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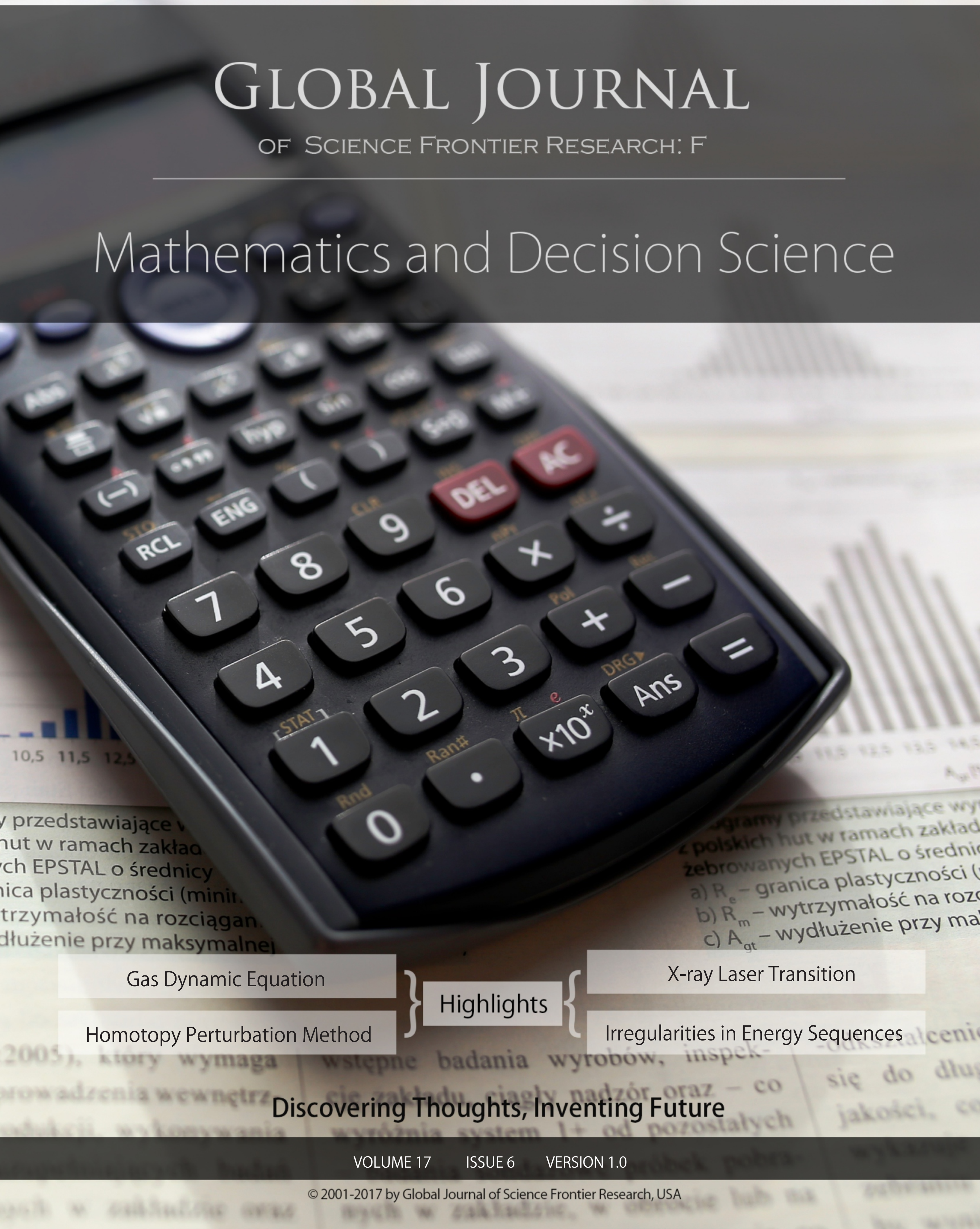

Highlights

- Gas Dynamic Equation
- Homotopy Perturbation Method
- X-ray Laser Transition
- Irregularities in Energy Sequences

Discovering Thoughts, Inventing Future

VOLUME 17 ISSUE 6 VERSION 1.0

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A New Approximation to Standard Normal Distribution Function

By Malki Abderrahmane & Boukhetala Kamel

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Abstract- This paper, presents three news-improved approximations to the Cumulative Distribution Function (C.D.F.). The first approximation improves the accuracy of approximation given by Polya (1945). In this first new approximation, we reduce the maximum absolute error (M.A.E.) from, 0.00314 to 0.00103. For this first new approximation, K. M. Aludaat and M. T. Alodat (2008) was reduce the (M.A.E.) from, 0.00314 to 0.001972. The second new approximation improve Tocher's approximation, we reduce the (M.A.E.) from, 0.166 to 0.00577. For the third new approximation, we combined the two previous approximations. Hence, this combined approximation is more accurate and its inverse is hard to calculate. This third approximation reduce the (M.A.E.) to be less than $2.232e-004$. The two improved previous approximations are less accurate, but his inverse is easy to calculate. Finally, we give an application to the third approximation for pricing a European Call using Black-Scholes Model.

Keywords: cumulative distribution function, normal distribution, maximum absolute error.

GJSFR-F Classification: MSC 2010: 11N60

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A New Approximation to Standard Normal Distribution Function

Malki Abderrahmane [α] & Boukhetala Kamel [σ]

Abstract- This paper, presents three news-improved approximations to the Cumulative Distribution Function (C.D.F.). The first approximation improves the accuracy of approximation given by Polya (1945). In this first new approximation, we reduce the maximum absolute error (M.A.E.) from, $\mathbf{0.00314}$ to $\mathbf{0.00103}$. For this first new approximation, K. M. Aludaat and M. T. Alodat (2008) was reduce the (M.A.E.) from, $\mathbf{0.00314}$ to $\mathbf{0.001972}$. The second new approximation improve Tocher's approximation, we reduce the (M.A.E.) from, $\mathbf{0.166}$to $\mathbf{0.00577}$. For the third new approximation, we combined the two previous approximations. Hence, this combined approximation is more accurate and its inverse is hard to calculate. This third approximation reduce the (M.A.E.) to be less than $\mathbf{2.232e-004}$. The two improved previous approximations are less accurate, but his inverse is easy to calculate. Finally, we give an application to the third approximation for pricing a European Call using Black-Scholes Model.

Keywords: cumulative distribution function, normal distribution, maximum absolute error.

I. Introduction

The cumulative distribution function (CDF) play an important role in financial mathematics and especially in pricing options with Black-Scholes Model. The European option pricing call given by Black-Scholes Model is

$$C(S,K,T,r,\sigma) = S\Phi(d) - Ke^{-rT}\Phi(d-\sigma\sqrt{T}) \tag{1}$$

Where
$$d = \frac{ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{2}$$

S, the current price, K the exercise price, r interest rate, T time option and σ volatility. The cumulative distribution function (C.D.F.) is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{t^2}{2}}dt \tag{3}$$

The (C.D.F) have not a closed form. His evaluation is an expensive task. For evaluate the (CDF) at a point z we need compute the integral under the probability density function (PDF) given by $\varphi(t) = e^{-0.5t^2}/\sqrt{2\pi}$.

In much research, we find approximations, with closed forms, for the area under the standard normal curve. Otherwise, we need consulting Tables of cumulative standard normal probabilities. Hence, in the literature, we find several approximations to this function from polya (1945) to Yerukala (2015). For this raison; we use some approximations to this C.D.F. (Polya's approximation and Tocher's approximation).

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II. Improving Polya's Approximation

We consider the case of, $z \geq 0$. (For $z < 0, \Phi(z) = 1 - \Phi(-z)$).
The original Polya's approximation given by:

$$\Phi_{Polya}(z) \approx \frac{1}{2}\left\{1 + \sqrt{1 - e^{-az^2}}\right\}, \text{ where } a = \frac{2}{\pi}. \tag{2.1}$$

The Maximum Absolute Error (M.A.E.)

$$\text{M.A.E.}_{Polya} = \max_z \left|\Phi_{Polya}(z) - N(z)\right| = 0.003138181653387. \tag{2.2}$$

K.M.Aludaat and M.T.Alodat (2008) proposed the same formula with $a = \sqrt{\frac{\pi}{8}}$ instead of $a = \frac{2}{\pi}$. They have

$$\text{M.A.E.}_{Aludaat} = \max_z |\Phi_{Aludaat}(z) - \Phi(z)| = 0.001971820656170\,.$$

In this paper, we write the formula (2.1) and (2.2) as

$$\Phi_{Malki}(z) \approx a + b\sqrt{1 - e^{-cz^2}} \tag{2.3}$$

Hence, we search the parameters a, b *and* c that

$$\text{M.A.E.}_{Malki} = \max_z |\Phi_{Malki}(z) - \Phi(z)| \tag{2.4}$$

Was the smallest possible using the following algorithm?

1) $h = 0.00001; H = 20h;\ Er = 0.00314;$
2) $a_0 = 0.5, b_0 = 0.5, c_0 = \frac{2}{\pi},$
3) $for\ a = a_0 - H{:}\,h{:}\,a_0 + H\ for\ b = b_0 - H{:}\,h{:}\,b_0 + H$
4) $for\ c = c_0 - H{:}\,h{:}\,c_0 + H; M = a + b\sqrt{1 - e^{-cz^2}},$
5) $e = \max_z |\text{M} - N(z)|;\ if(Er > e)\ Er = e; A = a; B = b; C = c; end;$
6) $a_0 = A, b_0 = B, c_0 = C.$
7) Repeat 3) to 6) until convergence

Using our algorithm, we find the best parameters

$$a^* = 0.50103;\ b^* = 0.49794\,;\ c^* = 0.62632 \tag{2.4}$$

Hence the best formula is

$$\Phi_{Malki\,1}(z) = 0.50102976 + 0.49794047\sqrt{1 - e^{-0.626317743\ z^2}}, \tag{2.5}$$

Note that the absolute error as function of z variable noted by

$$\text{E(z)} = |\Phi_{\text{Malki 1}}(\text{z}) - \Phi(\text{z})| = 0.001029767666887 \tag{2.6}$$

Figure one, shows the graph of Absolute Error for Polya, Aludaat and Malki1 as function of $-5 \leq z \leq 5$.

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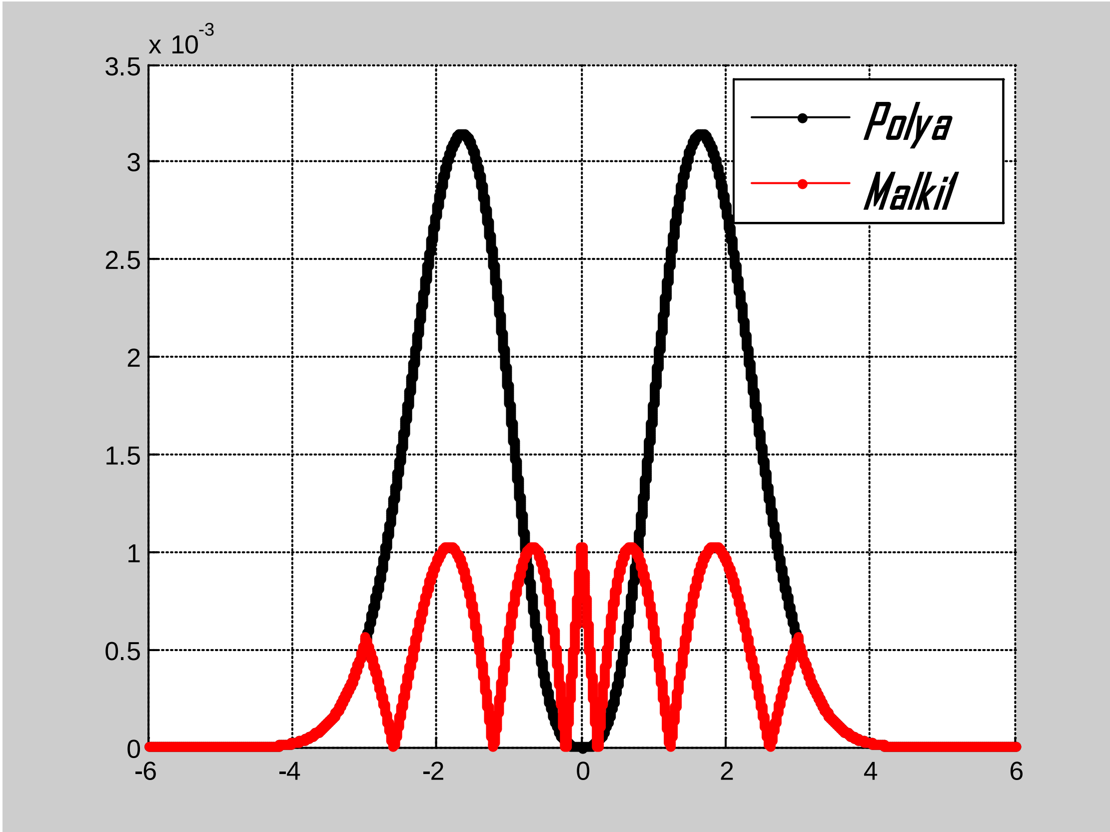

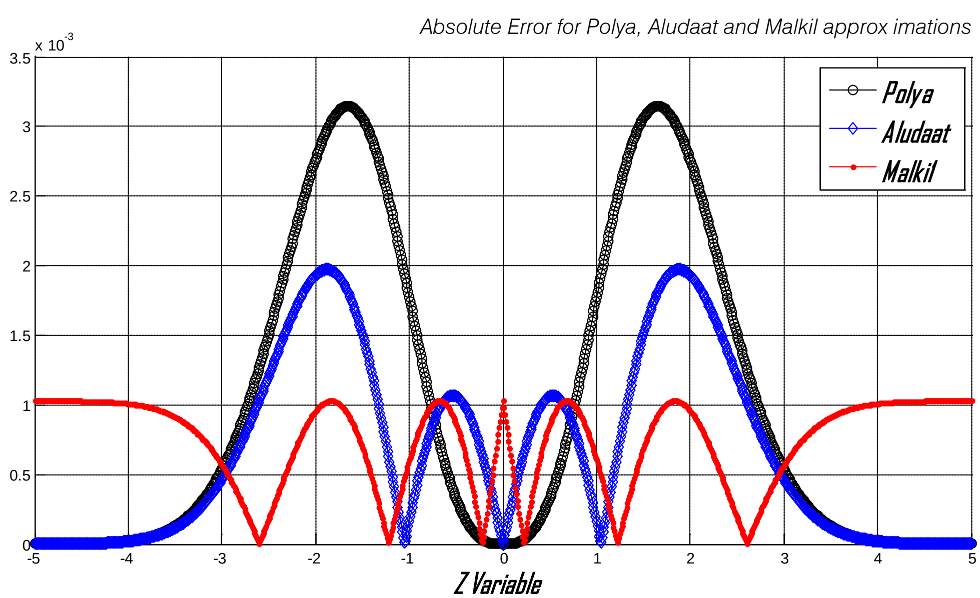

Figure 1: Comparison of absolute error for Polya, Aludaat and Malki1

III. Improving Tocher's Approximation

The Original Tocher's approximation is $\Phi_{Tocher}(z) = 1/(1 + e^{-\sqrt{\frac{2}{\pi}}z})$ with

$$\max_z |\Phi_{Tocher}(z) - \Phi(z)| = 0.165811983691380 \approx 0.166. \tag{3.1}$$

This approximation have the form:

$$\Phi_{Malki\,2}(z) = \frac{\mathrm{a}}{\mathrm{b}+e^{-\mathrm{c}z}} \tag{3.2}$$

Hence, we search the parameters a, b *and* c that

$$\mathrm{M.A.E.}_{Malki\,2} = \max_z |\mathrm{N}_{Malki\,2}(z) - N(z)| \tag{3.3}$$

Was the smallest possible using the following algorithm?

1) $h = 0.00001; H = 20h;\ Er = 0.166;$
2) $a_0 = 1, b_0 = 1, c_0 = \sqrt{\frac{2}{\pi}},$
3) $for\ a = a_0 - H{:}\,h{:}\,a_0 + H\ for\ b = b_0 - H{:}\,h{:}\,b_0 + H$
4) $for\ c = c_0 - H{:}\,h{:}\,c_0 + H; M = \frac{a}{b+e^{-cz}},$
5) $e = \max_z |\mathrm{M} - \Phi(z)|;\ if(Er > e)\ Er = e; A = a; B = b; C = c; end;$
6) $a_0 = A, b_0 = B, c_0 = C.$
7) Repeat 3) to 6) until convergence

Using our algorithm, we find the best parameters

$$a^* = 0.97186;\ b^* = 0.96628;\ c^* = 1.69075 \tag{3.4}$$

Hence the best-improved formula for Tocher's approximation is

$$\Phi_{Malki\,2}(z) = \frac{0.97186}{0.96628 + e^{-1.69075\,z}} \tag{3.5}$$

$$\max_z |\Phi_{Malki\,2}(z) - N(z)| = 0.005774676414954 \tag{3.6}$$

Figure 2 gives the curves of original absolute error and the new absolute error

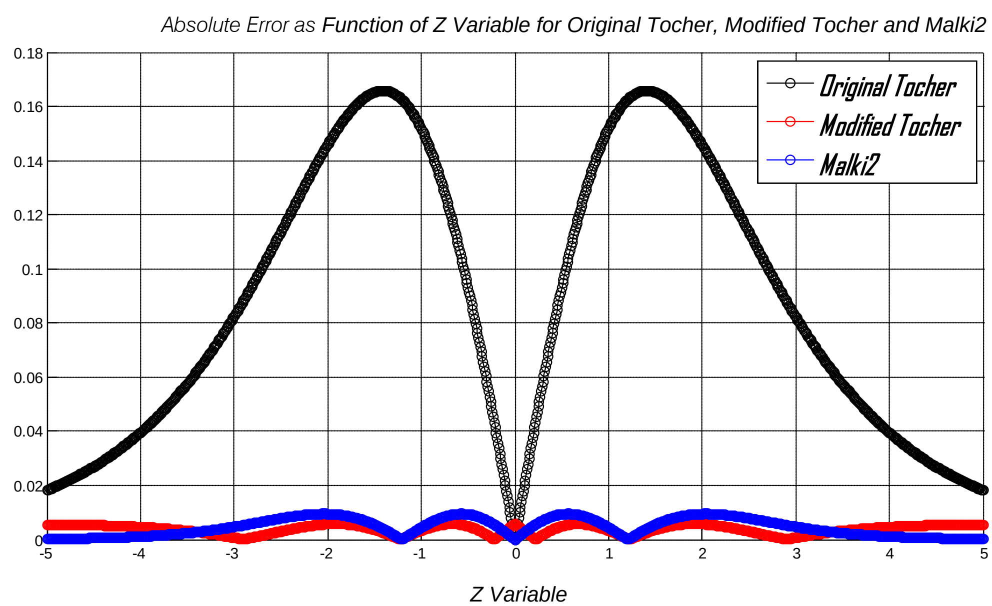

Figure 2: Comparison of Absolute Error for Original Tocher, Modified Tocher and Malki2 as function of z variable $(-5 \leq z \leq 5)$

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IV. Combined Formula

As the third new approximation formula, we consider the two previous formula

$$\Phi_{Malki\,1}(z) = 0.50103 + 0.49794\sqrt{1 - e^{-0.62632\,z^2}},$$

and,
$$\Phi_{Malki\,2}(z) = \frac{0.97186}{0.96628 + e^{-1.69075\,z}}.$$

Hence, we consider the third new formula as

$$\Phi_{Malki\,3}(z) = \omega\Phi_{Malki\,1}(z) + (1-\omega)\Phi_{Malki\,2}(z),\ \text{for},\ (\,0 \le \omega \le 1) \tag{4.1}$$

We search the optimum parameter ω that the

$$\text{M.A.E.}_{Malki\,3} = \max_z|\Phi_{Malki\,3}(z) - \Phi(z)|$$

Was the smallest possible. We find optimum parameter $\omega^* = 0.16$

The new third approximation is

$$\Phi_{Malki\,3}(z) = 0.16\Phi_{Malki\,1}(z) + 0.84\Phi_{Malki\,2}(z) \tag{4.2}$$

The adjusted formula is

$$\Phi_{\text{Malki 3}}(\text{z}) = \frac{0.1544976}{0.96568 + \text{e}^{-1.68975\,\text{z}}} + 0.4212652 + 0.4189696\sqrt{1 - \text{e}^{-0.62642\,\text{z}^2}} \tag{4.3}$$

For, this approximation we have:

$$\max_{\text{z}}|\Phi_{\text{Malki 3}}(\text{z}) - \Phi(\text{z})| = 2.231943559627414\text{e} - 004 \tag{4.4}$$

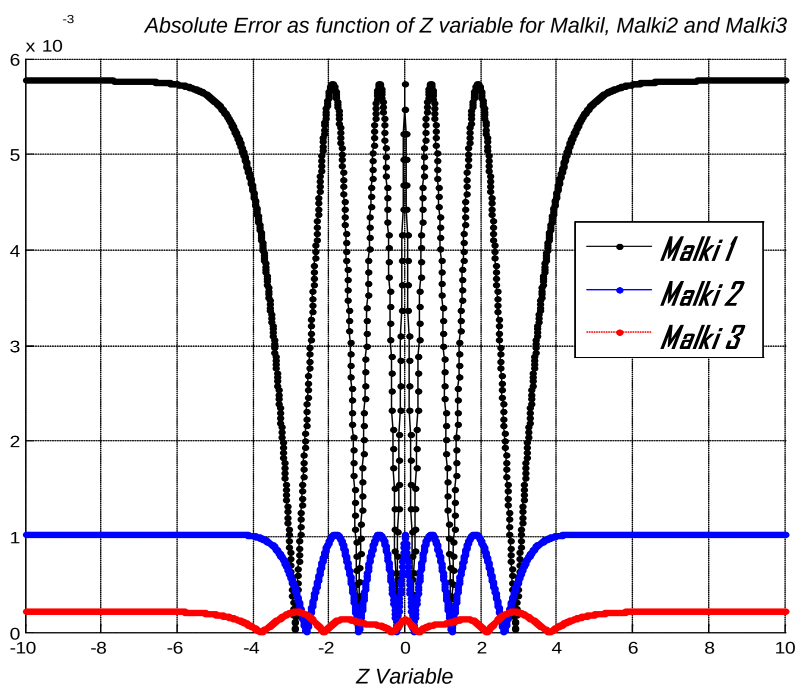

Figure 3

V. Application with Black-Scholes Model

For

$$S = 35; K = 30; r = 0.065; T = 1.2;\ \sigma = 0.35; \tag{5.1}$$

To calculate a Call European option we compute

$$d = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = 0.797198914562755 \quad (5.2)$$

And
$$d - \sigma\sqrt{T} = 0.413793124309139 \quad (5.3)$$

Hence

$$C = S\Phi(d) - Ke^{-rT}\Phi\left(d - \sigma\sqrt{T}\right) = 9.228813813962439 \quad (5.4)$$

Using, $\Phi_{Malki\,3}$ we have

$$C_3 = S\Phi_{Malki\,3}(d) - Ke^{-rT}\Phi_{Malki\,3}\left(d - \sigma\sqrt{T}\right) = 9.231095739041432 \quad (5.5)$$

The absolute error is $|C - C_3| \leq 0.0023$ (5.6).

VI. Conclusion

We have proposed three approximations to the cumulative distribution function of the standard normal distribution. The first approximation improve the Polya's formula in accuracy. The second new approximation improve the accuracy of Tocher's formula. The third formula is a combination of the two previous formula. The M.A.E. for the first approximation is 0.00103. The M.A.E. for the second approximation is 0.00577. For the third approximation the M.A.E. is less than $2.232\text{e}-004$. Finally, we insert an application to option pricing of a Call European option based on Black-Scholes formula.

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Irregularities in Energy Sequences of Ni- Like Ions. Applications to Wavelengths Calculations of the Self-Photo Pumped $4f^1p_1 - 4d^1P_1$ X-Ray Laser Transition

By Ivanova E. P.
Institute of Spectroscopy of RAS

Abstract- The energies of the Ni-like ions with Z=36-51 are calculated. The energies of the low $3d_{3/2}4d_{3/2}$ *[J=1]* and upper $3d_{3/2}4f_{5/2}$ *[J=1]* working levels of the self-photopumped X-ray laser are analyzed along the sequence. We studied irregularities in the crossing Z-points of each working level with another level of the same parity. The irregularities are the features of the energy levels and of the radiative transition probabilities (RTP) associated with this level. They result in a decrease in the radiative transition probability (RTP) in one transition channel and an enhancement of RTP in another channel. The list of elements that lase on the self-photopumped transition can be extended much further than originally known. We present the first calculation of the wavelengths of this transition in Ni-like sequence to $Z = 79$ using the relativistic perturbation theory with a zero approximation model potential.

Keywords: spectroscopy of multicharged ions, radiative-collisional X-ray lasers, energy levels and radiative transition probabilities.

GJSFR-F Classification: *MSC 2010: 00A69*

IRREGULARITIESINENERGYSEQUENCESOFNILIKEIONSAPPLICATIONSTOWAVELENGTHSCALCULATIONSOFTHESELFPHOTOPUMPED4F1P14D1P1XRAYLASERTRANSITION

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Ivanova E. P.

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Keywords: spectroscopy of multicharged ions, radiative-collisional X-ray lasers, energy levels and radiative transition probabilities.

I. Introduction

Effect of oscillator strength transfer in the neon isoelectronic sequence with application to X-ray laser modeling was considered by us in [1]. This study suggests that Z points at which interacting levels are close to each other may be important for modeling emission spectra of dense plasma. Later the authors of [2] where the level crossing in the Ni-like sequence and associated irregularities in the functions of energies and probabilities of radiative transitions in the range Z = 74-84 were studied arrived at the same conclusion. From this, the conclusion about the possible incorrect identification of levels in their crossing regions follows. Here we review the experiments on the study of self-photo pumped X-ray laser in Ni- like ions in order to determine possible irregularities in the sequences of working levels. Another challenge is to detect misidentifications in the working levels energies.

