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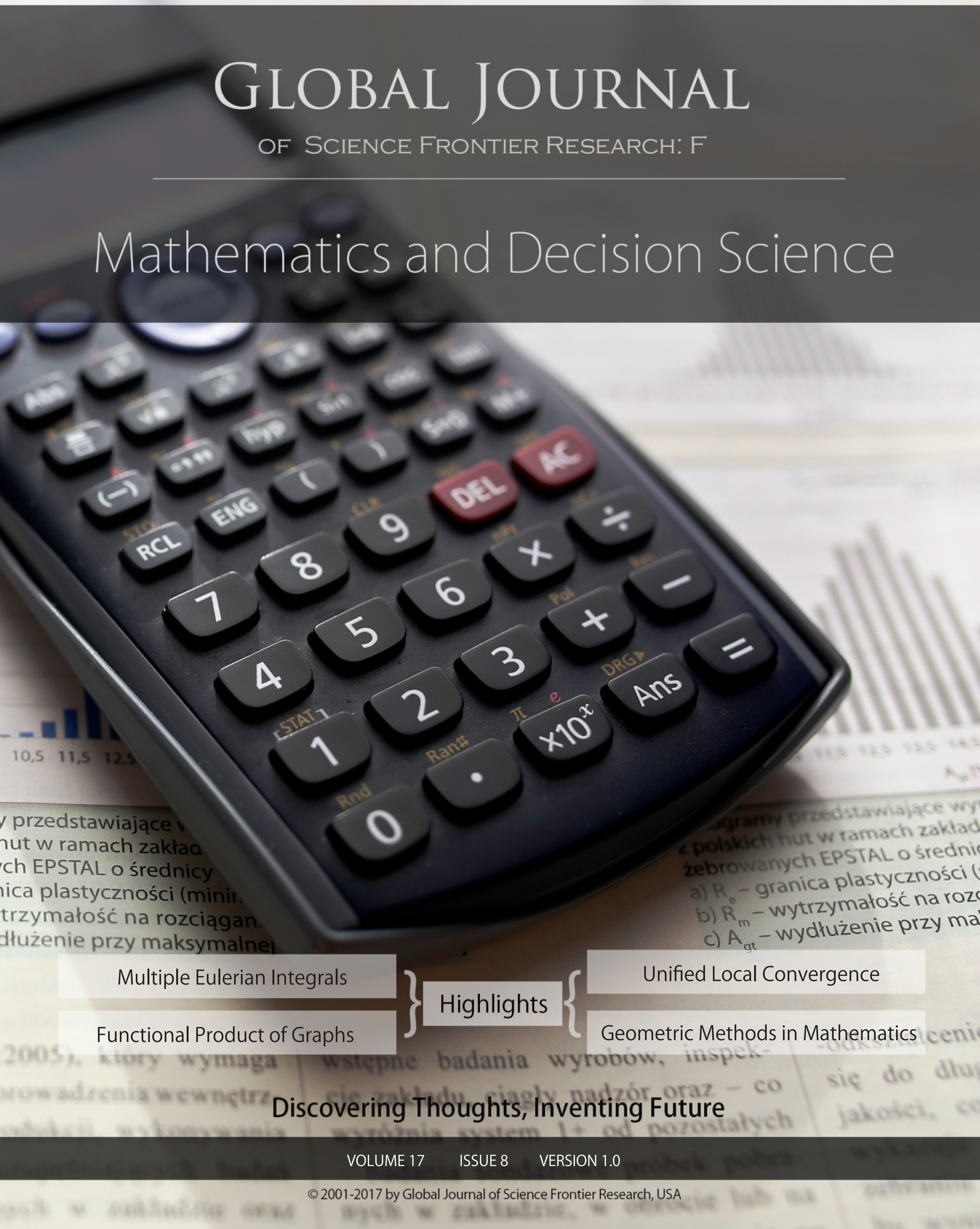

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Mathematics and Decision Science

Multiple Eulerian Integrals

Functional Product of Graphs

Highlights

Unified Local Convergence

Geometric Methods in Mathematics

Discovering Thoughts, Inventing Future

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On a General Class of Multiple Eulerian Integrals with Multivariable A-Functions

By Frederic Ayant

Abstract- Recently, Raina and Srivastava and Srivastava and Hussain have provided closed-form expressions for a number of an Eulerian integral involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerian integrals concerning the product of two multivariable A-functions defined by Gautam et Asgar[4], a class of multivariable polynomials and the extension of the Hurwitz-Lerch Zeta function. These integrals will serve as a fundamental formula from which one can deduce numerous useful integrals.

Keywords: multivariable A-function, multiple eulerian integral, the class of polynomials, the extension of the hurwitz-lerch zeta function, srivastava-daoust polynomial, A-function of one variable.

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ONAGENERALCLASSOFMULTIPLEEULERIANINTEGRALSWITHMULTIVARIABLEAFUNCTIONS

Strictly as per the compliance and regulations of:

On a General Class of Multiple Eulerian Integrals with Multivariable A-Functions

Frederic Ayant

Abstract- Recently, Raina and Srivastava and Srivastava and Hussain have provided closed-form expressions for a number of an Eulerian integral involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerian integrals concerning the product of two multivariable A-functions defined by Gautam et Asgar[4], a class of multivariable polynomials and the extension of the Hurwitz-Lerch Zeta function. These integrals will serve as a fundamental formula from which one can deduce numerous useful integrals.

Keywords: multivariable A-function, multiple eulerian integral, the class of polynomials, the extension of the hurwitz-lerch zeta function, srivastava-daoust polynomial, A-function of one variable.

I. Introduction and Prerequisites

The well-known Eulerian Beta integral [6]

$$\int_a^b (z-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(Re(\alpha)>0, Re(\beta)>0, b>a)$$

Is a basic result of evaluation of numerous other potentially useful integrals involving various special functions and polynomials. The mathematicians Raina and Srivastava [7], Saigo and Saxena [8], Srivastava and Hussain [11], Srivastava and Garg [10] et cetera have established some Eulerian integrals involving the various general class of polynomials, Meijer's G-function and Fox's H-function of one and more variables with general arguments. Recently, several Authors study some multiple Eulerian integrals, see Bhargava [2], Goyal and Mathur [5], Ayant [1] and others. The aim of this paper, we obtain general multiple Eulerian integrals of the product of two multivariable A-functions defined by Gautam et al [4], a class of multivariable polynomials [10] and the extension of the Hurwitz-Lerch Zeta function.

The last function noted $\phi(z,\mathfrak{s},a)$ is introduced by Srivastava et al ([15], eq.(6.2), page 503) as follows:

$$\phi^{(\rho_1,\cdots,\rho_{\mathbf{p}},\sigma_1,\cdots,\sigma_{\mathbf{q}})}_{(\lambda_1,\cdots,\lambda_{\mathbf{p}},\mu_1,\cdots,\mu_{\mathbf{q}})}(z;\mathfrak{s},a) = \sum_{R=0}^{\infty}\frac{\prod_{j=1}^{\mathbf{p}}(\lambda_j)_{R\rho_j}}{(a+R)^{\mathfrak{s}}\prod_{j=1}^{\mathbf{q}}(\mu_j)_{R\sigma_j}}\times\frac{z^R}{R!}$$

with : $\mathbf{p},\mathbf{q}\in\mathbb{N}_0, \lambda_j\in\mathbb{C}(j=1,\cdots,\mathbf{p}), a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^* \ (j=1,\cdots,\mathbf{q}), \rho_j,\sigma_k\in\mathbb{R}^+$

$(j=1,\cdots,\mathbf{p}; k=1,\cdots,\mathbf{q})$

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Ref

6. I.S. Gradsteyn and I.M. Ryxhik, Table of integrals, series and products: Academic press, New York 1980.

where $\Delta > -1$ when $\mathfrak{s}, z \in \mathbb{C}$; $\Delta = -1$ and $\mathfrak{s} \in \mathbb{C}, when|z| < \bigtriangledown^*$, $\Delta = -1$ and $Re(\chi) > \frac{1}{2}$ when $|z| = \bigtriangledown^*$

$$\bigtriangledown^* = \prod_{j=1}^{\mathbf{p}} \rho_j^{\rho_j} \prod_{j=1}^{\mathbf{q}} \sigma_j^{\sigma_j} \,; \Delta = \sum_{j=1}^{\mathbf{q}} \sigma_j - \sum_{j=1}^{\mathbf{p}} \rho_j; \chi = \mathfrak{s} + \sum_{j=1}^{\mathbf{q}} \mu_j - \sum_{j=1}^{\mathbf{p}} \lambda_j + \frac{\mathbf{p}-\mathbf{q}}{2}$$

We shall call these conditions the conditions (f) and $\bar{A}_R = \dfrac{\prod_{j=1}^{\mathbf{p}} (\lambda_j)_{R\rho_j}}{(a+R)^{\mathfrak{s}} \prod_{j=1}^{\mathbf{q}} (\mu_j)_{R\sigma_j}}$

The multivariable A-function is a generalization of the multivariable H-function studied by Srivastava and Panda [13,14]. The A-function of r-variables is defined and represented in the following manner.

$$A(z_1, \cdots, z_r) = A^{m,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r} \left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_j; A_j^{(1)}, \cdots, A_j^{(r)})_{1,p} : (\mathrm{c}_j^{(1)}, C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j; B_j^{(1)}, \cdots, B_j^{(r)})_{1,q} : (\mathrm{d}_j^{(1)}, D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, D_j^{(r)})_{1,q_r} \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.3}$$

where $\phi(s_1, \cdots, s_r), \theta_i(s_i), i = 1, \cdots, r$ are given by:

$$\psi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \sum_{i=1}^{r} B_j^{(i)} s_i) \prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} A_j^{(i)} s_j)}{\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} A_j^{(i)} s_j) \prod_{j=m+1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} B_j^{(i)} s_j)}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + C_j^{(i)} s_i) \prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - D_j^{(i)} s_i)}{\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - C_j^{(i)} s_i) \prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + D_j^{(i)} s_i)}$$

Here $m, n, p, m_i, n_i, p_i, c_i \in \mathbb{N}^*; i = 1, \cdots, r \,; a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if:

$$|arg(\Omega_i) z_k| < \frac{1}{2} \eta_k \pi, \xi^* = 0, \eta_i > 0 \tag{1.4}$$

$$\Omega_i = \prod_{j=1}^{p} \{A_j^{(i)}\}^{A_j^{(i)}} \prod_{j=1}^{q} \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i} \{D_j^{(i)}\}^{D_j^{(i)}} \prod_{j=1}^{p_i} \{C_j^{(i)}\}^{-C_j^{(i)}} ; i = 1, \cdots, r \tag{1.5}$$

$$\xi_i^* = Im(\sum_{j=1}^{p} A_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)}); i = 1, \cdots, r \tag{1.6}$$

$$\eta_i = Re\left(\sum_{j=1}^{n} A_j^{(i)} - \sum_{j=n+1}^{p} A_j^{(i)} + \sum_{j=1}^{m} B_j^{(i)} - \sum_{j=m+1}^{q} B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \right) \tag{1.7}$$

for $i = 1, \cdots, r$

Ref

14. H.M. Srivastava and R.Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. (1976), 265-274.

Notes

If all the poles of (1.6) are simple, then the integral (1.4) can be evaluated with the help of the residue theorem to give

$$A(z_1,\cdots,z_r)=\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!} \tag{1.8}$$

where ϕ and ϕ_i are defined by

$$\phi=\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}S_k\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}S_k\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}S_k\right)}$$

and

$$\phi_i=\frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}S_k\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}S_k\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}S_k\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}S_k\right)},i=1,\cdots,r$$

where

$$S_k=\eta_{G_k,g_k}=\frac{d_{g_k}^{(k)}+G_k}{\delta_{g_k}^{(k)}}\text{ for }k=1,\cdots,r$$

which is valid under the following conditions: $\epsilon_{M_k}^{(k)}[p_j^{(k)}+p_k']\neq\epsilon_j^{(k)}[p_{M_k}+g_k]$

We shall note $A(z_1,\cdots,z_r)=A_1(z_1,\cdots,z_r)$ and

$$A(z_1',\cdots,z_s')=A_{p',q':p_1',q_1';\cdots;p_s',q_s'}^{m',n':m_1',n_1';\cdots;m_s',n_s'}\left(\begin{array}{c|c} z_1' & (a'_j;A_j'^{(1)},\cdots,A_j'^{(s)})_{1,p'}:(c'^{(1)}_j,C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(s)},C_j'^{(s)})_{1,p_s'} \\ \cdot & \\ \cdot & \\ \cdot & \\ z_s' & (b'_j;B_j'^{(1)},\cdots,B_j'^{(s)})_{1,q'}:(d'^{(1)}_j,D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},D_j'^{(s)})_{1,q_s'}\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k'^{t_k}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

where $\zeta(t_1,\cdots,t_s)$, $\phi_i(t_i)$, $i=1,\cdots,s$ are given by :

$$\phi'(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{m'}\Gamma(b_j'-\sum_{i=1}^{s}B_j'^{(i)}t_i)\prod_{j=1}^{n'}\Gamma(1-a_j'+\sum_{i=1}^{s}A_j'^{(i)}t_j)}{\prod_{j=n'+1}^{p'}\Gamma(a_j'-\sum_{i=1}^{s}A_j'^{(i)}t_j)\prod_{j=m'+1}^{q'}\Gamma\left(1-b_j'+\sum_{i=1}^{s}B_j'^{(i)}t_j\right)}$$

$$\theta_i'(t_i)=\frac{\prod_{j=1}^{n_i'}\Gamma(1-c_j'^{(i)}+C_j'^{(i)}t_i)\prod_{j=1}^{m_i'}\Gamma(d_j'^{(i)}-D_j'^{(i)}t_i)}{\prod_{j=n_i'+1}^{p_i'}\Gamma(c_j'^{(i)}-C_j'^{(i)}t_i)\prod_{j=m_i'+1}^{q_i'}\Gamma(1-d_j'^{(i)}+D_j'^{(i)}t_i)}$$

Here $m',n',p',m_i',n_i',p_i',c_i'\in\mathbb{N}^*;i=1,\cdots,r;a_j',b_j',c_j'^{(i)},d_j'^{(i)},A_j'^{(i)},B_j'^{(i)},C_j'^{(i)},D_j'^{(i)}\in\mathbb{C}$

The multiple integral defining the A-function of r variables converges absolutely if:

$$|arg(\Omega_i')z_k'|<\frac{1}{2}\eta_k'\pi,\xi'^{*}=0,\eta_i'>0 \tag{1.10}$$

$$\Omega_i' = \prod_{j=1}^{p'}\{A_j'^{(i)}\}^{A_j'^{(i)}}\prod_{j=1}^{q'}\{B_j'^{(i)}\}^{-B_j'^{(i)}}\prod_{j=1}^{q_i'}\{D_j'^{(i)}\}^{D_j'^{(i)}}\prod_{j=1}^{p_i'}\{C_j'^{(i)}\}^{-C_j'^{(i)}}; i=1,\cdots,s \tag{1.11}$$

$$\xi_i'^{*} = Im(\sum_{j=1}^{p'}A_j'^{(i)} - \sum_{j=1}^{q'}B_j'^{(i)} + \sum_{j=1}^{q_i'}D_j'^{(i)} - \sum_{j=1}^{p_i'}C_j'^{(i)}); i=1,\cdots,s \tag{1.12}$$

$$\eta_i' = Re\left(\sum_{j=1}^{n'}A_j'^{(i)} - \sum_{j=n'+1}^{p'}A_j'^{(i)} + \sum_{j=1}^{m'}B_j'^{(i)} - \sum_{j=m'+1}^{q'}B_j'^{(i)} + \sum_{j=1}^{m_i'}D_j'^{(i)} - \sum_{j=m_i'+1}^{q_i'}D_j'^{(i)} + \sum_{j=1}^{n_i'}C_j'^{(i)} - \sum_{j=n_i'+1}^{p_i'}C_j'^{(i)}\right)$$

$$i=1,\cdots,s \tag{1.13}$$

Srivastava and Garg [10] introduced a class of multivariable polynomials as follows

$$S_L^{h_1,\cdots,h_u}[z_1,\cdots,z_u] = \sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}(-L)_{h_1R_1+\cdots+h_uR_u}B(L;R_1,\cdots,R_u)\frac{z_1^{R_1}\cdots z_u^{R_u}}{R_1!\cdots R_u!} \tag{1.14}$$

The coefficients $B(L;R_1,\cdots,R_u)$ are arbitrary real or complex constants.

We shall note

$$B_u = \frac{(-L)_{h_1R_1+\cdots+h_uR_u}B(L;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$$

II. Integral Representation of Generalized Hypergeometric Function

The following generalized hypergeometric function regarding multiple integrals contour is also required [12, page 39 eq.30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles $\Gamma(A_j+s_1+\cdots+s_r)$ of are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (2.1) is easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

The equivalent form of Eulerian beta integral is given by (1.1):

III. Main Integral

We shall note:

$X = m_1', n_1'; \cdots ; m_s', n_s'; 1,0; \cdots ; 1,0; 1,0; \cdots ; 1,0$

$Y = p_1', q_1'; \cdots ; p_s', q_s'; 0,1; \cdots ; 0,1; 0,1; \cdots ; 0,1$

$$\mathbb{A} = [1+\sigma_i^{(1)} - \sum_{k'=1}^{u}R_{k'}\rho_i''^{(1,k')} - \sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(1,k)} - \theta_i^{(1)}R;\rho_i'^{(1,1)},\cdots,\rho_i'^{(1,s)},\tau_i^{(1,1)},\cdots,\tau_i^{(1,l)},1,0,\cdots,0]_{1,s},\cdots,$$

Ref

10. H.M. Srivastava and M. Garg, Some integrals involving general class of polynomials and the multivariable H function. Rev. Roumaine. Phys. 32 (1987) 685-692.

Notes

$$\Big[1+\sigma_i^{(T)}-\sum_{k'=1}^{u}R_{k'}\rho_i''^{(T,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(T,k)}-\theta_i^{(T)}R;\rho_i'^{(T,1)},\cdots,\rho_i'^{(T,s)},\tau_i^{(T,1)},\cdots,\tau_i^{(T,l)},1,0,\cdots,0\Big]_{1,s},$$

$$[1-A_j;0,\cdots,0,1,\cdots,1,0,\cdots,0]_{1,P},$$

$$\Big[1-\beta_i-\sum_{k'=1}^{u}R_{k'}\eta_i''^{(k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\eta_i^{(k)}-R\lambda_i;\eta_i'^{(1)},\cdots,\eta_i'^{(s)},\theta_i^{(1)},\cdots,\theta_i^{(l)},1,\cdots,1,0,\cdots,0\Big]_{1,s}$$

$$\mathbf{A}=(a_j';A_j'^{(1)},\cdots,A_j'^{(s)},0,\cdots,0,0,\cdots,0)_{1,p'}:(c_j'^{(1)},C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(r)},C_j'^{(s)})_{1,p_s'}$$

$$(1,0),\cdots,(1,0);(1,0),\cdots,(1,0)$$

$$\mathbb{B}=[1+\sigma_i^{(1)}-\sum_{k'=1}^{u}R_{k'}\rho_i''^{(1,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(1,k)}-\theta_i^{(1)}R;\rho_i'^{(1,1)},\cdots,\rho_i'^{(1,s)},\tau_i^{(1,1)},\cdots,\tau_i^{(1,l)},0,\cdots,0]_{1,s},\cdots,$$

$$\Big[1+\sigma_i^{(T)}-\sum_{k'=1}^{u}R_{k'}\rho_i''^{(T,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(T,k)}-\theta_i^{(T)}R;\rho_i'^{(T,1)},\cdots,\rho_i'^{(T,s)},\tau_i^{(T,1)},\cdots,\tau_i^{(T,l)},0,\cdots,0\Big]_{1,s},$$

$$[1-B_j;0,\cdots,0,1,\cdots,1,0,\cdots,0]_{1,Q},$$

$$\Big[1-\alpha_i-\beta_i-\sum_{k'=1}^{u}(\delta_i''^{(k')}+\eta_i''^{(k')})R_{k'}-\sum_{k=1}^{r}(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}-(\zeta_i+\lambda_i)R;$$

$$(\delta_i'^{(1)}+\eta_i'^{(1)}),\cdots,(\delta_i'^{(s)}+\eta_i'^{(s)}),(\mu_i^{(1)}+\theta_i^{(1)}),\cdots,(\mu_i^{(l)}+\theta_i^{(l)}),1,\cdots,1]_{1,s}$$

$$\mathbf{B}=(b_j';B_j'^{(1)},\cdots,B_j'^{(s)},0,\cdots,0,0,\cdots,0)_{1,q'}:\ (\mathrm{d}_j'^{(1)},D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},D_j'^{(s)})_{1,q_s'};$$

$$(0,1),\cdots,(0,1);(0,1),\cdots,(0,1)$$

We have the following multiple Eulerian integrals, we obtain the A-function of $(r+l+T)-$variables.

