $(n-1)\alpha^2 > 2\sum_{a=1}^{q} g(A_a e_i, A_a e_i)$.

(b) expanding Ricci soliton if $(n-1)\alpha^2 < 2\sum_{a=1}^{q} g(A_a e_i, A_a e_i)$.

Now, let $A = 0$, using relation (2.22), it follows that

$$Ric(e_i, e_j) = -2\sum_{a=1}^{q} g(A_a e_i, A_a e_j), \quad i, j = 1, ..., n-1, \tag{3.13}$$

$$Ric(\xi, \xi) = 0, \tag{3.14}$$

$$Ric(e_i, \xi) = 0, \qquad i = 1, ..., n-1. \tag{3.15}$$

CR-submanifolds of maximal CR dimension M^n $(n \geq 3)$ is considered in a complex space form $\mathbb{C}^{\frac{n+p}{2}}$ satisfying Ricci soliton equation with potential vector field $f\xi$. From relations (3.13)-(3.15) and (3.7)-(3.9), Ricci soliton equation (3.4) is equivalent to

$$df(\xi) = \lambda, \tag{3.16}$$

$$df(e_i) = 0, \qquad (i = 1, ..., n-1), \tag{3.17}$$

$$(-\lambda)\delta_{ij} - 2\sum_{a=1}^{q} g(A_a e_i, A_a e_j) = 0, (i, j = 1, ..., n-1). \tag{3.18}$$

Using the relation (3.18), it follows $\lambda = -2\sum_{a=1}^{q} g(A_a e_i, A_a e_i)$ and hence

Theorem 3.3. Let M^n be a CR-submanifolds of maximal CR dimension of complex space form $\mathbb{C}^{\frac{n+p}{2}}$ with $A = 0$. Then a Ricci soliton (M, g, V, λ) with potential field $V := f\xi$ is expanding Ricci soliton.

Let M^n $(n \geq 3)$ is a CR-hypersurface in a complex space form $\overline{M}^{\frac{n+1}{2}}$. We assume that the shape operator A with respect to N has exactly two distinct eigenvalues α and β. The following lemma holds[6]

Lemma 3.4. Let $\overline{M}^{\frac{n+1}{2}}$ be a Kähler manifold of constant holomorphic sectional curvature $4k$, with $k \neq 0$. If the shape operator A has exactly two distinct eigenvalues, then ξ ia an eigenvector of A.

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By the lemma above, let $A\xi = \alpha\xi$. Differentiating $A\xi = \alpha\xi$ covariantly and the second relation (2.19) imply

$$(\nabla_X A)\xi = \alpha\varphi AX - A\varphi AX + (X\alpha)\xi$$

The Codazzi equation is obtained as

$$(\nabla_\xi A)X = k\varphi X + \alpha\varphi AX - A\varphi AX + (X\alpha)\xi$$

Since $\nabla_\xi A$ is self-adjoint, we conclude the relation:

$$\begin{aligned} 0 &= -2g(A\varphi AX, Y) + 2kg(\varphi X, Y) + \alpha g((A\varphi + \varphi A)X, Y) \\ &+ (X\alpha)\eta(Y) - (Y\alpha)\eta(X) \end{aligned} \tag{3.19}$$

Substituting Y for ξ in (3.19) and using of the fact that α is an eigenvalue of A, it follow that $(X\alpha) = \eta(X)\xi\alpha$. Similarly by substituting X for ξ in (3.19), we get $(Y\alpha) = \eta(Y)\xi\alpha$. It follows

$$2A\varphi AX - 2k\varphi X = \alpha(A\varphi + \varphi A)X \tag{3.20}$$

We assume that $AX = \beta X$ for any vector field $X \in D, \|X\| = 1$. Then

$$A\varphi X = \frac{(\alpha\beta + 2k)}{(2\beta - \alpha)}\varphi X. \tag{3.21}$$

Therefore, φX is an eigenvector corresponding to the eigenvalue

$$\gamma = \frac{(\alpha\beta + 2k)}{(2\beta - \alpha)} \tag{3.22}$$

As A has exactly two distinct eigenvalues, we have the following three cases:
If $\alpha = \beta$, we conclude that $\gamma = \frac{(\alpha^2+2k)}{(\alpha)}$ and $traceA = \frac{n\alpha^2+k(n-1)}{\alpha}$.
Since the shape operator A is self-adjoint, for any $X, Y \in D$

$$\alpha g(\varphi X, Y) = g(AX, \varphi Y) = g(X, A\varphi Y) = \gamma g(X, \varphi Y) \tag{3.23}$$

therefore, $\alpha = \beta = \gamma$, which is a contradiction since the shape operator A with respect to N has exactly two distinct eigenvalues.