Self-photo pumped (SPP) x-ray lasers (XRL) in Ni-like ions were presented in 1996 [3] as an alternative approach to the standard radiative collisional scheme for inversion creation. We use the term SPP following the name given in literature. This is really collisionally pumped laser assisted by radiation trapping. Both schemes for Ni-like ions are shown in Figure1. This new class of SPP in Ni-like XRL was first investigated theoretically in [4] where high gain was predicted for the $4f\ ^1P_1 - 4d\ ^1P_1$ transition in Mo^{14+} at 22.0 nm. It was supposed that preplasma was created by a nanosecond pulse

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followed by a picosecond pulse to control the temperature and density in plasma and to achieve high gain. This wavelength was calculated using the multiconfiguration Dirac-Fock atomic physics code by Grant and co-workers in the extended average level mode [5]. In the experiment [6] the Ni-like SPP XRL on the $4f\,^1P_1 - 4d\,^1P_1$ transition was demonstrated in Ni- like Zr, Nb, and Mo, and the measured wavelengths for these ions were presented. For Mo^{14+} a gain of 13 cm^{-1} was measured at 22.6 nm for a target up to 1 cm long [6]. The wavelengths of this transition for ions from Z = 36 to 54 were predicted in [6] using the experimental data of this work to provide small corrections to their calculations. In the experiment [7], the progress in the optimization and understanding of the collisional pumping of X-ray lasers using an ultrashort subpicosecond heating pulse was reported. Time integrated and time resolved lasing signals at the standard $4d\,^1S_0 - 4p\,^1P_1$ XRL line in Ni-like Ag was studied in detail. Under specific irradiation conditions, strong lasing was obtained on the SPP $4f\,^1P_1 - 4d\,^1P_1$ transition at 16.1 nm. The strong lasing on the SPP transition in Mo^{14+} was also observed with very modest (less than 1 J) pump energy at high repetition rate [8]. Recently lasing on the SPP $3d\,^1P_1 - 3p\,^1P_1$ laser line has been observed for Ne-like V, Cr, Fe, and Co, as well as for Ni-like Ru, Pd, and Ag [9]. A strong dependence on delay between the main and second prepulses was found: the optimum delay shifts towards smaller delays with increasing atomic number *Z*. Accurate wavelength measurements and calculations are shown to be in excellent agreement. The experiment [9] has demonstrated that the list of elements that lase on the SPP transitions can be extended much further than originally known.

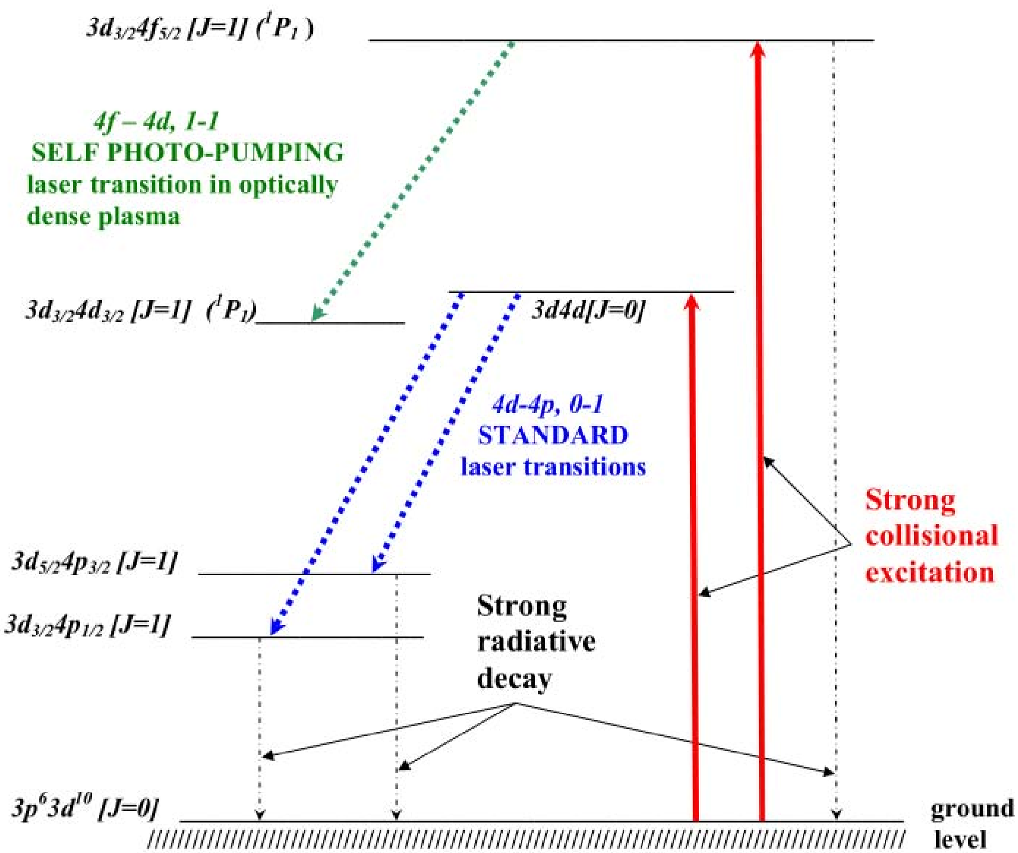

Fig. 1: The diagram of three XRL transitions in Ni-like ions

Ref

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Many authors have investigated the spectra of Ni-like ions using vacuum spark, laser produced plasma and electron beam ion trap as light sources [10-16]. The $3d^94d$ and $3d^94f$ configurations have been analyzed in the Rb X – Mo XV sequence [12-13]. In [12-13], these configurations were investigated using parameter extrapolations within the Generalized-Least-Squares (GLS) method. This method was used in [14,15] to predict for $3d^94d$, $3d^94f$ configuration energy levels in Cd XXI and Ag XX. GLS predictions of $3d^94d$, $3d^94f$ energy levels in the Zr XIII – Pd XIX sequence are tabulated in [16].

Ref

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Note that lasing wavelength (λ_{las}) in Mo^{14+} was determined theoretically [4] and in the experiment [6] using one and the same atomic physics code [5], but results for λ_{las} were somewhat different (by 4Å). The $3d_{3/2}4f_{5/2}$ $[J=1]$ upper working level has the largest oscillator strength and RTP to the $3d^{10}$ ground level. This fact allows it to achieve high precision in this level energy measurement along the Ni-like sequence up to high Z ˜ 84; in some ions, the energy of the transition to the ground state was accurate up to the fourth significant digit. The wavelengths of resonant radiative transitions in heavy Ni-like ions were calculated by us to Z = 83 in [17]. Moreover, in [17] the wavelengths (for Z within 79 –82) were predicted with the same accuracy, although they have not yet been measured experimentally.

In the present paper, we analyze the smoothness of the working energy levels of SPP XRL along the Ni-like sequence. We have found some irregularities in Ni- like sequence energies in the region Z=42 (Mo^{14+}) and in the region Z = 49 (In^{21+}) for the upper $3d_{3/2}4f_{5/2}$ $[J=1]$ working level. The causes of irregularities are studied.

The principle purpose of this paper is to predict the wavelengths of SPP XRL lines in Ni-like ions with Z ≤ 79. The calculations are performed by the Relativistic Perturbation Theory with Model zero approximation Potential (RPTMP). The fundamental principles of the RPTMP approach are given in [18] where energy levels of the $3p^63d^94l$, $3p^53d^{10}4l$, $(l=0,1)$ configurations and radiative transition rates to the $3p^63d^{10}$ ground state in the *Kr IX* ion are calculated. The stability of calculations on the approximation used was discussed in [18]. Energy levels of the odd and even states with $J=1$ of the Ni- like ions with Z = 36-51 are given in Tables 1 and 2.

Table 1: Energy levels (in 1000 см$^{-1}$) of the odd states with $J=1$ of the Ni- like ions with Z = 36-51

level	36	37	38	39	40	41	42	43
$3d_{5/2}4p_{3/2}$	855.9	1005.5	1175.7	1366.2	1557.8	1758.9	1971.7	2.210.0+
$3d_{3/2}4p_{1/2}$	869.6	1020.4	1191.7	1383.1	1575.7	1777.7	1991.5	2189.2+
$3d_{3/2}4p_{3/2}$	875.3	1027.6	1200.7	1394.4	1589.5	1794.4	2011.5	2233.8
$3d_{5/2}4f_{5/2}$	1310.8	1516.9	1743.8	1990.5	2238.1	2495.1	2771.8	3033.7
$3d_{5/2}4f_{7/2}$	1318.7	1526.3	1755.4	2004.5	2254.7	2514.3	2793.5	3060.0
$3d_{3/2}4f_{5/2}$	1332.2	1.545.2	1779.7	2034.9	2292.0	2559.1	2843.9	3121.3
$3p_{3/2}4s_{1/2}$	1650.9	1868.7	2077.8	2309.0	2541.3	2782.4	3035.0	3292.3
$3p_{1/2}4s_{1/2}$	1712.0	1939.7	2159.6	2403.5	2649.6	2905.8	3175.3	3451.0
$3p_{3/2}4d_{3/2}$	2044.8	2305.1	2256.9	2831.1	3106.7	3391.7	3688.7	3990.8
$3p_{3/2}4d_{5/2}$	2055.4	2317.2	2570.4	2846.0	3122.9	3409.4	3707.8	4011.6
$3p_{1/2}4d_{3/2}$	2116.3	2387.8	2651.6	2939.6	3230.2	3531.3	3846.2	4167.8

level	44	45	46	47	48	49	50	51
$3d_{5/2}4p_{3/2}$	2440.9	2682.8	2937.5	3205.2	3480.4	3.767.1	4064.5	4372.8
$3d_{3/2}4p_{1/2}$	2419.1	2659.7	2913.1	3179.1	3452.4	3737.0	4031.8	4337.2
$3d_{3/2}4p_{3/2}$	2468.9	2715.4	2975.2	3248.6	3530.0	3823.4	4128.1	4444.3
$3d_{5/2}4f_{5/2}$	3321.4	3617.3	3924.2	4247.3	4576.1	4916.4	5267.1	5627.2
$3d_{5/2}4f_{7/2}$	3350.0	3649.4	3960.0	4286.7	4619.5	4964.0	5319.1	5683.8
$3d_{3/2}4f_{5/2}$	3419.9	3728.2	4048.4	4383.3	4723.1	5094.1+	5454.3	5828.8
$3p_{3/2}4s_{1/2}$	3561.8	3842.3	4135.8	4442.1	4757.8	5065.3+	5404.8	5751.8
$3p_{1/2}4s_{1/2}$	3740.8	4043.4	4360.9	4692.7	5034.2	5389.3	5757.3	6139.8
$3p_{3/2}4d_{3/2}$	4305.9	4632.5	4972.3	5325.9	5687.3	6060.8	6445.5	6842.3
$3p_{3/2}4d_{5/2}$	4328.3	4656.6	4998.2	5353.7	5717.2	6092.9	6480.1	6879.4
$3p_{1/2}4d_{3/2}$	4504.2	4854.0	5219.2	5600.1	5991.3	6397.0	6816.4	7250.9

Table 2: Energy levels (in 1000 см$^{-1}$) of the even states with *J=1* of the Ni- like ions with Z = 36-51

level	36	37	38	39	40	41	42	43
$3d_{3/2}4s_{1/2}$	706.2	840.9	995.9	1171.0	1347.3	1532.7	1729.8	1931.4
$3d_{5/2}4d_{3/2}$	1088.4	1263.0	1458.1	1673.1	1889.4	2115.1	2352.6	2594.8
$3d_{5/2}4d_{5/2}$	1098.9	1275.3	1472.1	1688.8	1906.8	2134.3	2373.4	2617.4
$3d_{3/2}4d_{5/2}$	1106.4	1284.2	1482.7	1701.4	1921.6	2151.6	2400.8+	2648.7
$3d_{3/2}4d_{3/2}$	1110.1	1288.5	1487.5	1706.7	1927.6	2158.2	2393.6+	2640.6
$3p_{3/2}4p_{1/2}$	1793.6	2.025.4	2248.6	2494.2	2740.6	2996.2	3263.4	3535.2
$3p_{3/2}4p_{3/2}$	1811.0	2045.6	2271.6	2520.6	2770.7	3030.5	3302.4	3579.5
$3p_{1/2}4p_{1/2}$	1862.1	2104.2	2338.5	2597.0	2857.6	3128.4	3412.6	3703.1
$3p_{1/2}4p_{3/2}$	1.874.4	2119.3	2356.7	2618.9	2883.5	3159.0	3448.2	3744.5
$3p_{3/2}4f_{5/2}$	2258.9	2.549.8	2832.0	3136.7	3442.4	3757.5	4092.6	4413.0
$3d_{3/2}4s_{1/2}$	2145.1	2369.7	2607.0	2856.4	3113.3	3381.3	3659.5	3948.6
$3d_{5/2}4d_{3/2}$	2849.6	3115.3	3393.8	3685.5	3984.6	4295.3	4616.5	4948.6
$3d_{5/2}4d_{5/2}$	2873.9	3141.5	3421.8	3715.5	4016.7	4329.5	4653.1	4987.6
$3d_{3/2}4d_{5/2}$	2909.5	3181.8	3467.4	3766.8	4074.3	4394.0	4725.1	5067.6
$3d_{3/2}4d_{3/2}$	2900.6	3172.0	3456.5	3754.8	4060.9	4379.1	4708.5	5049.2
$3p_{3/2}4p_{1/2}$	3819.4	4114.3	4422.2	4742.5	5070.2	5409.0	5758.3	6118.8
$3p_{3/2}4p_{3/2}$	3869.5	4171.0	4486.0	4814.4	5150.9	5499.4	5859.3	6231.3
$3p_{1/2}4p_{1/2}$	4007.8	4325.1	4657.5	5.004.5	5361.2	5731.5	6114.8	6512.4
$3p_{1/2}4p_{3/2}$	4055.5	4379.9	4719.8	5075.4	5441.4	5821.8	6216.2	6625.7
$3p_{3/2}4f_{5/2}$	4760.5	5116.2	5483.5	5867.5	6257.7	6660.0	7073.3	7497.2

II. Features of Lower and Upper Working Levels of SPP XRL along the Ni-Like Sequence

The schematic diagram of three strong XRL transitions is shown in Fig. 1: two of them are standard *3d4d [J=0]* – *$3d_{5/2}4p_{3/2}$ [J=1]* and *3d4d [J=0]* – *$3d_{3/2}4p_{1/2}$[J=1]* transitions. The classifications of lower working levels in Fig. 1 are valid for $Z > 42$. The *$3d_{5/2}4p_{3/2}$ [J=1]* level is the lower working level of an XRL for the entire nickel isoelectronic sequence, the *$3d_{3/2}4p_{1/2}$[J=1]* level is the lower working level for heavy ions starting with Z = 62. The third *$3d_{3/2}4p_{3/2}$[J=1]* level decays to the ground state significantly weaker than two mentioned above, and does not provide a significant gain. In our recent work [19], the energies of standard XRL transitions in ions of the Ni-like sequence with Z ≤ 79 are refined by RPTMP calculations. The calculated energies of

Ref

19. Ivanova E.P. 2016 Wavelengths of the *4d – 4p*, 0 – 1 X-ray laser transitions in Ni-Like ions *Int. Journal of Advanced Research in Physical Science* **3** 34-40

the two standard $4d$–$4p$, J= 0–1 XRL transitions are corrected by extrapolation of the experimental differentials of XRL transition energies $dE_Z^{las} = E_Z^{las} - E_{Z-1}^{las}$; i.e., the differences between transition energies of neighboring ions, which weakly depend on Z (especially, in the region $Z \leq 50$). It is proven that the accuracy for the final results for large Z is within the experimental error.

The $3d_{3/2}4f_{5/2}$ $[J=1]$ – $3d_{3/2}4d_{3/2}$ $[J=1]$ transition is optically self-photopumped XRL in all Ni-like ions, the positions of working levels vary with respect to other levels along the sequence. Based on our previous studies of XRL [20-22], it can be argued that there are at least four principal differences between standard and self photo-pumped mechanisms:

(i) In the standard scheme, the upper working level is populated by strong monopole electron collisions; in the SPP scheme it is populated by strong dipole electron collisions, which means large oscillator strength and effective photoabsorbtion.

(ii) Effective SPP XRL is possible only in optically thick plasma (large electron density n_e and diameter d), while the standard XRL is possible both in optically thick and in optically thin plasma over a wide range of n_e and d.

(iii) In the SPP, the upper working level is quickly emptied due to the large radiative decay rate. Therefore, in this scheme, a laser effect is short-lived; maximum XRL duration may be a few tens of picoseconds. A standard XRL can operate in quasi-continuous mode (under certain conditions).

(iv) In the SPP, the lower and upper working levels does not change their classification along the Ni-like sequence, in the standard scheme the upper working level changes its classification: the $3d_{5/2}4d_{5/2}$ $[J = 0]$ state is dominant in the classification of the upper working level at $Z \leq 51$, and the $3d_{3/2}4d_{3/2}$ $[J = 0]$ state is dominant for $Z > 51$ [19].

Below we demonstrate the irregularities in the sequence of both the lower and the upper working levels of SPP XRL. Crossing of each working level with another level causes the irregularities. Level crossing is accompanied by a strong interaction at certain Z points. Fig. 2a shows the scaled energies along Z of the $3d_{3/2}4d_{3/2}$ $[J=1]$ lower working level and $3d_{3/2}4d_{5/2}$ $[J=1]$ level close to it. In addition to the energy levels calculated here, Fig. 2a also shows the corresponding experimental values [16]. Reference [16] does not indicate classification of $3d4d$ $[J=1]$ levels, their classification was made earlier in [13]. Note that theoretical and experimental classifications are identical. There are some differences between theoretical and experimental energies, typically few units in the 4th-5th digits. These differences are conditioned by the shift of the theoretical list of energy levels as a whole, but this shift does not affect the accuracy of λ_{las}. The energy levels in Fig.2a are scaled by dividing by $(Z\text{-}23)^2$, so that the behavior of the third and forth significant digits can be observed. At the beginning of the sequence, the $3d_{3/2}4d_{3/2}$ $[J=1]$ level is above the $3d_{3/2}4d_{5/2}$ $[J=1]$ level. The crossing of these levels is in the range $41 < Z < 42$ (shown by arrows). The crossing of the corresponding experimental energy levels occurs at exactly the same Z value. At Z = 42, one can observe the "repulsion" of levels caused by their interaction; the "repulsion" is a feature of theoretical and experimental data. Note, that repulsion can be seen due to energy scaling; in fact, the repulsion value is about few thousand of cm^{-1}, i.e. few units in the forth digit for the $3d_{3/2}4d_{5/2}$ $[J=1]$ level.

Ref

16. Churilov S.S., Ryabtsev A.N., and Wyart J.-F. 2005 Analysis of the 4-4 transition in the Ni-like Kr IX. *Physica Scripta* **71** 457-463

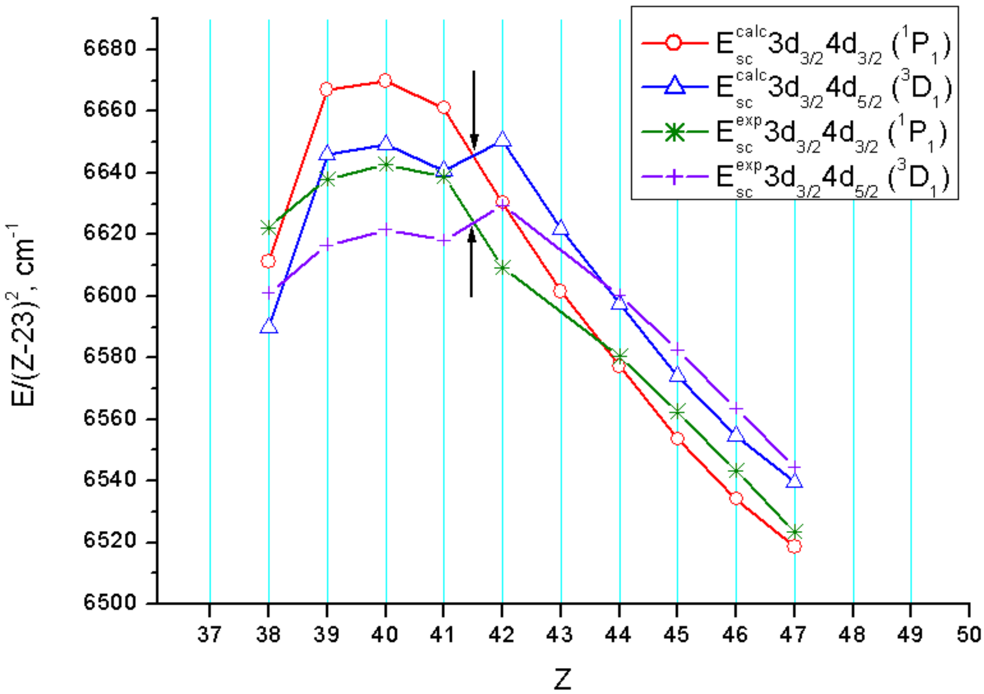

Fig. 2a: The crossing of low working $3d_{3/2}4d_{3/2}$ $[J=1]$ energy level with $3d_{3/2}4d_{5/2}$ $[J=1]$ energy level in Ni-like sequence, shown for the theoretical and experimental scaled energies along Z

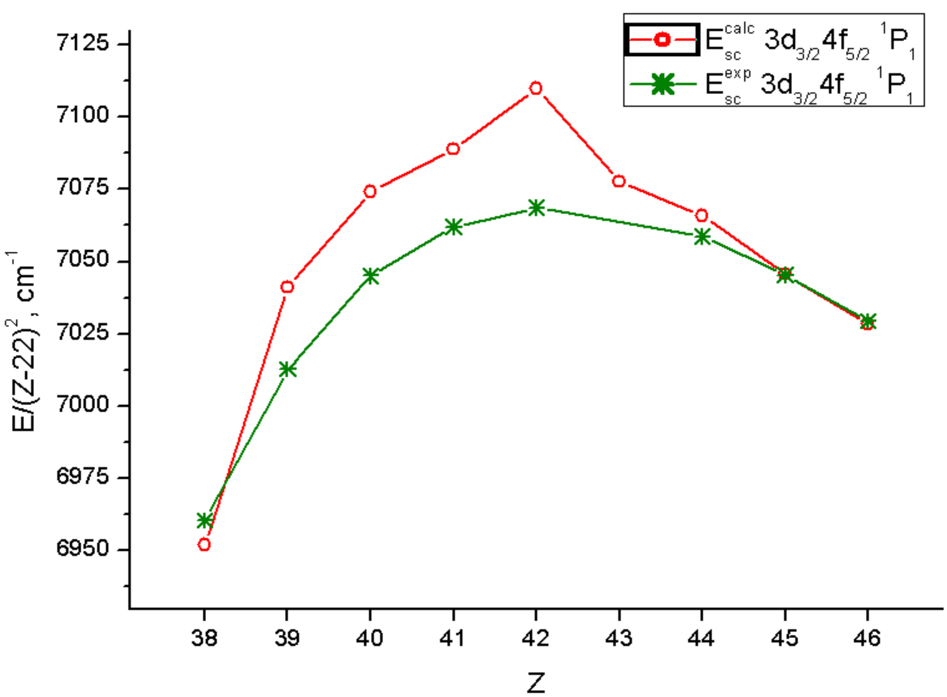

Fig. 2b: Features of theoretical upper working level $3d_{3/2}4f_{5/2}$ $[J=1]$, shown by scaled energy along Z in comparison with correspondent experimental data

Notes

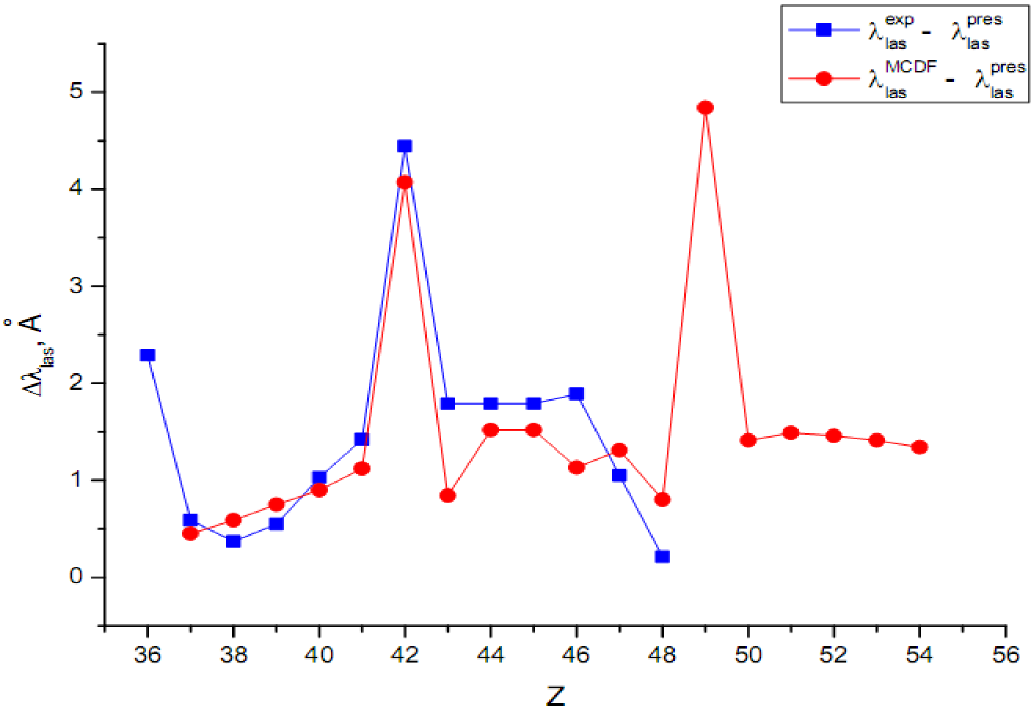

Fig. 3: Difference between experimental, predicted from [6] and calculated here, λ_{las} of SPP XRL transition in Ni-like ions

In Fig. 2b, we can see hard to explain behavior of the $3d_{3/2}4f_{5/2}$ *[J=1]* upper working level in the region of $Z = 42$. The features of this level will be considered below in more detail; however, it is important to note here that the energy structure of odd states in the range Z = 40-49 exhibits extremely high instability caused by the interaction of levels with each other, which rapidly changes with Z. (Note, that the calculation in simpler approximations without accounting for $3p_{1/2}$, $3p_{3/2}$ orbital's, in principle, confirms the results shown in Figures 2a, 2b). In this case at hand, we understand the instability as the ambiguity of the calculation of eigenvectors and eigenenergies. As a result, the calculation in the same approximation leads to different energies of a certain level. The deviation from the smooth curve in Fig. 2a ~10000 cm^{-1}; however, such a value leads to a sufficiently large deviation from the corresponding experimental values of λ_{las} at $Z = 42$, shown in Fig. 3.