Theorem

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$A_1\begin{pmatrix} \mathrm{z}_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{pmatrix}A_2\begin{pmatrix} \mathrm{z}_1'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(s)}}(v_i-x_i)^{\eta_i'^{(s)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,s)}}}\right]\end{pmatrix}$$

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$$S_L^{h_1,\cdots,h_u}\begin{pmatrix} z_1''\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ z_u''\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(u)}}(v_i-x_i)^{\eta_i''^{(u)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,u)}}}\right]\end{pmatrix}$$

$$\phi_{(\lambda_1,\cdots,\lambda_{\mathbf{p}},\mu_1,\cdots,\mu_{\mathbf{q}})}^{(\rho_1,\cdots,\rho_{\mathbf{p}},\sigma_1,\cdots,\sigma_{\mathbf{q}})}\left[\prod_{j=1}^t\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right];\mathfrak{s},a\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^l g_k\prod_{i=1}^t\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^t\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^W\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^T\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{R=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^r\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^r g_k}}{\prod_{k=1}^r\delta_{G^{(k)}}^{(k)}\prod_{k=1}^r g_k!}\bar{A}_Rb_u z_1''^{R_1}\cdots z_u''^{R_u}E_{ij}$$

$$A_{sT+P+2s+p',sT+Q+s+q':Y}^{m',sT+P+2s+n':X}\left(\begin{array}{c|c} \mathrm{z}_1'w_1 & \mathbb{A},\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_1'w_s & \cdot \\ \mathrm{g}_1W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{g}_lW_l & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{G}_T & \mathbb{B},\ \mathbf{B}\end{array}\right) \tag{3.1}$$

where

$$E_{ij}=\frac{1}{\prod_{i=1}^t\prod_{j=1}^W(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^u\rho_i''^{(j,k')}R_{k'}+\sum_{k=1}^r\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}R}}$$

$$\times\frac{\prod_{i=1}^t(v_i-u_i)^{\sum_{k'=1}^u(\delta_i''^{(k')}+\eta_i''^{(k')})R_{k'}+\sum_{k=1}^r(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}+(\zeta_i+\lambda_i)R}}{\prod_{i=1}^t\prod_{j=W+1}^T(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^u\rho_i''^{(j,k')}R_{k'}+\sum_{k=1}^r\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}R}}$$

Notes

$$w_m=\prod_{i=1}^{t}\left[(v_i-u_i)^{\delta_i'^{(m)}+\eta_i'^{(m)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i'^{(j,m)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\rho_i'^{(j,m)}}\right],m=1,\cdots,s$$

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$$W_k=\prod_{i=1}^{t}\left[(v_i-u_i)^{\mu_i^{(k)}+\theta_i^{(k)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\right],k=1,\cdots,l$$

$$G_j=\prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right],j=1,\cdots,W$$

$$G_j=-\prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right],j=W+1,\cdots,T$$

$$\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}=\sum_{G_1,\cdots,G_r=1}^{m_1,\cdots,m_r}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

Provided that:

(A) $W\in[0,T];u_i,v_i\in\mathbb{R};i=1,\cdots,t$

(B)$min\{\delta_i^{(g)},\eta_i^{(g)},\delta_i'^{(h)},\eta_i'^{(h)},\delta_i''^{(k)},\eta_i''^{(k)},\zeta_i,\eta_i\}\geqslant 0;g=1,\cdots,r;i=1,\cdots,t;h=1,\cdots,s;k=1,\cdots,u$

$min\{\rho_i^{(j,g)},\rho_i'^{(j,h)},\rho_i''^{(j,k')},\theta_i^{(j)},\tau_i^{(j,k)}\}\geqslant 0;j=1,\cdots,T;i=1,\cdots,t;g=1,\cdots,r;h=1,\cdots,s;k'=1,\cdots,u;k=1,\cdots,l$

(C) $\sigma_i^{(j)}\in\mathbb{R},U_i^{(j)},V_i^{(j)}\in\mathbb{C},z_{i'},z_{j'},z''_{k'},g_k,G_j\in\mathbb{C};i=1,\cdots,t;j=1,\cdots,T;i'=1,\cdots,r;$

$j'=1,\cdots,s;k'=1,\cdots,u;k=1,\cdots,l$

(D)$max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right]<1,i=1,\cdots,s;j=1,\cdots,W$ and

$$max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right]<1,i=1,\cdots,s;j=W+1,\cdots,T$$

(E) $\left|arg\left(z_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i^{(j,k)}}\right)\right|<\frac{1}{2}\eta_k\pi,k=1,\cdots,r$, where

$$\eta_i=Re\left(\sum_{j=1}^{n}A_j^{(i)}-\sum_{j=n+1}^{p}A_j^{(i)}+\sum_{j=1}^{m}B_j^{(i)}-\sum_{j=m+1}^{q}B_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}+\sum_{j=1}^{n_i}C_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\right)$$

$$-\delta_i^{(k)}-\eta_i^{(k)}-\sum_{j=1}^{T}\rho_i^{(j,k)}>0$$

$$\left|arg\left(z_i'\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i'^{(j,k)}}\right)\right|<\frac{1}{2}\eta_k'\pi, k=1,\cdots,s \text{ , where}$$

$$\eta_i'=Re\left(\sum_{j=1}^{n'}A_j'^{(i)}-\sum_{j=n'+1}^{p'}A_j'^{(i)}+\sum_{j=1}^{m'}B_j'^{(i)}-\sum_{j=m'+1}^{q'}B_j'^{(i)}+\sum_{j=1}^{m_i'}D_j'^{(i)}-\sum_{j=m_i'+1}^{q_i'}D_j'^{(i)}+\sum_{j=1}^{n_i'}C_j'^{(i)}-\sum_{j=n_i'+1}^{p_i'}C_j'^{(i)}\right)$$

$$-\delta_i'^{(k)}-\eta_i'^{(k)}-\sum_{j=1}^{T}\rho_i'^{(j,k)}>0$$

(F) $Re\left(\alpha_i+\zeta_iR+\sum_{j=1}^{r}\delta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\delta_i'^{(k)}\min_{1\leqslant j\leqslant m_k'}Re\left(\frac{d_j'^{(k)}}{D_j'^{(k)}}\right)>0$ and

$Re\left(\beta_i+\lambda_iR+\sum_{j=1}^{r}\eta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\eta_i'^{(k)}\min_{1\leqslant j\leqslant m_k'}Re\left(\frac{d_j'^{(k)}}{D_j'^{(k)}}\right)>0$ for $i=1,\cdots,t$

(G). $P\leqslant Q+1$ The equality holds, when, also

either $P>Q$ and $\sum_{k=1}^{l}\left|g_k\left(\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\tau_i^{(j,k)}}\right)\right|^{\frac{1}{Q-P}}<1 \qquad (u_i\leqslant x_i\leqslant v_i; i=1,\cdots,t)$

or $P\leqslant Q$ and $\max_{1\leqslant k\leqslant l}\left[\left|\left(g_k\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{-\tau_i^{(j,k)}}\right)\right|\right]<1 \qquad (u_i\leqslant x_i\leqslant v_i; i=1,\cdots,t)$

(H) the conditions (f) are satisfied

Proof

To establish the formula (3.1), we first express the extension of the Hurwitz-Lerch Zeta function, the class of multivariable polynomials $S_L^{h_1,\cdots,h_u}[.]$ and the multivariable A-function $A_1(z_1,\cdots,z_r)$ in series with the help of (1,2), (1,14) and (1.18) respectively, use integral contour representation with the help of (1.9) for the multivariable A-function $A_2(z_1',\cdots,z_s')$ occurring on its left-hand side and use the integral contour representation with the help of (2.1) for the Generalized hypergeometric function ${}_PF_Q(.)$. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now we write:

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} \tag{3.2}$$

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where $K_i^{(j)} = \upsilon_i^{(j)} - \theta_i^{(j)} R - \sum_{l=1}^{r} \rho_i^{(j,l)} \eta_{G_l,g_l} - \sum_{l=1}^{s} \rho_i'^{(j,l)} \psi_l - \sum_{l=1}^{v} \rho_i''^{(j,v)} K_l$ where $i = 1, \cdots, t; j = 1, \cdots, T$

and express the factors occurring in R.H.S. Of (3.1) in terms of following Mellin-Barnes integral contour, we obtain:

$$\prod_{j=1}^{W}(U_i^{(j)}x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right] \frac{1}{(2\pi\omega)^W}\int_{L_1'} \cdots \int_{L_W'} \prod_{j=1}^{W}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)} + \zeta_j')$$

$$\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i - u_i)}{(u_iU_i^{(j)} + V_i^{(j)})}\right]^{\zeta_j'}\Bigg] \mathrm{d}\zeta_1' \cdots \mathrm{d}\zeta_W' \tag{3.3}$$

and

$$\prod_{j=W+1}^{T}(U_i^{(j)}x_i + V_i^{(j)})^{K_i^{(j)}} = \prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i + V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right] \frac{1}{(2\pi\omega)^{T-W}}\int_{L_{W+1}'} \cdots \int_{L_T'} \prod_{j=W+1}^{T}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)} + \zeta_j')$$

$$\prod_{j=W+1}^{T}\left[-\frac{(U_i^{(j)}(v_i - x_i)}{(v_iU_i^{(j)} + V_i^{(j)})}\right]^{\zeta_j'}\Bigg] \mathrm{d}\zeta_{W+1}' \cdots \mathrm{d}\zeta_T' \tag{3.16}$$

We apply the Fubini's theorem for multiple integral. Finally evaluating the innermost **x**-integral with the help of (1.1) and reinterpreting the multiple Mellin-Barnes integrals contour regarding the multivariable A-function of $(r+l+T)-$ variables, we obtain the formula (3.7).

IV. Particular Cases

a) Srivastava-Daoust polynomial [9]

If $$B(L; R_1, \cdots, R_u) = \frac{\prod_{j=1}^{\bar{A}}(a_j)_{R_1\theta_j' + \cdots + R_u\theta_j^{(u)}} \prod_{j=1}^{B'}(b_j')_{R_1\phi_j'} \cdots \prod_{j=1}^{B^{(u)}}(b_j^{(u)})_{R_u\phi_j^{(u)}}}{\prod_{j=1}^{\bar{C}}(c_j)_{R_1\psi_j' + \cdots + R_u\psi_j^{(u)}} \prod_{j=1}^{D'}(d_j')_{R_1\delta_j'} \cdots \prod_{j=1}^{D^{(u)}}(d_j^{(u)})_{R_u\delta_j^{(u)}}}$$

we have

Corollary 1

$$\int_{u_1}^{v_1} \cdots \int_{u_t}^{v_t} \prod_{i=1}^{t}\left[(x_i - u_i)^{\alpha_i - 1}(v_i - x_i)^{\beta_i - 1}\prod_{j=1}^{T}(U_i^{(j)}x_i + V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$F^{1+\bar{A}:B';\cdots;B^{(u)}}_{\bar{C}:D';\cdots;D^{(u)}}\left(\begin{array}{c} \mathrm{z}_1'' \prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i + V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_u'' \prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(u)}}(v_i-x_i)^{\eta_i''^{(u)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i + V_i^{(j)}\right)^{\rho_i''^{(j,u)}}}\right] \end{array}\right)$$

Ref

9. H.M. Srivastava H.M. and M.C. Daoust, Certain generalized Neumann expansions associated with Kampé de Fériet function. Nederl. Akad. Wetensch. Proc. Ser A72 = Indag Math 31(1969), 449-457.

$$A_1\left(\begin{array}{c} z_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ z_r\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{array}\right) A_2\left(\begin{array}{c} z'_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ z'_s\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(s)}}(v_i-x_i)^{\eta_i'^{(s)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,s)}}}\right]\end{array}\right)$$

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$$\phi_{(\lambda_1,\cdots,\lambda_{\mathbf{p}},\mu_1,\cdots,\mu_{\mathbf{q}})}^{(\rho_1,\cdots,\rho_{\mathbf{p}},\sigma_1,\cdots,\sigma_{\mathbf{q}})}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right];\mathfrak{s},a\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{R=0}^{\infty}\sum_{R_1,\cdots,R_u=0}^{h_1R_1+\cdots h_uR_u\leqslant L}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!}\bar{A}_R b'_u z_1''^{R_1}\cdots z_u''^{R_u}E_{ij}$$

$$A_{sT+P+2s+p',sT+Q+s+q':Y}^{m',sT+P+2s+n':X}\left(\begin{array}{c|c} z'_1w_1 & \mathbb{A},\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z'_1w_s & \cdot \\ g_1W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ g_lW_l & \cdot \\ G_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ G_T & \mathbb{B},\ \mathbf{B}\end{array}\right) \tag{4.1}$$

where $b'_u=\dfrac{(-L)_{h_1R_1+\cdots+h_uR_u}B(L;R_1,\cdots,R_u)}{R_1!\cdots R_u!}$, $B[L;R_1,\ldots,R_u]$ *is defined by (4.1)*

The validity conditions are the same that (3.1).

b) A-function of one variable

If $r=s=1$, the multivariable A-functions reduce to A-functions of one variable defined by Gautam et Asgar [3]. We have.

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$A_1\left(\mathrm{z}\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\right)A_2\left(\mathrm{z'}\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\right)$$

$$S_L^{h_1,\cdots,h_u}\begin{pmatrix}\mathrm{z}_1''\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_u''\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(u)}}(v_i-x_i)^{\eta_i''^{(u)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,u)}}}\right]\end{pmatrix}$$

$$\phi_{(\lambda_1,\cdots,\lambda_{\mathbf{p}},\mu_1,\cdots,\mu_{\mathbf{q}})}^{(\rho_1,\cdots,\rho_{\mathbf{p}},\sigma_1,\cdots,\sigma_{\mathbf{q}})}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right];\mathfrak{s},a\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{k''=0,or1}^{\infty_*}\sum_{R=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{g_1=0}^{\infty}\phi_1\frac{z_k^{\eta_{G_1,g_1}}(-)^{g_1}}{\delta_{G^{(1)}}^{(1)}g_1!}\bar{A}_Rb_uz_1''^{R1}\cdots z_u''^{R_u}E_{ij}$$

$$A_{sT+P+2s+p',sT+Q+s+q':Y_1}^{m,sT+P+2s+n':X_1}\left(\begin{array}{c|c}\mathrm{z'w}_1 & \mathbb{A}_1,\ \mathbf{A}_1\\ \mathrm{g}_1W_1 & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{g}_lW_l & \cdot\\ \mathrm{G}_1 & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{G}_T & \mathbb{B}_1,\ \mathbf{B}_1\end{array}\right) \tag{4.2}$$

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3. B.P. Gautam, A.S. Asgar and A.N.Goyal. The A-function. Revista Mathematica. Tucuman (1980).

The validity conditions are the same that (3.1) with $r=s=1$. The quantities $\phi_1, V_1, W_1, \mathbb{A}_1, \mathbb{B}_1, \mathbf{A}_1, \mathbf{B}_1$ are equal to $\phi_k, V, W, \mathbb{A}, \mathbb{B}, \mathbf{A}, \mathbf{B}$ respectively for $r=s=1$.

Remark: By the similar procedure, the results of this document can be extended to the product of any finite number of multivariable A-functions and a class of multivariable polynomials defined by Srivastava and Garg [10].

V. Conclusion

Our main integral formula is unified in nature and possesses manifold generality. It acts a capital formula and using various particular cases of the multivariable A-function, a general class of multivariable polynomials and the generalization of the Hurwitz-Lerch Zeta function, we can obtain a large number of other integrals involving simpler special functions and polynomials of one and several variables.

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Functional Product of Graphs and Multiagent Systems

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Abstract- In this work, the concepts of functional product of graphs and equitable total coloring were used to propose a model of connection among the multiagent systems. We show how to generate a family of regular graphs that admits a range vertex coloringof order Δ with $\Delta + 1$ colors, denominated harmonic graphs. We prove that the harmonic graphs do not have cut vertices. We also show that the concept of equitable total coloring can be used to elaborate parallel algorithms that are independent of the network topology. Finally, we show a model of connection among multiagent systems (MAS) based on the use of harmonic graphs as a support for the construction of P2P overlay network topologies used for the communication among these systems.

Keywords: *functional product of graphs, harmonics graphs, multiagent system.*

GJSFR-F Classification: *MSC 2010: 05C50*

FUNCTIONALPRODUCTOFGRAPHSANDMULTIAGENTSYSTEMS

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Functional Product of Graphs and Multiagent Systems

A. R. G. Lozano [α], A. S. Siqueira [σ] & S. R. P. Mattos [ρ]

Abstract- In this work, the concepts of functional product of graphs and equitable total coloring were used to propose a model of connection among the multiagent systems. We show how to generate a family of regular graphs that admits a range vertex coloringof order Δ with Δ + 1 colors, denominated harmonic graphs. We prove that the harmonic graphs do not have cut vertices. We also show that the concept of equitable total coloring can be used to elaborate parallel algorithms that are independent of the network topology. Finally, we show a model of connection among multiagent systems (MAS) based on the use of harmonic graphs as a support for the construction of P2P overlay network topologies used for the communication among these systems.

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I. Introduction

Historically, product graphs, more specifically the cartesian product graphs, have been widely used as the topology of interconnection networks. Classical topologies such as mesh, hyper-star, star-cube, hypercube, and torus are obtained through the cartesian product of graphs. Currently, the concept of interconnection networks (physical structures) does not have the same relevance as before, However, the concept of multi agent systems (MAS), in which two or more agents work together to perform certain tasks, has been increasingly gaining space and applicability [9, 16]. It is on this tripod (functional product of graphs, harmonic graphs, and multiagent systems) that this work is supported.

In this article, we prove that the functional product of graphs allows building harmonic graphs from any regular graph and that the harmonic graphs do not have cut vertices. We show that a family of harmonic graphs disposes of a scalable and recursive structure since, from an initial basic instance, it can expand dynamically its form maintaining properties, such as connectedness and regularity. We also show that the concept of equitable total coloring can be used to elaborate parallel algorithms that are independent of the network topology. Finally, we present a model of connection among MAS through the use of harmonic graphs as a support for the construction of these topologies. Therefore, the main contributions of this work are the theorems 3.3, 3.4, 3.5, and the application of harmonic graphs as P2P overlay network topologies for the communication among multi agent systems.

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This text is organized as follows: in section 2, we present the concept of the functional product of graphs, idea that generalizes the cartesian product of graphs. Section 3 approaches the construction of harmonic graphs. In section 4, we enter in the computational aspect, and we present the concepts of the agents and multi agent systems. We also highlight the advantages of implementing a peer-to-peer communication system in the communication of the agents of a MAS, and we present a model of connection among MAS using harmonic graphs. In section 6, we make the final considerations.

II. Functional Product of Graphs

We can find introductory concepts about graphs and coloring in [3] and [20]. More specific concepts about coloring, such as equitable total coloring and range coloring of order k, can be viewed in [6] and [5] respectively. After that, we present the concept of the functional product of graphs, which also appears in [10] and [12]. To provide a better understanding of this section, some definitions and primary notations are necessary.

a) Definitions and primary notations

- $\{u, v\}$ or uv denotes an edge of graph G, in which u and v are adjacent;
- $\Delta(G)$ or Δ, if there is no ambiguity, denotes the maximum degree of graph G;
- $F(X)$ denotes the set of all bijections of X in X;
- $D(G)$ denotes the digraph obtained by replacing each edge uv of graph G by arcs (u, v) and (v, u) while maintaining the same set of vertices;
- D denotes the set of the digraphs that satisfy the following conditions:
  1. (u, v) is an arc of the digraph if and only if (v, u) is also an arc of the digraph;
  2. No two arcs are alike.
- If $\vec{G}\in D$, $G(\vec{G})$ denotes the graph obtained by replacing each pair of arcs (u, v) and (v, u) of $\vec{G}$ by edge uv while maintaining the same set of vertices;
- If A is a set, $|A|$ denotes the cardinality of A;
- C_n denotes the cycle of n vertices;
- K_n denotes the complete graph of n vertices.