Now, if $\gamma = \alpha$, we conclude that $\alpha\beta - \alpha^2 = 2k$ and $traceA = \frac{n-1}{2}\beta + \frac{n+1}{2}\alpha$.
By the note the shape operator A is self-adjoint, we have

$$\alpha g(X, \varphi Y) = g(X, A\varphi Y) = g(AX, \varphi Y) = \beta g(X, \varphi Y) \tag{3.24}$$

therefore, $\gamma = \alpha = \beta$, which is a contradiction since the shape operator A with respect to N has exactly two distinct eigenvalues. Thus the multiplicity of the eigenvalue α corresponding to the eigenvector ξ is one.

Therefore, we suppose that the shape operator A has exactly two distinct eigenvalues, α, $\beta = \gamma$. Then it follows that $\beta^2 - \alpha\beta = k$ and $A\varphi = \varphi A$ and $traceA = \alpha + (n-1)\beta$.

Hence, by the relation (2.22), Ricci tensor related to a CR-hypersurface (M^n, g) is written as

$$Ric(e_i, e_j) = \{2kn + (n-1)\beta^2\}\delta_{ij}, \qquad (i, j = 1, ..., n-1), \tag{3.25}$$

$$Ric(\xi, \xi) = (n-1)(k + \beta^2), \tag{3.26}$$

$$Ric(e_i, \xi) = 0, \qquad (i = 1, ..., n-1), \tag{3.27}$$

We consider a CR-hypersurface M^n $(n \geq 3)$ in complex space form $\overline{M}^{\frac{n+1}{2}}(4k)$ that satisfying Ricci soliton

$$\frac{1}{2}\mathcal{L}_V g + Ric - \lambda g = 0 \tag{3.28}$$

with respect to potential vector field V on M for constant λ.
We put

$$V := f\xi, \qquad (f : M \to \mathbb{R}, f \neq 0) \tag{3.29}$$

Definition of Lie derivative and the second relation (2.19) imply

$$(\mathcal{L}_{f\xi}g)(X,Y) = df(X)\eta(Y) + df(Y)\eta(X). \tag{3.30}$$

We obtain

$$(\mathcal{L}_{f\xi}g)(\xi,\xi) = 2df(\xi), \tag{3.31}$$

$$(\mathcal{L}_{f\xi}g)(\xi,e_i) = df(e_i), \qquad (i = 1,...,n-1), \tag{3.32}$$

$$(\mathcal{L}_{fU}g)(e_i,e_j) = 0, \qquad (i,j = 1,...,n-1). \tag{3.33}$$

Using relations (3.25)-(3.27) and (3.31)-(3.33), Ricci soliton equation (3.28) follows

$$\lambda = 2kn + (n-1)\beta^2, \tag{3.34}$$

$$df(\xi) = k(n+1), \tag{3.35}$$

$$df(e_i) = 0, \qquad (i = 1,...,n-1), \tag{3.36}$$

Theorem 3.5. Let M be a CR-hypersurface of complex space form $\overline{M}^{\frac{n+1}{2}}(4k)$. If $k > 0$, then a Ricci soliton (M,g,V,λ) with potential field $V := f\xi$ is shrinking Ricci soliton.

Theorem 3.6. Let M be a CR-hypersurface of complex space form $\overline{M}^{\frac{n+1}{2}}(4k)$ with $k < 0$.

a) If $|k| > \frac{(n-1)\beta^2}{2n}$. Then a Ricci soliton (M,g,V,λ) with potential field $V := f\xi$ is expanding Ricci soliton.

b)If $|k| < \frac{(n-1)\beta^2}{2n}$. Then a Ricci soliton (M,g,V,λ) with potential field $V := f\xi$ is shrinking Ricci soliton.

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