At the point Z = 42, λ_{las} calculated here is ~222 Å that is smaller than the experimental and theoretical values of [4] by 4 Å. In the recent experiment [9], the delay time between preliminary and main pump pulses was optimized to achieve the maximum yield of the X-ray laser. In fact, the electron density was optimized in [9]. X-ray lasing occurs in the Ni-like ion ionization mode, so that the lasing times on both transitions were restricted to the ionization time of Ni-like ion to the Co-like state. Time resolved measurements in [9] allowed high-accuracy wavelength measurements of the SPP and standard X-ray laser lines. Thus, the calculations of the previous work [6] were confirmed: $\lambda_{las} \approx 22.61$ nm in Ni-like molybdenum (Mo^{14+}, Z=42). Our calculations are performed for an isolated atom. Based on the studies performed, it can be argued that the interaction of levels at the point Z = 42 is so strong that the energy levels $3d_{3/2}4d_{3/2}$ *[J=1]*, $3d_{3/2}4d_{5/2}$ *[J=1]* in dense hot plasma can differ significantly from the corresponding energy levels in an isolated atom.

Ref

4. Nilsen J., Design of a picosecond laser-driven Ni-like Mo x-ray laser near 20 nm. 1997 *J. Opt. Soc. Am. B* **14** 1511-1514

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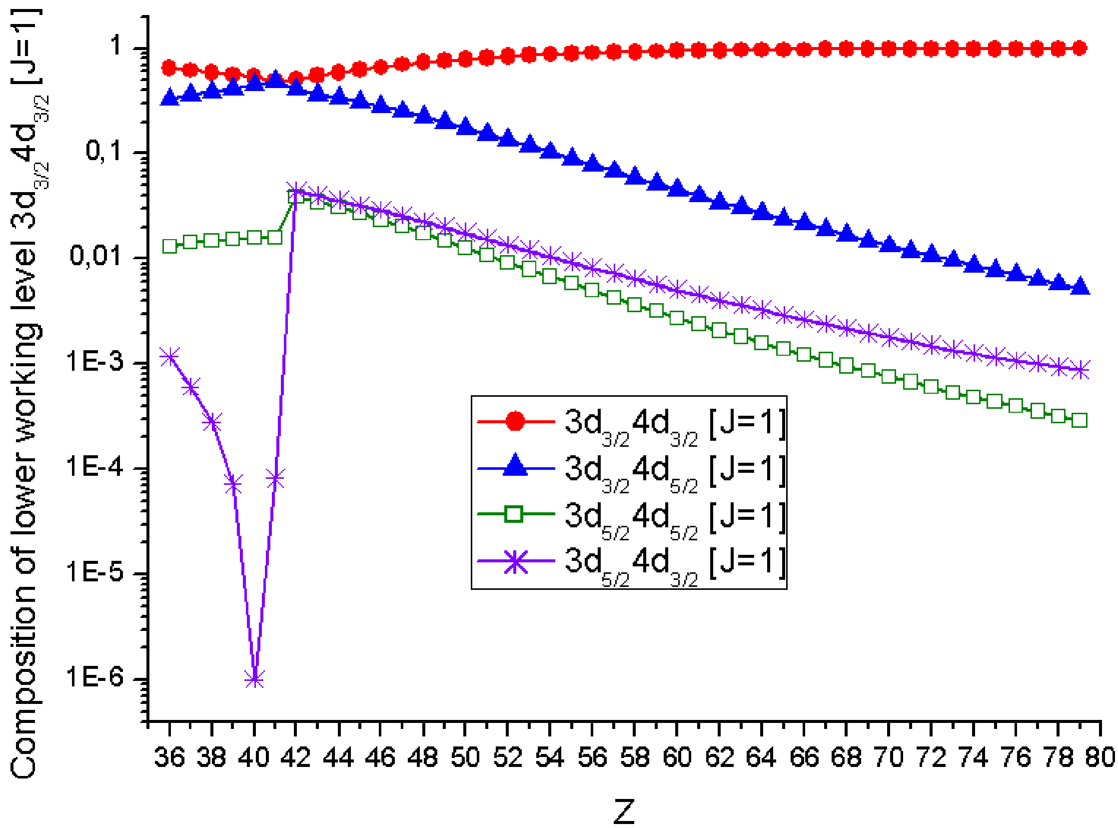

Fig. 4: Composition of low working level $3d_{3/2}4d_{3/2}$ $[J=1]$ along Ni-like sequence on a logarithmic scale

The problem is related to the composition of the $3d_{3/2}4d_{3/2}$ $[J=1]$ working level, which indicates the strength of level interaction. It is shown in Fig. 4 for all $3d4d$ $[J=1]$ levels in Ni-like ions with Z = 36 – 79. Figure 4 shows that contributions of the $3d_{3/2}4d_{3/2}$ $[J=1]$ and $3d_{3/2}4d_{5/2}$ $[J=1]$ levels are almost equal at Z = 42, which could lead to levels' misidentification. Theoretical energies of these levels at $Z = 42$ are 2393554 cm^{-1} and 2400846 cm^{-1} (51% and 41%, respectively, are the contributions to the $3d_{3/2}4d_{3/2}$ $[J=1]$ low working level). The contributions of these levels in [13] are 45% and 34%, and the energies are 2385902 cm^{-1} and 2393229 cm^{-1} respectively. (We note that the theoretical list of energies of Ni- like ions in the range of small Z is shifted as a whole by 5000-8000 cm^{-1}). Fig. 4 demonstrates the rapid restructuring of lower working level compositions: so that the $3d_{5/2}4d_{3/2}$ $[J=1]$ level contribution increases by five orders of magnitude in the range $Z = 40 - 42$.

Ref

13. Ryabtsev A.N., Churilov S.S., Nilsen J., Li Yu, Dunn J., Osterheld A.L. 1999 Additional analysis of Ni-like ions spectra. *Optics and Spectroscopy* **87** 197-202 (Russians)

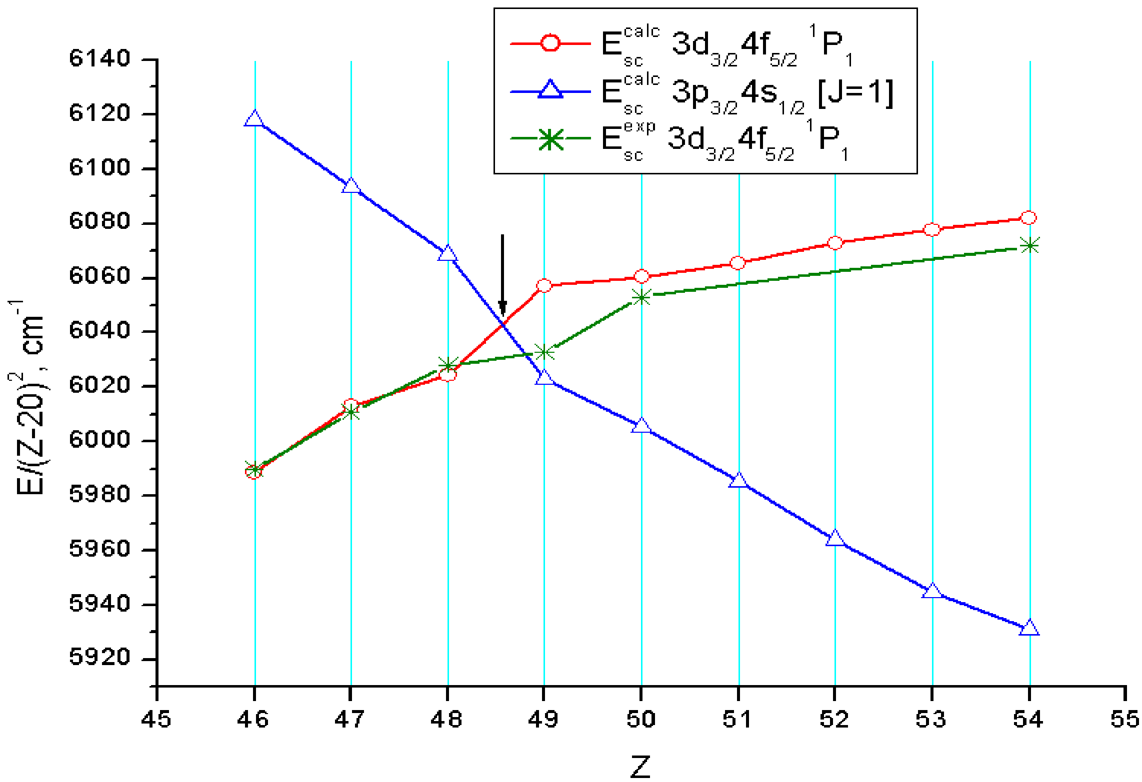

Fig. 5: Crossing of upper working $3d_{3/2}4f_{5/2}$ $[J=1]$ energy level with $3p_{3/2}4s_{1/2}$ $[J=1]$ energy level in Ni-like sequence, shown by scaled energy values along Z. The corresponding experimental values for $3d_{3/2}4f_{5/2}$ $[J=1]$ energies are also shown

Figure 5 shows the scaled energies along Z of the $3d_{3/2}4f_{5/2}$ $[J=1]$ upper working level and the close $3p_{3/2}4s_{1/2}$ $[J=1]$ level. Crossing of these levels occurs in the range $48 < Z < 49$. At $Z = 49$ one can see the "repulsion" of levels caused by their interaction; the "repulsion" is a feature of theoretical data. In Fig.5, the corresponding experimental energies for the $3d_{3/2}4f_{5/2}$ $[J=1]$ level are shown [16]. Unfortunately, we have no available data on the experimental $3p_{3/2}4s_{1/2}$ $[J=1]$ levels in the Z region under consideration. The value $Z = 49$ is the point of an abrupt jump (irregularity) in spectroscopic constants of the $3d_{3/2}4f_{5/2}$ $[J=1]$ upper working level and the $3p_{3/2}4s_{1/2}$ $[J=1]$ level crossing it, caused by the strong interaction of these levels at this value of Z. This interaction is shown in Fig. 6, where we can see the $3d_{3/2}4f_{5/2}$ $[J=1]$ level composition. The interaction of levels at the point $Z = 49$ leads to the so-called effect of oscillator strength transfer we considered in [1] for the Ne-like sequence. At this point, the rate of radiative processes abruptly changes: the probabilities of the transition from the $3d_{3/2}4f_{5/2}$ $[J=1]$ level to the ground state and to the state of the lower working level slightly decrease. At the same time, these probabilities for the $3p_{3/2}4s_{1/2}$ $[J=1]$ level increase by an order of magnitude and become almost equal in magnitude to the corresponding values of the $3d_{3/2}4f_{5/2}$ $[J=1]$ level. These effects are clearly expressed in figures 7a, 7b showing the RTP to the ground state from $3d_{3/2}4f_{5/2}$ $[J=1]$ and $3p_{3/2}4s_{1/2}$ $[J=1]$ levels. Thus, it can be assumed that there was incorrect identification at the point $Z = 49$ when extrapolating the upper working level in [6], and the $3p_{3/2}4s_{1/2}$ $[J=1]$ level which is close to the $3d_{3/2}4f_{5/2}$ $[J=1]$ level in energy was used as the upper working level (see Figure 5). If this assumption is correct, $\lambda_{las} \sim 144.7$ Å for $Z = 49$, which is identical to [6]. When using our value for $3d_{3/2}4f_{5/2}$ $[J=1]$, λ_{las} ~ 140.0 Å (here the energy jump shown in Fig. 5 is taken into

Ref

1. Ivanova E.P., Grant I. 1998 Oscillator strength anomalies in the neon isoelectronic sequence with applications to x-ray laser modeling. *J. of Phys. B: At. Mol. Opt. Phys.* **31**, 2871-2883

account). Another argument in favor of the incorrect identification in [6], are large jumps of the differential $d\lambda_{las}(Z) = \lambda_{las}(Z) - \lambda_{las}(Z-1)$ in the range Z = 47-50.

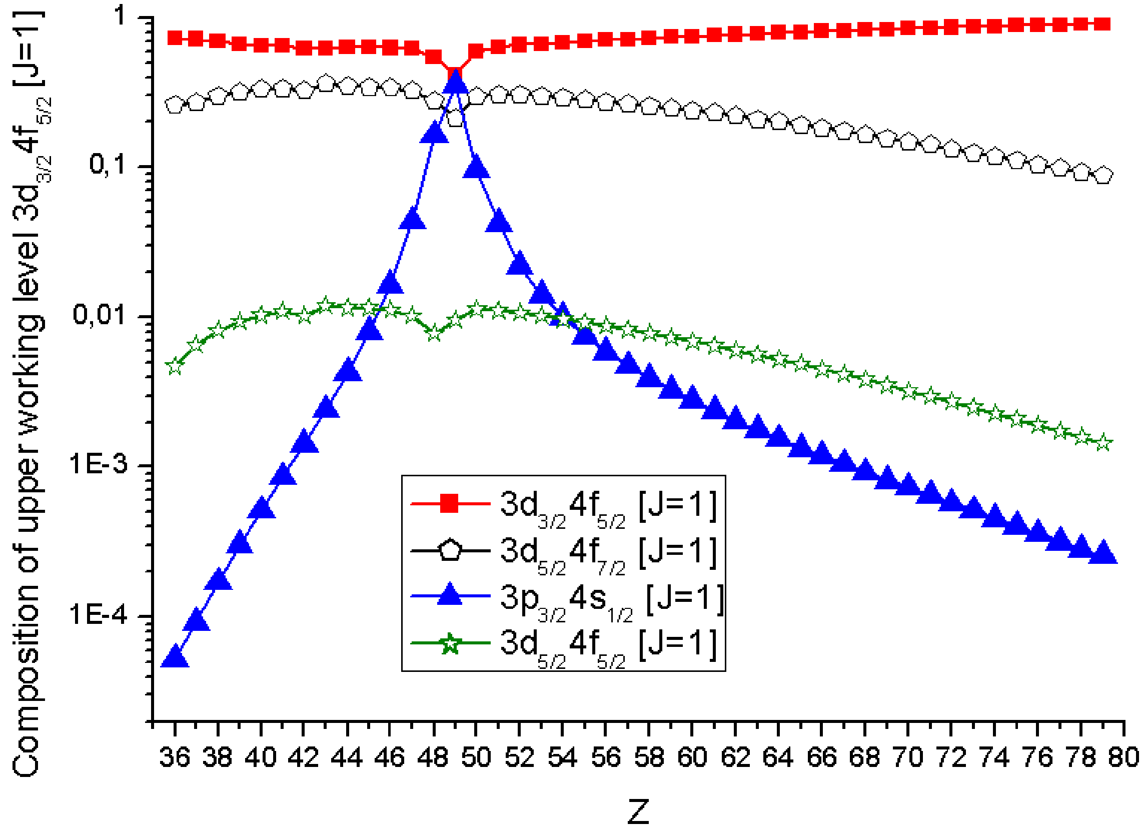

Fig. 6: Composition of upper working level $3d_{3/2}4f_{5/2}$ $[J=1]$ along Ni-like sequence on a logarithmic scale

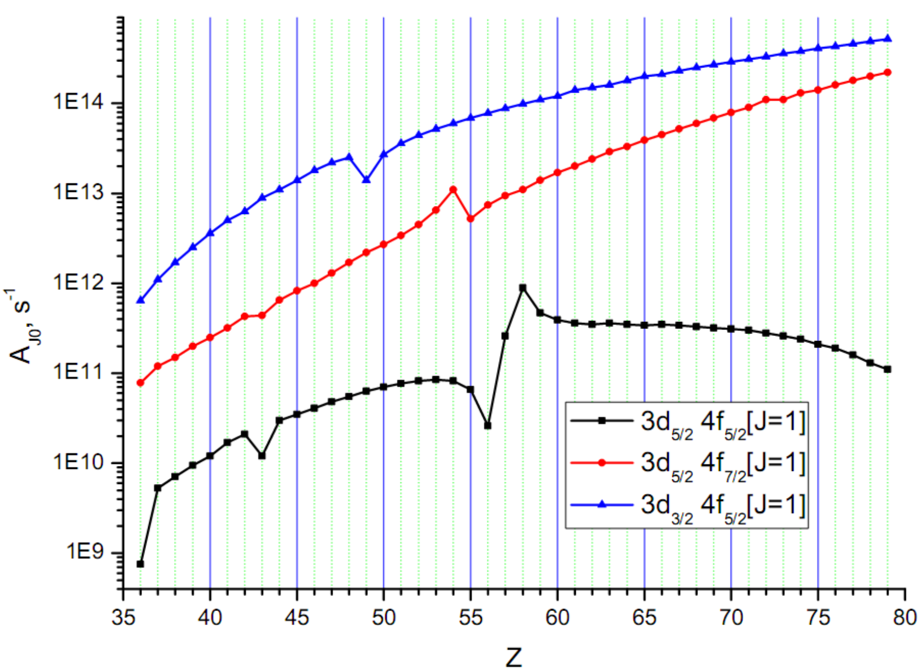

Fig. 7a: Radiative transition probability from the upper working level $3d_{3/2}4f_{5/2}$ $[J=1]$ and the levels $3d_{5/2}4f_{5/2}$ $[J=1]$, $3d_{5/2}4f_{7/2}$ $[J=1]$ to the ground level $^1S_0$.

Ref

6. Nilsen J., Dunn J., Osterheld A.L., Li Yu 1999 Lasing on the self-photopumped nickel-like $4f\ ^1P_1 - 4d\ ^1P_1$ x-ray transition. *Phys. Rev.* A **60** R2677-80

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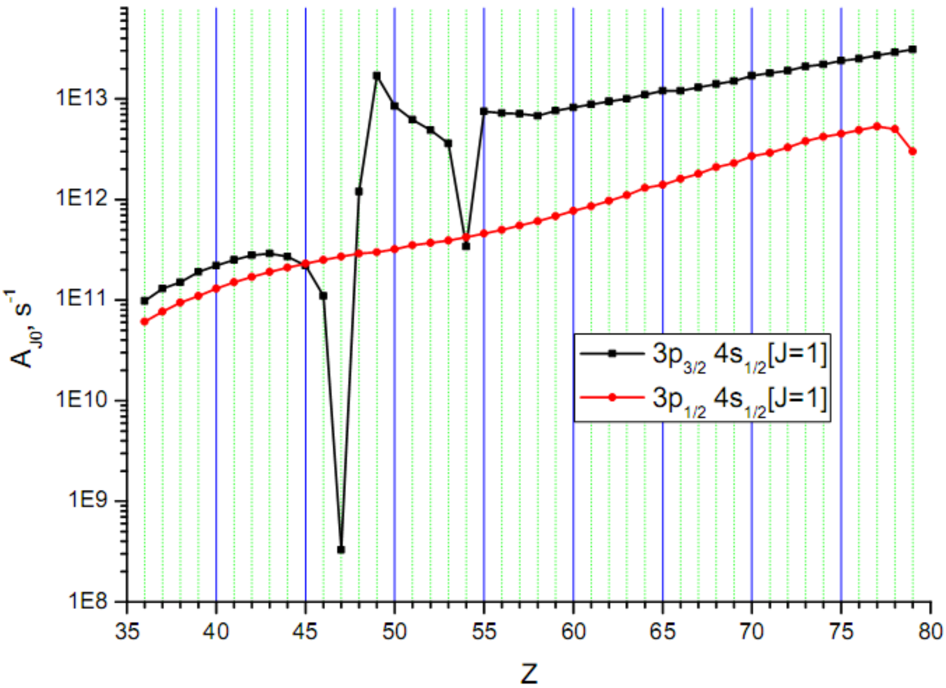

Fig. 7b: Radiative transition probability from the levels $3p_{3/2}4s_{1/2}$ [J=1] and $3p_{1/2}4s_{1/2}$ [J=1] to the ground level $^1S_0$

III. Wavelengths of the Self-Photopumped Nickel-Like $4f^1P_1 \rightarrow 4d^1P_1$ X-Ray Laser Transitions

A comparison of the wavelengths of the self-photopumped nickel-like $4f^1P_1 \rightarrow 4d^1 P_1$ X-ray laser transitions, we calculated by the RPTMP method with corresponding experimental values, shown in Figure 3, exhibits a deviation of ≤ 1% in the range Z = 37-46. For $Z \geq 48$ Å, our results are identical to experimental data with an accuracy of several units in the fourth significant digit.

Table 3: Energy levels (10^3 cm^{-1}) of W XLVII. Comparison of present calculations with experimental data [23, 24] and with calculations by GRASP92 [25]

Configuration	Term		Experiment	Present Work	GRASP92
$3p^63d^{10}$	$^1S_0$		0.0	0.0	0.0
$3p^63d^94s$	(5/2,1/2)		12601.5	12600.1	
			12616.44	12615.2	12591.1
$3p^63d^94s$	(3/2,1/2)		13138.66	13137.8	13110.8
			13148.2	13147.4	13120.7
$3p^63d^94p$	(5/2,1/2)		13379.05	13357.5	
			13388.20	13366.3	
$3p^63d^94p$	(3/2,1/2)		13916.27	13894.8	
			13940.6	13922.4	13930.6
$3p^63d^94p$	(5/2,3/2)		14229.0	14234.9	14221.0
$3p^63d^94p$	(3/2,3/2)		14751.0	14756.2	14741.1
$3p^63d^94d$	(3/2,3/2)			15935.9	15924.2
$3p^63d^94d$	(5/2,5/2)		15556.1	15561.3	15550.2
			15610.2	15614.9	15605.0

$3p^53d^{10}4s$	(3/2,1/2)		16247.0	16258.9	
$3p^63d^94d$	(3/2,3/2)		16256.2	16284.7	16282.9
$3p^63d^94f$	(5/2,7/2)		17045.9	17042.2	17030.6
$3p^63d^94f$	(3/2,5/2)		17574.7 17580.3*	17586.5	17585.6
$3p^53d^{10}4s$	(1/2,1/2)		*[18727]*	18726.4	18724.4
$3p^53d^{10}4d$	(3/2,3/2)		19044.4	19041.8	19057.5
$3p^53d^{10}4d$	(3/2,5/2)		19244.5	19234.8	19244.1
$3p^53d^{10}4f$	(3/2,7/2)		20589.0	20600.1	20613.8
$3p^53d^{10}4d$	(1/2,3/2)		21561.0	21547.0	21614.6

Data from [24]

Two values of Z are exceptions: (i) the calculation instability point at $Z = 42$ and (ii) the point Z = 49 where the $3d_{3/2}4f_{5/2}$ *[J=1]* and $3p_{3/2}4s_{1/2}$ *[J=1]* states are probably incorrectly identified in the calculation by the MCDF method in [6].

Table 4: Wavelengths ($\lambda_{las,}$ Å) of the $3d_{3/2}4f_{5/2}$ $(^1P_1) - 3d_{3/2}4d_{3/2}$ $(^1P_1)$ SPP laser transitions in Ni-like sequence calculated by RPTMP

Z	λ_{las}
50	134.08
51	128.12
52	122.54
53	117.39
54	112.66
55	108.36
56	104.295
57	100.51
58	96.98
59	93.68
60	90.57
61	87.65
62	84.89
63	82.28
64	79.81
65	77.47
66	75.23
67	73.08
68	71.06
69	69.11
70	67.25
71	65.47
72	63.75
73	62.10
74	60.51
75	58.97
76	57.48
77	56.04
78	54.64
79	53.23

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6. Nilsen J., Dunn J., Osterheld A.L., Li Yu 1999 Lasing on the self-photopumped nickel-like $4f\ ^1P_1 - 4d\ ^1P_1$ x-ray transition. *Phys. Rev.* A **60** R2677-80

We estimated the accuracy of the calculation of the energies of the upper and lower working states for high Z using experimental measurements of various studies. As an example, we compare the experimental energies for $Z = 74$ (W^{46+}) obtained using the Super EBIT (electron beam ion trap) [23-24], presented in Table 1. There are also listed the theoretical results calculated using the MCDF method called the Grasp92 [25]. Here we do not present earlier calculations of other authors. We also note the impossible comparison to the other calculations [26] in view of the level identification entanglement in this paper.