Definition 2.1. The digraphs $\vec{G}_1(V_1, E_1)$ and $\vec{G}_2(V_2, E_2)$ are said to be functionally connected by the applications f_1: $E_1 \to F(V_2)$ and f_2: $E_2 \to F(V_1)$ if f_1 and f_2 are such that:

1. For every arc $(u,v)\in E_1$, if $(v,u)\in E_1$, then $f_1((u,v))=(f_1((v,u)))^{-1}$;
2. For every $(x,y)\in E_2$, if $(y,x)\in E_2$, then $f_2((x,y))=(f_2((y,x)))^{-1}$;
3. For every pair of arcs $(u,v)\in E_1$ and $(x,y)\in E_2$, it has that $f_2((x,y))(u)\neq v$ or $f_1((u,v))(x)\neq y$.

The applications f_1 and f_2 are called linking applications.

Definition 2.2. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be graphs, if $D(G_1)$ and $D(G_2)$ are functionally connected by applications f_1: $E(D(G_1))\to F(V_2)$ and f_2: $E(D(G_2))\to F(V_1)$, then the graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to be *functionally connected* by the same applications.

Definition 2.3. Let $\vec{G}_1(V_1,E_1)$ and $\vec{G}_2(V_2, E_2)$ be *digraphs* functionally connected by applications f_1: $E_1\to F(V_2)$ and f_2: $E_2\to F(V_1)$, the *functional product* of digraph $\vec{G}_1$ by digraph $\vec{G}_2$ according to f_1 and f_2, denoted by $(\vec{G}_1, f_1)\ x(\vec{G}_2, f_2)$, is digraph $\vec{G}^*(V^*,E^*)$ defined by:

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6. Fu, H. Some results on equalized total coloring. *Congressus Numerantium*(1994), 111–120.

- $V^* = V_1 \times V_2$.
- $((u,x),(v,y)) \in E^*$ *if and only if one of following conditions is true:*
  1. $(u, v) \in E_1$ and $f_1((u, v))(x) = y$;
  2. $(x, y) \in E_2$ and $f_2((x, y))(u) = v$.

Notes

Definition 2.4. Let $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *be graphs functionally connected by applications* $f_1: E(D(G_1)) \rightarrow F(V_2)$ *and* $f_2: E(D(G_2)) \rightarrow F(V_1)$, *the functional product of graph* G_1 *by graph* G_2, *denoted by* $(G_1, f_1) \times (G_2, f_2)$, *is graph* G $(\vec{G}^* (V^*, E^*))$, *such that* $\vec{G}^* (V^*, E^*) = (D(G_1), f_1) \times (D(G_2), f_2)$.

Figures 1, 2 and 3 present the functional product between two paths P_3. The linking applications f_1 *and* f_2 are defined by $f_1(x)=r_2$ for every edge $x \in E_1$ and $f_2(y)=r_1$ for every edge $y \in E_2$, in which $r_1(v_i) = v_{i+1(mod3)}$ and $r_2(v_i) = v_{i+2(mod3)}$, with $i \in \{1, 2, 3\}$. Figure 1 makes reference to the definitions 2.1 and 2.2 while figures 2 and 3 illustrate the definitions 2.3 and 2.4 respectively.

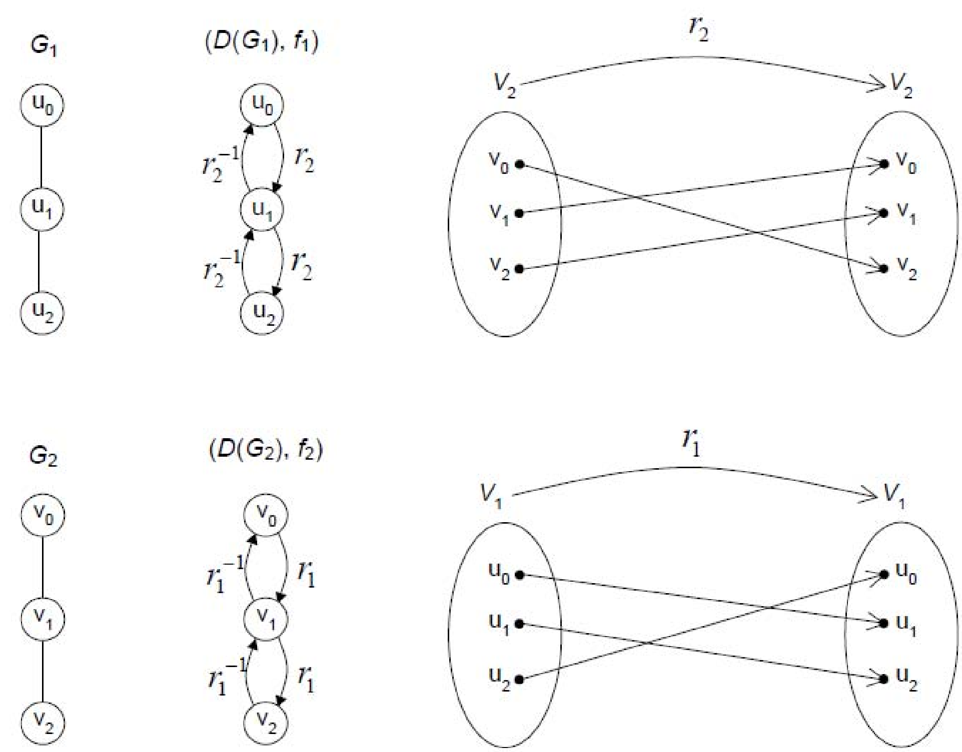

Figure 1: Graphs G_1 and G_2, the respective digraphs, and associated bijections r_1 and r_2

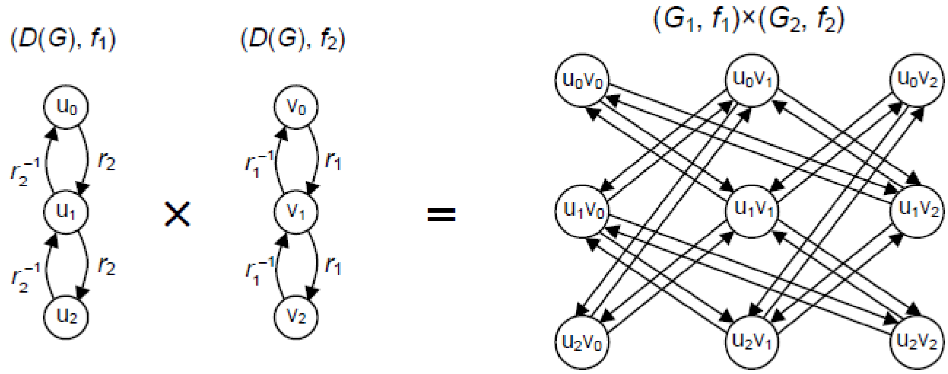

Figure 2: Functional product among digraphs $D(G_1)$ and $D(G_2)$ according to f_1 and f_2

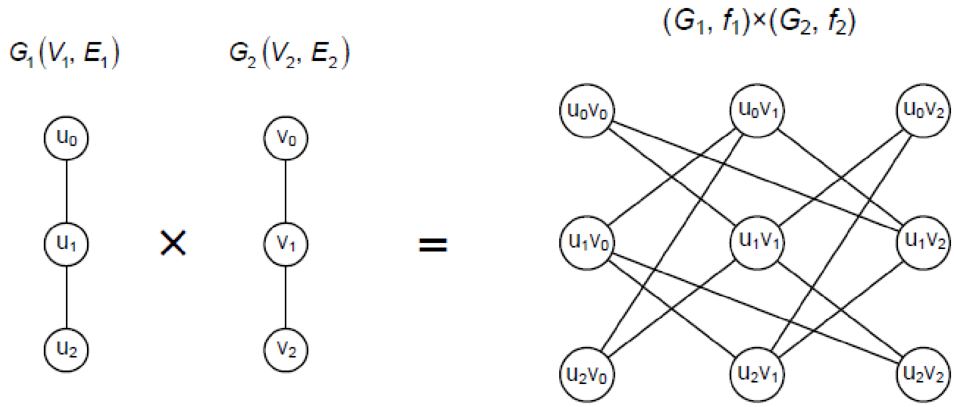

Figure 3: Functional product among graphs G_1 and G_2 according to f_1 and f_2

III. Harmonic Graphs

In this section, we presented the main contributions of this paper, the Theorems 3.3 and 3.4, which show how to build harmonic graphs from the functional product of graphs, and the Theorem 3.5 that proves that harmonic graphs do not have *articulation points*. To provide a better comprehension of this results, we enunciate some important concepts, the Petersen theorem, described in [20], and the result that guarantees the extension of a range coloring of order Δ to an equitable total coloring, which also appears in [11].

Definition 3.5. Let $G(V, E)$ be a graph, $C=\{c_1, c_2, c_3 \ldots c_p\}$ be a set of colors, with $p \in \mathbb{N}$ and a natural number k, such that $k \leq \Delta(G)$, an application $f: V \rightarrow C$ is a range vertices coloring of order k of G if for every $v \in V$, it has that $d(v) < k$, then $|c(N(v))| = d(v)$, otherwise $|c(N(v))| \geq k$, such that $|c(N(v))|$ is the cardinality of the set of colors used in the neighborhood of v [5].

Definition 3.6. A regular graph $G(V, E)$ is said to be harmonic if it admits a range vertices coloring of order Δ with $\Delta + 1$ colors [11].

Definition 3.7. A vertex in a connected graph is an articulation point or a cut vertex if by removing it, the graph becomes disconnected [4].

Theorem 3.1. If $G(V, E)$ is a 2k-regular graph, then G is 2-factorized [20].

Ref

20. Yap, H. *Total colorings of graphs*. Springer, Berlin, 1996.

Theorem 3.2. Let G (V, E) a regular graph and c:V → C = {1, 2, 3, . . . Δ + 1} a range coloring of order Δ of vertices of G, then the natural extension from c to G is an equitable total coloring [11].

The following theorem shows how to generate a harmonic graph from any regular graph and its complement.

Theorem 3.3. For every regular graph G and its complement G^*, there are linking applications f_1 and f_2, such that (G, f_1) x(G^*, f_2) is a harmonic graph.

Proof. Initially, note that for every regular graph G, if $n = |V(G)|$ is odd, then $\Delta(G)$ and $\Delta(G')$ are even. If $n = |V(G)|$ is even, then $\Delta(K_n)$ is odd and, as $\Delta(K_n) = \Delta(G) + \Delta(G')$, it has that $\Delta(G)$ or $\Delta(G')$ is even. Suppose that $\Delta(G')$ is even, by theorem 3.1, there is a decomposition in 2-factors of G'. Let $F_1, F_2, F_3, \ldots, F_t$ be the 2-factors of decomposition of G', each 2-factors F_i is replaced by anoriented cycle, and we define the application $b: V(F) \to V(F)$ such that if $(u,v) \in E(F)$, then $b(u) = v$.

Note that b is a bijection, and each 2-factors have associated a bijection of vertices of G. The application f_1 associates the identity to every pair of arcs associated to the edges of G. The application f_2 associates the bijection b to every arc of the cycle. In the cycle, in the opposite direction, we associate the reverse bijection. Now, if $V(G) = \{v_0, v_1, v_2, \ldots, v_p\}$, we give the color p to each vertex in the form (x, vp). By construction, the coloring obtained in (G,f_1) x(G',f_2) is a range coloring of order Δ with $\Delta+1$ colors. If $\Delta(G')$ is odd, then $\Delta(G)$ is even. Therefore, just change the positions of G *and* G', in the previous reasoning, to obtain the desired result. Then, (G,f_1) x(G',f_2) is a harmonic graph.

Theorem 3.4. Let G and G^* be a regular graph and its complement, if $\Delta(G^*)$ is even, then for any graph G', such that $\Delta(G^*) = \Delta(G')$, there are linking applications f_1 and f_2, such that (G, f_1) x (G', f_2) is a harmonic graph.

Proof. Just note that both G' and H decompose themselves in the same amount of 2-factors. Let $F_1, F_2, F_3, \ldots, F_t$ be 2-factors of decomposition of G', $r_1, r_2, \ldots, r_t$ be the associated bijections, and $K_1, K_2, \ldots, K_t$ be the 2-factors of decomposition of H, which will be replaced by oriented cycles $O_1, O_2, \ldots, O_t$., the application f_1 makes the identity correspond to all the edges of G. The application f_2 makes the bijection r_i correspond to each oriented arc O_i, and the bijection r^{-1} corresponds to the arc of opposite direction, for every $i \in 1, 2, \ldots, t$. Now, if $V(G) = \{v_1, v_2, \ldots, v_p\}$, we give the color p to each vertex in the form (x, v_p). Again, by construction, the coloring obtained in (G, f_1) x (H, f_2) is arange coloring of order Δ with $\Delta+1$ colors. So, (G, f_1) x (H, f_2) is a harmonic graph.

Theorem 3.5. Harmonic graphs do not have cut vertices.

Proof. Let $G(V, E)$ be a harmonic graph and c: $V \to C = \{0,1,2,\ldots,\Delta\}$ be a range coloring of order Δ of the vertices of G, suppose by absurdity that G has a cut vertex $u \in V$ and, without losing generality, suppose that the vertex u was colored with the color 0. Let $G'(V', E')$ be one of the connected components obtained by removing u of graph G, observe that the colors of $C - \{0\} = \{1, 2, \ldots, \Delta\}$ are used the same number of times in G' because, in a range coloring of order Δ, all of the adjacent vertices are colored with distinct colors, so given two arbitrary colors $i \in C - \{0\}$ and $j \in C - \{0\}$, every vertex of V', with the color i, has one and only one neighbor with the color j. Denote by V_i' the set of vertices

Ref

11. Lozano, A. R. G., Siqueira, A. S., Friedmann, C., and Jurkiewicz, S. Relationship between equitable total coloring and range coloring in some regular graphs. *Pesquisa Operacional 36*, 1(2016), 101–111.

of G' colored with the color $i \in C$, let $q = |V_i|$, $i \in C - \{0\}$ if $|V_0'| = q$, then all of the vertices of V' colored with colors different from 0 have a neighbor in V_0', so none of them can be neighbor of u, which is an absurdity. If $|V_0'| < q$, then it exists at least Δ vertices of G' with color other than 0 that do not have neighbors in V_0'. But, the number of neighbors of u in G' is less than Δ, so it exists vertices of V' with color other than 0 that do not have neighbor with color 0, which is an absurdity.
From the previous Theorem, it is obtained, immediately, the following corollary.

Corollary 3.1 Let u, $v \in V$ be any two vertices of $G(V, E)$, if G is a harmonic graph, then it exists a cycle in G that contains u and v.

IV. Multiagent Systems and Peer-to-Peer Communication System

According to Russel and Norvig[16], "an agent is anything that can be viewed as perceiving its environment through sensors and acting upon that environment through actuators". According to Lesser [9]" Multi agent systems are computational systems in which two or more agents interact or work together to perform some set of tasks or to satisfy some set of goals". The investigation of multi agent systems is focused on the development of computational principles and models to construct, analyze, and implement forms of interaction and coordination of agents in small or large-scale societies [9].

A peer-to-peer system implements an abstract overlay network on top of the network topology. The overlay network is a "virtual" network and the peers are connected to each other through logical connections, in which all of them should cooperate among themselves providing part of its resources on behalf of the accomplishment of a certain service [2].The objective of a peer-to-peer (P2P) system is to share computational resources through direct communication among its components therefore any device can access directly the resources of other devices of the system without any centralized control [2].

The combination between peer-to-peer network and multi agent systems has presented great solutions for the realization of applications that expand themselves on the internet. In [7], these two technologies were used to create an intelligent peer-to-peer infrastructure, which allows a dynamic network of intelligent agents while it manages several ways of discovering, cooperating, and executing efficiently computational resources. RETSINA [8,19] is a MAS infrastructure that uses the P2P Gnutella[4] network and some protocols based on DHT to extend the discovery services. ZHANG [21] proposes a peer-to-peer multi agent system that supports the execution of tasks of electronic commerce facilitating a dynamic selection of partners and allowing the use of heterogeneous agents.

The structured P2P overlay networks are characterized by a well-defined topology. Peers are positioned in a controlled way and the resources are distributed in a deterministic way making their location in the overlay network more efficient. Currently, we find several topologies implementing overlays networks on P2P systems. For example, Pastry [15] and Tapestry [22] are mesh-based, Chord [18] implements ring topology, and CAN [13] the d-dimensional torus.

V. Model of Connection among Multiagent Systems

In this paper, *multiagent systems connection* is a linking between two or more multiagent systems, in such way that agents of one MAS can communicate with agents of another MAS. It enables access to services, resource sharing, and guarantees the joint work.

[4]It is a network of file-sharing used mainly for the exchange of songs, movies and software [1].

Ref

16. Russel, S., and Norvig, P. *Artificial Intelligence: A Modern Approach.* Prentice Hall Series in Artificial Intelligence, New Jersey, 2004.

Notes

The reasons to amplify a computational structure range from opening a new sector of a company to the necessity of sharing servers interconnected to the internet to supply the demand of online sales during the launch of a product or on special dates, such as Christmas, for example.

In the proposed model, multiagent systems are overlaid by a P2P infrastructure that guarantees the interaction/communication among MAS agents. Therefore, each MAS agent corresponds to a peer of the P2P overlay network. The P2P overlay network architecture is represented by a graph $G(V, E)$, in which the vertices are the peers and the edges are the bonds(links) between the peers. Consequently, a vertex of the graph corresponds to a MAS agent, and graph $G(V, E)$ represents a multiagent system. Figure 4 illustrates an example in which three MAS are intended to be connected.

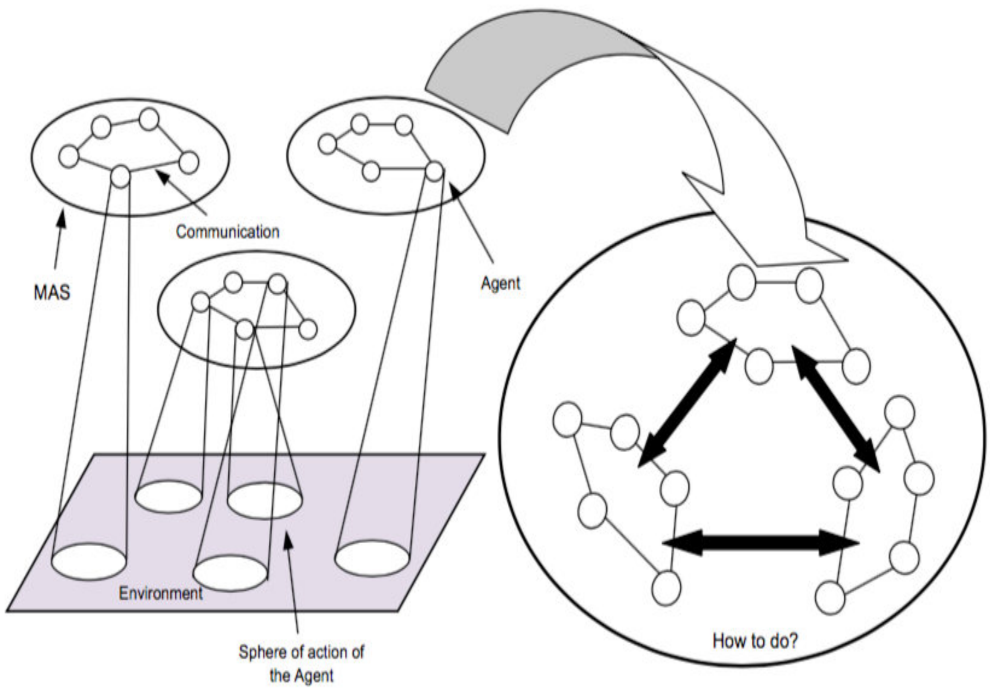

Figure 4: Structure of three MAS (left); Connection among three MAS (right). Adapted from Reis [14]

Initially, we are going to show how the theorem 3.4 allows making the connection among three multiagent systems. A graph C_5 will be used to represent the topology of each MAS to be connected, and a graph C_3 will be used to describe the type of connection, in other words, the way of making the connection among the three multiagent systems. Note that the graphs C_5 and C_3 satisfy the conditions of the theorem 3.4, so the harmonic graph illustrated in figure 5 can be constructed.