Good agreement between experimental and theoretical results for the energy levels in Table 3 may be noted: the maximum deviation is two units in the fourth significant digit. For the problem under study, it is important to ascertain the high accuracy of the calculation of the upper and lower working levels. For the experimental energy of the $3d_{3/2}4f_{5/2}$ $[J=1]$ level, Table 3 gives two values: one obtained in the experiments [23] and the other later [24], respectively. The difference with our calculation is 6 units in the fifth significant digit. We did not find the experimental energy of the $3d_{3/2}4d_{3/2}$ $[J=1]$ lower working level for high Z in the literature. The energies of two other states of the *3d4d* configuration with $J = 1$, 2, given in Table 1 also agree with high accuracy, which indirectly confirms the calculation reliability.

Wavelengths of the $3d_{3/2}4f_{5/2}$ $(^1P_1) - 3d_{3/2}4d_{3/2}$ $(^1P_1)$ SPP laser transitions in Ni-like sequence calculated by RPTMP are listed in Table 4. The data on λ_{las} (see Table 4) are obtained *a priori*, no fittings were used. The error can be several units in the fourth significant digit.The precision wavelengths of laser transitions are necessary, in particular, to determine ions in which intense laser emission is possible at wavelength for which multilayer mirrors (MM) with high reflectance are developed.

IV. Conclusion

It is generally accepted that the energy levels of atomic system/multicharged ion are identified unambiguously. It means that atomic energy values are practically independent on plasma sources. Present studies of the irregularities in energy sequences led to a theoretical observation of an extraordinary phenomenon; it could be called "instability of the multicharged ion state". The calculations directly indicates the ion Mo^{14+} (Ni-like state of Mo) in which the $3d_{3/2}4d_{3/2}$ $[J=1]$ level energy defined in typical laboratory source might significantly differs from that defined in a source of very small ion density. The ultra high vacuum is maintained in EBITs (electron beam ion trap) apparatus. Thereby EBITs are used to investigate the fundamental properties of highly charged ions.

The crossing region of each working level with another level is characterized by their strong effect on each other, which can cause strong instability of the energy structure in the crossing region. In such regions, jumps in functions of energy levels and probabilities of radiative transition on Z are possible (see Figures 7a and 7b). From this, the conclusion about the possible incorrect identification of levels in their crossing regions follows.

The SPP XRL can be very sensitive to external fields. It is implied that even an insignificant change in the plasma density can affect the emission spectrum. An indirect confirmation of this can be the remarkable phenomenon (see Fig. 4) where a rapid increase in the contribution of the $3d_{5/2}4d_{3/2}$ $[J=1]$ level to the composition of the lower working level is demonstrated. In the interval $Z = 40$-42, the contribution of this level increases by five orders of magnitude. A similar pattern is observed in Fig. 6 where the contribution of $3p_{3/2}4s_{1/2}$ $[J=1]$ also rapidly increases to $Z = 49$, where this level strongly

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23. Kramida A.E., Shirai T. 2009 Energy levels and spectral lines of tungsten. W III through W LXXIV. *At. Data Nucl. Data Tables* **95** 305-474

interacts with the upper working level. In this case, the oscillator strength is transferred from the upper working level to the $3p_{3/2}4s_{1/2}$ *[J=1]* level.

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On The Applications of Transmuted Inverted Weibull Distribution

By Ogunde Adebisi Ade, Fatoki Olayode & Oseghale Osezuwa Innocent
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Abstract- In this paper, we introduce a new distribution called transmuted inverted Weibull distribution (TIWD).The new distribution was used in analyzing bathtub failure rates lifetime data.We consider the standard transmuted inverted Weibull distribution (TIWD) that generalizes the standard inverted Weibull distribution (IWD), the new distribution has two shape parameters. The moments, median, survival function, hazard function, maximum likelihood estimators, least-squares estimators, fisher information matrix and asymptotic confidence intervals were obtained. A real data set is analyzed and it is observed that the (TIWD) distribution can provide a better fitting than (IWD) distribution.

Keywords: *moments, bathtub failure rate, shape parameters.*

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On the Applications of Transmuted Inverted Weibull Distribution

Ogunde Adebisi Ade [α], Fatoki Olayode [σ] & Oseghale Osezuwa Innocent [ρ]

Abstract- In this paper, we introduce a new distribution called transmuted inverted Weibull distribution (TIWD).The new distribution was used in analyzing bathtub failure rates lifetime data.We consider the standard transmuted inverted Weibull distribution (TIWD) that generalizes the standard inverted Weibull distribution (IWD), the new distribution has two shape parameters. The moments, median, survival function, hazard function, maximum likelihood estimators, least-squares estimators, fisher information matrix and asymptotic confidence intervals were obtained. A real data set is analyzed and it is observed that the (TIWD) distribution can provide a better fitting than (IWD) distribution.

Keywords: moments, bathtub failure rate, shape parameters.

I. Introduction

The inverted Weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates. Ref. [11] explained the flexibility of the three parameters inverted Weibull distribution and its interested properties. Ref. [1]studied the properties of the inverted Weibull distribution and its application to failure data. Ref. [2] introduced the exponentiated Weibull distribution as generalization of the standard Weibull distribution and applied the new distribution as a suitable model to the bus-motor failure time data. Ref. [3] reviewed the exponentiated Weibull distribution with new measures. Ref. [4] studied the properties of transmuted Weibull distribution. Ref. [6] proposed andstudied the various structural properties of the transmuted Rayleigh distribution. Ref. [7] introduced the transmuted modified Weibull distribution. Transmuted Lomax distribution is presented by Ref. [8]. Ref. [9] introduces transmuted Pareto distribution. Transmuted Generalized Linear Exponential Distribution introduced by Ref. [10].

In this article we use transmutation map approach suggested by Shaw *et al.* [5] to define a new model which generalizes the inverted Weibull model. We will call the generalized distribution as the transmuted inverted Weibull distribution. According to the Quadratic Rank Transmutation Map (QRTM), approach the cumulative distribution function(cdf) satisfy the relationship

$$F(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2 |\lambda| \leq 1 \quad (1)$$

The probability density function (PDF) is given by

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$$f(x) = \frac{dF}{dx} = [(1+\lambda) - 2\lambda G(x)]g(x) \tag{2}$$

Where $g(x)$ is the *pdf* of the base line distribution.

II. Transmuted Inverted Weibull Distribution (TIWD)

We say that the random variable X has a standard inverted Weibull distribution (IWD) if its distribution function takes the following form:

$$G(x) = e^{-x^{-\beta}} \qquad x > 0,\ \beta > 0 \tag{3}$$

The pdf of Inverted Weibull distribution is given by

$$g(x) = \beta x^{-\beta} e^{-x^{-\beta}} \tag{4}$$

Now using (3) and (4) we have the cdf of a (TIWD) given by

$$F(x) = e^{-x^{-\beta}} \left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right] \tag{5}$$

Where $|\lambda| \leq 1$ is simply the transmutation parameter of the distribution function of the standard inverted Weibull distribution (IWD). Here λ and β are the shape parameters. Therefore, the probability density function is:

$$f(x) = \beta x^{-(\beta+1)} e^{-x^{-\beta}} \left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right] \tag{6}$$

Note that for $\lambda = 0$, we have the pdf of a standard inverted Weibull distribution. Figure 1 illustrates some of the possible shapes of the density function of (TIWD) for selected parameters.

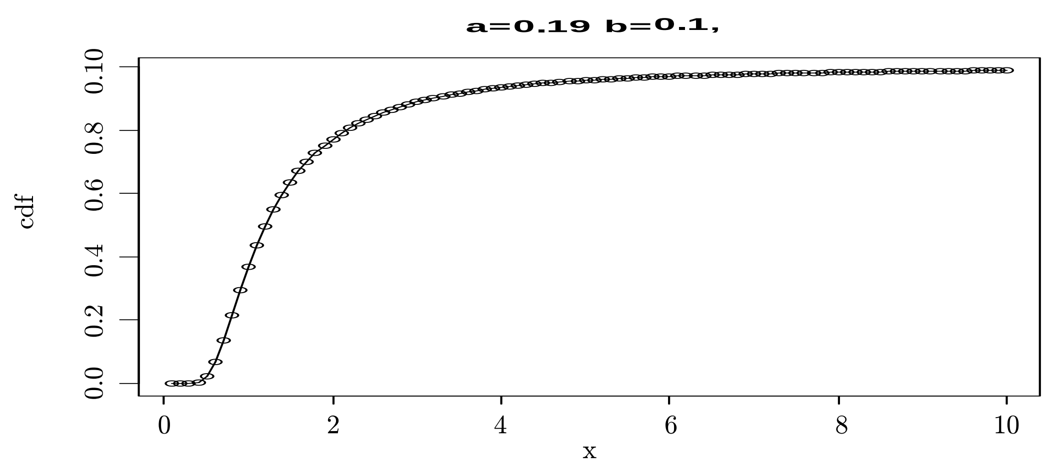

Fig. 1: The graph of the cdf of TIWD

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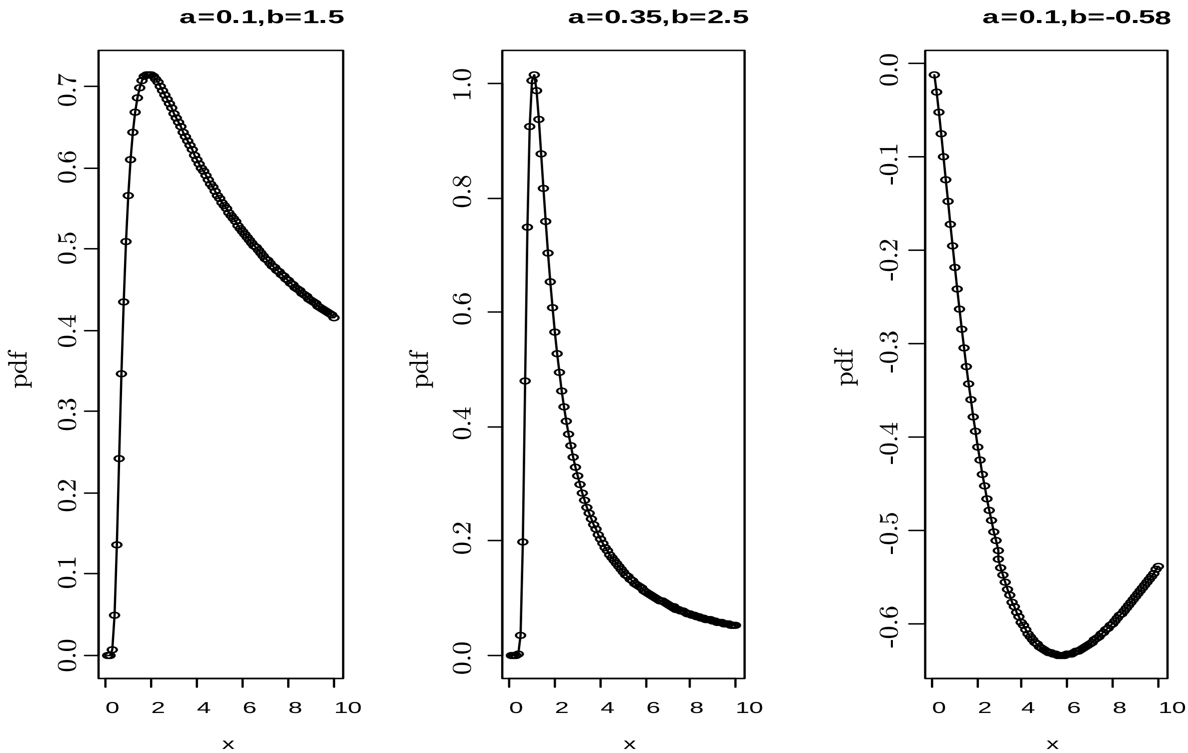

Fig. 2: The graph of the pdf of TIWD

III. Moments, Mean, Variance, Median and Quantile of (TIWD)

In this section we shall present the moments and qunatiles for the (TIWD). The k^{th}order moments, for $< \beta$, of (TIWD) can be obtained as follows for a random variable X,

$$E(X)^k = \int_{-\infty}^{\infty} x^k f(x)d \tag{7}$$

Putting eq. (6) in eq. (7), we have

$$E(X)^k = \beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-x^{-\beta}} \left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right] dx \tag{8}$$

Expanding eq. (8) and splitting it into three, we obtain

$$P_1 = \beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-x^{-\beta}} dx \tag{9}$$

$$P_2 = \lambda\beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-x^{-\beta}} dx \tag{10}$$

$$P_3 = -2\lambda\beta \int_{-\infty}^{\infty} x^k x^{-(\beta+1)} e^{-2x^{-\beta}} dx \tag{11}$$

If we let $u = x^{-\beta}$ in eq. (9) and eq. (10) and also let $u = 2x^{-\beta}$ in eq. (11), then we have

$$P_1 = -\Gamma(1 - \frac{k}{\beta}) \tag{12}$$

$$P_2 = -\lambda\Gamma(1-\frac{k}{\beta}) \tag{13}$$

$$P_3 = \left(\frac{1}{2}\right)^{-\frac{k}{\beta}} \lambda\Gamma\left(1-\frac{k}{\beta}\right) \tag{14}$$

It should be noted that $\beta < k$

Then combining eq.(12), eq.(13) and eq.(14) we obtain the k^{th} moment of (TIWD), we have

$$\mu_k = E(X)^k = \left(1-\frac{k}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{k}{\beta}}-1\right]-1\right\} \tag{15}$$

Using eq. (15), we obtain the $1^{st}, 2^{nd}, 3^{rd}, and\ 4^{th}$ moment for $k = 1, 2, 3, 4$

$$\mu_1 = \Gamma\left(1-\frac{1}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{1}{\beta}}-1\right]-1\right\} \tag{16}$$

$$\mu_2 = \Gamma\left(1-\frac{2}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{2}{\beta}}-1\right]-1\right\} \tag{17}$$

$$\mu_3 = \Gamma\left(1-\frac{3}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{3}{\beta}}-1\right]-1\right\} \tag{18}$$

$$\mu_4 = \Gamma\left(1-\frac{4}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{4}{\beta}}-1\right]-1\right\} \tag{19}$$

The mean of (TIWD) is the first moment about the origin (μ_1) which corresponds to eq. (16)

And the variance of (TIWD) can be obtained using the relation

$$V(X) = \mu_2 - (\mu_1)^2 \tag{20}$$

Inserting eq. (16) and eq. (17) in eq. (20) we have

$$V(X) = \Gamma\left(1-\frac{2}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{2}{\beta}}-1\right]-1\right\} - \left[\Gamma\left(1-\frac{1}{\beta}\right)\left\{\lambda\left[\left(\frac{1}{2}\right)^{-\frac{1}{\beta}}-1\right]-1\right\}\right]^2 \tag{21}$$

The quantile function for the (TIWD) is given as,

$$x_q = \left[-ln\frac{\lambda-\sqrt{\lambda(\lambda+4q)}}{2\lambda}\right]^{-\frac{1}{\beta}} \tag{22}$$

The lower quartile, median and the upper quartile of the (TIWD) can be obtained by letting $q = 0.25, q = 0.5$ and $q = 0.75$ respectively in eq. (22).

Table 1: Moments based values for TIWD

β	λ	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	$\hat{\mu}_4$
−1.0	−3.0	0.50	2.50	9.75	43.50
−1.0	−3.5	0.75	3.25	12.30	54.75
−1.0	−4.0	1.00	4.00	15.00	66.00
−1.0	−4.5	1.25	4.75	17.63	386.25

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Table 2: Moments based measures of the TIWD

β	λ	*Mean*	*Variance*	*CV*	*CS*	*CK*
−1.0	−3.0	0.50	2.25	3.17	2.47	6.96
−1.0	−3.5	0.75	2.69	2.19	2.10	5.18
−1.0	−4.0	1.00	3.00	1.73	1.89	4.12
−1.0	−4.5	1.25	3.19	1.43	1.70	17.12

Notes

IV. Reliability Analysis

The survival function, also known as the reliability function in engineering, is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some certain mechanical wearing system with time. It is the conditional probability that the system will survive beyond a specified time. The (TIWD) can be a useful model to characterize failure time of a given system because of the analytical structure. The reliability function $R(t)$, which is the probability of an item not failing prior to time t, is defined by $R(t) = 1-F(t)$. The reliability function of a (TIWD) is given by

The reliability function define as,$R(x) = 1 - F(x)$, for (TIWD) is given by:

$$R(x) = 1 - e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right] \tag{23}$$

The other characteristic of interest of a random variable is the hazard rate function also known as instantaneous failurerate defined by

$$h(x) = \frac{f(x)}{1-F(x)} \tag{24}$$

Is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time *t:* The hazard rate function for a (TIWD) is given by

$$h(x) = \frac{\beta x^{-(\beta+1)} e^{-x^{-\beta}}\left[1+\lambda-2\lambda e^{-x^{-\beta}}\right]}{1-e^{-x^{-\beta}}\left[(1+\lambda)-\lambda e^{-x^{-\beta}}\right]} \tag{25}$$

For $\lambda = 0$, it represents the hazard of the standard inverted Weibull distribution. The graph (fig.3) drawn below depict the graph of the hazard rate function for TIWD for various values of the parameters.

Recently, it is observed that the reversed hazard function plays an important role in the reliability analysis; see Gupta and Gupta (2007). The reversed hazard function of the (TIWD) is given as

$$R(x) = \beta x^{-(\beta+1)}\left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right] \tag{26}$$

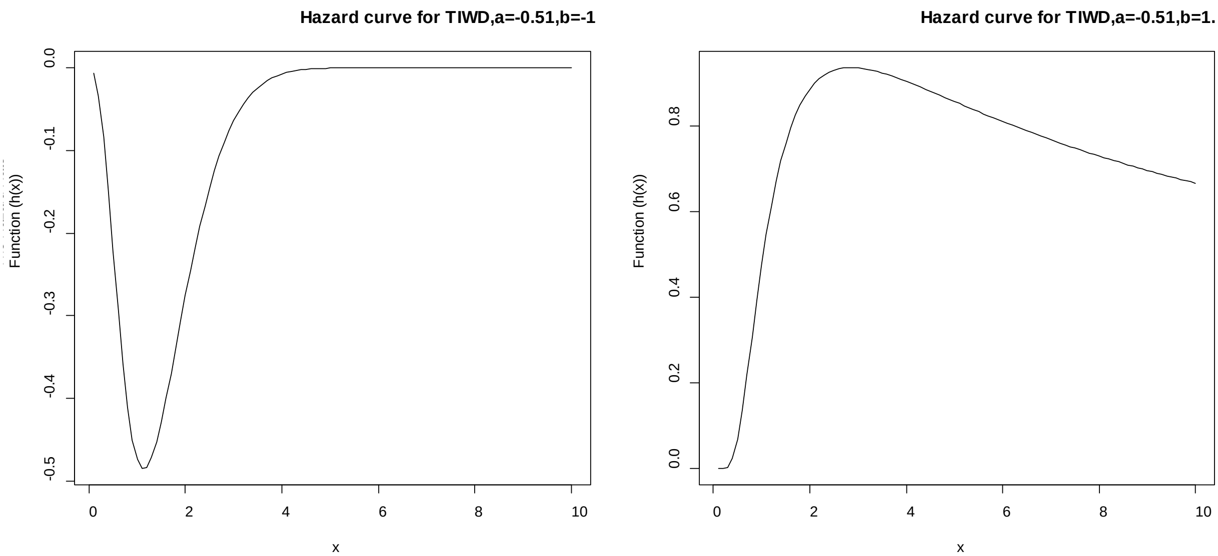

Fig. 3: The graph of hazard rate function for TIWD

V. Moment Generating Function of (TIWD)

The moment generating function of a random variable x is defined by

$$M_t(x) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{27}$$

The above expression can further be simplify as

$$M_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k f(x)dx \tag{28}$$

Since,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^k x^k}{k!} \tag{29}$$

Inserting eq. (15) in eq. (28) we have

$$M_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(1 - \frac{k}{\beta}\right) \left\{ \lambda \left[\left(\frac{1}{2}\right)^{-\frac{k}{\beta}} - 1 \right] - 1 \right\} \tag{30}$$

The above expression is the moment generating function of (TIWD).

VI. Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. We know that if $X_{(1)}, X_{(2)}, . . . , X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, . . , X_n$ from a continuous population with cdf $F_X(X)$ *and pdf* $f_X(x)$.

Then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1 - F_X(x)]^{n-j} \tag{31}$$

For $j = 1, 2, 3, \ldots, n$

Notes

We have from (5) and (6) the pdf of the j^{th}order inverted Weibull random variable X_j given by

$$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!}\beta x^{-\beta} e^{-x^{-\beta}} \left[e^{-x^{-\beta}}\right]^{j-1} \left[1 - e^{-x^{-\beta}}\right]^{n-j} \tag{32}$$

If we consider the series expansion,

$$(1-z)^m = \sum_{j=0}^{\infty}(-1)^i \binom{m}{i} z^i \tag{33}$$

Applying eq. (33) in eq. (31) will yield

$$f_{X_{(j)}} = \frac{n!}{(j-1)!\,(n-j)!}\beta x^{-\beta} \sum_{i=0}^{\infty}(-1)^i \binom{n-j}{i} \left[e^{-x^{-\beta}}\right]^{i+j} \tag{34}$$

Therefore, the pdf of the n^{th}order inverted Weibull random variable X_n is given by

$$f_{X_{(n)}} = n!\,\beta x^{-\beta} \left[e^{-x^{-\beta}}\right]^n \tag{35}$$

Now we provide the distribution of the order statistics for transmuted inverted Weibull random variable. The pdf of the j^{th} order statistic for the transmuted inverted Weibull random variable is given by

$$f_{X_{(j)}} = \frac{n!}{(j-1)!\,(n-j)!}\beta x^{-(\beta+1)} e^{-x^{-\beta}} \Big[1+\lambda - 2\lambda e^{-x^{-\beta}}\Big] \left[e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right]\right]^{j-1} \left[1 - e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right]\right]^{n-j}$$

Using the relation in eq. (32) the above expression can be simplified as

$$f_{X_{(j)}} = \frac{n!}{(j-1)!\,(n-j)!}\beta x^{-(\beta+1)} e^{-x^{-\beta}} \Big[1+\lambda - 2\lambda e^{-x^{-\beta}}\Big] \sum_{j=0}^{\infty}(-1)^i \binom{n-j}{i} \left[e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right]\right]^{i+j-1} \tag{36}$$

Therefore the pdf of the largest order statistic $X_{(n)}$ for the transmuted inverted Weibull random variable is given by

$$f_{X_{(n)}} = n!\,\beta x^{-(\beta+1)} e^{-x^{-\beta}} \left[1+\lambda - 2\lambda e^{-x^{-\beta}}\right] \left[e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right]\right]^{n-1} \tag{37}$$

And for the first order we have

$$f_{X_{(1)}} = \beta x^{-(\beta+1)} e^{-x^{-\beta}} \left[1+\lambda - 2\lambda e^{-x^{-\beta}}\right] \tag{38}$$

VII. Entropy

The entropy of a random variable X with probability density $f(x)$ is a measure ofvariation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Renyi, A. (1961) introduced the Renyi entropy defined as

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$$I_R(\delta) = \frac{1}{1-\delta} log\left\{\int_0^\infty f(x)^\delta dx\right\} \tag{39}$$

Where $\delta > 0$ and $\delta \neq 1$. the integral in $I_R(\delta)$ of the TIWD$(x;\ \lambda, \beta)$ can be define as

$$\int_0^\infty f(x)^\delta dx = \int_0^\infty \left\{x^{-(\beta+1)} e^{-x^{-\beta}} \left[1+\lambda-2\lambda e^{-x^{-\beta}}\right]\right\}^\delta dx \tag{40}$$

This can be simplify as

$$\beta^\delta \int_0^\infty x^{-\delta(\beta+1)} e^{-\delta x^{-\beta}} \left[1-\lambda(2e^{-x^{-\beta}}-1\right]^\delta dx$$

$$= \beta^\delta \sum_{k=1}^{\infty} (-1)^k \binom{\delta}{k} \lambda^k \int_0^\infty x^{-\delta(\beta+1)} e^{-\delta x^{-\beta}} \left(2e^{-x^{-\beta}}-1\right)^k dx \tag{41}$$

But $\left(2e^{-x^{-\beta}}-1\right)^k = (-1)^k \left(1-2e^{-x^{-\beta}}\right)^k$, therefore the RHS equation (41) can be simplify as

$$\beta^\delta \sum_{k=1}^{\infty} \sum_{l=1}^{k} (-1)^{2k} (-1)^l \binom{\delta}{k} \lambda^k \binom{k}{l} 2^l \int_0^\infty x^{-\delta(\beta+1)} e^{-(\delta+l)x^{-\beta}} dx \tag{42}$$

Letting $P = x^{-\beta(\delta+l)}$, then $x = P^{\frac{-1}{\beta(\delta+l)}}$ and $dx = -\frac{P^{\frac{-1}{\beta(\delta+l)}-1}}{\beta(\delta+l)} dP$

Then substituting in the above integral function we have

$$-\frac{1}{\beta(\delta+l)} \int_0^\infty P^{\frac{\delta-1-\beta l}{\beta(\delta+l)}} e^{-P} dP = -\frac{1}{\beta(\delta+l)} \Gamma\left(\frac{\delta-1-\beta l}{\beta(\delta+l)}+1\right) \tag{43}$$

Then substituting equation (43), (42) in (39), we have the entropy of TIWD as

$$I_R(\delta) = \frac{-1}{1-\delta} log\left\{\beta^\delta \sum_{k=1}^{\infty} \sum_{l=1}^{k} (-1)^{2k} (-1)^l \binom{\delta}{k} \lambda^k \binom{k}{l} 2^l \frac{1}{\beta(\delta+l)} \Gamma\left(\frac{\delta-1-\beta l}{\beta(\delta+l)}+1\right)\right\} \tag{44}$$

VIII. Parameters Estimation

In this section, we discuss the maximum likelihood estimators of the two-parameter transmuted inverted Weibull distribution and their fisher information matrix as well as asymptotic confidence intervals.

a) Maximum likelihood estimators and fisher information matrix

If $x_1, x_2, . . ., x_n$ is a random sample from transmuted inverted Weibull distribution given by (6), then the Likelihood function (L) becomes:

$$L = \prod_{i=1}^{n} f(x_i, \lambda, \beta) \tag{45}$$

By substituting from equation (6) into Equation (39), we get

Notes

Notes

$$L = \prod_{i=1}^{n} \beta x^{-(\beta+1)} e^{-x^{-\beta}} \left[1 + \lambda - 2\lambda e^{-x^{-\beta}}\right] \tag{46}$$

Then the log – likelihood function becomes

$$l = \ln(\beta) - (\beta+1)\sum_{i=1}^{n} ln\,(x_i) - \sum_{i=1}^{n} x_i^{-\beta} + \sum_{i=1}^{n} \ln(1 + \lambda - 2\lambda e^{-x_i^{-\beta}}) \tag{47}$$

And the score vector is given as

$$\frac{dl}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} ln\,(x_i) + \sum_{i=1}^{n} x_i^{-\beta}\ln(x_i) + 2\lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} e^{-x_i^{-\beta}}}{(1 + \lambda - 2\lambda e^{-x_i^{-\beta}})} \tag{48}$$

$$\frac{dl}{d\lambda} = \sum_{i=1}^{n} \frac{\left(1 - 2e^{-x_i^{-\beta}}\right)}{(1 + \lambda - 2\lambda e^{-x^{-\beta}})} \tag{49}$$

Therefore, the MLEs of λ and β which maximize equation (47) must satisfy the nonlinear normal equations given by:

$$\frac{dl}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} ln\,(x_i) + \sum_{i=1}^{n} x_i^{-\beta}\ln(x_i) + 2\lambda \sum_{i=1}^{n} \frac{x_i^{-\beta} e^{-x_i^{-\beta}}}{(1 + \lambda - 2\lambda e^{-x_i^{-\beta}})} = 0 \tag{50}$$

$$\frac{dl}{d\lambda} = \sum_{i=1}^{n} \frac{\left(1 - 2e^{-x_i^{-\beta}}\right)}{(1 + \lambda - 2\lambda e^{-x^{-\beta}})} = 0 \tag{51}$$

The maximum likelihood estimator $\hat{\Theta} = \left(\hat{\lambda}, \hat{\beta}\right)'$ of $\Theta = (\lambda, \beta)'$ is obtained by setting the score vector to zero and solving the nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function given in (47).

b) Least square Estimates

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the ordered sample of size n from TIWD. Then the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right] = \frac{i}{n+1};\ i = 1,2,3,..n \tag{52}$$

Then the least square estimates of $\hat{\lambda}_{LS}$ and $\hat{\beta}_{LS}$ of λ and β are obtained by minimizing $\mathcal{Z}$ giving as

$$\mathcal{Z}(\lambda,\beta) = \sum_{i=1}^{n} \left(e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right] - \frac{i}{n+1}\right)^2 \tag{53}$$

Therefore the $\hat{\lambda}_{LS}$ and $\hat{\beta}_{LS}$ of λ and β can be obtained as the solution of the following equations

$$\frac{\delta\mathcal{Z}(\lambda,\beta)}{\delta\lambda} = \sum_{i-1}^{n} -\left\{e^{-x^{-\beta}}\right\}^2 \left(e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right] - \frac{i}{n+1}\right) = 0 \tag{54}$$

$$\frac{\delta\mathcal{Z}(\lambda,\beta)}{\delta\beta} = \sum_{i-1}^{n} \left\{\beta x^{-`(\beta+1)} e^{-x^{-\beta}}(1+\lambda)\right\} \left(e^{-x^{-\beta}}\left[(1+\lambda) - \lambda e^{-x^{-\beta}}\right] - \frac{i}{n+1}\right) = 0 \tag{55}$$

IX. Asymptotic Confidence Bounds

In this section, we derive the asymptotic confidence intervals of these parameters when$\lambda > 0$ and $\beta > 0$ and as the MLEs of the unknown parameters λ and β cannot be obtained in closed forms, by using variance covariance matrix I_0^{-1} see Lawless(2003), where I_0^{-1} is the inverse of the observed information matrix

$$I_0^{-1} = \begin{bmatrix} \frac{\partial^2 l}{\partial \lambda^2} & \frac{\partial^2 l}{\partial \lambda \partial \beta} \\ \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \beta^2} \end{bmatrix} \tag{56}$$

Thus

$$I_0^{-1} = \begin{bmatrix} var\hat{\lambda} & cov(\hat{\lambda}, \hat{\beta}) \\ cov(\hat{\lambda}, \hat{\beta}) & var\hat{\beta} \end{bmatrix} \tag{57}$$

The derivatives in $I_0^{-1}$are given as follows:

$$\frac{d^2 l}{d\lambda^2} = -\sum_{i=1}^{n} \left(\frac{1 - 2e^{-x_i^{-\beta}}}{1 + \lambda - 2\lambda e^{-x^{-\beta}}} \right)^2 \tag{58}$$

$$\frac{d^2 l}{d\beta^2} = -\frac{n}{\beta} - \sum_{i=1}^{n} x_i^{-\beta} (lnx)^2 + \sum_{i=1}^{n} \frac{x_i^{-\beta} \, lnx e^{-x_i^{-\beta}} \left[\left(x_i^{-\beta} - 1 \right) - 2x_i^{-\beta} e^{-x_i^{-\beta}} \right]}{\left(1 + \lambda - 2\lambda e^{-x_i^{-\beta}} \right)^2} \tag{59}$$

$$\frac{d^2 l}{d\lambda d\beta} = \sum_{i=1}^{n} \frac{2x_i^{-\beta} \, lnx e^{-x_i^{-\beta}} \left[\left(1 + \lambda - 2\lambda e^{-x^{-\beta}} \right) - \lambda \left(1 - 2e^{-x_i^{-\beta}} \right) \right]}{\left(1 + \lambda - 2\lambda e^{-x_i^{-\beta}} \right)^2} \tag{60}$$

We can derive the $(1 - \delta)100\%$ confidence intervals of the parameters λ and β by using variance covariance matrix as in the following forms

$$\hat{\lambda} \pm Z_{\frac{\delta}{2}} \sqrt{var\hat{\lambda}} \;\; , \;\; \hat{\beta} \pm Z_{\frac{\delta}{2}} \sqrt{var\hat{\beta}}$$

Where $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right)^{th}$ percentile of the standard normal distribution.

X. Applications

In this section, we use two real data sets to show that the TIWD can be a better model than one based on the IWD and the exponentiated inverted Weibull distribution (EIWD). In the first application, we consider a data set of the tensile strength of 100 observation of carbon fibers, the data was obtained from see Ref.[12]. These data are: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88,

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12. Andrews, D. F. & Herzberg, A. M. (1985), Data: A Collection of Problems from Many Fields for the Student and Research Worker, Springer Series in Statistics, New York.

2.82, 2.05, 3.65. Table 3.0and Table 6.0 gives the exploratory data analysis of the data considered, Table 4.0 and Table 7.0 gives the maximum likelihood estimates of the parameters with their standard error in parenthesis and Table 5.0 and Table 8.0 gives the criteria for comparison. Fig. 4 and fig. 5 represents the empirical density and the cumulative density of the data considered.

Notes

Table 3: Descriptive Statistics on Breaking stress of Carbon fibres

Min	Lower quartile	Median	Upper quartile	Mean	Max.	Skewness	Kurtosis	Range
0.390	1.840	2.700	3.220	2.640	5.560	0.37378	0.17287	5.17

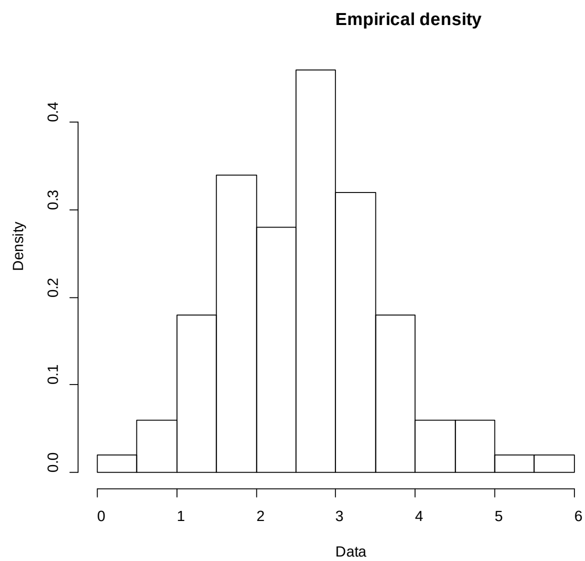

Fig. 4: The graph of the Empirical density and the cumulative distribution function of the carbon data

Table 4: Estimated parameters of the TIWD, EIWD and IWD

Mode I	Estimates		$l(\hat{\theta})$
$TIWD$ (λ, β)	-0.8965 (0.0908)	3.1175 (0.2668)	-44.571
$EIWD$ (θ, β)	0.2485 (1.9685)	0.2344 (2.8876)	-46.853
IWD (β)	- $(-)$	2.806 (2.723)	-58.482

Table 5: Measures of Goodness of Fit

Mode I	$K-S$	AD	W	AIC	BIC	HQIC	CAIC
$\boldsymbol{TIWD}$	0.4591	6.3175	1.1915	93.142	97.428	94.827	93.827
$\boldsymbol{EIWD}$	0.5173	5.7414	1.0727	97.907	101.993	99.392	97.907
$\boldsymbol{IWD}$	0.4359	6.0615	1.1388	118.963	121.107	119.806	119.029

Also, we consider a data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. This data are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.For previous studies with these data sets see Andrews and Herzberg [12]

Table 6: Summary of data on fatigue fracture of Kevlar 373/epoxy at 90 % stress level

Min	Lower quartile	Median	Upper quartile	Mean	Max.	Variance	Skewness	Kurtosis	Range
0.0251	0.09048	1.7361	2.2960	1.9590	9.0960	2.4774	1.9406	8.1608	9.0709

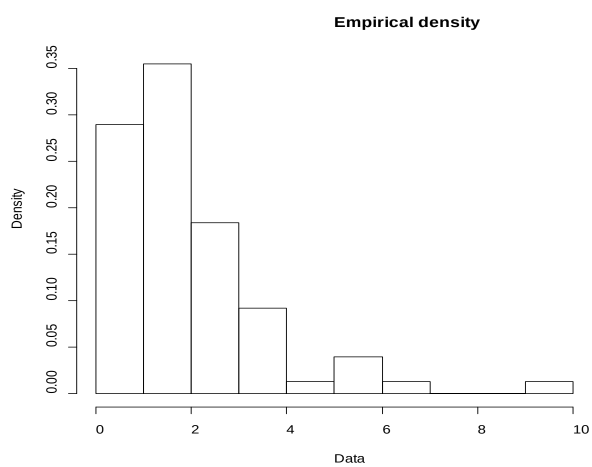

Fig. 7: The graph of the Empirical density and the cumulative distribution function of the Kevlar 373/epoxy data

Table 7: Estimated parameters of the TIWD, EIWD and IWD

Mode I	Estimates		$l(\hat{\theta})$
$TIWD$ (λ, β)	0.7074 (03994)	0.6903 (0.0575)	-152.483
$EIWD$ (θ, β)	0.8608 (0.1088)	0.7588 (0.0541)	-153.539
IWD (β)	- (–)	0.7322 (0.0474)	-154.278

Table 8: Measures of Goodness of Fit

Mode I	$K-S$	AD	W	AIC	BIC	HQIC	CAIC
$TIWD$	0.2440	4.9831	0.8506	308.967	313.128	310.829	308.967
$EIWD$	0.2290	5.9661	1.0351	311.078	315.740	312.941	311.243
IWD	0.2291	5.2691	0.9036	310.556	312.887	311.487	310.610

We employ the statistical tools for model comparison such as Kolmogorov-Smirnov (K-S) statistics, Anderson Darling statistic (AD), crammer von misses statistic

Ref

12. Andrews, D. F. & Herzberg, A. M. (1985), Data: A Collection of Problems from Many Fields for the Student and Research Worker, Springer Series in Statistics, New York.

(W), Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Hannan Quinine information criterion (HQIC) and Bayesian information criterion o choose the best possible model for the data sets among the competitive models.The selection criterion is that the lowest AIC, CAIC, BIC and HQIC correspond to the best fit model.

Notes

XI. Conclusion

Among the models considered the best model is the transmuted inverted Weibull distribution for the two data sets.

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A Litle Note on the Twisted/Warped Product Metrics and the Hopf Conjecture

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GJSFR-F Classification: *MSC 2010: F3C20, F3C50*

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4. O. Perdomo: A *note on Hopf's conjecture* . Universidad del Val, Cali-Colombia.

A Litle Note on the Twisted/Warped Product Metrics and the Hopf Conjecture

Adama Thiandoum α, Athoumane Niang σ & Abdoulaye Sarr ρ

Abstract- This work defines some notions on the twisted, warped product metric on the manifold $\mathbb{S}^2 \times \mathbb{S}^2$ before prouving that Hopf conjecture is true in the class of this metrics.

Keywords: twisted, warped product metrics, sectional curvature, hopf conjecture.

I. Introduction

Let g_0 be the riemannian metric induced by $\mathbb{R}^6$ on $M = \mathbb{S}^2 \times \mathbb{S}^2$ and let T_pM denoted the tangent vector space of M at p. For any 2-plane $\sigma \subset T_pM$ spanned by two tangent vector fields, we define a function $K(\sigma)$ called the sectional curvature of σ. We will say that the riemannian manifold M has positive sectional curvature if for every point $p \in M$ the sectional curvature $K(\sigma)$ of every 2-plane $\sigma \subset T_pM$ is positive. An example of such manifolds are the n dimensional spheres $\mathbb{S}^n$ with the metric induced by $\mathbb{R}^{n+1}$. In this case its sectional curvature is equal to 1 for any 2-plane $\sigma \subset T_pM$. In general the question of deciding if a given manifold admits a riemannian metric with positive sectional curvature is a difficult one. One of the Hopf conjecture can be stated as follows: The product manifold $\mathbb{S}^2 \times \mathbb{S}^2$ does not admit any riemannian metrics with strictly positive sectional curvature. For the relevance of this problem, we refer to the [4], [5] and [6].

A direct computation shows that when we endowed $\mathbb{S}^2 \times \mathbb{S}^2$ with the riemannian metric g_0, then for any $(a,b) \in \mathbb{S}^2 \times \mathbb{S}^2$ the sectional curvature of a plane spaned by a vector of the form $(v,0) \in T_{(a,b)}(\mathbb{S}^2 \times \mathbb{S}^2)$ and a vector of the form $(0,w) \in T_{(a,b)}(\mathbb{S}^2 \times \mathbb{S}^2)$, with $v, w \in \mathbb{R}^3$, is zero, we also found out that the sectional curvature for the planes $T_{(a,b)}\mathbb{S}^2 \times \mathbb{S}^2 \cap \mathbb{R}^3 \times \{0\}$ and $T_{(a,b)}\mathbb{S}^2 \times \mathbb{S}^2 \cap \{0\} \times \mathbb{R}^3$ is 1, for any other plane the sectional curvature is a number between 0 and 1 ([4]).

The notion of warped product manifolds was introduced by Bishop and O'Neil [7] for contructing manifolds of negative curvature. Later this notion has been extented to be twisted product manifolds [[9], [8]].

More precisely, let (M_1, g_1) and (M_2, g_2) be two-riemannian manifolds of dimension m_1 and m_2 (respectively) and let $\pi_1 : M_1 \times M_2 \longmapsto M_1$ and $\pi_2 : M_1 \times M_2 \longmapsto M_2$ be the canonical projections. Also let $f : M_1 \times M_2 \longmapsto \mathbb{R}^+$ be a smooth function. Then the twisted product $M = M_1 \times_f M_2$ of the manifolds (M_1, g_1) and (M_2, g_2) with twisting

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function f, is defined to be the product manifold $M = M_1 \times M_2$ with the riemannian tensor $g = g_1 \oplus f^2 g_2$ given by $g = \pi_1^* g_1 + f^2 \pi_2^* g_2$. Moreover, if the function f only depends on the points of M_1, that is $f : M_1 \longmapsto \mathbb{R}^+$, then $M = M_1 \times_f M_2$ is called the warped product manifold of (M_1, g_1) and (M_2, g_2) with warping function f. In particular, if $f = 1$, then $M = M_1 \times_1 M_2$ is the product manifold of (M_1, g_1) and (M_2, g_2) with the product metric defined by $g = \pi_1^* g_1 + \pi_2^* g_2$.

Now, let $(\mathbb{S}^2 \times_f \mathbb{S}^2, g)$be a twisted product manifold with twisting function $f$and let $h = log(f)$. In this paper, we prove the following mains results.

Theorem 1.1. For any smooth function $h : \mathbb{S}^2 \times \mathbb{S}^2 \longmapsto \mathbb{R}$, the twisted product manifold $(\mathbb{S}^2 \times \mathbb{S}^2, g)$ has no positive sectional curvature.

Corollary 1.1. For any smooth function $h : \mathbb{S}^2 \longmapsto \mathbb{R}$, the warped product manifold $(\mathbb{S}^2 \times \mathbb{S}^2, g)$ has no positive sectional curvature.

II. Preliminaries

Let M_1 and M_1 be two riemannian manifolds, and let $f > 0$ be smooth fonction on M_1, the warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ furnished with the riemannian metric

$$g = \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2, \tag{1}$$

where π_1 and π_2 are canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. This notion has been extented to several forms. Let (M, g_1) and (M_2, g_2) be riemannian manifolds, $h : M_1 \times M_2 \longmapsto \mathbb{R}$ a positive and smooth function, g a riemannian metric on the manifold $M = M_1 \times M_2$ and assume that the canonical foliation L_1 and L_2 intersects perpendicularly everywhere. Then g is the metric tensor of

(1) a twisted product $M_1 \times_h M_2$ if only if L_1 is a totally geodesic foliation and L_2 is a totally umbilic foliation,

(2) a warped product $M_1 \times_h M_2$ if only if L_1 is a totally geodesic foliation and L_2 is a spheric foliation,

(3) a usual product of riemannian manifolds if only if L_1 and L_2 are totally geodesic foliations.

let (M, g_1) and (M_2, g_2) be riemannian manifolds with the Levi-Civita connections ∇^1 and ∇^2, respectively, and let ∇ denote the Levi-Civita connection of the twisted product manifold $M_1 \times_f M_2$ of (M_1, g_1) and (M_2, g_2) with twisting function f. Also, let $h = log(f)$ and let $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$ be the sets of lifts of vector fields on M_1 and M_2 to $M_1 \times M_2$, respectively.
We have the following proposition for a twisted product manifold.

Proposition 2.1. ([7], [9], [8]) Let $M = M_1 \times_f M_2$ be a twisted product manifold with the metric $g = g_1 \oplus f^2 g_2$ and let $X, Y \in \mathcal{L}(M_1)$ and $V, W \in \mathcal{L}(M_2)$. Then
1) $\nabla_X Y = \nabla^1_X Y$,
2) $\nabla_X V = \nabla_V X = X(h)V$,
3) $\nabla_V W = \nabla^2_V W + V(h)W + W(h)V - g(V, W)\nabla h$,
where $h = log(f)$ and ∇h is the gradient of the function h.

Notes

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We define the hessian $Hess_1h(X,Y) = XY(h) - (\nabla^1_X Y)(h)$, for $X, Y \in \mathcal{L}(M_1)$. If $X, Y \in \mathcal{L}(M_1)$ and $V \in \mathcal{L}(M_2)$, we have $XV(h) = VX(h)$ and the hessian $hessh$ of h on the manifold $M = M_1 \times_f M_2$ satisfies

$$Hessh(X,V) = XV(h) - X(h)V(h), \tag{2}$$

$$Hessh(X,Y) = Hess_1h(X,Y). \tag{3}$$

Notes

We have the following proposition for the riemannian curvature tensor of a twisted product manifold.

Proposition 2.2. *([7], [9], [8]) Let $M = M_1 \times_f M_2$ be a twisted product manifold and let $X, Y, Z \in \mathcal{L}(M_1)$, $V, W, U \in \mathcal{L}(M_2)$. Then, we have*

1) $R(X,Y)Z = R^1(X,Y)Z$,
2) $R(X,Y)V = 0$,
3) $R(V,W)U = R^2(V,W)U - |\nabla h|^2[g_2(V,U)W - g_2(W,U)V]$,
4) $R(X,W)V = [X(h)V(h) + Hessh(X,Y)]W - g(W,V)[X(h)\nabla h + H^h(X)]$,

where H^h is the hessian tensor of h on $M_1 \times_f M_2$, that is $Hessh(X,Y) = g(H^h(X),Y)$.

Lemma 2.1. *Let $M = M_1 \times_f M_2$ be a twisted product manifold. Then, for orthonormal vectors X and V, where $X \in \mathcal{L}(M_1)$, $V \in \mathcal{L}(M_2)$. The mixed sectional curvature is given by*

$$K(X,V) = g(R(X,V)V,X) = -[g(\nabla h, X)^2 + Hessh(X,X)], \tag{4}$$

where $h = log(f)$ and f positive smooth function.

Proof of the Lemma. Using the formula in 4) of the Proposition 2.2 with the definition of the sectional curvature in orthonormal mixed coordinate systems $\{X, V\}$, we obtain the result (4) of the lemma. □

III. Proof of the Theorem and the Corollary

In this section we prove the theorem and the corollary. The proof of the corollay is strongly related to that of the theorem, it is the same proof.

Proof of the Theorem. we will assume that the sectional curvature of the manifold (M,g) is everywhere positive.
We consider the case where M_1 and M_2 are the standard unit sphere $\mathbb{S}^2$, and the twisted product manifold $\mathbb{S}^2 \times_f \mathbb{S}^2$ with the twisting function f and $h = log(f)$. Let X and Y be orthonormal vectors, tangents to the left sphere $\mathbb{S}^2$, and V and W orthonormal vectors, tangents to the right sphere $\mathbb{S}^2$. Then, it readily follows from 1) and 3) of the Proposition 2.2 that the sectional curvatures $K(X,Y) = g(R(X,Y)Y,X)$ and $K(V,W) = g(R(V,W)W,V)$ are strictly positive, or still if for every unit tangent vector X one have

$$g(\nabla h, X)^2 + hessh(X,X) < 0. \tag{5}$$

Finally, Since M is a compact manifold, the smooth function $h : \mathbb{S}^2 \times \mathbb{S}^2 \longmapsto \mathbb{R}$ has a minimum $m = (m_1, m_2) \in \mathbb{S}^2 \times \mathbb{S}^2$. It is sufficient to show that at critical point m of h, we have $g(\nabla h, X) = 0$ and the mixed sectional curvature in (4) becomes

$$K(X, V) = -XXh. \tag{6}$$

Since the critical point m of h is a minimum then, at m we have that $XXh \geq 0$. And in this case, the sectional curvature $K(X, V)$ is not positive. This leads to a contradiction.

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Bounds on Vertex Zagreb Indices of Graphs

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Abstract- The vertex version of first and second Zagreb indices were introduced by Tavakoli et al. [2] are defined as $\overline{M}_1^*(G) = \sum_{\{u,v\} \subseteq V(G)} (d_G(u) + d_G(v))$ and $\overline{M}_2^*(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u)d_G(v)$. In this paper, we obtained the relation between the first and second vertex Zagreb indices of graphs using some well-known results.

Keywords: *degree, zagreb index, topological index.*

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Bounds on Vertex Zagreb Indices of Graphs

K. Pattabiraman [α] & M. Seenivasan [σ]

Abstract- The vertex version of first and second Zagreb indices were introduced by Tavakoli et al. [2] are defined as $\overline{M}_1^*(G) = \sum_{\{u,v\}\subseteq V(G)} (d_G(u)+d_G(v)$ and $\overline{M}_2^*(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u)d_G(v)$. In this paper, we obtained the relation between the first and second vertex Zagreb indices of graphs using some well-known results.

Keywords: degree, zagreb index, topological index.

I. Introduction

A graph can be recognized by a numeric number, a polynomial, a sequence of numbers or a matrix which represents the whole graph, and these representations are aimed to be uniquely defined for that graph. A topological index is a numeric quantity associated with a graph which characterize the topology of graph and is invariant under graph automorphism. There are some major classes of topological indices such as distance based topological indices, degree based topological indices and counting related polynomials and indices of graphs. Among these classes degree based topological indices are y and particularly in chemistry. In more precise way, a topological index $Top(G)$ of a graph G, is a number with the property that for every graph H isomorphic to G, $Top(G) = Top(H)$. The topological indices are graph invariants which has been used for examine quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR) extensively in which the biological activity or other properties of molecules are correlated with their chemical structures, see [4].

For a (molecular) graph G, The *first Zagreb index* $M_1(G)$ is the equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb index* $M_2(G)$ is the equal to the sum of the products of the degrees of pairs of adjacent vertices, that is, $M_1(G) = \sum_{u\in V(G)} d_G^2(u) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$, $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$, where $d_G(v)$ is a degree of a vertex v in G. There are various study of different versions of Zagreb indices. One of the modified versions of classical Zagreb indices, the vertex version of first and second Zagreb indices were introduced by Tavakoli et al. [2] are defined as $\overline{M}_1^*(G) = \sum_{\{u,v\}\subseteq V(G)} (d_G(u) + d_G(v))$ and $\overline{M}_2^*(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u)d_G(v)$.