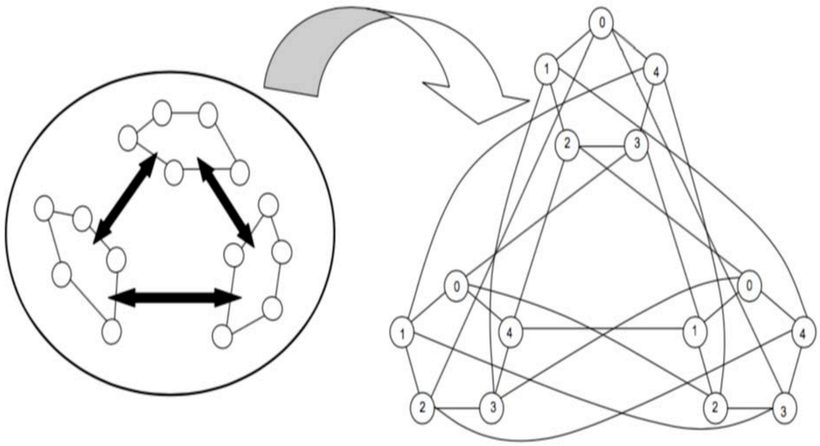

Figure 5: Process of construction of a range colored harmonic graph of order 4 with 5 colors

Now, we are going to show how to expand the structure of a MAS from the application of the theorem 3.3. Figures 6 and 7 illustrate this construction. Again, we used a graph C_5 as an example.

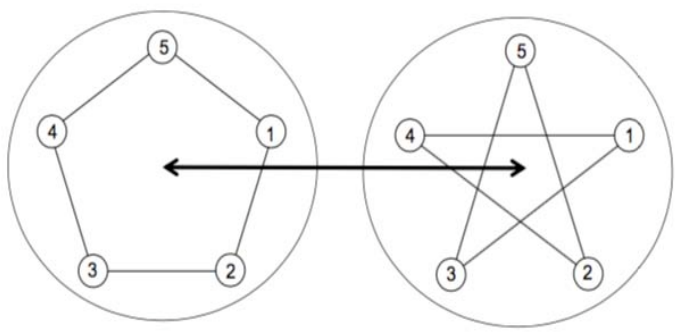

Figure 6: Graph G and its complement G^*(topology of the MAS)

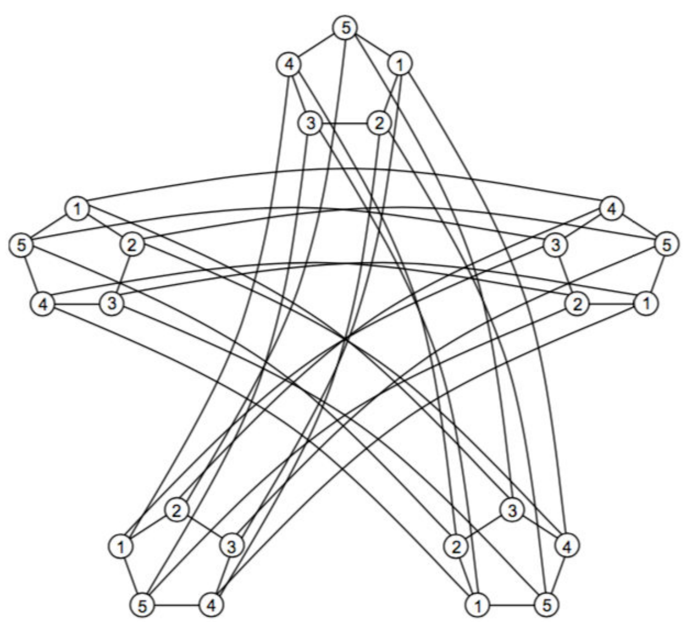

Figure 7: Resulting range colored harmonic graph of order 4 with 5 colors

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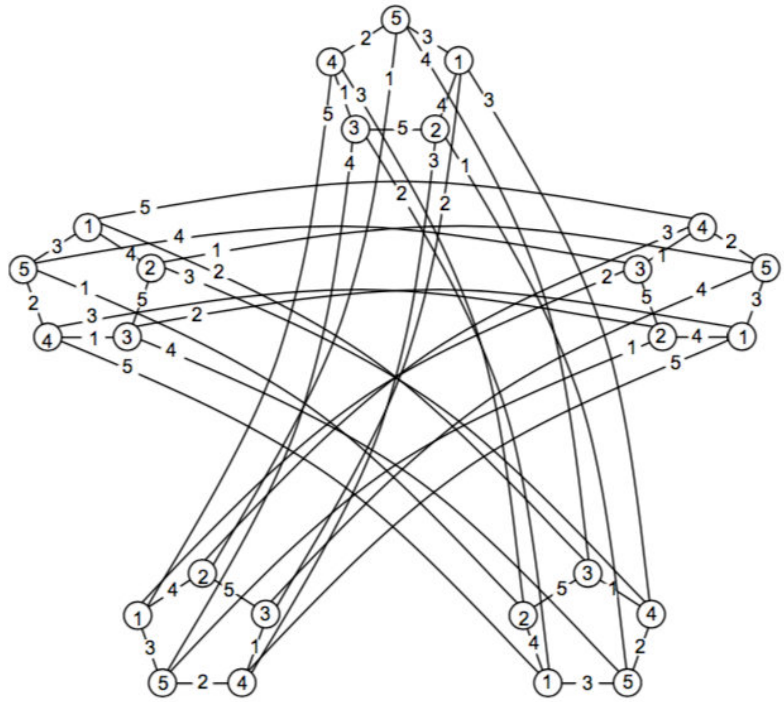

Figure 8: Harmonic graph total and equitably colored with 5 colors (resulting model of the multiagent system)

a) Parallel Algorithm of Complete Exchange

In this section, we present an algorithm of complete exchange that aims to guarantee the interaction /communication among every present agent of the P2P overlay network. In an algorithm that involves a complete exchange of information, each processor has information, and it is necessary that every processor knows all of the information. The algorithm bellow does not intend to be optimum, its objective is to show that from the total coloring it is possible to build algorithms that are independent of the topology.

Algorithm1: Complete Exchange of Information

```
Entrance: total and equitably colored graph G
  Variables: k: entire (total number of necessary colors)
           x[k]: array of color
           aux: color
start
 | Step 1:
 | For i= 1 up to k make
 |  | x[i] = i
 | end
end
start
 | Step 2:
 | for i= 1 a k make
 |  | Transmit the information through whichever edge with the color i, in
 |  | the direction of the vertex in which the color has the highest rate
 |  | in the array x[k]
 | end
end
start
 | Step 3:
 | aux:
 | x[k]
 | For i= k going down to 1 make
 |  | x[i] = x[i- 1]
 |  | x[1] = aux
 | end
end
start
 | Step 4:
 | repeat
 |  | the steps 2 and 3
 | until all of the vertices receive all the information;
end
```

Figures 9, 10, 11, 12, and 13 detail the functioning of the algorithm. Figure 9a shows a graph C_6 (MAS topology) in which the numbering of the vertices and edges represents an equitable total coloring, and the letters above the vertices symbolize the information contained in these vertices (agents). As a way to facilitate the comprehension of the algorithm, in the following figures, the numbering of the vertices indicates the exchanges of indexes of the *array* of color proposed in step 3.

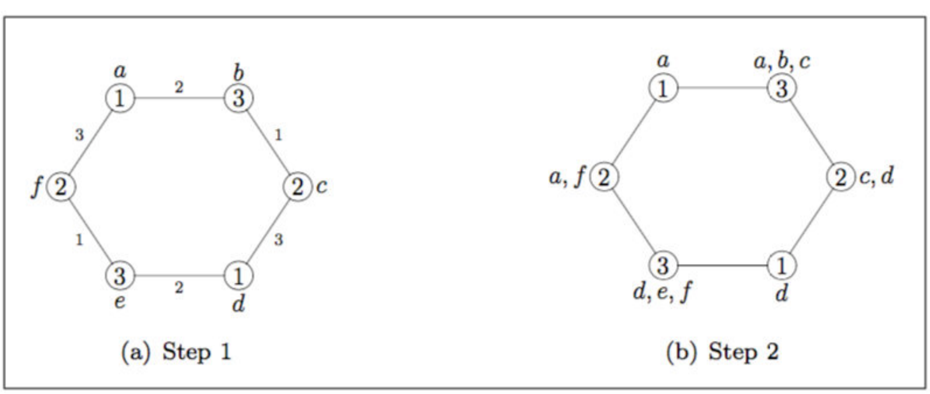

Figure 9: Step-by-step of the Algorithm (original position)

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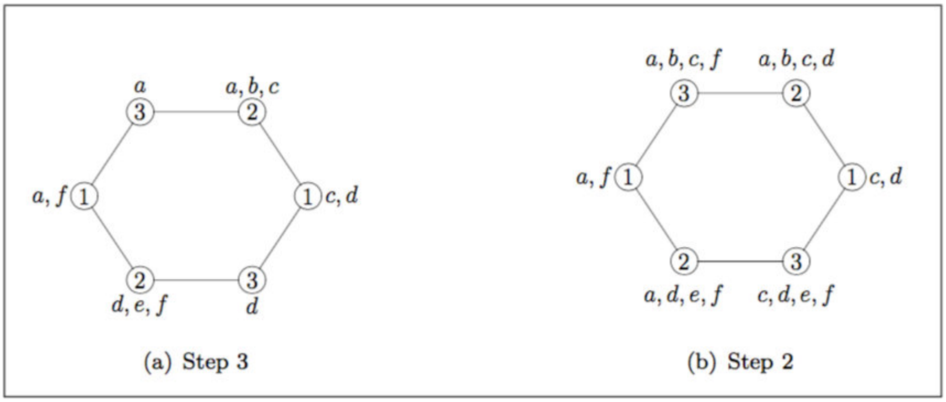


Figure 10: Step-by-step of the Algorithm (1st exchange)

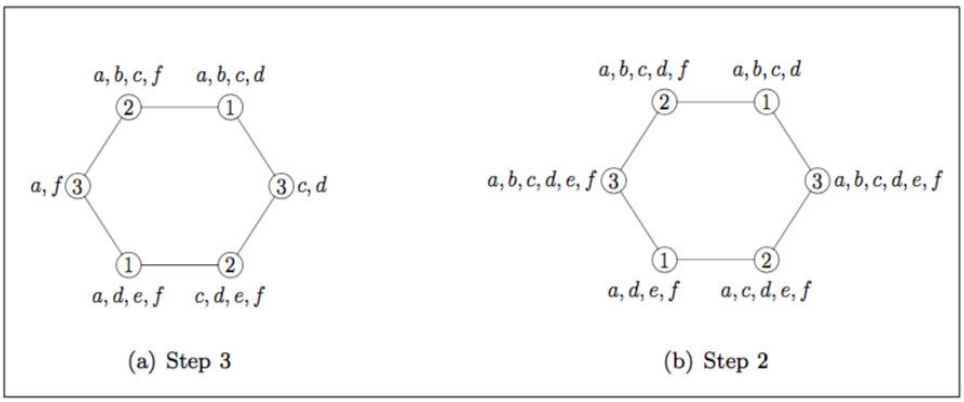


Figure 11: Step-by-step of the Algorithm (2nd exchange)

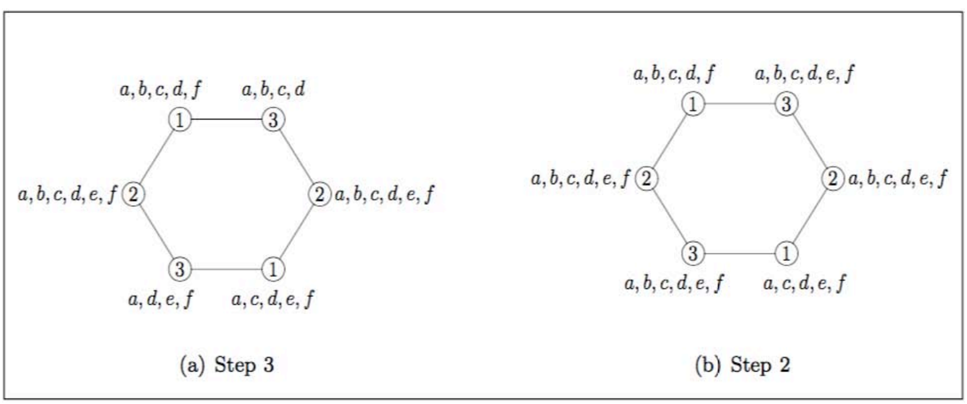
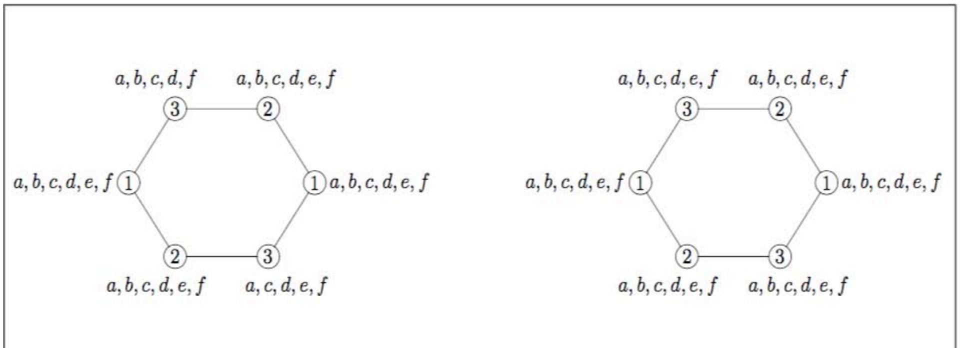


Figure 12: Step-by-step of the Algorithm (3rd exchange)



(a) Step 3 (b) Step 2

Figure 13: Step-by-step of the Algorithm (4th exchange)

b) Algorithm analysis

Theorem 5.6. Let x and y be adjacent vertices of graph $G(V, E)$ associated to a network, vertex y will receive the information of x in a maximum of k repetitions of steps 2 and 3 of the algorithm.

Proof. If two vertices are adjacent, then they cannot have the same color, Consequently, they have distinct numbers. Step 2 guarantees that the exchange of information will always be in the direction of the vertex from the smallest to the biggest index in the array of color. With the exchange of the index of the array of color proposed by step 3, after at least k exchanges, every vertex will have at least once a bigger index than its neighbor. Therefore, in a maximum of k repetitions of steps 2 and 3, two adjacent vertices exchange information.

Theorem 5.7. If d is the length of the longest path between any two vertices of graph G, then in the $d\,.k^2$ steps all of the vertices will receive all the information.

Proof. Note that the algorithm executes k steps in each passage through step 2 since k colors were used in the total coloring of the graph. At each exchange in the indices of the array of color predicted by step 3, we return to step 2 therefore, after k exchanges, we have $k.k = k^2$ steps performed by the algorithm. As demonstrated previously, after k repetitions of steps 2 and 3, in other words, $k.\ k = k^2$ steps of the algorithm, any two adjacent vertices exchange information. In practice, it means that, after k^2 steps of the algorithm, a given vertex has the information of all of its neighbors. Therefore, the following steps of the algorithm guarantee that non-adjacent vertices exchange information. If d is the length of the longest way between any two vertices of graph G, then, after $d.k^2$ steps of the algorithm, all of the vertices will have received all of the system information.

c) Advantages of the Proposed Model

The implementation of topologies that admit equitable total coloring can make the processing more efficient because it allows a natural division of the network resources, in which at least $\frac{t}{\Delta+2}$ processors or connections can be used simultaneously, let t be the number of elements of the graph associated to the network, i.e., number of vertices plus number of edges and Δ the maximum degree of the graph.

An equitable total coloring obtained by the natural extension of the range coloring of order Δ with $\Delta + 1$ colors guarantees that neighbor vertices always have distinct colors. In the case of the graph having an even maximum degree, the equitable total coloring is obtained with $\Delta + 1$ colors [17], fact that provides to the topology a processing optimization because in every given moment or the processor (vertex of color i) is executing

Ref

17. Siqueira, A. S. *Coloração Total Equilibrada em Subfamílias de Grafos Regulares.* Tese de D.Sc., COPPE/UFRJ, Rio de Janeiro, Brasil, 2011.

a task or it is receiving information from one of its neighbors through one of its links (edge of color j), in other words, in every moment, every processor of the system is being activated. In this sense, the use of algorithms based on equitable total coloring does not allow the existence of idle processors in any step of the computation.

Notes

Moreover, it is not difficult to verify that topologies such as mesh, hypercube, and torus do not always admit a range vertex coloring of order Δ with $\Delta + 1$ colors. In contrast, among the topologies obtained by the cartesian product, the hypercube is the most scalable and the only one that allows recursive increase while preserves its original structure. Under this perspective, harmonic graphs, besides being scalable and having a recursive structure, they also present as an advantage the fact that they admit an equitable total coloring obtained by the natural extension of a range coloring of order Δ with Δ+ 1 colors.

VI. Conclusions

In this article, we used the concept of the functional product of graphs to build a family of regular graphs that admits a range vertex coloring Δ with $\Delta + 1$ colors, denominated harmonic graphs. We also proved that the harmonic graphs do not have cut vertices. We showed that the family of harmonic graphs offers advantages in its implementation, as P2P overlay network topology for the communication among MAS because it disposes of a scalable and recursive structure since, from an initial basic instance, it can expand its form dynamically maintaining properties, such as connectedness and regularity. Moreover, a topology based on harmonic graphs offers security against failures, since it does not have cut vertices. We also showed that, from the concept of equitable total coloring, the confection of parallel algorithms could be done generically, which guarantees a natural division of the resources of a network of connections. Finally, we presented a model of connection among multiagent systems based on the use of harmonic graphs as a support for the construction of P2P overlay network topologies used for the communication among the MAS.

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Notes

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On a General Class of Multiple Eulerian Integrals with Multivariable Aleph-Functions

By Frederic Ayant

Abstract- Recently, Raina and Srivastava [5] and Srivastava and Hussain [12] have provided closed-form expressions for a number of a Eulerian integral involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerians integral involving the product of two multivariable Aleph-functions, a class of multivariable polynomials and the general sequence of functions. These integrals will serve as a capital formula from which one can deduce numerous integrals.

Keywords: multivariable aleph-function, multiple eulerian integral, class of polynomials, the sequence of functions, aleph-function of two variables, multivariable h-function, aleph-function of one variable.

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ONAGENERALCLASSOFMULTIPLEEULERIANINTEGRALSWITHMULTIVARIABLEALEPHFUNCTIONS

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Abstract- Recently, Raina and Srivastava [5] and Srivastava and Hussain [12] have provided closed-form expressions for a number of a Eulerian integral involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerians integral involving the product of two multivariable Aleph-functions, a class of multivariable polynomials and the general sequence of functions. These integrals will serve as a capital formula from which one can deduce numerous integrals.

Keywords: multivariable aleph-function, multiple eulerian integral, class of polynomials, the sequence of functions, aleph-function of two variables, multivariable h-function, aleph-function of one variable.

I. Introduction and Prerequisites

The well-known Eulerian Beta integral [6]

$$\int_a^b (z-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(Re(\alpha)>0, Re(\beta)>0, b>a)$$

is a basic result of evaluation of numerous other potentially useful integrals involving various special functions and polynomials. The mathematicians Raina and Srivastava [7], Saigo and Saxena [8], Srivastava and Hussain [14], Srivastava and Garg [13] et cetera have established some Eulerian integrals involving a various general class of polynomials, Meijer's G-function and Fox's H-function of one and more variables with general arguments. Recently, several Author study some multiple a Eulerian integrals, see Bhargava [4], Goyal and Mathur [5], Ayant [3] and others. In this paper we obtain general multiple Eulerians integral of the product of two multivariable Aleph-functions, a general class of multivariable polynomials [12] and the general sequence of functions.

For this study, we need the following series formula for the general sequence of functions introduced by Agrawal and Chaubey [1] and was established by Salim [9].

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}] = \sum_{w,v,u,t',e,k_1,k_2} \psi(w,v,u,t',e,k_1,k_2)x^R$$

where

$$\psi(w,v,u,t',e,k_1,k_2) = \frac{(-)^{t'+w+k_2}(-v)_u(-t')_e(\alpha)_t l^n}{w!v!u!t'!e!l'_n k_1!k_2!}\frac{\mathfrak{s}^{w+k_1}F^{\gamma n-t'}}{(1-\alpha-t')_e}(-\alpha-\gamma n)_e(-\beta-\delta n)_v$$

$$g^{v+k_2}h^{\delta n-v-k_2}(v-\delta n)_{k_2}E^{t'}\left(\frac{pe+\tau w+\lambda+qu}{l}\right)_n \tag{1.3}$$

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and $\sum_{w,v,u,t',e,k_1,k_2} = \sum_{w=0}^{\infty}\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{t'=0}^{n}\sum_{e=0}^{t'}\sum_{k_1,k_2=0}^{\infty}$

The infinite series on the right-hand side of (1.3) is convergent and $R = ln + qv + pt' + \tau w + \tau k_1 + k_2 q$

We shall note $R_n^{\alpha,\beta}[x; E, F, g, h; p, q; \gamma; \delta; e^{-sx^{\tau}}] = R_n^{\alpha,\beta}(x)$

The class of multivariable polynomials defined by Srivastava [29], is given in the following manner:

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \quad (1.4)$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary real or complex constants.

We shall note $a_v = \frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]$

The Aleph-function of several variables is an extension the multivariable I-function defined by Sharma and Ahmad [11], itself is a generalization of G and H-functions of several variables studied by Srivastava et Panda [16,17]. The multiple Mellin-Barnes integrals occurring in this paper will refer to as the multivariable Aleph-function of r-variables throughout our present study and will be defined and represented as follows (see Ayant [2]).

We have

$$\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot & \cdot \\ \cdot & \\ \cdot & \cdots\cdots\cdots\cdots , \\ \mathrm{z}_r & \end{array}\right.$$

$$[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;$$
$$[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}] : [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;$$

$$\left.\begin{array}{c}[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (1.5)$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]}$$

Ref

11. C.K. Sharma and S.S. Ahmad, On the multivariable I-function. Acta ciencia Indica Math, 1994 vol 20,no2, 113-116.

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For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0 \quad (1.6)$$

With $k = 1, \cdots, r, i = 1, \cdots, R\ , i^{(k)} = 1, \cdots, R^{(k)}$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions concerning the multivariable Aleph-function.

If all the poles of (1.8) are simples, then the integral (1.6) can be evaluated with the help of the residue theorem to give

$$\aleph(z_1, \cdots, z_r) = \sum_{G_k=1}^{m_k} \sum_{g_k=0}^{\infty} \phi \frac{\prod_{k=1}^{r} \phi_k z_k^{\eta_{G_k,g_k}} (-)^{\sum_{k=1}^{r} g_k}}{\prod_{k=1}^{r} \delta_{G^{(k)}}^{(k)} \prod_{k=1}^{r} g_k!}$$

where

$$\phi = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(i)} S_k)}{\sum_{i=1}^{R} [\tau_i \prod_{j=\mathfrak{n}+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} S_k) \prod_{j=1}^{q_k} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} S_k)]}$$

$$\phi_k = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} S_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} S_k)}{\sum_{i^{(i)}=1}^{R^{(i)}} [\tau_{i^{(i)}} \prod_{j=m_i+1}^{q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} S_k) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} S_k)]}$$

and

$$S_k = \eta_{G_k,g_k} = \frac{d_{g_k}^{(k)} + G_k}{\delta_{g_k}^{(k)}} \text{ for } k = 1, \cdots, r$$

which is valid under the following conditions: $\epsilon_{M_k}^{(k)}[p_j^{(k)} + p_k'] \neq \epsilon_j^{(k)}[p_{M_k} + g_k]$

We shall note $\aleph(z_1, \cdots, z_r) = \aleph_1(z_1, \cdots, z_r)$ and

$$\aleph(z_1', \cdots, z_s') = \aleph^{0,n':m_1',n_1',\cdots,m_s',n_s'}_{p_i',q_i',\iota_i;r':p'_{i(1)},q'_{i(1)},\iota_{i(1)};r^{(1)};\cdots;p'_{i(s)},q'_{i(s)};\iota_{i(s)};r^{(s)}} \left(\begin{array}{c|c} \mathrm{z}_1' & [(\mathrm{u}_j; \mu_j^{(1)}, \cdots, \mu_j^{(s)})_{1,n'}], \\ \cdot & \cdot \\ \cdot & \\ \cdot & \cdots\cdots\cdots\cdots\cdots\cdots , \\ \mathrm{z}_s' & \end{array} \right.$$

$$[\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots, \mu_{ji}^{(s)})_{n'+1,p_i'}] : [(\mathrm{a}_j^{(1)}); \alpha_j^{(1)})_{1,n_1'}], [\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{n_1'+1,p_i'^{(1)}}]; \cdots ;$$

$$[\iota_i(v_{ji}; \upsilon_{ji}^{(1)}, \cdots, \upsilon_{ji}^{(s)})_{m'+1,q_i'}] : [(\mathrm{b}_j^{(1)}); \beta_j^{(1)})_{1,m_1'}], [\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{m_1'+1,q_i'^{(1)}}]; \cdots ;$$

Ref

2. F.Y.Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31 No.3 (2016), 142-154.

$$\left.\begin{array}{c}[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,n_s'}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n_s'+1,p_i'^{(s)}}]\\ \cdot\\ [(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,m_s'}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m_s'+1,q_i'^{(s)}}]\end{array}\right)=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k'^{\,t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \quad (1.8)$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{n'}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=n'+1}^{p_i'}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{q_i'}\Gamma(1-\upsilon_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]}$$

and $$\phi_k(t_k)=\frac{\prod_{j=1}^{m_k'}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{n_k'}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=m_k'+1}^{q_{i^{(k)}}'}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=n_k'+1}^{p_{i^{(k)}}'}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]}$$

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.14) is obtained by extension of the corresponding conditions for multivariable H-function given by as:

$|argz_k'|<\frac{1}{2}B_i^{(k)}\pi$ Where

$$B_i^{(k)}=\sum_{j=1}^{n'}\mu_j^{(k)}-\iota_i\sum_{j=n'+1}^{p_i'}\mu_{ji}^{(k)}-\iota_i\sum_{j=1}^{q_i'}\upsilon_{ji}^{(k)}+\sum_{j=1}^{n_k'}\alpha_j^{(k)}-\iota_{i^{(k)}}\sum_{j=n_k'+1}^{p_{i^{(k)}}'}\alpha_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k'}\beta_j^{(k)}-\iota_{i^{(k)}}\sum_{j=m_k'+1}^{q_{i^{(k)}}'}\beta_{ji^{(k)}}^{(k)}>0 \quad (1.9)$$

with $k=1,\cdots,s;i=1,\cdots,r';i^{(k)}=1,\cdots,r^{(k)}$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form:

$$\aleph(z_1,\cdots,z_s)=0(\,|z_1|^{\alpha_1'},\cdots,|z_s|^{\alpha_s'}\,),max(\,|z_1|,\cdots,|z_s|\,)\to 0$$

$$\aleph(z_1,\cdots,z_s)\;=0(\,|z_1|^{\beta_1'},\cdots,|z_s|^{\beta_s'}\,),min(\,|z_1|,\cdots,|z_s|\,)\to\infty$$

Where $k=1,\cdots,z:\alpha_k'=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m_k'$ and

$$\beta_k'=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n_k'$$

We shall note $\aleph(z_1',\cdots,z_s')=\aleph_2(z_1',\cdots,z_s')$

II. Integral Representation of Generalized Hypergeometric Function

The following generalized hypergeometric function regarding multiple integrals contour is also required [15, page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\,{}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (2.1)$$

Ref

2. F.Y.Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31 No.3 (2016), 142-154.

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (2.1) is easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$
The equivalent form of Eulerian beta integral is given by (1.1):

III. Main Integral

Notes

We shall note:

$$V=m_1',n_1';\cdots;m_s',n_s';1,0;\cdots;1,0;1,0;\cdots;1,0$$

$$W=p_{i^{(1)}}',q_{i^{(1)}}',\iota_{i^{(1)}};r^{(1)};\cdots;p_{i^{(s)}}',q_{i^{(s)}}',\iota_{i^{(s)}};r^{(s)};\ 0,1;\cdots;0,1;0,1;\cdots;0,1$$

$$\mathbb{A}=[1+\sigma_i^{(1)}-\sum_{k'=1}^{v}K_{k'}\rho_i''^{(1,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(1,k)}-\theta_i^{(1)}R;\rho_i'^{(1,1)},\cdots,\rho_i'^{(1,s)},\tau_i^{(1,1)},\cdots,\tau_i^{(1,l)},1,0,\cdots,0]_{1,s},\cdots,$$

$$[1+\sigma_i^{(T)}-\sum_{k'=1}^{v}K_{k'}\rho_i''^{(T,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(T,k)}-\theta_i^{(T)}R;\rho_i'^{(T,1)},\cdots,\rho_i'^{(T,s)},\tau_i^{(T,1)},\cdots,\tau_i^{(T,l)},1,0,\cdots,0]_{1,s},$$

$$[1-A_j;0,\cdots,0,1,\cdots,1,0,\cdots,0]_{1,P},$$

$$[1-\alpha_i-\sum_{k'=1}^{v}K_{k'}\delta_i''^{(k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\delta_i^{(k)}-R\zeta_i;\delta_i'^{(1)},\cdots,\delta_i'^{(s)},\mu_i^{(1)},\cdots,\mu_i^{(l)},1,\cdots,1,0,\cdots,0]_{1,s},$$

$$[1-\beta_i-\sum_{k'=1}^{v}K_{k'}\eta_i''^{(k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\eta_i^{(k)}-R\lambda_i;\eta_i'^{(1)},\cdots,\eta_i'^{(s)},\theta_i^{(1)},\cdots,\theta_i^{(l)},1,\cdots,1,0,\cdots,0]_{1,s}$$

$$\mathbf{A}=\{(u_j;\mu_j^{(1)},\cdots,\mu_j^{(s)},0,\cdots,0,0\cdots,0)_{1,n'}\},\{\iota_i(u_{ji};\mu_{ji}^{(1)},\cdots,\mu_{ji}^{(s)},0,\cdots,0,0\cdots,0)_{n'+1,p_i'}\}:$$

$$;\{(a_j^{(1)};\alpha_j^{(1)})_{1,n_1'},\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{n_1'+1,p_{i^{(1)}}'}\};\cdots;\{(a_j^{(s)};\alpha_j^{(s)})_{1,n_s'},\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{n_s'+1,p_{i^{(s)}}'}\};$$

$$(1,0),\cdots,(1,0);(1,0),\cdots,(1,0)$$

$$\mathbb{B}=[1+\sigma_i^{(1)}-\sum_{k'=1}^{v}K_{k'}\rho_i''^{(1,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(1,k)}-\theta_i^{(1)}R;\rho_i'^{(1,1)},\cdots,\rho_i'^{(1,s)},\tau_i^{(1,1)},\cdots,\tau_i^{(1,l)},0,\cdots,0]_{1,s},\cdots,$$

$$[1+\sigma_i^{(T)}-\sum_{k'=1}^{v}K_{k'}\rho_i''^{(T,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(T,k)}-\theta_i^{(T)}R;\rho_i'^{(T,1)},\cdots,\rho_i'^{(T,s)},\tau_i^{(T,1)},\cdots,\tau_i^{(T,l)},0,\cdots,0]_{1,s},$$

$$[1-B_j;0,\cdots,0,1,\cdots,1,0,\cdots,0]_{1,Q},$$

$$[1-\alpha_i-\beta_i-\sum_{k'=1}^{v}(\delta_i''^{(k')}+\eta_i''^{(k')})K_{k'}-\sum_{k=1}^{r}(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}-(\zeta_i+\lambda_i)R;$$

$$(\delta_i'^{(1)}+\eta_i'^{(1)}),\cdots,(\delta_i'^{(s)}+\eta_i'^{(s)}),(\mu_i^{(1)}+\theta_i^{(1)}),\cdots,(\mu_i^{(l)}+\theta_i^{(l)}),1,\cdots,1]_{1,s}$$

$$\mathbf{B}=\{\iota_i(v_{ji};v_{ji}^{(1)},\cdots,v_{ji}^{(s)},0,\cdots,0,0,\cdots,0)_{m'+1,q_i'}\}:\{(b_j^{(1)};\beta_j^{(1)})_{1,m_1'},\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{m_1'+1,q'_{i^{(1)}}}\};\cdots;$$

$$\{(b_j^{(s)};\beta_j^{(s)})_{1,m_s'},\iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{m_s'+1,q'_{i^{(s)}}}\}\quad;(0,1),\cdots,(0,1);(0,1),\cdots,(0,1)$$

We have the following multiple Eulerian integrals, we obtain the Aleph-function of $(r+l+T)-$ variables.

Theorem

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$\aleph_1\begin{pmatrix} \mathrm{z}_1\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{pmatrix}\aleph_2\begin{pmatrix} \mathrm{z}_1'\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s'\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i'^{(s)}}(v_i-x_i)^{\eta_i'^{(s)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,s)}}}\right]\end{pmatrix}$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \mathrm{z}_1''\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_v''\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right]\end{pmatrix}R_n^{\alpha,\beta}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi'(w,v,u,t',e,k_1,k_2)$$

Notes

$$\aleph^{0,sT+P+2s+n':V}_{U_{sT+P+2s;sT+Q+s}:W}\left(\begin{array}{c|c} \mathrm{z}'_1 w_1 & \mathbb{A},\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_s w_s & \cdot \\ \mathrm{g}_1 W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{g}_l W_l & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{G}_T & \mathbb{B},\ \mathbf{B} \end{array}\right) \tag{3.1}$$

where $U_{st+P+2s;sT+Q+s} = sT+P+2s+p'_i, sT+Q+s+q'_i, \iota_i; r'$

$$\psi'(w,v,u,t',e,k_1,k_2) = \frac{\psi(w,v,u,t,e,k_1,k_2)}{\prod_{i=1}^{t}\prod_{j=1}^{W}(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^{v}\rho_i^{\prime\prime(j,k')}K_{k'}+\sum_{k=1}^{r}\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}R}}$$

$$\times\frac{\prod_{i=1}^{t}(v_i-u_i)^{\sum_{k'=1}^{v}(\delta_i^{\prime\prime(k')}+\eta_i^{\prime\prime(k')})K_{k'}+\sum_{k=1}^{r}(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}+(\zeta_i+\lambda_i)R}}{\prod_{i=1}^{t}\prod_{j=W+1}^{T}(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^{v}\rho_i^{\prime\prime(j,k')}K_{k'}+\sum_{k=1}^{r}\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}R}}$$

$$w_m = \prod_{i=1}^{t}\left[(v_i-u_i)^{\delta_i^{\prime(m)}+\eta_i^{\prime(m)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i^{\prime(j,m)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\rho_i^{\prime(j,m)}}\right], m=1,\cdots,s$$

$$W_k = \prod_{i=1}^{t}\left[(v_i-u_i)^{\mu_i^{(k)}+\theta_i^{(k)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\right], k=1,\cdots,l$$

$$G_j = \prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=1,\cdots,W$$

$$G_j = -\prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=W+1,\cdots,T$$

$$\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty} = \sum_{G_1,\cdots,G_r=1}^{m_1,\cdots,m_r}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

Provided that:

(A) $W\in[0,T]; u_i, v_i\in\mathbb{R}; i=1,\cdots,t$

(B) $min\{\delta_i^{(g)},\eta_i^{(g)},\delta_i^{\prime(h)},\eta_i^{\prime(h)},\delta_i^{\prime\prime(k)},\eta_i^{\prime\prime(k)},\zeta_i,\eta_i\}\geqslant 0; g=1,\cdots,r; i=1,\cdots,t; h=1,\cdots,s; k=1,\cdots,v$

$min\{\rho_i^{(j,g)},\rho_i^{\prime(j,h)},\rho_i^{\prime\prime(j,k')},\theta_i^{(j)},\tau_i^{(j,k)}\}\geqslant 0; j=1,\cdots,T; i=1,\cdots,t; g=1,\cdots,r; h=1,\cdots,s; k'=1,\cdots,v, k=1,\cdots,l$

(C) $\sigma_i^{(j)} \in \mathbb{R}, U_i^{(j)}, V_i^{(j)} \in \mathbb{C}, z_{i'}, z_{j'}, z''_{k'}, g_k, G_j \in \mathbb{C}; i=1,\cdots,t; j=1,\cdots,T; i'=1,\cdots,r;$

$j'=1,\cdots,s; k'=1,\cdots,v; k=1,\cdots,l$

(D) $max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right] < 1, i=1,\cdots,s; j=1,\cdots,W$ and

$$max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right] < 1, i=1,\cdots,s; j=W+1,\cdots,T$$

(E) $\left|arg\left(z_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i^{(j,k)}}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} - \delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^{T}\rho_i^{(j,k)} > 0$$

$$\left|arg\left(z'_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i'^{(j,k)}}\right)\right| < \frac{1}{2}B_i^{(k)}\pi \text{, where}$$

$$B_i^{(k)} = \sum_{j=1}^{n'}\mu_j^{(k)} - \iota_i\sum_{j=n'+1}^{p'_i}\mu_{ji}^{(k)} - \iota_i\sum_{j=1}^{q'_i}\upsilon_{ji}^{(k)} + \sum_{j=1}^{n'_k}\alpha_j^{(k)} - \iota_{i^{(k)}}\sum_{j=n'_k+1}^{p'_{i^{(k)}}}\alpha_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m'_k}\beta_j^{(k)} - \iota_{i^{(k)}}\sum_{j=m'_k+1}^{q'_{i^{(k)}}}\beta_{ji^{(k)}}^{(k)}$$

$$-\delta_i'^{(k)} - \eta_i'^{(k)} - \sum_{j=1}^{T}\rho_i'^{(j,k)} > 0$$

(F) $Re\left(\alpha_i+\zeta_iR+\sum_{j=1}^{r}\delta_i^{(j)}\eta_{G_j,g_j}\right) + \sum_{k=1}^{s}\delta_i'^{(k)}\min_{1\leqslant j\leqslant m'_k} R\left(\frac{b_j^{(k)}}{\beta_j^{(k)}}\right) > 0$ and

$$Re\left(\beta_i+\lambda_iR+\sum_{j=1}^{r}\eta_i^{(j)}\eta_{G_j,g_j}\right) + \sum_{k=1}^{s}\eta_i'^{(k)}\min_{1\leqslant j\leqslant m'_k} Re\left(\frac{b_j^{(k)}}{\beta_j^{(k)}}\right) > 0 \text{ for } i=1,\cdots,t$$

(G) $P \leqslant Q+1$. The equality holds, also,

either $P>Q$ and $\sum_{k=1}^{l}\left|g_k\left(\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\tau_i^{(j,k)}}\right)\right|^{\frac{1}{Q-P}} < 1 \qquad (u_i \leqslant x_i \leqslant v_i; i=1,\cdots,t)$

Notes

or $\quad P \leqslant Q$ and $\max\limits_{1\leqslant k\leqslant l}\left[\left|\left(g_k\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{-\tau_i^{(j,k)}}\right)\right|\right]<1 \quad (u_i\leqslant x_i\leqslant v_i; i=1,\cdots,t)$

Proof

Notes

To establish the formula (3.1), we first express the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ in series with the help of (1.4), the multivariable Aleph-function $\aleph_1(z_1,\cdots,z_r)$ in serie with the help of (1.7), the sequence of functions in series with the help of (1.4), use integral contour representation with the help of (1.8) for the multivariable Alephfunction $\aleph_2(z'_1,\cdots,z'_s)$ occurring in its left-hand side and use the integral contour representation with the help of (2.1) for the Generalized hypergeometric function ${}_PF_Q(.)$.

Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now we write:

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}$$

where $K_i^{(j)}=\upsilon_i^{(j)}-\theta_i^{(j)}R-\sum_{l=1}^{r}\rho_i^{(j,l)}\eta_{G_l,g_l}-\sum_{l=1}^{s}\rho_i^{\prime(j,l)}\psi_l-\sum_{l=1}^{v}\rho_i^{\prime\prime(j,v)}K_l$ where $i=1,\cdots,t; j=1,\cdots,T$

and express the factors occurring in R.H.S. of (3.1) regarding the following Mellin-Barnes integrals contour, we obtain:

$$\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^W}\int_{L'_1}\cdots\int_{L'_W}\prod_{j=1}^{W}\Big[\Gamma(-\zeta'_j)\Gamma(-K_i^{(j)}+\zeta'_j)$$

$$\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i-u_i)}{(u_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta'_j}\Big]\,\mathrm{d}\zeta'_1\cdots\mathrm{d}\zeta'_W \tag{3.3}$$

and

$$\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^{T-W}}\int_{L'_{W+1}}\cdots\int_{L'_T}\prod_{j=W+1}^{T}\Big[\Gamma(-\zeta'_j)\Gamma(-K_i^{(j)}+\zeta'_j)$$

$$\prod_{j=W+1}^{T}\left[-\frac{(U_i^{(j)}(v_i-x_i)}{(v_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta'_j}\Big]\,\mathrm{d}\zeta'_{W+1}\cdots\mathrm{d}\zeta'_T \tag{3.4}$$

We apply the Fubini's theorem for multiple integrals. Finally evaluating the innermost **x**-integral with the help of (1.1) and reinterpreting the multiple Mellin-Barnes integrals contour in terms of multivariable Aleph-function of $(r+l+T)$–variables, we obtain the formula (3.1).