Another topological index, defined as sum of cubes of degrees of all the vertices was also introduced in the same paper, where the first and second Zagreb indices were introduced [14]. Furtula and Gutman in [13] recently investigated this index and named this index as *forgotten topological index* (or) F-index and showed that the predictive ability of this index is almost similar to that of first Zagreb index and for the entropy and acetic factor, both of them yield correlation coefficients greater than 0.95. The F-index of a graph G is defined as $F = F(G) = \sum_{u\in V(G)} d_G^3(u) = \sum_{uv\in E(G)} (d_G^2(u) + d_G^2(v))$.

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For the survey on theory and application of Zagreb indices, see [11]. Feng et al.[10] have given a sharp bounds for the Zagreb indices of graphs with a given matching number. Khalifeh et al. [3] have obtained the Zagreb indices of the Cartesian product, composition, join, disjunction and symmetric difference of graphs. Ashrafi et al. [9] determined the extremal values of Zagreb coindices over some special class of graphs. Hua and Zhang [12] have given some relations between Zagreb coindices and some other topolodical indices. In [2], the vertex version of Zagreb indices for the generalized hierarchical product of two connected graphs are computed. The vertex Zagreb indices of different types of operations of graphs are computed by De [1]. In this sequence, we have obtained the relation between the first and second vertex Zagreb indices of graphs.

II. MAIN RESULTS

In this section, we present the relation between the first and second vertex Zagreb indices of graph. The *inverse degree* of a graph G is the sum of reciprocal of the vertex degrees, that is, $ID(G) = \sum\limits_{u \in V(G)} \frac{1}{d_G(u)}$.

Lemma 2.1. [15]*Let* $\overrightarrow{x} = (x_1, x_2, \ldots, x_N)$ *and* $\overrightarrow{y} = (y_1, y_2, \ldots, y_N)$ *be sequences of real numbers* $\overrightarrow{z} = (z_1, z_2, \ldots, z_N)$ *and* $\overrightarrow{w} = (w_1, w_2, \ldots, w_N)$ *be nonnegative sequences, then*

$$\sum_{i=1}^{N} w_i \sum_{i=1}^{N} z_i x_i^2 + \sum_{i=1}^{N} z_i \sum_{i=1}^{N} w_i y_i^2 \geq 2 \sum_{i=1}^{N} z_i x_i \sum_{i=1}^{N} w_i y_i. \tag{2.1}$$

In particular, if z_i *and* w_i *are positive, then the equality holds in* (2.1) *if and only if* $\overrightarrow{x} = \overrightarrow{y} = \overrightarrow{k}$, *where* $\overrightarrow{k} = (k, k, \ldots, k)$ *is a constant sequence.* ■

Theorem 2.2. Let G *be a connected graph with* n *vertices and* m *edges. Then*

$$2\overline{M}_1^*(G)\overline{M}_2^*(G) \leq (n-1)ID(G) + \frac{(n-1)\delta\Delta}{\delta + \Delta} + \frac{(n-1)(n-2)\Delta}{4} + 2mM_1(G) - F(G) \tag{2.2}$$

with equality if and only if G *is regular.*

Proof. Note that each i in Lemma 2.1 corresponds a vertex pair $\{u_i, u_j\}$ such that $N = \frac{n(n-1)}{2}$. setting $z_i = w_i = \frac{1}{x_i y_i}$ and each x_i is replaced by $\frac{1}{d_G(u_i)d_G(u_j)}$ and each y_i is replaced by $\frac{1}{d_G(u_i)+d_G(u_j)}$, then we obtain

$$\sum_{\{u_i,u_j\}\subseteq V(G)} d_G(u_i)d_G(u_j)\big(d_G(u_i)+d_G(u_j)\big) \sum_{\{u_i,u_j\}\subseteq V(G)} \left(\frac{d_G(u_i)+d_G(u_j)}{d_G(u_i)d_G(u_j)} + \frac{d_G(u_i)d_G(u_j)}{d_G(u_i)+d_G(u_j)}\right)$$
$$\geq 2 \sum_{\{u_i,u_j\}\subseteq V(G)} \big(d_G(u_i)+d_G(u_j)\big) \sum_{\{u_i,u_j\}\subseteq V(G)} d_G(u_i)d_G(u_j). \tag{2.3}$$

One can observe that $\frac{2}{\Delta} \leq \frac{1}{d_G(u_i)} + \frac{1}{d_G(u_j)} \leq \frac{2}{\delta}$, it immediately follows that $\frac{d_G(u_i)d_G(u_j)}{d_G(u_i)+d_G(u_j)} \leq \frac{\Delta}{2}$. Also $\frac{1}{\delta} + \frac{1}{\Delta} \leq \frac{1}{d_G(u_i)} + \frac{1}{\delta}$, we have $\frac{d_G(u_i)\delta}{d_G(u_i)+\delta} \leq \frac{\delta\Delta}{\delta+\Delta}$. Let v_k be the minimum degree vertex in G. Then

$$\sum_{\{u_i,u_j\}\subseteq V(G)} \left(\frac{d_G(u_i)+d_G(u_j)}{d_G(u_i)d_G(u_j)} + \frac{d_G(u_i)d_G(u_j)}{d_G(u_i)+d_G(u_j)}\right) = \sum_{\{u_i,u_j\}\subseteq V(G)} \frac{d_G(u_i)+d_G(u_j)}{d_G(u_i)d_G(u_j)} + \sum_{\{u_i,u_j\}\subseteq V(G)} \frac{d_G(u_i)d_G(u_j)}{d_G(u_i)+d_G(u_j)}$$
$$= \sum_{\{u_i,u_j\}\subseteq V(G)} \left(\frac{1}{d_G(u_i)} + \frac{1}{d_G(u_j)}\right) + \sum_{\{u_i,u_k\}\subseteq V(G)} \frac{d_G(u_i)\delta}{d_G(u_i)+\delta}$$

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$$+\sum_{\substack{\{u_i,u_j\}\subseteq V(G)\\ u_j\neq u_k}}\frac{d_G(u_i)d_G(u_j)}{d_G(u_i)+d_G(u_j)}$$

$$\leq \sum_{u_i\in V(G)}^{n}\frac{n-1}{d_G(u_i)}+\frac{(n-1)\delta\Delta}{\delta+\Delta}+\Big(\frac{n(n-1)}{2}-(n-1)\Big)\frac{\Delta}{2}$$

$$=(n-1)ID(G)+\frac{(n-1)\delta\Delta}{\delta+\Delta}+\frac{(n-1)(n-2)\Delta}{4}, \quad (2.4)$$

where $ID(G)$ is the inverse degree index of G.

Since $\sum_{u_i\in V(G)} d_G(u_i)=2m$, then we have

$$\sum_{\{u_i,u_j\}\subseteq V(G)} d_G(u_i)d_G(u_j)\big(d_G(u_i)+d_G(u_j)\big) \;\leq\; \sum_{u_i\in V(G)} d_G(u_i)^2\big(2m-d_G(u_i)\big).$$

From the definitions of first Zagreb index and F-index of G, we obtain

$$\sum_{\{u_i,u_j\}\subseteq V(G)} d_G(u_i)d_G(u_j)\big(d_G(u_i)+d_G(u_j)\big) \;\leq\; 2mM_1(G)-F(G). \quad (2.5)$$

Using (2.4) and (2.5) in (2.3), we obtain the required result in (2.2).

By Lemma 2.1, the equality in (2.2) holds if $\frac{1}{d_G(u_i)d_G(u_j)}=\frac{1}{d_G(u_i)d_G(u_k)}=\frac{1}{d_G(u_i)+d_G(u_j)}+\frac{1}{d_G(u_i)+d_G(u_k)}$ for any three vertices u_i, u_j and u_k in G. This implies G is regular. ■

Proposition 2.3. Let p be the number of pendent vertices in G. Then $F(G)\geq\delta M_1(G)+p(1-\delta)$.

Proof. Since p is the number of pendent vertices in G,

$$F(G)=\sum_{u_i\in V(G)} d_G(u_i)^3=p+\sum_{\substack{u_i\in V(G)\\ d_G(u_i)\neq 1}} d_G(u_i)^3 \;\geq\; p+\delta\sum_{\substack{u_i\in V(G)\\ d_G(u_i)\neq 1}} d_G(u_i)^2$$

$$= p+\delta(M_1(G)-p).$$

■

Using Proposition 2.3 in Theorem 2.2, we obtain the following corollary.

Corollary 2.4. Let G be a connected graph with n vertices and m edges. If p is the number of pendent vertices in G, then

$$2\overline{M}_1^*(G)\overline{M}_2^*(G)\leq(n-1)ID(G)+\frac{(n-1)\delta\Delta}{\delta+\Delta}+\frac{(n-1)(n-2)\Delta}{4}+(2m-\delta)M_1(G)+p(\delta-1)$$

with equality if and only if G is regular. ■

Lemma 2.5.(Radon's inequality) For real numbers $p>0, a_1.a_2,\dots,a_N\geq 0$ and $b_1,b_2,\dots,$ $b_N>0$, the following inequality holds:

$$\sum_{i=1}^{N}\frac{a_i^{p+1}}{b_i^{p}}\geq\frac{\Big(\sum_{i=1}^{N}a_i\Big)^{p+1}}{\Big(\sum_{i=1}^{N}b_i\Big)^{p}}.$$

■

Next we obtain the another relation between the first and second vertex Zagreb indices of graphs.

Theorem 2.6. Let G be a connected graph with n vertices and m edges. Then $\frac{\left(M_1(G)\right)^2}{M_2(G)} \leq \frac{n(n-1)(\delta+\Delta)^2}{2\delta\Delta}$ with equality if and only if G is regular.

Proof. For each i in Lemma 2.5 corresponds a vertex (u_i, u_j) with $N = \frac{n(n-1)}{2}$ and $p = 1$, setting each $a_i = d_G(u_i) + d_G(u_i)$ and $b_i = d_G(u_i)d_G(u_i)$, it follows that

$$\sum_{(u_i,u_j)\subseteq V(G)} \frac{\left(d_G(u_i) + d_G(u_i)\right)^2}{d_G(u_i)d_G(u_i)} \geq \frac{\left(\sum_{(u_i,u_j)\subseteq V(G)} d_G(u_i) + d_G(u_i)\right)^2}{\sum_{(u_i,u_j)\subseteq V(G)} d_G(u_i)d_G(u_i)}. \tag{2.6}$$

The equation (2.6) is equivalent to

$$\sum_{(u_i,u_j)\subseteq V(G)} \left(\sqrt{\frac{d_G(u_i)}{d_G(u_j)}} + \sqrt{\frac{d_G(u_j)}{d_G(u_i)}}\right)^2 \geq \frac{\left(\overline{M}_1^*(G)\right)^2}{\overline{M}_2^*(G)}.$$

It has been proved in [15] that

$$\left(\sqrt{\frac{d_G(u_i)}{d_G(u_j)}} + \sqrt{\frac{d_G(u_j)}{d_G(u_i)}}\right)^2 \leq \frac{(\delta+\Delta)^2}{\delta\Delta}.$$

Hence

$$\frac{\left(M_1(G)\right)^2}{M_2(G)} \leq \sum_{(u_i,u_j)\subseteq V(G)} \frac{(\delta+\Delta)^2}{\delta\Delta} = \frac{n(n-1)(\delta+\Delta)^2}{2\delta\Delta}. \tag{2.7}$$

Equality in (2.7) holds if and only if G is regular. ■

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A Reliable Technique for Solving Gas Dynamic Equation using Natural Homotopy Perturbation Method

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Abstract- In this paper, a new analytical technique called Natural transform homotopy perturbation method has been successfully applied to obtain exact solution of nonlinear gas dynamic equation.

Application of the method to three test modelling problems from mathematical physics lead to sequence which tends to the exact solution of the problems.

The solution procedure shows the reliability of the method and is high accuracy evident.

Keywords: gas dynamic equation, homotopy perturbation method, natural transform method.

GJSFR-F Classification: MSC 2010: 26E70

ARELIABLETECHNIQUEFORSOLVINGGASDYNAMICEQUATIONUSINGNATURALHOMOTOPYPERTURBATIONMETHOD

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Adesina. K. Adio

Abstract- In this paper, a new analytical technique called Natural transform homotopy perturbation method has been successfully applied to obtain exact solution of nonlinear gas dynamic equation.

Application of the method to three test modelling problems from mathematical physics lead to sequence which tends to the exact solution of the problems.

The solution procedure shows the reliability of the method and is high accuracy evident.

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I. Introduction

Gas dynamics is a science in the branch of fluid dynamics concerned with studying the motion of gases and its effects on physical systems, based on the principles of fluid mechanics and thermodynamics.

The science arises from the studies of gas flows, often around or within physical bodies. Examples of these studies include but not limited to choked flows in nozzles and valves, shock waves around jets, aerodynamic heating on atmospheric reentry vehicles and flow of gas fuel within a jet engine.

The equations of gas dynamics (Aminikah & Jamalian, 2013) are mathematical expressions based on the physical laws of conservation, namely, the laws of conservation of mass, conservation of momentum, conservation of energy and so forth.

The nonlinear equations of ideal gas dynamics are applicable for three types of nonlinear waves like shock fronts, rare factions and contact discontinuities.

Different types of gas dynamics equations in physics have been solved using Fourier transform A domian decomposition method (Ramezanpour, et al, 2013), homotopy analysis method (Jafari, et al, 2009), finite difference scheme (Rasulov & Karaguler, 2003), Reduced differential transform method (Keskin & Oturanc, 2010), Reconstruction of Variational Iteration method (Nikkar, 2012), Homotopy perturbation method (Jafari, et al, 2008), Variational iterative method (Jafari, et al, 2008), El-zaki transform homotopy perturbation method (Bhadane & Pradhan, 2013), Natural decomposition method (Maitama & Sabuwa, 2014), Variational homotopy perturbation method (Matinfar & Raeisi, 2011), Differential transform method, Modified homotopy perturbation method (Mohiuddin, 2015), Homotopy perturbation transform method (Singh, et al, 2012) and so on.

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In this study, we construct solution of gas dynamic equation by using a combination of Natural transform and homotopy perturbation methods (NHPM).

In one spatial dimension, the inviscid equations of gas dynamics can be written in the conservative form (Keskin & Oturanc, 2010) as:

$$v_t(x,t)-v(x,t)+\frac{1}{2}v_x^{\,2}(x,t)+v^2(x,t)=f(x,t)\,;\;\; 0\le x\le 1,\; t>0 \tag{1.1}$$

$$v(x,0)=g(x) \tag{1.2}$$

Notes

II. Analysis of the Method

For the purpose of illustration of the methodology of the proposed method, we write the gas dynamic equation in the standard operator form:

$$Dv(x,t)+Rv(x,t)+Nv(x,t)=f(x,t) \tag{2.1}$$

With the following initial conditions:

$$v(x,0)=g(x)\;\;;\;\; v_t(x,0)=h(x) \tag{2.2}$$

Where $D(v(x,t))=v_t(x,t)$ is a linear operator which has partial derivatives; $R(v(x,t))=v(x,t)$; $N(v(x,t))=\frac{1}{2}v_x^{\,2}(x,t)+v^2(x,t)$is a nonlinear term and $g(x)$ is an in homogenousterm.

Applying the Natural transform to equation (2.1) subject to the given initial condition, we have

$$N^+[Dv(x,t)]+N^+[Rv(x,t)]+N^+[Nv(x,t)]=N^+[f(x,t)] \tag{2.3}$$

using differentiation property of Natural transform and above initial conditions, we have

$$V(x,s,v)=\frac{1}{s}g(x)+\frac{v}{s^2}h(x)-\frac{v^2}{s^2}\mathrm{N}^+[RV(x,t)]-\frac{v^2}{s^2}\mathrm{N}^+[NV(x,t)]+\frac{v^2}{s^2}\mathrm{N}^+[f(x,t)] \tag{2.4}$$

Operating with the inverse Natural transform on both sides of equation (2.3), we have

$$v(x,t)=F(x,t)-\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^+[Rv(x,t)+\mathrm{N}v(x,t)]\right] \tag{2.5}$$

where $F(x,t)$ represents the term arising from the source term and the prescribed initial condition.

Now, applying the homotopy perturbation method (HPM)

$$v(x,t)=\sum_{n=0}^{\infty}P^n v_n(x,t) \tag{2.6}$$

and the nonlinear term can be decomposed as

$$Nv(x,t)=\sum_{n=0}^{\infty}P^n H_n(v) \tag{2.7}$$

where $H_n(v)$ are He's polynomials and can be evaluated using the following formula:

$$H_n(v_1, v_2, \ldots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial P^n}\left[N\sum_{j=0}^{n} P^J V_J\right]_{p=0} \quad ; \; n = 0, 1, 2, \ldots \tag{2.8}$$

Substituting equations (2.6) and (2.7) in (2.5); we have

Notes

$$\sum_{n=0}^{\infty} P^n v_n(x,t) = F(x,t) - P\left(\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[R\sum_{n=0}^{\infty} P^n v_n(x,t) + \sum_{n=0}^{\infty} P^n H_n(v)\right]\right]\right) \tag{2.9}$$

which is the coupling of the Natural transform and the homotopy perturbation method using

He's polynomials.

Comparing the coefficient of same powers of P, we obtain the following approximations:

$$\begin{aligned}
&P^0 : v_0(x,t) = F(x,t)\\
&P^1 : v_1(x,t) = -\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[Rv_0(x,t) + H_0(v)\right]\right]\\
&P^2 : v_2(x,t) = -\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[Rv_1(x,t) + H_1(v)\right]\right]\\
&P^3 : v_3(x,t) = -\mathrm{N}^{-1}\left[\frac{v^2}{s^2}\mathrm{N}^{+}\left[Rv_2(x,t)\right] + H_2(v)\right]
\end{aligned} \tag{2.10}$$

and so on.

Thus, the series solution of equation (2.1) is

$$v(x,t) = \lim_{k\to\infty}\sum_{n=0}^{k} v_n(x,t) \tag{2.11}$$

III. Experimental Evaluation

In this section; we consider the following nonlinear homogenous and nonhomogenous gas dynamics equations:

Example 3.1: Consider the nonlinear homogenous gas dynamicequation [3-7]:

$$v_t(x,t) + \frac{1}{2}v^2{}_x(x,t) - v(x,t) + v^2(x,t) = 0 \tag{3.1}$$

with initial condition $$v(x,0) = g(x) = e^{-x} \tag{3.2}$$

applying the natural transform on both sides of equation (3.1) subject to the initial condition (3.2), we have

$$v(x,s) = \frac{e^{-x}}{s} + \frac{v}{s}\mathrm{N}^{+}\left[v(1-v) - \frac{1}{2}(v^2)_x\right] \tag{3.3}$$

The inverse of natural transform implies that

$$v(x\ t)=e^{-x}+\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v(1-v)-\frac{1}{2}\left(v^{2}\right)_{x}\right]\right] \tag{3.4}$$

Now, we apply the homotopy perturbation method to get

$$\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)=e^{-x}+P\left(\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)\left(1-\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)\right)-\frac{1}{2}\left(\left(\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)\right)^{2}\right)_{x}\right]\right]\right) \tag{3.5}$$

Comparing the coefficients of like powers of P in eqn (3.5); we obtain the following approximations.

$$P^{0}:\qquad v_{0}(x,t)=e^{-x}$$

$$P^{1}:\qquad v_{1}(x,t)=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_{0}(1-v_{0})-\frac{1}{2}\left(v_{0}{}^{2}\right)_{x}\right]\right]=e^{-x}t$$

$$P^{2}:\qquad v_{2}(x,t)=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_{1}(1-v_{1})-\frac{1}{2}\left(v_{1}{}^{2}\right)_{x}\right]\right]=e^{-x}\frac{t^{2}}{2!}$$

$$P^{3}:\qquad v_{3}(x,t)=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_{2}(1-v_{2})-\frac{1}{2}\left(v_{2}{}^{2}\right)_{x}\right]\right]=e^{-x}\frac{t^{3}}{3!}$$

and so on.

Therefore, the solution $v(x,t)$ is given by

$$v(x,t)=e^{-x}\left(1+t+\frac{t^{2}}{2!}+\frac{t^{3}}{3!}+\frac{t^{4}}{4!}+\quad...\right)=e^{t-x} \tag{3.6}$$

Obtained upon using the Taylor expansion for e^{t}. Higher number of iterations lead to convergence to the exact solution e^{t-x}. This result is similar to what obtained using HPTM [16], RDTM [7], VIM [5], RVIM [7], ADM [13], ETHPM [2], HPM [4], VHPM [10], MHPM [11] and NDM[9].

Example 3.2: Consider the nonlinear, nonhomogenous gas dynamic equation [2,12,13]

$$v_{t}(x,t)+\frac{1}{2}v_{x}{}^{2}(x,t)-v(x,t)+v^{2}(x,t)=-e^{t-x} \tag{3.7}$$

with initial condition $$v(x,0)=g(x)=1-e^{-x} \tag{3.8}$$

Applying the natural transform on both sides of equation (3.7) subject to the initial condition (3.8), we have

$$v(x,s)=\frac{1-e^{-x}}{s}-\frac{ve^{-x}}{s(s-v)}+\frac{v}{s}\left[\mathrm{N}^{+}\left[v(1-v)-\frac{1}{2}\left(v^{2}\right)_{x}\right]\right] \tag{3.9}$$

The inverse of Natural transform implies that:

Ref

4. Jafari, H.,Zabihi, M., Saidy, M., 2008.Application of homotopy perturbation method for solving gas dynamic equation. Applied Mathematical Sciences. 2, 2393-2396.

$$v(x,t)=\left(1-e^{t-x}\right)+\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v(1-v)-\frac{1}{2}\left(v^{2}\right)_{x}\right]\right] \quad (3.10)$$

Since $$\mathrm{N}^{-1}\left[\frac{ve^{-x}}{s(s-v)}\right]=e^{-x}\,\mathrm{N}^{-1}\left[\frac{v}{s(s-v)}\right]=e^{-x}\left(e^{t}-1\right)$$

Now, we apply the homotopy perturbation method to get:

$$\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)=\left(1-e^{t-x}\right)+P\left(\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)\left(1-\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)\right)-\frac{1}{2}\left(\left(\sum_{n=0}^{\infty}P^{n}v_{n}(x,t)\right)^{2}\right)_{x}\right]\right]\right) \quad (3.11)$$

Comparing the coefficients of like powers of P in eqn (3.11), we obtain the following approximations:

$$P^{0}: v_{0}(x,t)=1-e^{t-x}$$

$$P^{1}: v_{1}(x,t)=\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_{0}(1-v_{0})-\frac{1}{2}\left(v_{0}^{\,2}\right)_{x}\right]\right]=0$$

$$P^{2}: v_{2}(x,t)=N^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_{1}(1-v_{1})-\frac{1}{2}\left(v_{1}^{\,2}\right)_{x}\right]\right]=0$$

$$P^{3}: v_{3}(x,t)=N^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_{2}(1-v_{2})-\frac{1}{2}\left(v_{2}^{\,2}\right)_{x}\right]\right]=0$$

and so on.
Therefore, the solution $v(x,t)$ is given by

$$v(x,t)=1-e^{t-x} \quad (3.12)$$

Similar to results obtained using VIM [5], RVIM [12], ADM [13], ETHPM [2], VHPM [10]and NDM [9].

Example 3.3: Consider the non-homogenous, nonlinear gas dynamic equation [1,3]

$$v_{t}(x,t)+v(x,t)v_{x}(x,t)-v(x,t)+v^{2}(x,t)=-e^{t-x} \quad (3.13)$$

with initial condition $$v(x,0)=1-e^{-x} \quad (3.14)$$

Applying the natural transform on both sides of eqn (3.13) subject to the initial condition (3.14)we have

$$v(x,s)=\frac{1-e^{-x}}{s}-\frac{ve^{-x}}{s(s-v)}+\frac{v}{s}\mathrm{N}^{+}\left[v(1-v)-vv_{x}\right] \quad (3.15)$$

The inverse of Natural transform implies that

$$v(x,t)=1-e^{t-x}+\mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v(1-v)-vv_{x}\right]\right] \quad (3.16)$$

Ref

12. Nikkar, A., 2012. A new approach for solving gas dynamic equation. Acta Technica Corviniensis Bulletin of Engineering. 4, 113-116.

Now, we apply the homotopy perturbation method to get:

$$\sum_{n=0}^{\infty} P^n v_n(x,t) = 1 - e^{t-x} + P\left(N^{-1}\left[\frac{v}{s}N^{+}\left[\sum_{n=0}^{\infty} P^n v_n(x,t)\left(1-\sum_{n=0}^{\infty} P^n v_n(x,t)\right) - \sum_{n=0}^{\infty} P^n v_n(x,t)\left(\sum_{n=0}^{\infty} P^n v_n(x,t)\right)_x\right]\right]\right) \tag{3.17}$$

Comparing the coefficients of like powers of P in eqn (3.17), we obtain the following approximations

$$P^0 : v_0(x,t) = 1 - e^{t-x}$$

$$P^1 : v_1(x,t) = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_0(1-v_0) - v_0 v_{0x}\right]\right] = 0$$

$$P^2 : v_2(x,t) = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_1(1-v_1) - v_1 v_{1x}\right]\right] = 0$$

$$P^3 : v_3(x,t) = \mathrm{N}^{-1}\left[\frac{v}{s}\mathrm{N}^{+}\left[v_2(1-v_2) - v_2 v_{2x}\right]\right] = 0$$

and so on.

Therefore, the solution $v(x,t)$ is given by

$$v(x,t) = 1 - e^{t-x} \tag{3.18}$$

Which is the exact solution of the problem.

IV. Conclusion

In the present work, we proposed a combination of Natural transform and Homotopy perturbation methods and successfully applied it to study the homogenous and non-homogenous cases of nonlinear gas dynamics equation.

The method gave closed form solution of the equations, with high accuracy, using the initial conditions and can be considered as a reliable refinement of existing techniques.

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Notes

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Asymptotic Solutions of Second Order Equations with Holomorphic Coefficients with Degeneracies and Laplace's Equations on A Manifold with A Cuspidal Singularity

By M. V. Korovina

Abstract- In this paper, we construct the asymptotics for second order linear differential equations with higher-order singularity for the case where the principle symbol has multiple roots. In addition, we solve the problem of constructing asymptotic solutions of Laplace's equation on a manifold with a second order cuspidal singularity.

Keywords: differential equations with cuspidal, singularitus, laplas-borel transformation, resurgent function, laplace's equation.

GJSFR-F Classification: MSC 2010: 44A10

ASYMPTOTICSOLUTIONSOFSECONDORDEREQUATIONSWITHHOLOMORPHICCOEFFICIENTSWITHDEGENERACIESANDLAPLACESEQUATIONSONAMANIFOLDWITHACUSPIDALSINGULARITY

Strictly as per the compliance and regulations of:

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Asymptotic Solutions of Second Order Equations with Holomorphic Coefficients with Degeneracies and Laplace's Equations on a Manifold with a Cuspidal Singularity

M. V. Korovina

Abstract- In this paper, we construct the asymptotics for second order linear differential equations with higher-order singularity for the case where the principle symbol has multiple roots. In addition, we solve the problem of constructing asymptotic solutions of Laplace's equation on a manifold with a second order cuspidal singularity.

Keywords: differential equations with cuspidal, singularitus, laplas-borel transformation, resurgent function, laplace's equation.

I. Introduction

This paper is devoted to asymptotic expansions for solutions to equations with higher-order degeneracies, namely, to equations of the form

$$H\left(r, -\frac{1}{k} r^{k+1}\frac{d}{dr}, x, -i\frac{\partial}{\partial x}\right)u = 0 \tag{1}$$

where

$$H\left(r, r^{k+1}\frac{d}{dr}, x, -i\frac{\partial}{\partial x}\right) = \sum_{j=0}^{2}\sum_{l=1}^{2} a_{jl}(x,r)\left(-i\frac{\partial}{\partial x}\right)^{j}\left(r^{k+1}\frac{d}{dr}\right)^{l}$$

$k-$integer non-negative number, $a_{jl}(x,r)$ holomorphic coefficients in the neighbourhood of zero in variable r. Here $r \in C$ and x belongs to a compact manifold without edge. Such equations are referred to as equations with cuspidal singularitus of order *k+1*, *for* $k=0$, such singularitus are said to be conical. The case of conical singularitus was studied dy Kondratev in [1]. Here we consider the case of cuspidal singularitus. Note that any linear differential equations of second order with holomorphic coefficients with singularitus in one of the variables is representable in the form (1). Laplace's equation on a manifold with cuspidal singularity is a typical example of such an equation.

In the first part of the article we construct asymptotic solutions of ordinary differential equations of second order with coefficients $a_i(r), i=0,1,2$ which is holomorphic in some neighborhood of the point $r=0$

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$$a_2(r)\left(\frac{d}{dr}\right)^2 u(r)+a_1(r)\left(\frac{d}{dr}\right)u(r)+a_0(r)u(r)=0. \tag{2}$$

$a_i(r), i=0,1,2$ are is holomorphic in some neighborhood of the point $r=0$

In paper [5] received asymptotic solutions of equation

$$H\left(r,-r^2\frac{d}{dr}\right)u=0, \tag{3}$$

where $H(r,p)=\sum_{i=0}^{n}a_i(r)p^i$, with the roots of principle operator symbol $H_0(p)=H(0,p)$ being simple. The asymptotic solutions of the above equation have the form

$$u=\sum_{i=1}^{n}e^{\alpha_i/r}r^{\sigma_i}\sum_{k=0}^{\infty}a_i^k r^k \tag{4}$$

where $\alpha_i, i=1,\ldots,n$-are the roots of $H(0,p)$ and σ_i and a_i^k –are some complex numbers. However, if the asymptotic expansion has at last two terms corresponding to values α_1 and α_2 with distinct real parts (to be definite, we assume that $\mathrm{Re}\,\alpha_1>\mathrm{Re}\,\alpha_2$), then it becomes quite difficult to interpret the rights hand part of (4). The point is that all terms of the ftrst element corresponding to the value α_1 (the dominant component) have a higher order as $r\to 0$ then any term of the second (the recessive element). If the argument r moves in the complex plane, then the role of the components can be changed. Therefore, to interpret the expansion (4), one should sum the (not necessarily convergent) series (3), the analysis of asymptotic expansions of solutions of equations (1) requires the introduction of regular summation method for divergent series for the construction of uniform asymptotic expansions of solutions with respect to the variable r.

In paper [2] and [5] author examined the conditions of infinite continuable for Laplace-Borel k-transforms of solutions to these equations and proved their continuability along any path on the Riemann surface not passing through a certain discrete set of points depending on the function, the exact definition of resurgent function is given in below.

Based on the concept of resurgent function first introduced by J. Ecalle [3], apparatus for summing expressions of the form (4), based on the Borel-Laplace transformation is called resurgent analysis. The fundamentals of resurgent analysis and of the Borel-Laplace transform are based on can be found in [4].

In articles [7], [8] asymptotic solutions of equations

$$\left(\frac{d}{dx}\right)^2 u(x)+a_1(x)\left(\frac{d}{dx}\right)u(x)+a_0(x)u(x)=0$$

are constructed in the neighborhood of infinity, provided that the coefficients $a_i(x)$ are holomorphic in the neighborhood of infinity. This equation is reduced to the second order equations with cuspidal singularitus in the neighborhood of the point $r=0$, by substituting $x=\frac{1}{r}$, which is a private case of tasks which are considering in that paper, that is equations of any order singularitus.

Ref

2. Korovina M. V., Existence of Resurgent Solutions for Equatons with Higher –Order Degenerations, Differ. Uravn, 2011, vol. 47, no.3, pp. 349-357.

In the second part of the paper we consider partial differential equations with cuspidal singularitus. As an example we construct asymptotic solutions of Laplace's equation on a manifold with a second order caspidal singularity.

The paper continues the research into asymptotic behaviour of solutions to equations with singularit carried out in a series of articles [2], [5], [6] and so on.

In [5], asymptotic expansions for solutions to equations of type (1) are constructed for $k=1$, and, in [6], they are constructed for $k>1$ if the roots of the principle operator symbol are simple. The problem of asymptotic solutions in the case of multiple roots is much more complicated and is still an open problem. This paper proposes a method for obtaining asymptotic solutions of second order equations with higher-order singularity in the case of multiple roots. The method is also applicable to some types of higher-order ordinary or partial differential equations.

a) Basic definitions

In this section we introduce some notions of resurgent analysis for further use.

Let $S_{R,\varepsilon}$ denote the sector $S_{R,\varepsilon}=\{r \,|\, -\varepsilon<\arg r<\varepsilon, |r|<R\}$. We say that the function f analytic at $S_{R,\varepsilon}$ has at most k-exponential growth if there exist nonnegative constants C and α such that the inequality

$$|f|<Ce^{a\frac{1}{|r|^k}}$$

holds in the sector $S_{R,\varepsilon}$.

Let $E_k(S_{R,\varepsilon})$ denote the space of holomorphic functions of k-exponential growth, and let $E(\tilde{\Omega}_{R,\varepsilon})$ denote the space of holomorphic functions of exponential growth in $\tilde{\Omega}_{R,\varepsilon}$.The domain $\tilde{\Omega}_{R,\varepsilon}$ is shown in Fig1 . $E(C)$ will denote the space of entire functions of exponential growth.

The Laplace-Borel k-transform of the function $f(r)\in E_k(S_{R,\varepsilon})$ is given by

$$B_k f=\int_0^{r_0} e^{-p/r^k} f(r)\frac{dr}{r^{k+1}}.$$

We can show that $B_k : E_k(S_{R,\varepsilon})\to \mathrm{E}(\tilde{\Omega}_{R,\varepsilon})/\mathrm{E(C)}$. The inverse -transform is defined by

$$B_k^{-1}\tilde{f}=\frac{k}{2\pi i}\int_{\tilde{\gamma}} e^{p/r^k}\tilde{f}(p)\,dp.$$

where $\tilde{\gamma}$ is shown in Fig. 1.

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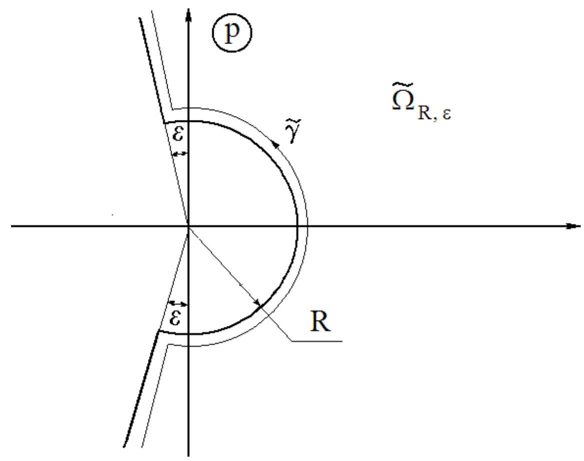

Fig. 1: The Laplace-Borel transform domain of holomorphism and the reverse transform calculation

We can now give the definition of a k-resurgent function.

Definition 1. The function $\widetilde{f}$ is called k-endlessly continuable, if for any R there exists a discrete set of points Z_R in C such that the function $\widetilde{f}$ can be analytically continued from the initial domain along any path of length$<$R not passing through Z_R.

Definition 2. The element f of the space $E_k(S_{R,\varepsilon})$ is called a k-resurgent function, if its Borel k-transform $\widetilde{f} = B_k f$ is endlessly continuable.

II. Asymptotic Solutions of Ordinary Differential Equations

In this section we consider second-order homogeneous ordinary differential equations with cuspidal singularitus, that is the equations of the form

$$H(r, \frac{1}{n} r^{n+1} \frac{d}{dr}) = 0,$$

where the symbol $H(r,p)$ is second-order polynomial in p with holomorphic coefficients. In [2] it is proved that solutions of these equations are resurgent functions as formulated in Definition 2. Here we assume that the principle symbol $H_0(p) = H(0,p)$ has a multiple root. In the case of simple roots of a principle symbol such equations are discussed in [6]. In other words, we consider the equations of the form

$$\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right)^2 u + a_1(r)\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right) u + a_0(r) u + r v(r)\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right)^2 u = 0, \qquad (5)$$

where $a_i(r)$ are holomorphic coefficients.

$$a_1(r) = b r^k + r^{k+1} b_1(r),$$
$$a_0(r) = c r^p + r^{p+1} c_1(r)$$

Here $b_1(r), c_1(r)$ are holomorphic functions, with non-zero c and *b.*

Thus we rewrite equation (5) as

Ref

6. Korovina M. V. Asymptotic Solutions for Equatons with Higher – Order Degenerations, Doklady Mathematics Vol. 437, No. 3, 2011 p. 302-304.

$$\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)^2 u + br^k\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)u + cr^p u + r^{k+1}b_1(r)\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)u +$$
$$+ r^{p+1}c_1(r)u + rv(r)\left(\frac{1}{n}r^{n+1}\frac{d}{dr}\right)^2 u = 0. \tag{6}$$

Since for any $m: 0 < m < n+1$ the relation

$$\left(r^{n+1}\frac{d}{dr}\right)^2 = mr^{n+m}\left(r^{n-m+1}\frac{d}{dr}\right) + r^{2m}\left(r^{n-m+1}\frac{d}{dr}\right)^2,$$

holds, then equation (6) can be rewritten in the form

$$r^{2m}\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)^2 u + \frac{m}{n}r^{n+m}\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u + br^{k+m}\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u + cr^p u +$$
$$+ r^{k+1+m}b_1(r)\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u + r^{p+1}c_1(r)u + r^{2m+1}v(r)\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)^2 u + \frac{m}{n}r^{m+n+1}v(r)\left(\frac{1}{n}r^{n-m+1}\frac{d}{dr}\right)u = 0 \tag{7}$$

The following two cases are considered.

1. $\frac{p}{2} \geq n$ and $k \geq n$.

   This is the simplest case. We set $m = n$, then equation (7) has the form

$$r^{2n}\left(\frac{1}{n}r\frac{d}{dr}\right)^2 u + r^{2n}\left(\frac{1}{n}r\frac{d}{dr}\right)u + br^{k+n}\left(\frac{1}{n}r\frac{d}{dr}\right)u + cr^p u +$$
$$+ r^{k+n+1}b_1(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u + r^{p+1}c_1(r)u + r^{2n+1}v(r)\left(\frac{1}{n}r\frac{d}{dr}\right)^2 u + r^{2n+1}v(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u = 0 \tag{8}$$

Dividing (8) by r^{2n}, we obtain the equation

$$\left(\frac{1}{n}r\frac{d}{dr}\right)^2 u + \left(\frac{1}{n}r\frac{d}{dr}\right)u + br^{k-n}\left(\frac{1}{n}r\frac{d}{dr}\right)u + cr^{p-2n}u +$$
$$+ r^{k-n+1}b_1(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u + r^{p-2n+1}c_1(r)u + rv(r)\left(\frac{1}{n}r\frac{d}{dr}\right)^2 u + +rv(r)\left(\frac{1}{n}r\frac{d}{dr}\right)u = 0$$

This is an equation with a conic singularity. In other words, it is not the case of a cuspidal singularity. This case is have been studied extensively (e.g. [1]). It is reasonable to search for the solutions of these equations in weighted Sobolev spaces $H^{k,\sigma}(0,\infty)$. For nonhomogeneous equations a solution can be represented in the form of conormal asymptotics. For homogeneous equations, it is identically equal to zero.

2. Let $\frac{p}{2} < n$ or $k < n$, and if $k < \frac{p}{2}$, then, similarly to the conic case, we set $m = k$ in (7). Divide the equation (7) by r^{2k} and multiplying by $\left(\frac{n}{n-k}\right)^2$ we obtain the equation

Ref

1. Kondratev V.A. Boundary Value Problems for Elliptic Equations in Domains with Conical or Angular Points, Tr. Mosk. Mat. Obs.,1967, vol,16, pp 209-292.

$$\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)^2 u+\frac{n}{n-k}r^{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+b\frac{n}{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+c\frac{n^2}{(n-k)^2}r^{p-2k}u+$$

$$+r\,b_1(r)\frac{n}{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+r^{p+1-2k}\left(\frac{n}{n-k}\right)^2 c_1(r)u+$$

$$+r\,v(r)\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)^2 u+\frac{n}{n-k}r\,v(r)\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)^2 u=0$$

The principle symbol in this case is equal to $H_0(p)=p^2+b\dfrac{n}{n-k}p$, and has simple roots. In [6], the solution of this equation is shown to belong to the space $E_{n-k}(S_{R,\varepsilon})$. Asymptotic solutions for this equation has the form

$$r^{\sigma_0}v_0(r)\exp\left(\sum_{i=1}^{n-k-1}\frac{\alpha^0{}_{n-k-1}}{r^{n-k-i}}\right)+r^{\sigma_1}v_1(r)\exp\left(-\frac{bn}{(n-k)r^{n-k}}+\sum_{i=`1}^{n-k-1}\frac{\alpha^1{}_{k-i}}{r^{n-k-i}}\right).$$

In this case $v_i(r), i=0,1$ denotes,in general, divergent series

$$v_i(r)=\sum_{j=1}^{\infty}v_j^i r^j, i=0,1$$

and $\sigma_i(r), i=0,1$ and $\alpha^j{}_i$ denote corresponding numbers. The method to calculate them is outlined in [6].

Now assume that $\frac{p}{2}<k$ then, we set $m=\frac{p}{2}$, in equation (7) and divide it by r^p, thus obtain the equation

$$\left(\frac{1}{n}r^{n-\frac{p}{2}+1}\frac{d}{dr}\right)^2 u+\frac{p}{2n}r^{n-\frac{p}{2}}\left(\frac{1}{n}r^{n-\frac{p}{2}+1}\frac{d}{dr}\right)u+br^{k-\frac{p}{2}}\left(\frac{1}{n}r^{n-\frac{p}{2}+1}\frac{d}{dr}\right)u+cu+$$

$$+r^{k+1-\frac{p}{2}}b_1(r)\left(\frac{1}{n}r^{n-\frac{p}{2}+1}\frac{d}{dr}\right)u+r\,c_1(r)u+rv(r)\left(\frac{1}{n}r^{n-\frac{p}{2}+1}\frac{d}{dr}\right)^2 u+\frac{p}{2n}r^{n+1-q}v(r)\left(\frac{1}{n-q}r^{n-\frac{p}{2}+1}\frac{d}{dr}\right)u=0 \tag{9}$$

If $p=2q$ is an even number, then the equation takes the form

$$\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)^2 u+\frac{q}{n-q}r^{n-q}\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)u+$$

$$+br^{k-q}\frac{n}{n-q}\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)u+c\left(\frac{n}{n-q}\right)^2 u+$$

$$+r^{k+1-q}b_1(r)\frac{n}{n-q}\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)u+r\left(\frac{n}{n-q}\right)^2 c_1(r)u+$$

$$+rv(r)\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)^2 u+\frac{q}{n-q}r^{n-q+1}v(r)\left(\frac{1}{n-q}r^{n-q+1}\frac{d}{dr}\right)u=0$$

Ref

6. Korovina M. V. Asymptotic Solutions for Equations with Higher – Order Degenerations, Doklady Mathematics Vol. 437, No. 3, 2011 p. 302-304.

Then the symbol is equal to $H_0(p) = p^2 + \left(\frac{n}{n-q}\right)^2 c$. Its roots are simple. The solution of this equation belongs to the space $E_{n-q}(S_{R,\varepsilon})$. In this case, as shown in the article [6], the asymptotics has the form

$$r^{\sigma_1} v_1(r) \exp\left(-\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}} + \sum_{i=1}^{n-q-1} \frac{\alpha^1{}_{n-q-i}}{r^{n-q-i}} \right) + r^{\sigma_2} v_0(r) \exp\left(\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}} + \sum_{i=1}^{n-q-1} \frac{\alpha^2{}_{n-q-i}}{r^{n-q-i}} \right).$$

If p is odd, we make the change (of) $x = r^{\frac{1}{2}}$ and by substituting

$$r^{n-\frac{p}{2}+1}\frac{d}{dr} = \frac{1}{2}\left(x^{2n-p+1}\frac{du}{dx} \right)$$

into equation (9) we obtain

$$\left(\frac{1}{2n-p} x^{2n+1-p} \frac{d}{dx} \right)^2 u + \frac{p}{2n-p} x^{2n-p} \left(\frac{1}{2n-p} x^{2n+1-p} \frac{d}{dx} \right) u + b x^{2k-p} \frac{2n}{2n-p} \left(\frac{1}{2n-p} x^{2n+1-p} \frac{d}{dx} \right) u +$$

$$+ \left(\frac{2n}{2n-p} \right)^2 cu + x^{2k+2-p} b_1(x^2) \frac{2n}{2n-p} \left(\frac{1}{2n-p} x^{2n+1-p} \frac{d}{dx} \right) u +$$

$$+ x^2 \left(\frac{2n}{2n-p} \right)^2 c_1(x^2) u + x^2 v(x^2) \left(\frac{1}{2n-p} x^{2n+1-p} \frac{d}{dx} \right)^2 u + \frac{2p}{2n-p} x^{2n+2-p} v(x^2) \left(\frac{1}{2n-p} x^{2n+1-p} \frac{d}{dx} \right)^2 u = 0$$

In this case the principle symbol is equal to $H_0(p) = p^2 + \left(\frac{2n}{2n-p}\right)^2 c$. The asymptotic solution of the last equation has the form

$$\exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p)x^{2n-p}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^1{}_i}{x^{2n-p-i}} \right) x^{\sigma_1} v_0(x) + \exp\left(\frac{2\pi i\sqrt{c}}{(2n-p)x^{2n-p}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^2{}_{ii}}{x^{2n-p-i}} \right) x^{\sigma_2} v_1(x) =$$

$$= \exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^1{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}} \right) r^{\frac{\sigma_1}{2}} v_0\left(r^{\frac{1}{2}} \right) + \exp\left(\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^2{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}} \right) r^{\frac{\sigma_2}{2}} v_1\left(r^{\frac{1}{2}} \right)$$

Since

$$v_1\left(r^{\frac{1}{2}} \right) = \sum_{i=1}^{\infty} v_i^1 r^{\frac{i}{2}} = \sum_{i=1}^{\infty} v_{2i}^1\, r^i + \sum_{i=0}^{\infty} v_{2i+1}^1\, r^{\frac{i+1}{2}} = \sum_{i=1}^{\infty} v^1{}_{2i} r^i + r^{\frac{1}{2}} \sum_{i=0}^{\infty} v^1{}_{2i+1} r^i,$$

then the asymptotic solution can be rewritten in the form

Ref

6. Korovina M. V. Asymptotic Solutions for Equatons with Higher – Order Degenerations, Doklady Mathematics Vol. 437, No. 3, 2011 p. 302-304.

$$\exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^1{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right)r^{\frac{\sigma_1}{2}}\left(\tilde{v}_1^1(r)+r^{\frac{1}{2}}\tilde{v}_1^2(r)\right)+$$

$$+\exp\left(\frac{2\pi i\sqrt{c}}{(2n-p)r^{n-\frac{p}{2}}}+\sum_{i=1}^{2n-p-1}\frac{\alpha^2{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right)r^{\frac{\sigma_2}{2}}\left(\tilde{v}_2^1(r)+r^{\frac{1}{2}}\tilde{v}_2^2(r)\right).$$

Here $\tilde{v}_j^1=\sum_{i=1}^{\infty}v_{2i}^jr^i, \tilde{v}_j^2=\sum_{i=1}^{\infty}v_{2i+1}^jr^i,\ j=1,2.$ In this case we have obtained a new asymptotic type, namely the asymptotics with nonintegral degrees r in their exponents.

Let us consider the last case. Suppose $k=\frac{p}{2}<n$. We set $m=k=\frac{p}{2}$ and dividing by r^{2m} we obtain and multiplying on $\left(\frac{n}{n-k}\right)^2$ then

$$\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)^2u+\frac{n}{n-k}r^{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+b\frac{n}{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+\left(\frac{n}{n-k}\right)^2cu+$$

$$+rb_1(r)\frac{n}{n-k}\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u+r\left(\frac{n}{n-k}\right)^2c_1(r)u+$$

$$+rv(r)\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)^2u+\frac{n}{n-k}r^{n-k+1}v(r)\left(\frac{1}{n-k}r^{n-k+1}\frac{d}{dr}\right)u=0$$

The principle symbol in this case has the form $H_0(p)=p^2+b\frac{n}{n-k}p+c\left(\frac{n}{n-k}\right)^2$. If $b\neq 2\sqrt{c}$, then the polynomial $p^2+b\frac{n}{n-k}p+c\left(\frac{n}{n-k}\right)^2$ has two roots c_1, c_2 and the asymptotics has the form

$$r^{\sigma_0}v_0(r)\exp\left(\frac{c_1}{r^{n-k}}+\sum_{i=1}^{n-k-1}\frac{\alpha^0{}_i}{r^{n-k-i}}\right)+r^{\sigma_1}v_1(r)\exp\left(\frac{c_2}{r^{n-k}}+\sum_{i=1}^{n-k-1}\frac{\alpha^1{}_i}{r^{n-k-i}}\right)$$

If $b=2\sqrt{c}$, namely $p^2+b\frac{n}{n-k}p+c\left(\frac{n}{n-k}\right)^2$, then by replacing $\tilde{u}=e^{\frac{n}{n-k}\frac{\sqrt{c}}{r}}u$ we obtain the equation with a multiple root at zero. The asymptotic solution of this equation is constructed in the same way as is shown above and depends on the degrees of degeneracy of the functions $b_1(r), c_1(r)$.