IV. Particular Cases

a) Multivariable H functions

Here, the multivariable Aleph-functions $\aleph_1$ and $\aleph_2$ reduce to multivariable H-functions defined by Srivastava and Panda [16,17], we have the following Eulerian integral:

Corollary 1

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$H_1\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{array}\right) H_2\left(\begin{array}{c} \mathrm{z}'_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_s\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(s)}}(v_i-x_i)^{\eta_i'^{(s)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,s)}}}\right]\end{array}\right)$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right]\end{array}\right) R_n^{\alpha,\beta}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi'(w,v,u,t',e,k_1,k_2)$$

Ref

17. H.M. Srivastava and R.Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. (1976), 265-274.

$$H^{0,sT+P+2s+n':V}_{sT+P+2s+p',sT+Q+s+q':W'}\left(\begin{array}{c|c} \mathrm{z}'_1 w_1 & \mathbb{A},\ \mathbf{A}' \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_s w_s & \cdot \\ \mathrm{g}_1 W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{g}_l W_l & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{G}_T & \mathbb{B},\ \mathbf{B}' \end{array}\right) \tag{4.1}$$

where $W = p'_1, q'_1; \cdots ; p'\ \ q'; 0\ 1; \cdots ; 0\ 1; 0\ 1; \cdots ; 0\ 1$ and the validity conditions are the same that (3.1) for $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}}, \iota_i, \iota_{i^{(1)}}, \cdots, \iota_{i^{(s)}} \to 1$ and $R = R^{(1)} = \cdots = R^{(r)} = r' = r^{(1)} = \cdots = r^{(s)} .= 1$ (condition 1)

The quantities $\mathbf{A}', \mathbf{B}'$ are equal respectively to A, B with the conditions 1.

b) Aleph-functions of two variables

If $r = s = 2$, the multivariable Aleph-functions reduce to Aleph-functions of two variables defined by Sharma [10].

Corollary 2

$$\int_{u_1}^{v_1} \cdots \int_{u_t}^{v_t} \prod_{i=1}^{t} \left[(x_i - u_i)^{\alpha_i - 1} (v_i - x_i)^{\beta_i - 1} \prod_{j=1}^{T} (U_i^{(j)} x_i + V_i^{(j)})^{\sigma_i^{(j)}} \right]$$

$$\aleph_1 \left(\begin{array}{c} \mathrm{z}_1 \prod_{i=1}^t \left[\frac{(x_i-u_i)^{\delta_i^{(1)}} (v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_2 \prod_{i=1}^t \left[\frac{(x_i-u_i)^{\delta_i^{(2)}} (v_i-x_i)^{\eta_i^{(2)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i^{(j,2)}}} \right] \end{array} \right) \aleph_2 \left(\begin{array}{c} \mathrm{z}'_1 \prod_{i=1}^t \left[\frac{(x_i-u_i)^{\delta_i'^{(1)}} (v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i'^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}'_2 \prod_{i=1}^t \left[\frac{(x_i-u_i)^{\delta_i'^{(2)}} (v_i-x_i)^{\eta_i'^{(2)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i'^{(j,2)}}} \right] \end{array} \right)$$

$$S^{\mathfrak{M}_1, \cdots, \mathfrak{M}_v}_{N_1, \cdots, N_v} \left(\begin{array}{c} \mathrm{z}''_1 \prod_{i=1}^t \left[\frac{(x_i-u_i)^{\delta_i''^{(1)}} (v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i''^{(j,1)}}} \right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}''_v \prod_{i=1}^t \left[\frac{(x_i-u_i)^{\delta_i''^{(v)}} (v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^T \left(U_i^{(j)} x_i + V_i^{(j)}\right)^{\rho_i''^{(j,v)}}} \right] \end{array} \right) R_n^{\alpha,\beta} \left[\prod_{j=1}^{t} \left[\frac{(x_i - u_i)^{\zeta_i} (v_i - x_i)^{\lambda_i}}{\prod_{j=1}^{T} \left(U_i^{(j)} x_i + V_i^{(j)} \right)^{\theta_i^{(j)}}} \right] \right]$$

Ref

10. K. Sharma, On the integral representation and applications of the generalized function of two variables, International Journal of Mathematical Engineering and Sciences, 3(1) (2014), 1-13.

$$
{}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l} g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t
$$

$$
=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]
$$

$$
\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{G_2=1}^{m_2}\sum_{g_1,g_2=0}^{\infty}\phi_2\frac{\prod_{k=1}^{2}\phi_{2k}z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{2}g_k}}{\prod_{k=1}^{2}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{2}g_k!}\,{}_vz_1''^{K_1}\cdots z_v''^{K_v}\psi'(w,v,u,t',e,k_1,k_2)
$$

$$
\aleph_{U_{sT+P+2s;sT+Q+s}:W_2}^{0,n'+sT+P+2s:V_2}\left(\begin{array}{c|c}
\mathrm{z}_1'w_1 & \mathbb{A}_2,\ \mathbf{A}_2\\
\cdot & \cdot\\
\cdot & \cdot\\
\mathrm{z}_2'w_2 & \cdot\\
\mathrm{g}_1W_1 & ;\\
\cdot & \cdot\\
\cdot & \cdot\\
\mathrm{g}_lW_l & \cdot\\
\mathrm{G}_1 & \cdot\\
\cdot & \cdot\\
\cdot & \cdot\\
\mathrm{G}_T & \mathbb{B}_2,\ \mathbf{B}_2
\end{array}\right) \tag{4.2}
$$

The validity conditions are the same that (3.1) with $r=s=2$. The quantities $\phi_2,\phi_{2k},V_2,W_2,\mathbb{A}_2,\mathbb{B}_2,\mathbf{A}_2,\mathbf{B}_2$ are equal to $\phi,\phi_k,V,W,\mathbb{A},\mathbb{B},\mathbf{A},\mathbf{B}$ respectively for $r=s=2$.

c) Aleph-function of one variable

Corollary 3

If $r=s=1$, the multivariable Aleph-functions reduce to Aleph-functions of one variable defined by Sudland [18].

$$
\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]
$$

$$
\aleph_1\left(\mathrm{z}\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\right)\aleph_2\left(\mathrm{z}'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\right)
$$

Ref

18. N. Südland, B. Baumann and T.F. Nonnenmacher T, Open problem: who knows about the Aleph-functions? Fract. Calc. Appl. Anal, 1(4) (1998): 401-402.

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} z''_1 \prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ z''_v \prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right] \end{pmatrix} R_n^{\alpha,\beta}\left[\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l} g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_t$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{w,v,u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{g_1=0}^{\infty}\phi_1\frac{z_k^{\eta_{G_1,g_1}}(-)^{g_1}}{\delta_{G^{(1)}}^{(1)}g_1!}a_v z_1''^{K_1}\cdots z_v''^{K_v}\psi'(w,v,u,t',e,k_1,k_2)$$

$$\aleph_{U_{sT+P+2s;sT+Q+s}:W_1}^{0,n'+sT+P+2s:V_1}\left(\begin{array}{c|c} \text{z'w}_1 & \mathbb{A}_1,\ \mathbf{A}_1 \\ \text{g}_1W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \text{g}_lW_l & \cdot \\ \text{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \\ \text{G}_T & \mathbb{B}_1,\ \mathbf{B}_1 \end{array}\right) \tag{4.3}$$

The validity conditions are the same that (3.1) with $r=s=1$. The quantities $\phi_1,V_1,W_1,\mathbb{A}_1,\mathbb{B}_1,\mathbf{A}_1,\mathbf{B}_1$ are equal to $\phi_k,V,W,\mathbb{A},\mathbb{B},\mathbf{A},\mathbf{B}$ respectively for $r=s=1$ and

$$U_{st+P+2s;sT+Q+s}=sT+P+2s+p'_{i^{(1)}},sT+Q+s+q'_{i^{(1)}},\iota_{i^{(1)}};r^{(1)}.$$

Remark: By the similar procedure, the results of this document can be extended to the product of any finite number of multivariable Aleph-functions and class of multivariable polynomials defined by Srivastava [12].

V. Conclusion

Our main integral formula is unified in nature and possesses manifold generality. It acts a fundamental expression and using various particular cases of the multivariable Aleph-function, the class of multivariable polynomials and a general sequence of functions, we can obtain a large number of other integrals involving simpler special functions and polynomials of one and several variables.

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12. H.M. Srivastava, A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), 183-191.

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Notes

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On Some Geometric Methods in Mathematics and Mechanics

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Abstract- We give a survey of geometric methods used in papers and books of V.I. Arnold and V.V. Kozlov. They are methods of different normal forms, of some polyhedra, of small denominators and asymptotic expansions.

Keywords: geometric methods, normal form, asymptotic expansion, Painlevé equations, Hamiltonian system.

GJSFR-F Classification: *MSC 2010: 05B35*

ONSOMEGEOMETRICMETHODSINMATHEMATICSANDMECHANICS

On Some Geometric Methods in Mathematics and Mechanics

Alexander D. Bruno

Abstract- We give a survey of geometric methods used in papers and books of V.I. Arnold and V.V. Kozlov. They are methods of different normal forms, of some polyhedra, of small denominators and of asymptotic expansions.

Keywords: *geometric methods, normal form, asymptotic expansion, Painlevé equations, Hamiltonian system.*

I. Introduction

In the paper Khesin et al. (2012) there was given a short description of main achievements of V.I. Arnold. Below in Sections 2-4, we give some additions to several Sections of this paper. Here in Sections 5, 6, 8, 9 we discuss two kinds of normal forms in publications by V.I. Arnold and by V.V. Kozlov.

Logarithmic branching of solutions to Painlevé equations is discussed here in Section 7.

In this article all the formulas denoted as (n*) refer to the formulas in the the cited papers.

II. On the Last Paragraph of Page 381 in Khesin et al. (2012) Devoted to Small Divisors

Arnold's Theorem on the stability of the stationary point in the Hamiltonian system with two degrees of freedom in Arnold (1963) had the wrong formulation (see Bruno (1972, § 12, Section IVd)). Then V.I. Arnold Arnold (1968) added one more condition in his Theorem, but its proof was wrong because it used the wrong statement (see Bruno (1985, 1986)). All mathematical world was agreed with my critics except V.I. Arnold. On the other hand, in the first proof of the same Theorem by J. Moser Moser (1968) there was a similar mistake (see Bruno (1972, § 12, Section IVe)). But in Siegel et al. (1971) J. Moser corrected his proof after my critics, published in Bruno (1972, § 12, Section IVe).

Concerning the KAM theory. In 1974 I developed its generalization via normal forms Bruno (1974, 1980, 1989, Part II). But up-to-day almost nobody understands my generalization.

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III. On the Last Paragraph of Page 384 in Khesin et al. (2012) Concerning Higher-Dimensional Analog of the Continued Fraction

The paper Lauchand (1993) "Polyèdre d'Arnol'd et ..." by G. Lachaud (1993) was presented to C.R. Acad. Sci. Paris by V.I. Arnold. When I saw the article, I published the article Bruno et al. (1994) "Klein polyhedra ..." (1994), because so-called "Arnold polyhedra" were proposed by F. Klein one hundred years early. Moreover, they were introduced by B.F. Scubenko (1991) as well. In 1994–2000 me and V.I. Parusnikov studied Klein polyhedra from algorithmic point of view and found that they do not give a basis for algorithm generalizing the continued fraction. So in 2003, I proposed another approach and another unique polyhedron, which give a basis for the generalization in 3 and any dimension (see Bruno (2005a,b); Bruno et al. (2009); Bruno (2010a,b, 2015c)). Now there are a lot of publications on the Klein polyhedra and their authors following after V.I. Arnold wrongly think that the publications are on the generalization of the continued fraction.

IV. On the Last Two Paragraphs of Page 395 in Khesin et al. (2012) Devoted to Newton Polygon

In that text, the term "Newton polygon" must be replaced by "Newton polyhedron". In contemporary terms, I. Newton introduced *support* and one *extreme edge* of the *Newton open polygon* for one polynomial of two variables. The full Newton open polygon was proposed by V. Puiseux (1850) and by C. Briot and T. Bouquet (1856) for one ordinary differential equation of the first order. Firstly a polyhedron as the convex hull of the support was introduced in my paper Bruno (1962) for an autonomous system of n ODEs. During 1960–1969 V.I. Arnold wrote three reviews on my works devoted to polygons and polyhedrons for ODEs with sharp critics "of the geometry of power exponents" (see my book Bruno (2000, Ch. 8, Section 6)). Later (1974) he introduced the name "Newton polyhedron", made the view that it is his invention and never gave references on my works. Now I have developed "Universal Nonlinear Analysis" which allows to compute asymptotic expansions of solutions to equations of any kind (algebraic, ordinary differential and partial differential) Bruno (2015a).

V. On Non-Hamiltonian Normal Form

In my paper Bruno (1964) and my candidate thesis "Normal form of differential equations" (1966) I introduced normal forms in the form of power series. It was a new class of them. Known before normal forms were either linear (Poincare, 1879) Poincare (1879) or polynomial (Dulac, 1912) Dulac (1912). An official opponent was A.N. Kolmogorov. He estimated very high that new class of normal forms. V.I. Arnold put my normal form into his book Arnold (1978, 1998, § 23) without reference to my publication and named it as "Poincare-Dulac normal form". So, readers of his book attributed

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my normal form to Arnold. I saw several articles where my normal form was named as Arnold's.

VI. On Canonical Normalizing Transformation

Notes

In Arnold et al. (1988, Ch. 7, § 3, Subsection 3.1) a proof of Theorem 7 is based on the construction of a generating function $F = \langle P, q\rangle + S_l(P, q)$ in mixed coordinates P, q. Transformation from old coordinates P, Q to new coordinates p, q is given by the formulae

$$p = \frac{\partial F}{\partial q}, \quad Q = \frac{\partial F}{\partial p}. \tag{1}$$

Here $S_l(P, q)$ is a homogeneous polynomial in P and q of order l. According to (1), the transformation from coordinates P, Q to coordinates p, q is given by infinite series, which are results of the resolution of the implicit equations (1). Thus, the next to the last sentence on page 272 (in Russian edition) "The normalizing transformation is constructed in the polynomial form of order $L - 1$ in phase variables" is wrong. Indeed that property has the normalizing transformation computed by the Zhuravlev-Petrov method Bruno et al. (2006).

VII. On Branching of Solutions of Painlevé Equations

In Kozlov et al. (2013, Ch. I, § 4, example 1.4.6) the Painlevé equations are successive considered. In particularly, there was find the expansion

$$x(\tau) = \tau^{-1} \sum_{k=0}^{\infty} x_k \tau^k \tag{2}$$

of a solution to the fifth Painlevé equation. The series (2) is considered near the point $\tau = 0$. After the substitution $\tau = \log t$, we obtain the series

$$x(t) = \log^{-1} t \sum_{k=0}^{\infty} x_k \log^k t, \tag{3}$$

which has a sense near the point $t = 1$, where $\log t = 0$. However, from the last expansion (3) authors concluded that $t = 0$ is the point of the logarithmic branching the solution $x(t)$. It is wrong, because the expansion (3) does not work for $t = 0$ as $\log 0 = \infty$ and the expansion (3) diverges. That mistake is in the first edition of the book Kozlov et al. (2013) (1996) and was pointed out in the paper Bruno et al. (2004a), but it was not corrected in the second "corrected" edition of the book.

A similar mistake is there in consideration of the sixth Painlevé equation. The expansion (2) was obtained for a solution to the sixth Painlevé equation in the same publication. After the substitution $\tau = \log(t(t-1))$, it takes the form

$$x(t) = \log^{-1}(t(t-1)) \sum_{k=0}^{\infty} x_k \log^k(t(t-1)).$$

As the expansion (1) has a sense near the point $\tau = 0$, the last expansion has a sense near points $t = (1 \pm \sqrt{5})/2$, because in them $t(t-1) = 0$ and $\tau = 0$. Thus, the conclusion in the book, that point $t = 0$ and $t = 1$ are the logarithmic branching points of the solution, is noncorrect. The mistake was point out in the paper Bruno et al. (2004b), but it was repeated in the second edition of the book Kozlov et al. (2013). Indeed solutions of the Painlevé equations have logarithmic branching, see Bruno et al. (2011, 2010c); Bruno (2015b).

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VIII. On Integrability of the Euler-Poisson Equations

In the paper Kozlov (1976) Theorem 1 on nonexistence of an additional analytic integral was applied in § 3 to the problem of motion of a rigid body around a fixed point. The problem was reduced to a Hamiltonian system with two degrees of freedom and with two parameters x, y. The system has a stationary point for all values of parameters. Condition on existence of the resonance $3:1$ was written as equation (6*) on parameters x, y. Then the second order form of the Hamiltonian function was reduced to the simplest form by a linear canonical transformation

$$(x_1, x_2, y_1, y_2) \to (q_1, q_2, p_1, p_2). \tag{4}$$

Condition of vanishing the resonant term of the fourth order in the obtained Hamiltonian function was written as equation (7*) on x, y. System of equations (6*) and (7*) was considered for

$$x > 0 \quad \text{and} \quad y > \frac{x}{x+1},$$

where the system has two roots

$$x = \frac{4}{3},\ y = 1 \text{ and } x = 7,\ y = 2. \tag{5}$$

They correspond to two integrable cases $y = 1$ and $y = 2$ of the initial problem. It was mentioned in Theorem 3. But in the whole real plane (x, y) the system of equations (6*) and (7*) has roots (5) and three additional roots

$$x = -\frac{16}{3}, y = 1; \qquad x = -\frac{17}{9}, y = 2; \tag{6}$$

$$x = 0, y = 9. \tag{7}$$

Roots (6) belong to integrable cases $y = 1$ and $y = 2$. But the root (7) is out of them. Indeed the transformation (4) is not defined for $x = 0$. If to make an additional analysis for $x = 0$, then for resonance $3:1$ one obtains two points: (7) and

$$x = 0, \quad y = \frac{1}{9}. \tag{8}$$

Notes

In both these points, the resonant term of the fourth order part of the Hamiltonian function vanishes. But points (7) and (8) are out of the integrable cases $y = 1$ and $y = 2$; they contradict to statement of Theorem 3 Kozlov (1976). The paper Kozlov (1976) was repeated in the book Kozlov (1996, Ch. VI, § 3, Section 3). A non-Hamiltonian study of the problem see in the paper Bruno (2007, Section 5). Nonintegrability at the points (7) and (8) was shown in Bruno (2014).

IX. On Normal Forms of Families of Linear Hamiltonian Systems

Real normal forms of families of linear Hamiltonian systems were given in Galin (1982, § 2), where formula (16*) wrongly indicated the normal form corresponding to the elementary divisor λ^{2l}: the third sum in the formula (16*) has to be omitted. The indicated mistake was reproduced in the first three editions of the book Arnold (1978, Appendix 6) by Arnold. Discussions of that see in the paper Bruno (1988) and in the book Bruno (1994, Ch. I, Section 6, Notes to Section 1.3).

X. Conclusions

Sections 2–4 were sent to Notices of the AMS for publication as a letter to the editor. But Editor S.G. Krantz rejected it. I consider that as one more case of the scientific censorship in the AMS.

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Unified Local Convergence for Some High order Methods with One Parameter

By Ioannis K. Argyros & Santhosh George

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Abstract- The aim of this paper is to extend the applicability of some Chebyshev-Halley-type method with one parameter for solving nonlinear equations under weaker than before hypotheses on the second derivative.

Keywords: *chebyshev-halley methods, banach space, local convergence.*

GJSFR-F Classification: *MSC 2010: 49M15, 47H17*

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Unified Local Convergence for Some High order Methods with One Parameter

Ioannis K. Argyros [α] & Santhosh George [σ]

Abstract- The aim of this paper is to extend the applicability of some Chebyshev-Halley-type method with one parameter for solving nonlinear equations under weaker than before hypotheses on the second derivative.

Keywords: chebyshev-halley methods, banach space, local convergence.

I. Introduction

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces and Ω be an open and convex subset of $\mathcal{B}_1$. The problem of finding a solution x^* of equation

$$F(x) = 0. \tag{1.1}$$

where $F : \Omega \longrightarrow \mathcal{B}_2$ is differentiable in the sense of Fréchet is an important problem in applied mathematics due its wide applications. Higher order methods like [2–17] are considered for approximating the solution x^* of (1.1). The convergence analysis of higher order methods, requires assumptions on higher order derivatives.

A typical example of (1.1), in which the Lipschitz-type condition on derivatives of order greater than two does not hold is the mixed Hammerstein type equation defined on $X = Y = C[0,1]$ by

$$x(s) = \int_0^1 K(s,t)(\frac{1}{2}x(t)^{\frac{5}{2}} + \frac{x(t)^2}{8})dt, \tag{1.2}$$

where the kernel K is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$K(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases} \tag{1.3}$$

Define $F : C[0,1] \longrightarrow C[0,1]$ by

$$F(x)(s) := x(s) - \int_0^1 K(s,t)(\frac{1}{2}x(t)^{\frac{5}{2}} + \frac{x(t)^2}{8})dt \tag{1.4}$$

and consider

$$F(x)(s) = 0. \tag{1.5}$$

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Then, we have that

$$F'(x)\mu(s) = \mu(s) - \int_0^1 K(s,t)(\frac{5}{8}x(t)^{\frac{3}{2}} + \frac{x(t)}{4})\mu(t)dt.$$

Notice that $x^*(s) = 0$ is one of the solutions of (1.1). Using (1.3), we obtain

$$\|\int_0^1 K(s,t)dt\| \le \frac{1}{8}. \tag{1.6}$$

Then, by (1.3)–(1.6), we have that

$$\|F'(x) - F'(y)\| \le \frac{1}{8}(5\|x-y\|^{\frac{1}{4}} + \|x-y\|). \tag{1.7}$$

Note that , F'' is not Lipschitz. Hence the results in [1–17] cannot be used to solve (1.5).

In this paper we study the local convergence of the method defined for each $n = 0, 1, 2...$ [7] by

$$\begin{aligned} x_{n+1} &= x_n - [I + \frac{1}{2}L_n G_n]F'(x_n))^{-1}F(x_n), \quad (1.8)\\ G_n &= I + \frac{\alpha}{2}L_n J_n \\ J_n &= (I - \frac{1}{2}L_n)^{-1} \\ L_n &= F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n), \end{aligned}$$

with $x_0 \in \Omega$ is an initial guess and $\alpha \in \mathbb{R}$. The semilocal convergence of method (1.8) was shown in [7] using hypotheses on F'' satisfying Lipschitz or Hölder or $\omega-$continuity conditions. Here we use $\omega-$type weaker hypotheses to study the local convergence not studied in [7].

The paper is structured as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies [2–17]. Special cases and numerical examples are presented in the concluding Section 3.

II. Local Convergence

The convergence shall be computed based on scalar parameters and functions. Let $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and nondecreasing with $w_0(0) = w(0) = 0$. Let ρ_0 stand for the smallest positive number satisfying

$$w_0(t) = 1. \tag{2.1}$$

Let also w, v, z be real continuous and nondecreasing function defined on the interval $[0, \rho_0)$ with $w(0) = 0$. Define functions q, h_q on interval $[0, \rho_0)$ by

$$q(t) = \frac{1}{2}\left(\frac{1}{1-w_0(t)}\right)^2 \int_0^1 v(\theta t)d\theta z(t)t$$

and

$$h_q(t) = q(t) - 1.$$

We have $h_q(0) = -1 < 0$ and $h_q(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. The application of the intermediate value theorem on $[0, \rho_0)$ guarantees the existence of solutions

Ref

7. J. A. Ezquero and M. A. Hernandez, On a class of iteration containing the Chebyshev and the Halley method, Publ. Math. Debrecen, 54 (1999), 403–415.

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for the equation $h_q(t) = 0$ in $(0, \rho_0)$. Let r_q stand for the smallest such solution. Moreover, define functions f and h_f on the interval $[0, r_q)$ by

$$f(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)} + \frac{q(t)(1+|\alpha-1|q(t))\int_0^1 v(\theta t)d\theta}{(1-w_0(t))(1-q(t))}$$

and

$$h_f(t) = f(t) - 1.$$

We also get $h_f(0) = -1 < 0$ and $h_f(t) \longrightarrow +\infty$ as $t \longrightarrow r_q^-$. Denote by r the smallest solution of equation $h_f(t) = 0$. Then, we have that for each $t \in [0, r)$

$$0 \le q(t) < 1 \tag{2.2}$$

and

$$0 \le f(t) < 1. \tag{2.3}$$

The set $B(u, \lambda) = \{x \in \mathcal{B}_1 : \|x - x_0\| < \lambda\}$ is called the closed ball in $\mathcal{B}_1$ with center $u \in \mathcal{B}_1$ and radius $\lambda > 0$, whereas $\bar{B}(u, \lambda)$ is its closure.

The local convergence analysis is based on the notation introduced previously.

Theorem 2.1 *Let $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchet differentiable operator. Suppose: there exist $x^* \in \Omega$, function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ continuous and nondecreasing with $w_0(0) = 0$ such that*

$$F(x^*) = 0, \;\; F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{2.4}$$

and for all $x \in D$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le w_0(\|x - x^*\|). \tag{2.5}$$

Let $\Omega_0 := \Omega \cap B(x^*, \rho_0)$. There exist functions $w, v, z : [0, \rho_0) \longrightarrow [0, +\infty)$ continuous, nondecreasing with $w(0) = 0$ such that

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le w(\|x - y\|), \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \le v(\|x - x^*\|) \tag{2.7}$$

$$\|F'(x^*)^{-1}F''(x)\| \le z(\|x - x^*\|) \tag{2.8}$$

and

$$\bar{B}(x^*, r) \subseteq \Omega, \tag{2.9}$$

where the radius r is defined previously. Then, iteration $\{x_n\}$produced for $x_0 \in B(x^*, r) - \{x^*\}$ by method (1.8) exists, lies in $B(x^*, r)$ and converges to x^*, so that

$$\|x_{n+1} - x^*\| \le f(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{2.10}$$

where the functions f is defined previously. Moreover, if there exists $r^* \ge r$ such that

$$\int_0^1 w_0(\theta r^*)d\theta < 1,$$

the x^* is the only solution of equation $F(x) = 0$ in $\Omega_1 = \Omega \cap \bar{B}(x^*, r^*)$.

Proof. We shall base the proof on mathematical induction. By hypothesis $x_0 \in B(x^*, r) - \{x^*\}$, (2.5) and the definition of r, we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq w_0(\|x_0 - x^*\|) < w_0(r) < 1. \tag{2.11}$$

The Banach perturbation lemma [1] in combination with (2.11) assert the existence of $F'(x_0)^{-1}$ and the estimate

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.12}$$

Next, we show the existence of x_1 as follows: By (2.4) we can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.13}$$

The point $x^* + \theta(x_0 - x^*) \in B(x^*, r)$, since $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| \leq \|x_0 - x^*\| < r$. Then, by (2.7) and (2.13) we get that

$$\|F'(x^*))^{-1}F(x_0)\| \leq \int_0^1 v(\theta\|x_0 - x^*\|)\|x_0 - x^*\|d\theta. \tag{2.14}$$

We need an upper bound on $\frac{1}{2}L_0$. Using (2.2), (2.8), (2.12) and (2.14), we obtain in turn that

$$\begin{aligned}
\frac{1}{2}\|L_0\| &\leq \frac{1}{2}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F''(x_0)\| \\
&\quad \times\|F'(x_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\leq \frac{1}{2}\left(\frac{1}{1 - w_0(\|x_0 - x^*\|}\right)^2 \int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|z(\|x_0 - x^*\|) \\
&= q(\|x_0 - x^*\|) \leq q(r) < 1,
\end{aligned} \tag{2.15}$$

so J_0^{-1} exists and

$$\||J_0^{-1}\| \leq \frac{1}{1 - q(\|x_0 - x^*\|)}. \tag{2.16}$$

Hence, x_1 is well defined by the first substep of method (1.8) for $n = 0$. By the definition of G_0 we can write

$$G_0 = I + \frac{\alpha}{2}L_0J_0 = [I + \frac{\alpha - 1}{2}L_0]J_0$$

so

$$\begin{aligned}
\|G_0\| &\leq \|I + \frac{\alpha - 1}{2}L_0\|\|J_0\| \\
&\leq \frac{1 + |\alpha - 1|q(\|x_0 - x^*\|)}{1 - q(\|x_0 - x^*\|)}.
\end{aligned} \tag{2.17}$$

Ref

1. I.K. Argyros, Computational theory of iterative methods. Series: Studies in Computational Mathematics, 15, Editors: C.K.Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

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Furthermore, using method (1.8) for $n = 0$, (2.3), (2.6), (2.12), (2.14), (2.16) and (2.17) we have in turn from the identity

$$\begin{aligned} x_1 - x^* &= x_0 - F'(x_0)^{-1}F(x_0) - x^* \\ &\quad -\frac{1}{2}L_0 G_0 F'(x_0)^{-1}F(x_0) \end{aligned} \tag{2.18}$$

that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|F'(x_0)^{-1}F(x^*)\| \\ &\times \left\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta \right\| \\ &\frac{1}{2}\|L_0\|\|G_0\|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\leq \left[\frac{\int_0^1 w(\theta\|x_0 - x^*\|)d\theta}{1 - w_0(\|x_0 - x^*\|)} \right. \\ &\left. + \frac{q(\|x_0 - x^*\|)(1 + |\alpha - 1|q(\|x_0 - x^*\|)\int_0^1 v(\theta\|x_0 - x^*\|)d\theta}{(1 - w_0(\|x_0 - x^*\|))(1 - q(\|x_0 - x^*\|))} \right] \|x_0 - x^*\| \\ &= f(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.19}$$

which shows (2.10) for $n = 0$ and $x_1 \in B(x^*, r)$. Simply replacing x_0, x_1 by x_k, x_{k+1} in the preceding estimates, we arrive at (2.10). In view of the estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r, \ c = f(\|x_0 - x^*\|) \in [0, 1),$$

we deduce that $\lim x_k = x^*$ and $x_{k+1} \in B(x^*, r)$. Finally, to show the uniqueness part, let $F(y^*) = 0$ with $y^* \in \Omega_1$. Define $Q = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Using (2.5), we get that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \leq \int_0^1 w_0(\theta r^*)d\theta < 1. \tag{2.20}$$

That is $Q^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Using the identity

$$0 = F(y^*) - F(x^*) = Q(y^* - x^*),$$

we conclude that $x^* = y^*$.

Remark 2.2 *1. In view of (2.6) and the estimate*

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|) \end{aligned}$$

condition (2.8) can be dropped and v can be replaced by

$$v(t) = 1 + w_0(t).$$

2. *Let $w_0(t) = L_0 t$, $w(t) = Lt$, $v(t) = M$ for some $L_0 > 0, L > 0$ and $M \geq 1$. In this special case, the results obtained here can be used for operators F satisfying autonomous differential equations [1, 3] of the form*

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. *The radius $\rho = \frac{2}{2L_0 + L}$, was shown by us to be the convergence radius of Newton's method [1, 3]*

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.21}$$

under the conditions (2.5)–(2.7). It follows from the definition of r that the convergence radius r of the method (1.8) cannot be larger than the convergence radius ρ of the second order Newton's method (2.21). As already noted in [1, 3] ρ is at least as large as the convergence ball given by Rheinboldt [14]

$$r_R = \frac{2}{3L}. \tag{2.22}$$

In particular, for $L_0 < L$ we have that

$$r_R < \rho$$

and

$$\frac{r_R}{\rho} \to \frac{1}{3} \quad as \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball ρ is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [15].

4. *It is worth noticing that method (1.8) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [3–17]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

III. Numerical Examples

The numerical examples are presented in this section.

Example 3.1 *Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, \Omega = \bar{B}(0,1), x^* = (0,0,0)^T$. Define function F on Ω for $w = (x, y, z)^T$ by*

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14. W.C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, In: Mathematical models and numerical methods (A.N.Tikhonov et al. eds.) pub.3, (1977), 129-142 Banach Center, Warsaw Poland.

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.6)-(2.8) conditions, we get $L_0 = e - 1, L = e^{\frac{1}{L_0}} = M$, *so* $w_0(t) = L_0 t = (e-1)t$, $w(t) = Lt = e^{\frac{1}{L_0}} t$ *and* $z(t) = v(t) = M = e^{\frac{1}{L_0}}$. *Then the parameters are*

$$r_q = 0.8770, \; r = 0.1496.$$

Example 3.2 *Let* $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, *the space of continuous functions defined on* $[0,1]$ *and be equipped with the max norm. Let* $\Omega = \overline{B}(0,1)$. *Define function* F *on* Ω *by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$, $M = 2$ *so* $w_0(t) = 7.5t, w(t) = 15t$, $v(t) = 2$ *and* $z(t) = 30$. *Then the parameters are*

$$r_q = 0.8000, \; r = 0.0108.$$

Example 3.3 *Returning back to the motivational example at the introduction of this study, we have* $w_0(t) = w(t) = \frac{1}{32}(5t^{3/2} + t)$ $v(t) = 1 + w_0(t)$ *and* $z(t) = \frac{3}{16}$. *Then the parameters are*

$$r_q = 2.5084, \; r = 1.3679.$$

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Analytical Solutions for a Horizontally Oscillated Semi-Submerged Cylinder

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GJSFR-F Classification: MSC 2010: 33C10

ANALYTICALSOLUTIONSFORAHORIZONTALLYOSCILLATEDSEMISUBMERGEDCYLINDER

Analytical Solutions for a Horizontally Oscillated Semi-Submerged Cylinder

Shamima Aktar [α] & M. Abul Kawser [σ]

Abstract- The Krylov-Bogoliubov-Mitropolskii (KBM) method is an enormously used technique to study the transient behavior of vibrating systems. In this work, an oscillating semi-submerged horizontal cylinder is considered in a liquid and the governing equation of this system deem to weakly nonlinear to find out the solutions. Analytical approximate solutions are investigated for obtaining the transient response of the system for undamped and damped oscillatory motions. The results obtained by the proposed technique for the different sets of initial conditions have found to be well suited with the numerical solutions.

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I. Introduction

Krylov and Bogoliubov [1] initiated a perturbation method to obtain approximate solution (oscillatory type) of the second order nonlinear differential system with a small nonlinearity

$$\ddot{x}+\omega_0^2 x=-\varepsilon f(x,\dot{x}) \tag{1}$$

where the over dots denote the differentiation with concerning t, $\omega_0>0$ and ε is a small parameter. This method has amplified and justified by Bogoliubov and Mitropolskii [2, 3]. Today the method is a well-known method as Krylov-Bogoliubov-Mitropolskii (KBM) [1, 2] in the literature of nonlinear oscillations. Popov [4] extended it to the following damped oscillatory system

$$\ddot{x}+c\dot{x}+\omega^2 x=-\varepsilon f(x,\dot{x}) \tag{2}$$

where $c>0$, $\omega>0$ and $c<2\omega$. It is to be noted that if $c\geq 2\omega$, the system (2) becomes non-oscillatory. First, Murty *et al.* [5, 6] used this method to obtain an approximate solution of (2) characterized by non-oscillatory processes. In the case of over-damped systems, we know the characteristic roots of the unperturbed equation of (2) become real, unequal and negative inequality. The roots of the unperturbed equation of (1) are purely imaginary. On the contrary, these are complex conjugate with the negative real part, when $c<2\omega$ (considered by Popov[4]). Sattar [7] found an approximate solution of (2) characterized by critical damping. An asymptotic solution proposed by Kawser and Akbar [8] for the third order critically damped nonlinear system. Kawser

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1. Krylov, N. N. and Bogoliubov N. N., Introduction to Nonlinear Mechanics, Princeton University Press, New Jersey, 1947.

and Sattar [9] suggested an asymptotic solution of a fourth order critically damped nonlinear system with pair-wise equal eigenvalues. Later, Kawser *et al.* [10] has developed a method for fourth order critically damped oscillatory nonlinear systems when the eigenvalues are complex and pair-wise equal. It has further extended by Kawser *et al.* [11] to fourth order critically undamped oscillatory nonlinear systems with pair-wise equal imaginary eigenvalues. Recently, Kawser *et al.* [12] presented a technique to obtain perturbation solutions of fifth order critically undamped nonlinear oscillatory systems with pair-wise equal imaginary eigenvalues.

In this paper, we have investigated the solutions of a horizontally semi-submerged cylinder in a liquid under oscillations due to the gravitational force for both oscillatory and damped oscillatory motions. So in these cases, the eigenvalues are imaginary and complex conjugate for undamped and damped motions respectively. For different sets of initial conditions the solutions show excellent coincidence with the numerical solutions obtained by the *Mathematica* program.

II. Formulation of the Problem

Suppose a half-submerged horizontal cylinder of radius (R) and length (l) is floating in a liquid. If we consider x is instantaneous displacement of a diametric plane from the equilibrium position.
Volume of cylinder from the bottom of height h is,

$$V_h = l\left[R^2 \cos^{-1}\left(\frac{R-h}{R}\right) - (R-h)\sqrt{2Rh-h^2}\right] \tag{3}$$

Now using equation (3), we obtain the volume of cylinder from the bottom of height R *i.e.* volume of half cylinder is, $V_R = \frac{1}{2}\pi R^2 l$

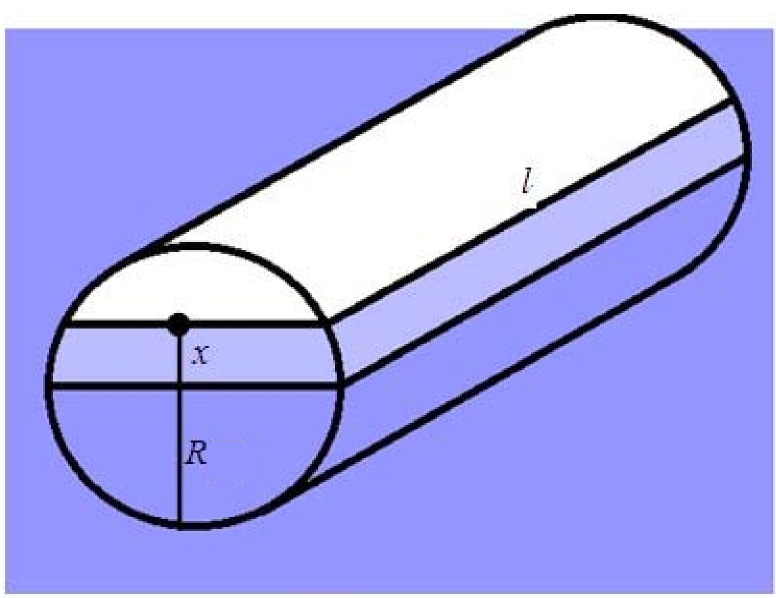

Figure 1: The oscillating semi-submerged cylinder in a liquid

And the volume of cylinder from the bottom of height $R+x$ is,

Ref

12. Abul Kawser, M., Mahafujur Rahaman, Md. and Nurul Islam, Md., Perturbation Solutions of Fifth Order Critically Undamped Nonlinear Oscillatory Systems with Pairwise Equal Eigenvalues, International Journal of Mathematics and Computation, Vol. 28, pp. 22-39, 2017.

$$V_{R+x} = l\left[R^2 \cos^{-1}\left(\frac{R-R-x}{R}\right) - (R-R-x)\sqrt{2R(R+x)-(R+x)^2}\right]$$

Thus the volume of the partial part from the diametric plane of height x is

Notes

$$V_x = V_{R+x} - V_R$$

$$= l\left[R^2 \cos^{-1}\left(-\frac{x}{R}\right) + x(R^2 - x^2)^{\frac{1}{2}} - \frac{\pi}{2}R^2\right]$$

$$= l\left[R^2\left\{\frac{\pi}{2} + \frac{x}{R} + \frac{1}{6}\frac{x^3}{R^3} + \frac{31}{40}\frac{x^5}{R^5} + \cdots\cdots\right\}\right.$$

$$\left. + xR\left\{1 - \frac{x^2}{2R^2} - \frac{x^4}{8R^4} - \cdots\cdots\right\} - \frac{\pi}{2}R^2\right]$$

As x is very small and R is large, so neglecting the terms higher than $\left(\frac{x}{R}\right)^3$, we get

$$V_x = l\left[R^2\left\{\frac{\pi}{2} + \frac{x}{R} + \frac{x^3}{6R^3}\right\} + xR\left(1 - \frac{x^2}{2R^2}\right) - \frac{\pi}{2}R^2\right]$$

$$= 2l\left[xR - \frac{x^3}{6R}\right]$$

Suppose m is the mass of the cylinder, m_1 is mass of the liquid occupied by the volume V_x, ρ is the density of the cylinder, 2ρ is the density of the liquid and g is the gravitational force. Suppose the semi-submerged cylinder is oscillating in the liquid without damping, then Newton's second law of motion gives

$$m\frac{d^2x}{dt^2} = -m_1 g$$

$$i.e., \frac{d^2x}{dt^2} + \frac{8g}{\pi R}x = \frac{4g}{3\pi R^3}x^3 \tag{4}$$

Also if the half-submerged cylinder is floating in the liquid under damping, then the equation of the system is given by

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \frac{8g}{\pi R}x = \frac{4g}{3\pi R^3}x^3 \tag{5}$$

where $2k$ is the damping constant.