Thus we have cuspidal degeneracies in the case where $\frac{p}{2}<n$ or $k<n$. The obtained results are written in the table

Notes

Notes

Theorem 1. Let $\frac{p}{2} < n$ or $k < n$ then

Table 1

Conditions	Space	Asymptotics
$k < \frac{p}{2}$	$u \in E_{n-k}\left(S_{R,\varepsilon}\right)$	$r^{\sigma_0} v_0(r) \exp\left(\sum_{i=1}^{n-k-1} \frac{\alpha^0{}_{n-k-1}}{r^{n-k-i}}\right) +$ $+ r^{\sigma_1} v_1(r) \exp\left(-\frac{bn}{r^{n-k}(n-k)} + \sum_{i=1}^{n-k-1} \frac{\alpha^1{}_{k-i}}{r^{n-k-i}}\right)$
$k > \frac{p}{2}$, $p = 2q, q \in N$	$u \in E_{n-q}\left(S_{R,\varepsilon}\right)$	$r^{\sigma_1} v_0(r) \exp\left(-\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}} + \sum_{i=1}^{n-q-1} \frac{\alpha^1{}_{n-q-i}}{r^{n-q-i}}\right) +$ $+ r^{\sigma_2} v_1(r) \exp\left(\frac{i\frac{n}{n-q}\sqrt{c}}{r^{n-q}} + \sum_{i=1}^{n-q-1} \frac{\alpha^2{}_{n-q-i}}{r^{n-q-i}}\right)$
$k > \frac{p}{2}$, p is odd	$u \in E_{n-\frac{p}{2}-1}\left(S_{R,\varepsilon}\right)$	$\exp\left(-\frac{2\pi i\sqrt{c}}{(2n-p) r^{n-\frac{p}{2}}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^1{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right) r^{\frac{\sigma_1}{2}} \left(\tilde{v}_1^1(r) + r^{\frac{1}{2}} \tilde{v}_1^2(r)\right) +$ $+ \exp\left(\frac{2\pi i\sqrt{c}}{(2n-p) r^{n-\frac{p}{2}}} + \sum_{i=1}^{2n-p-1} \frac{\alpha^2{}_i}{r^{n-\frac{p}{2}-\frac{i}{2}}}\right) r^{\frac{\sigma_2}{2}} \left(\tilde{v}_2^1(r) + r^{\frac{1}{2}} \tilde{v}_2^2(r)\right)$
$k = \frac{p}{2}$ and $b \neq 2\sqrt{c}$	$u \in E_{n-k-1}\left(S_{R,\varepsilon}\right)$	$r^{\sigma_0} v_0(r) \exp\left(\frac{c_1}{r^{n-k}} + \sum_{i=}^{n-k-1} \frac{\alpha^0{}_{n-k-1}}{r^{n-k-i}}\right) +$ $+ r^{\sigma_1} v_1(r) \exp\left(\frac{c_2}{r^{n-k}} + \sum_{i=2}^{n-k-1} \frac{\alpha^1{}_{k-i}}{r^{n-k-i}}\right)$

Now we proceed to examine the higher-order equations

$$\left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right)^k u + b_{k-1}(r) \left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right)^{k-1} u + \ldots + b_1(r) \left(\frac{1}{n} r^{n+1} \frac{d}{dr}\right) u + b_0(r) u = 0$$

where $b_i(r) = \sum_{j=m_i}^{\infty} c_i^j r^j, i = 0, \ldots, k-1$ are entire functions. We assume that $c_i^{m_i} \neq 0$. The number m_i will be called a degree of degeneracy of the coefficient $b_i(r)$. Suppose that zero is the root of this equation principle symbol and the degree of degeneracy of the coefficients $b_1(r)$ or $b_0(r)$ is no more than the degree of degeneracy for any of the coefficients $b_i(r)$, $i = 2, \ldots, k-1$. The above method is also applicable to this case. Specifically, the equation is to be divided by r^p, where $p = \min(m_0, m_1)$, then we have

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the equation with a symbol of the form $H\left(r, r^{n-\frac{p}{k}}\frac{d}{dr}\right)$, which is solved similarly to the previous one.

III. Partial Differential Equations

We consider a second-order partial differential equation with holomorphic coefficients. It can be represented in the form

$$H\left(r, -\frac{1}{k}r^{k+1}\frac{d}{dr}, x, -i\frac{\partial}{\partial x}\right)u = 0,$$

where x varies on some compact manifold without boundary. These equations can be interpreted as equations with respect to the functions with values in Banach spaces, namely

$$\hat{H}\left(r, -\frac{1}{k}r^{k+1}\frac{d}{dr}\right)u = 0, \tag{10}$$

where $\hat{H}: E_k\left(S_{R,\varepsilon}, B_1\right) \to E_k\left(S_{R,\varepsilon}, B_2\right)$. Here B_i, $i = 1,2$ denote some Banach spaces (e.g. the space $H^s(\Omega)$). The degree k of the function u grows exponentially for $r \to 0$, for fixed r and p

$$\hat{H}(r, p): B_1 \to B_2$$

is a bounded operator acting in Banach spaces which is polynomial dependent on p and holomorphic in the neighbourhood of zero.

In what follows, we will assume that the operator family $\hat{H}_0(p) = \hat{H}(0, p)$ is a *Fredholm* family. This implies that the operator $\hat{H}_0(p)$ is a Fredholm operator for each fixed p, and there exists $p_0 \in C$ such that the operator $\hat{H}_0(p_0)$ is invertible.

In addition to the requirement of the Fredholm property for the family $\hat{H}_0(p)$ we assume that there exists a cone in C containing the imaginary axis and does not contain the points of the spectrum of $\hat{H}_0(p)$ for a sufficiently large $|p|$. The spectrum of a Fredholm family is a set of points $p \in C$ such that $\hat{H}_0(p)$ is irreversible.

Suppose that the point $p_1 \in spec\hat{H}_0(p)$ *is such that the following conditions hold:*

1. The operator-valued function $\hat{H}^{-1}{}_0(p)$ has the second-order pole at the point p_1.
2. The dimensionality of the $\ker \hat{H}_0(p_1)$ is equal to one.

In this case theorem 1 is also true. The proof is analogous to that of Theorem 9 in [5] In this case the coefficients of the series $\sum_{j=1}^{\infty} v_j^i r^j$ are the elements of the space B_1.

Ref

5. Korovina M. V. and Shatalov V. E., Differential Equations with Degeneration and Resurgent Analysis, Differ. Uravn, 2010, vol. 46, no 9, p 1259-1277.

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IV. Laplace's Equation on a Manifold with a Caspidal Singularity

Notes

We consider the Laplace equation $\Delta u = 0$ on the 2D-Riemannian manifold with a second order caspidal singularity. The Riemannian metric induced from R^3 with the help of embedding which is defined as the surface by the rotational of the parabolic branch $y = r^2$ around axis $0r$. in R^3. We choose local coordinates r, φ on the manifold in the neighbourhood of zero. The metrics on this manifold is given by

$$ds^2 = dr^2 + 4r^2 dr^2 + r^4 d\varphi^2 = (1 + 4r^2)dr^2 + r^4 d\varphi^2,$$

hence

$$gradu = (A_1, A_2) = \left(\frac{1}{\sqrt{4r^2+1}} \frac{\partial u}{\partial r}, \frac{1}{r^2} \frac{\partial u}{\partial \varphi} \right).$$

and

$$divu = \frac{1}{h_1 h_2} \left(\frac{\partial}{\partial r}(h_2 A_1) + \frac{\partial}{\partial \varphi}(h_1 A_2) \right) = \frac{1}{r^2\sqrt{4r^2+1}} \frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r^2} \frac{\partial}{\partial \varphi} A_2.$$

Finally, for the Laplace operator we obtain

$$\Delta u = divgradu = \frac{1}{r^2\sqrt{4r^2+1}} \frac{\partial}{\partial r} \left(r^2 \frac{1}{\sqrt{4r^2+1}} \frac{\partial u}{\partial r} \right) + \frac{1}{r^4} \left(\frac{\partial}{\partial \varphi} \right)^2 u .$$

Thus Laplace's equation on a manifold has the form

$$\hat{H}u = \frac{1}{4r^2+1} \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{2r^2+1}{(4r^2+1)^2} \frac{\partial u}{\partial r} + \frac{1}{r^4} \frac{\partial^2 u}{\partial \varphi^2} = 0 \quad . \tag{11}$$

In other words

$$\hat{H} : E_k\left(S_{R,\varepsilon}, H^s(S_1)\right) \to E_k\left(S_{R,\varepsilon}, H^{s-2}(S_1)\right),$$

We rewrite equation (11) in the form

$$\left(r^2 \frac{d}{dr} \right)^2 u + \frac{\partial^2 u}{\partial \varphi^2} = -4r^2 \frac{\partial^2 u}{\partial \varphi^2} + \frac{4r^3}{(4r^2+1)} \left(r^2 \frac{\partial u}{\partial r} \right) \tag{12}$$

Obviously this case does not satisfy condition 2 who formulated in part 3, (see of the paper) Therefore, the results formulated there is not applicable. We perform the Laplace-Borel transform

$$p^2 \tilde{u} + \left(\frac{\partial}{\partial \varphi} \right)^2 \tilde{u} = -Br^3 a_1(r) \left(r^2 \frac{\partial}{\partial r} \right) \tilde{u} - 4 \frac{\partial^2}{\partial \varphi^2} Br^2 \tilde{u} + \tilde{f}(p, \varphi), \tag{13}$$

where $a_1(r) = \dfrac{r}{(4r^2+1)}$, and $\tilde{f}(p,\varphi)$ denotes a holomorphic function.

The principle symbol of this operator is equal $H(o,p)=p^2+\left(\frac{\partial}{\partial\varphi}\right)^2$. Therefore the corresponding operator family $\hat{H}_0(p)$ is a Fredholm one. The existence of resurgent solution of this equation follows from the results of [2], [3]. We denote $\hat{r}^n = Br^nB^{-1}$, as was done in [5]. If $a(r)$ is holomorphic in the neighbourhood of $r=0$, vanishing at the origin, and

$$a(r)=\sum_{k=1}^{\infty}a_k r^k \tag{14}$$

is its Taylor expansion, then the function

$$\tilde{a}(p)=\sum_{k=1}^{\infty}(-1)^k a_k \frac{p^{k-1}}{(k-1)!} \tag{15}$$

is called a *formal Laplace-Borel transform* of (14), then as shown in the article [5] the equation

$$B\circ a(r)\circ B^{-1}\tilde{f}=\int_{p_0}^{p}\tilde{a}(p-p')\tilde{f}(p')dp' \tag{16}$$

Equality (16) implies that the right-hand part of equation (13) can be transformed

$$-Br^2B^{-1}Bra_1(r)\left(r^2\frac{\partial}{\partial r}\right)\tilde{u}-4\frac{\partial^2}{\partial\varphi^2}Br^2\tilde{u}+\tilde{f}(p,\varphi)=$$

$$=-\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'\tilde{u}(p',\varphi)dp'-4\frac{\partial^2}{\partial\varphi^2}\hat{r}^2\tilde{u}(p',\varphi)+\tilde{f}(p,\varphi)$$

Equation (13) is ultimately transformed into following

$$p^2\tilde{u}+\left(\frac{\partial}{\partial\varphi}\right)^2\tilde{u}=-\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'\tilde{u}(p',\varphi)dp'-4\hat{r}^2\frac{\partial^2}{\partial\varphi^2}\tilde{u}(p,\varphi)+\tilde{f}(p,\varphi) \tag{17}$$

Decompose functions $\tilde{u}$ and $\tilde{f}$ into a series on eigen function of operator $\frac{\partial^2}{\partial\varphi^2}$

$$\tilde{u}(p,\varphi)=\sum_{k=-\infty}^{\infty}A_k(p)e^{ik\varphi}$$

$$\tilde{f}(p,\varphi)=\sum_{k=1}^{\infty}a_k(p)e^{ik\varphi} \tag{18}$$

Substituting (18) into equation, (17)we obtain

$$p^2\sum_{k=-\infty}^{\infty}A_k(p)e^{ik\varphi}-\sum_{k=1}^{\infty}k^2A_k(p)e^{ik\varphi}=$$

$$=-\hat{r}^3\int_{p_0}^{p}\tilde{a}(p-p')\sum_{k=-\infty}^{\infty}A_k(p')e^{ik\varphi}dp'+4\hat{r}^2\sum_{k=-\infty}^{\infty}k^2A_k(p)e^{ik\varphi}+\sum_{k=1}^{\infty}a_k(p)e^{ik\varphi}$$

Ref

3. J. Ecalle. Cinq applications des fonctions resurgentes, 1984. Preprint 84T 62, Orsay.

This equality is equivalent to the system of equations

$$\left(p^2-k^2\right)A_k(p)=-\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'A_k(p')dp'+4\hat{r}^2k^2A_k(p)+a_k(p),$$

Here k is an integer. The cases $k=0$ and $k\neq 0$ should be considered separately. In the first case the singular point of the function $A_0(p)$ is $p=0$, being a pole of second order. We solve this equation by the method of successive approximations.

$$A_0(p)=-\frac{2}{p^2}\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'\frac{A_0(p')}{2}dp'+\frac{1}{p^2}a_0(p) \tag{19}$$

Substitute $A_0(p)=\frac{1}{p^2}a_0(p)$ into the integral in right-hand part side of (19). Applying successive approximation method. Obtain the equality

$$\frac{1}{p^2}\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')\frac{1}{p'}a_0(p')dp'=\frac{1}{p^2}\hat{r}^2\int_{p_0}^{p}\frac{G(p,p')}{p'}dp'$$

where $G(p,p')=\tilde{a}(p-p')a_0(p')$. We represent this function as the series $G(p,p')=\sum_{i=0}^{\infty}l_i(p)p'^i$, then similarly to [5] we obtain

$$\frac{1}{p^2}\hat{r}^2\int_{p_0}^{p}\frac{\sum_{i=0}^{\infty}l_i(p)p'^i}{p'}dp'=\frac{1}{p^2}\int_0^{p_2}\int_{p_0}^{p_2}\left(b_1(p_1)\ln p_1+n(p_1)\right)dp_1dp_2=$$

$$=C_{-1}\frac{1}{p}+C_0\ln p+\sum_{i=1}^{\infty}C_ip^i\ln p+g(p)$$

where $g(p)$ denotes the function holomorphic in the zero neighbourhood, and C_i denotes corresponding constants. The proof of convergence of the successive approximation method is the same as analogous to the proof in [6]. Thus we obtain that the function $A_0(p)$ with an accuracy of a holomorphic summand in the neighbourhood of the point $p=0$ has the form

$$A_0(p)=\frac{M_{-2}}{p^2}+\frac{M_{-1}^0}{p}+\sum_{i=0}^{\infty}M_i^0p^i\ln p$$

Here by M_i^0 we denote corresponding constants.

Now we consider the second case $k\neq 0$. In this case the points $p=k$ and $p=-k$ are first-order poles of the function $A_k(p)$. Here we have the equation

$$A_k(p)=-\frac{1}{p^2-k^2}\hat{r}^2\int_{p_0}^{p}\tilde{a}(p-p')p'A_k(p')dp'+\frac{1}{p^2-k^2}4\hat{r}^2k^2A_k(p)+\frac{a_k(p)}{p^2-k^2}$$

Ref

5. Korovina M. V. and Shatalov V. E., Differential Equations with Degeneration and Resurgent Analysis, Differ. Uravn, 2010, vol. 46, no 9, p 1259-1277.

Equations of this form have been studied in [5]. The asymptotics of the function $A_k(p)$ in the neighbourhood of the singular point $p=k$, with an accuracy of a holomorphic function, will take the form

$$A_k(p)=\frac{M_{-1}^k}{p-k}+\sum_{j=0}^{\infty}M_j^k(p-k)^j\ln(p-k)$$

Here by M_i^k we denote corresponding constants. Thus we have demonstrated that, with an accuracy of a holomorphic function, the solution of equation (17) has the form

$$\tilde{u}(p,\varphi)=\sum_{k=-\infty}^{\infty}A_k(p)e^{ik\varphi}=\frac{M_{-2}}{p^2}+\sum_{k=-\infty}^{\infty}\left(\frac{M_{-1}^k}{p-k}+\sum_{j=0}M_i^k(p-k)^i\ln(p-k)\right)e^{ik\varphi}$$

Hence it follows that singular points of the function $\tilde{u}(p,\varphi)$ lie in any half-plane $\operatorname{Re} p>A>0$. In order to construct the solution of problem (17) we represent the function $\tilde{u}(p,\varphi)$ as a sum of two functions $\tilde{u}_-(p,\varphi)$ and $\tilde{u}_+(p,\varphi)$such that the singularities of the first one lie to the left of the line $\operatorname{Re} p>A$, and the singularities of the second one lie to the right of this line $\operatorname{Re} p>A$, where $\operatorname{Re} A\notin Z$. Then the solution can be constructed as the inverse transformation of the function $\tilde{u}_-(p,\varphi)$ (see [5]). To put it differently the equality

$$u(r,\varphi)=B^{-1}\tilde{u}_-(p,\varphi)=B^{-1}\left(\frac{M_{-2}}{p^2}+\sum_{k=-\infty}^{N}\left(\frac{M_{-1}^k}{p-k}+\sum_{j=0}^{\infty}M_j^k(p-k)^j\ln(p-k)\right)e^{i\varphi k}\right)$$

holds with an accuracy of an entire function.
Hence it follows that the asymptotic solution of Laplace's equation (13) has the form

$$u(r,\varphi)=\frac{C_{-1}}{r}+\sum_{k=-\infty}^{N}e^{\frac{k}{r}}e^{i\varphi k}\sum_{i=0}^{\infty}M_i^k r^i$$

We should note that the solution depends on the representation $\tilde{u}(p,\varphi)=\tilde{u}_-(p,\varphi)+\tilde{u}_+(p,\varphi)$. One solution differs from the other solutions by an operator kernel. Thus we have proved

Assertion. Any solution of equation of exponential growth (12) can be represented as

$$u(r,\varphi)=\frac{C_{-1}}{r}+\sum_j u_j(r,\varphi)+O\left(e^{-\frac{A}{r}}\right),$$

where A is an arbitrary positive number, and the sum contains a finite number of summands, each of which corresponding to a point $p=k$ located in the half-plane $\operatorname{Re} p>-A$, and $u_j(r)$ have asymptotic expansions

$$u_j(r,\varphi)=e^{\frac{j}{r}}e^{i\varphi jj}\sum_{l=0}^{\infty}M_l^k r^l,$$

Ref

5. Korovina M. V. and Shatalov V. E., Differential Equations with Degeneration and Resurgent Analysis, Differ. Uravn, 2010, vol. 46, no 9, p 1259-1277.

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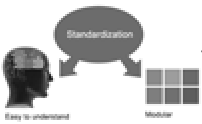

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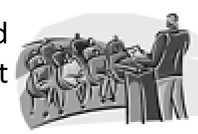

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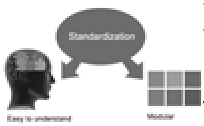

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Title: The title page must carry an instructive title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) wherever the work was carried out. The full postal address in addition with the e-mail address of related author must be given. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining and indexing.

Abstract, used in Original Papers and Reviews:

Optimizing Abstract for Search Engines

Many researchers searching for information online will use search engines such as Google, Yahoo or similar. By optimizing your paper for search engines, you will amplify the chance of someone finding it. This in turn will make it more likely to be viewed and/or cited in a further work. Global Journals Inc. (US) have compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Key Words

A major linchpin in research work for the writing research paper is the keyword search, which one will employ to find both library and Internet resources.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy and planning a list of possible keywords and phrases to try.

Search engines for most searches, use Boolean searching, which is somewhat different from Internet searches. The Boolean search uses "operators," words (and, or, not, and near) that enable you to expand or narrow your affords. Tips for research paper while preparing research paper are very helpful guideline of research paper.

Choice of key words is first tool of tips to write research paper. Research paper writing is an art.A few tips for deciding as strategically as possible about keyword search:

- One should start brainstorming lists of possible keywords before even begin searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in research paper?" Then consider synonyms for the important words.
- It may take the discovery of only one relevant paper to let steer in the right keyword direction because in most databases, the keywords under which a research paper is abstracted are listed with the paper.
- One should avoid outdated words.

Keywords are the key that opens a door to research work sources. Keyword searching is an art in which researcher's skills are bound to improve with experience and time.

Numerical Methods: Numerical methods used should be clear and, where appropriate, supported by references.

Acknowledgements: Please make these as concise as possible.

References

References follow the Harvard scheme of referencing. References in the text should cite the authors' names followed by the time of their publication, unless there are three or more authors when simply the first author's name is quoted followed by et al. unpublished work has to only be cited where necessary, and only in the text. Copies of references in press in other journals have to be supplied with submitted typescripts. It is necessary that all citations and references be carefully checked before submission, as mistakes or omissions will cause delays.

References to information on the World Wide Web can be given, but only if the information is available without charge to readers on an official site. Wikipedia and Similar websites are not allowed where anyone can change the information. Authors will be asked to make available electronic copies of the cited information for inclusion on the Global Journals Inc. (US) homepage at the judgment of the Editorial Board.

The Editorial Board and Global Journals Inc. (US) recommend that, citation of online-published papers and other material should be done via a DOI (digital object identifier). If an author cites anything, which does not have a DOI, they run the risk of the cited material not being noticeable.

The Editorial Board and Global Journals Inc. (US) recommend the use of a tool such as Reference Manager for reference management and formatting.

Tables, Figures and Figure Legends

Tables: Tables should be few in number, cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g. Table 4, a self-explanatory caption and be on a separate sheet. Vertical lines should not be used.

Figures: Figures are supposed to be submitted as separate files. Always take in a citation in the text for each figure using Arabic numbers, e.g. Fig. 4. Artwork must be submitted online in electronic form by e-mailing them.

Preparation of Electronic Figures for Publication

Even though low quality images are sufficient for review purposes, print publication requires high quality images to prevent the final product being blurred or fuzzy. Submit (or e-mail) EPS (line art) or TIFF (halftone/photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Do not use pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings) in relation to the imitation size. Please give the data for figures in black and white or submit a Color Work Agreement Form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution (at final image size) ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs) : >350 dpi; figures containing both halftone and line images: >650 dpi.

Color Charges: It is the rule of the Global Journals Inc. (US) for authors to pay the full cost for the reproduction of their color artwork. Hence, please note that, if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a color work agreement form before your paper can be published.

Figure Legends: Self-explanatory legends of all figures should be incorporated separately under the heading 'Legends to Figures'. In the full-text online edition of the journal, figure legends may possibly be truncated in abbreviated links to the full screen version. Therefore, the first 100 characters of any legend should notify the reader, about the key aspects of the figure.

6. AFTER ACCEPTANCE

Upon approval of a paper for publication, the manuscript will be forwarded to the dean, who is responsible for the publication of the Global Journals Inc. (US).

6.1 Proof Corrections

The corresponding author will receive an e-mail alert containing a link to a website or will be attached. A working e-mail address must therefore be provided for the related author.

Acrobat Reader will be required in order to read this file. This software can be downloaded

(Free of charge) from the following website:

www.adobe.com/products/acrobat/readstep2.html. This will facilitate the file to be opened, read on screen, and printed out in order for any corrections to be added. Further instructions will be sent with the proof.

Proofs must be returned to the dean at dean@globaljournals.org within three days of receipt.

As changes to proofs are costly, we inquire that you only correct typesetting errors. All illustrations are retained by the publisher. Please note that the authors are responsible for all statements made in their work, including changes made by the copy editor.

6.2 Early View of Global Journals Inc. (US) (Publication Prior to Print)

The Global Journals Inc. (US) are enclosed by our publishing's Early View service. Early View articles are complete full-text articles sent in advance of their publication. Early View articles are absolute and final. They have been completely reviewed, revised and edited for publication, and the authors' final corrections have been incorporated. Because they are in final form, no changes can be made after sending them. The nature of Early View articles means that they do not yet have volume, issue or page numbers, so Early View articles cannot be cited in the conventional way.

6.3 Author Services

Online production tracking is available for your article through Author Services. Author Services enables authors to track their article - once it has been accepted - through the production process to publication online and in print. Authors can check the status of their articles online and choose to receive automated e-mails at key stages of production. The authors will receive an e-mail with a unique link that enables them to register and have their article automatically added to the system. Please ensure that a complete e-mail address is provided when submitting the manuscript.

6.4 Author Material Archive Policy

Please note that if not specifically requested, publisher will dispose off hardcopy & electronic information submitted, after the two months of publication. If you require the return of any information submitted, please inform the Editorial Board or dean as soon as possible.

6.5 Offprint and Extra Copies

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Before start writing a good quality Computer Science Research Paper, let us first understand what is Computer Science Research Paper? So, Computer Science Research Paper is the paper which is written by professionals or scientists who are associated to Computer Science and Information Technology, or doing research study in these areas. If you are novel to this field then you can consult about this field from your supervisor or guide.

TECHNIQUES FOR WRITING A GOOD QUALITY RESEARCH PAPER:

1. Choosing the topic: In most cases, the topic is searched by the interest of author but it can be also suggested by the guides. You can have several topics and then you can judge that in which topic or subject you are finding yourself most comfortable. This can be done by asking several questions to yourself, like Will I be able to carry our search in this area? Will I find all necessary recourses to accomplish the search? Will I be able to find all information in this field area? If the answer of these types of questions will be "Yes" then you can choose that topic. In most of the cases, you may have to conduct the surveys and have to visit several places because this field is related to Computer Science and Information Technology. Also, you may have to do a lot of work to find all rise and falls regarding the various data of that subject. Sometimes, detailed information plays a vital role, instead of short information.

2. Evaluators are human: First thing to remember that evaluators are also human being. They are not only meant for rejecting a paper. They are here to evaluate your paper. So, present your Best.

3. Think Like Evaluators: If you are in a confusion or getting demotivated that your paper will be accepted by evaluators or not, then think and try to evaluate your paper like an Evaluator. Try to understand that what an evaluator wants in your research paper and automatically you will have your answer.

4. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

5. Ask your Guides: If you are having any difficulty in your research, then do not hesitate to share your difficulty to your guide (if you have any). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work then ask the supervisor to help you with the alternative. He might also provide you the list of essential readings.

6. Use of computer is recommended: As you are doing research in the field of Computer Science, then this point is quite obvious.

7. Use right software: Always use good quality software packages. If you are not capable to judge good software then you can lose quality of your paper unknowingly. There are various software programs available to help you, which you can get through Internet.

8. Use the Internet for help: An excellent start for your paper can be by using the Google. It is an excellent search engine, where you can have your doubts resolved. You may also read some answers for the frequent question how to write my research paper or find model research paper. From the internet library you can download books. If you have all required books make important reading selecting and analyzing the specified information. Then put together research paper sketch out.

9. Use and get big pictures: Always use encyclopedias, Wikipedia to get pictures so that you can go into the depth.

10. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right! It is a good habit, which helps to not to lose your continuity. You should always use bookmarks while searching on Internet also, which will make your search easier.

11. Revise what you wrote: When you write anything, always read it, summarize it and then finalize it.

12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

Segment Draft and Final Research Paper: You have to strictly follow the template of research paper. If it is not done your paper may get rejected.

- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.

- Do not give permission to anyone else to "PROOFREAD" your manuscript.

- Methods to avoid Plagiarism is applied by us on every paper, if found guilty, you will be blacklisted by all of our collaborated research groups, your institution will be informed for this and strict legal actions will be taken immediately.)
- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

Criterion for Grading a Research Paper (Compilation) by Global Journals Inc. (US)

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals Inc. (US).

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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