III. The Method

Consider a second order weakly nonlinear ordinary differential system

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon\, f(x, \dot{x}) \tag{6}$$

where over dots are used for the first and second derivatives of x concerning t; k is a non-negative constant, ε is a small parameter and $f(x, \dot{x})$ is the nonlinear function. Since the equation is second order, so, we shall get two eigenvalues for the damped oscillatory system and the eigenvalues are complex conjugate, *i.e.* $-k \pm i\lambda$ (say), where $\lambda = \sqrt{\omega^2 - k^2}$ and $\omega > k$, and for oscillatory systems *i.e.* $k = 0$, then the eigenvalues of system (6) are $\pm i\omega$.

When $\varepsilon = 0$, then the equation (6) becomes linear, and the solution of the corresponding linear equation (6) is

$$x(t,0) = e^{-kt}\left(a_0 \cos \lambda t + b_0 \sin \lambda t\right) \tag{7}$$

where a_0 and b_0 are arbitrary constants.

Now we seek a solution of (6) that reduces to (7) as the limit $\varepsilon \to 0$. We look for an asymptotic solution of (6) is

$$x(t,\varepsilon) = e^{-kt}\left(a \cos \lambda t + b \sin \lambda t\right) + \varepsilon u_1\left(a,b,t\right) + O(\varepsilon^2) \tag{8}$$

where *a, b* are slowly varying functions of time, t and satisfy the following first order differential equations:

$$\frac{da}{dt} = \varepsilon A_1\left(a,b,t\right) + \ldots\ldots$$

$$\frac{db}{dt} = \varepsilon B_1\left(a,b,t\right) + \ldots\ldots \tag{9}$$

Now differentiating (8) twice times, substituting for the derivatives $\dot{x}$, $\ddot{x}$ and x in (6). Now utilizing relations (9) and comparing the coefficients of various powers of ε, we get

$$e^{-kt}\left\{\left(\frac{\partial A_1}{\partial t} + 2\lambda B_1\right)\cos \lambda t + \left(-2\lambda A_1 + \frac{\partial B_1}{\partial t}\right)\sin \lambda t\right\}$$

$$+\frac{\partial^2 u_1}{\partial t^2} + 2k\frac{\partial u_1}{\partial t} + \omega^2 u_1 = -f^{(0)}\left(a,b,t\right) \tag{10}$$

where $f^{(0)} = f\left(x_0, \dot{x}_0\right)$ and $x_0 = e^{-kt}\left(a \cos \lambda t + b \sin \lambda t\right)$.

Notes

For the oscillatory system to obtain the solution, we have to put $k=0$ and replacing λ by ω in equation (10).
Thus for oscillatory system, we get

$$\left(\frac{\partial A_1}{\partial t}+2\omega B_1\right)\cos\omega t+\left(-2\omega A_1+\frac{\partial B_1}{\partial t}\right)\sin\omega t+\frac{\partial^2 u_1}{\partial t^2}+\omega^2 u_1=-f^{(0)}(a,b,t) \qquad (11)$$

Usually, equation (10) or (11) is solved for the unknown functions A_1 and B_1 under the assumption that u_1 does not contain first harmonic terms. We shall follow this assumption (early imposed by KBM [1, 3]) partially to obtain approximate solutions of nonlinear systems with large damping.

IV. Oscillatory Motion

For the oscillatory motion from equation (4), we have

$$\ddot{x}+\frac{8g}{\pi R}x=\varepsilon x^3 \qquad (12)$$

where $\varepsilon=\dfrac{4g}{3\pi R^3}$

Thus the solution of equation (12) is given by putting $k=0$ and replacing λ by ω in equation (8), we get

$$x(t,\varepsilon)=a\cos\omega t+b\sin\omega t+\varepsilon u_1(a,b,t) \qquad (13)$$

where $\omega=\sqrt{\dfrac{8g}{\pi R}}$.
Comparing equation (12) with the equation (6), we obtain

$$f(x,\dot{x})=x^3$$

Therefore, $f^{(0)}=[a\cos\omega t+b\sin\omega t]^3$

$$=\frac{3}{4}\left(a^3+ab^2\right)\cos\omega t+\frac{3}{4}\left(a^2b+b^3\right)\sin\omega t$$

$$+\left(\frac{1}{4}a^3-\frac{3}{4}ab^2\right)\cos 3\omega t+\left(\frac{3}{4}a^2b-\frac{1}{4}b^3\right)\sin 3\omega t \qquad (14)$$

Substituting $f^{(0)}$ from equation (14) into equation (11), we obtain

$$-2A_1\omega\sin\omega t+\frac{\partial A_1}{\partial t}\cos\omega t+2B_1\omega\cos\omega t+\frac{\partial B_1}{\partial t}\sin\omega t+\left(D^2+\omega^2\right)u_1$$

$$=\frac{3}{4}\left(a^3+ab^2\right)\cos\omega t+\frac{3}{4}\left(a^2b+b^3\right)\sin\omega t \qquad (15)$$

Ref

1. Krylov, N. N. and Bogoliubov N. N., Introduction to Nonlinear Mechanics, Princeton University Press, New Jersey, 1947.

$$+\left(\frac{1}{4}a^3-\frac{3}{4}ab^2\right)\cos 3\omega t+\left(\frac{3}{4}a^2b-\frac{1}{4}b^3\right)\sin 3\omega t$$

According to our assumption, u_1 does not contain first harmonic terms of $f^{(0)}$. The following equations can be obtained by comparing the coefficients of $\sin\omega t$ and $\cos\omega t$ are the higher argument terms of $\sin\omega t$ and $\cos\omega t$ as

$$\left(D^2+4\omega^2\right)A_1=-\frac{3\omega}{2}b^3-\frac{3\omega}{2}a^2b \tag{16}$$

$$\left(D^2+4\omega^2\right)B_1=\frac{3\omega}{2}a^3+\frac{3\omega}{2}ab^2 \tag{17}$$

$$\left(D^2+\omega^2\right)u_1=\left(\frac{1}{4}a^3-\frac{3}{4}ab^2\right)\cos 3\omega t+\left(\frac{3}{4}a^2b-\frac{1}{4}b^3\right)\sin 3\omega t \tag{18}$$

The solutions of the equations (16) to (18) are respectively

$$A_1=-\frac{3(b^3+a^2b)}{8\omega} \tag{19}$$

$$B_1=\frac{3(a^3+ab^2)}{8\omega} \tag{20}$$

$$u_1=\left(3ab^2-a^3\right)\cos 3\omega t+\left(b^3-3a^2b\right)\sin 3\omega t \tag{21}$$

Substituting the values of A_1, B_1 from equations (19) and (20) into equation (9), we obtain

$$\frac{da}{dt}=-\varepsilon\frac{3(b^3+a^2b)}{8\omega} \tag{22}$$

$$\frac{db}{dt}=\varepsilon\frac{3(a^3+ab^2)}{8\omega} \tag{23}$$

Thus for the transformation $a=c\cos\phi$ and $b=-c\sin\phi$, the equations (21) to (23) respectively become

$$u_1=-\frac{c^3}{32\omega^2}\cos(3\omega t+3\phi) \tag{24}$$

And $$\dot{c}=0$$

$$\dot{\phi}=-\frac{3\varepsilon c^2}{8\omega}$$

Notes

Or,
$$c = c_0 \tag{25}$$

$$\phi = \phi_0 - \frac{3\varepsilon c^2 t}{8\omega} \tag{26}$$

Thus by substituting $a = c\cos\phi$ and $b = -c\sin\phi$ into equation (13) and after simplification it becomes

Notes

$$x(t,\varepsilon) = c\cos(\omega t + \phi) + \varepsilon u_1 \tag{27}$$

Therefore, equation (27) represents the first order oscillatory solution of equation (12), where c, ϕ, u_1 are given by (25), (26) and (24).

V. Damped Oscillatory Motion

For the damped oscillatory motion, we have from equation (5)

$$\ddot{x} + 2k\dot{x} + \frac{8g}{\pi R}x = \varepsilon x^3 \tag{28}$$

where $\varepsilon = \dfrac{4g}{3\pi R^3}$

Comparing equation (28) with the equation (6), we obtain

$$f(x,\dot{x}) = x^3 \tag{29}$$

Therefore, $f^{(0)} = [e^{-kt}(a\cos\lambda t + b\sin\lambda t)]^3$

$$\begin{aligned} &= e^{-3kt}\left(a^3\cos^3\lambda t + 3a^2 b\sin\lambda t\cos^2\lambda t + 3ab^2\sin^2\lambda t\cos\lambda t + b^3\sin^3\lambda t\right) \\ &= e^{-3kt}\left[\frac{3}{4}\left(a^3 + ab^2\right)\cos\lambda t + \frac{3}{4}\left(a^2 b + b^3\right)\sin\lambda t\right. \\ &\left. + \left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right)\cos 3\lambda t + \left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right)\sin 3\lambda t\right] \end{aligned} \tag{30}$$

where $\lambda = \sqrt{\dfrac{8g}{\pi R} - k^2}$.

Substituting $f^{(0)}$ from equation (30) into equation (10), we obtain

$$\begin{aligned} &e^{-kt}\left(-2A_1\lambda\sin\lambda t + \frac{\partial A_1}{\partial t}\cos\lambda t + 2B_1\lambda\cos\lambda t + \frac{\partial B_1}{\partial t}\sin\lambda t\right) \\ &\left(D^2 + 2kD + \frac{8g}{\pi R}\right)u_1 = e^{-3kt}\left[\frac{3}{4}\left(a^3 + ab^2\right)\cos\lambda t + \frac{3}{4}\left(a^2 b + b^3\right)\sin\lambda t\right. \\ &\qquad \left. + \left(\frac{1}{4}a^3 - \frac{3}{4}ab^2\right)\cos 3\lambda t + \left(\frac{3}{4}a^2 b - \frac{1}{4}b^3\right)\sin 3\lambda t\right] \end{aligned} \tag{31}$$

Since, u_1 does not contain first harmonic terms, the following equations obtained by comparing the coefficients of $\sin\lambda t$ and $\cos\lambda t$ are the higher argument terms of $\sin\lambda t$ and $\cos\lambda t$ as

$$\left(D^2+4\lambda^2\right)A_1=-\frac{3}{2}e^{-2kt}\left\{k\left(a^3+ab^2\right)+\lambda\left(a^2b+b^3\right)\right\} \tag{32}$$

$$\left(D^2+4\lambda^2\right)B_1=\frac{3}{2}e^{-2kt}\left\{\lambda\left(a^3+ab^2\right)-k\left(a^2b+b^3\right)\right\} \tag{33}$$

$$\left(D^2+2kD+\frac{8g}{\pi R}\right)u_1=e^{-3kt}\cos 3\lambda t\left(\frac{1}{4}a^3-\frac{3}{4}ab^2\right)+e^{-3kt}\sin 3\lambda t\left(\frac{3}{4}a^2b-\frac{1}{4}b^3\right) \tag{34}$$

The solutions of the equations (32) to (34) are respectively

$$A_1=-\frac{3e^{-2kt}\left\{k\left(a^3+ab^2\right)+\lambda\left(a^2b+b^3\right)\right\}}{8\left(k^2+\lambda^2\right)} \tag{35}$$

$$B_1=\frac{3e^{-2kt}\left\{-k\left(a^2b+b^3\right)+\lambda\left(a^3+ab^2\right)\right\}}{8\left(k^2+\lambda^2\right)} \tag{36}$$

$$u_1=\frac{e^{-3kt}}{16(k^4+5k^2\lambda^2+4\lambda^4)}\Big[(a^3-3ab^2)\{(k^2-2\lambda^2)\cos 3\lambda t-3k\lambda\sin 3\lambda t\} \tag{37}$$

$$+(3a^2b-b^3)\{3k\lambda\cos 3\lambda t+(k^2-2\lambda^2)\sin 3\lambda t\}\Big]$$

Substituting the values of A_1, B_1 from equations (35) and (36) into equation (9), we obtain

$$\frac{da}{dt}=-\varepsilon\frac{3e^{-2kt}\left\{k\left(a^3+ab^2\right)+\lambda\left(a^2b+b^3\right)\right\}}{8\left(k^2+\lambda^2\right)} \tag{38}$$

$$\frac{db}{dt}=\varepsilon\frac{3e^{-2kt}\left\{-k\left(a^2b+b^3\right)+\lambda\left(a^3+ab^2\right)\right\}}{8\left(k^2+\lambda^2\right)} \tag{39}$$

Therefore, under the transformation, $a=c\sin\phi$ and $b=-c\sin\phi$ equations (37) to (39) respectively become

$$u_1=\frac{c^3e^{-3kt}}{16(k^4+5k^2\lambda^2+5\lambda^4)}[(k^2-2\lambda^2)\cos\left(3\lambda t+3\phi\right)-3k\lambda\sin\left(3\lambda t+3\phi\right)] \tag{40}$$

$$\dot{\phi}=-\frac{3\lambda\varepsilon c^2e^{-2kt}}{8(k^2+\lambda^2)}$$

Notes

Notes

$$\dot{c} = -\frac{3k\varepsilon c^3 e^{-2kt}}{8(k^2+\lambda^2)}$$

Or,

$$\phi = \phi_0 + \frac{3\lambda\varepsilon c^2}{16k\left(k^2+\lambda^2\right)}\left(e^{-2kt}-1\right) \tag{41}$$

$$c = c_0 + \varepsilon\frac{3c_0{}^3}{16\left(k^2+\lambda^2\right)}\left(e^{-2kt}-1\right) \tag{42}$$

Thus by substituting $a_1 = c\cos\phi$ and $b_1 = -c\sin\phi$ into equation (8) and after simplification it becomes

$$x(t,\varepsilon) = ce^{-kt}\cos\left(\lambda t+\phi\right)+\varepsilon u_1 \tag{43}$$

Therefore, equation (43) represents a first order damped oscillatory solution of equation (28), where c, ϕ, u_1 are given by (42), (41) and (40).

VI. Results and Discussion

To test the accuracy of our results, we match our results with the numerical results obtained by the *Mathematica* program for the different sets of initial conditions. Firstly, $x(t,\varepsilon)$ has been computed by analytic solution (27) for undamped motion in which c, ϕ are calculated by (25), (26) and u_1 is obtained from (24) together with three sets of initial conditions, which are obtained for different radius of the cylinder and gravitational force, $g = 9.8\ ms^{-2}$. The corresponding numerical solutions that computed by the *Mathematica* program for various values of time, t and all the results are showed in the *Figure 2 to Figure 4.* The numerical results for damped oscillatory motion obtained by the *Mathematica* program for same initial conditions are assimilating with the perturbation results. Here, $x\left(t,\varepsilon\right)$ has been computed by asymptotic solution (43), where c, ϕ are calculated by (42), (41) and u_1 is obtained from (40) with the same initial conditions, when $g = 9.8\ ms^{-2}$ and different values of damping constant $2k$. The comparative results of numerical and perturbation for various values of t are showed graphically in the *figure 5 to figure 7.*

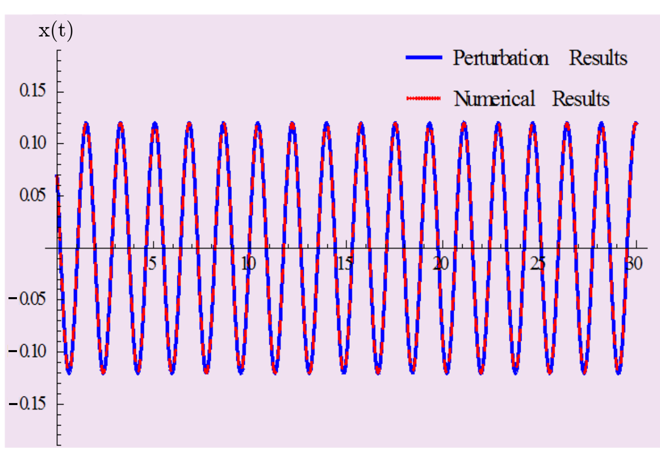

Figure 2: Comparison between perturbation and numerical results for $R = 2\ m,\ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.12\ m,\ \phi_0 = 55°$

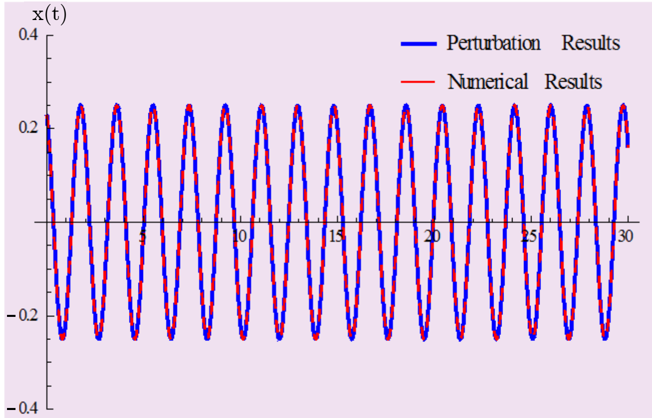

Figure 3: Comparison between perturbation and numerical results for $R = 2.2\ m,\ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.25\ m\,,\ \phi_0 = 25°$

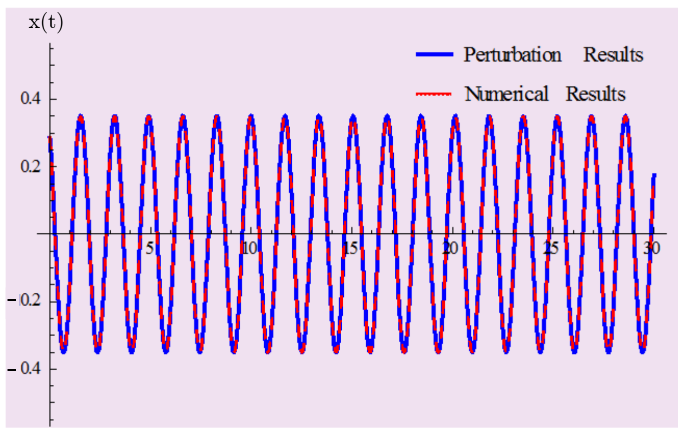

Figure 4: Comparison between perturbation and numerical results for $R = 1.8\ m,\ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.35\ m\,,\ \phi_0 = 35°$

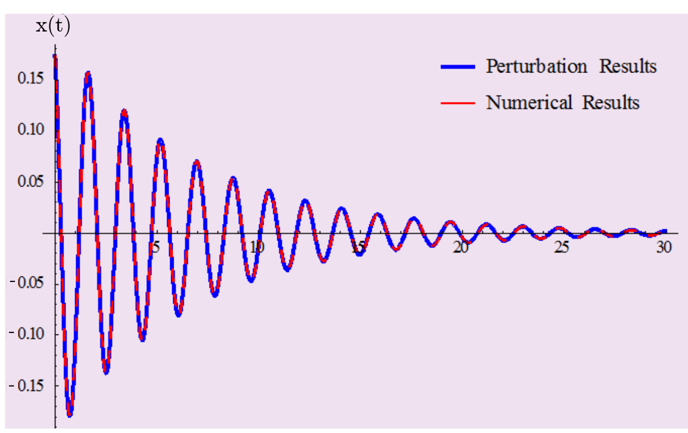

Figure 5: Comparison between perturbation and numerical results for $R = 2.0\ m,\ k = 0.15\ s^{-1},\ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.20\ m\,,\ \phi_0 = 30°$

Notes

Notes

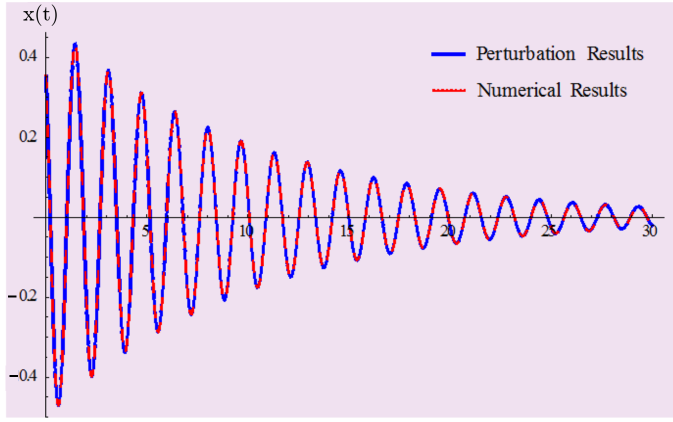

Figure 6: Comparison between perturbation and numerical results for $R = 1.7\ m,\ k = 0.10\ s^{-1},\ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 0.50\ m,\ \phi_0 = 45°$

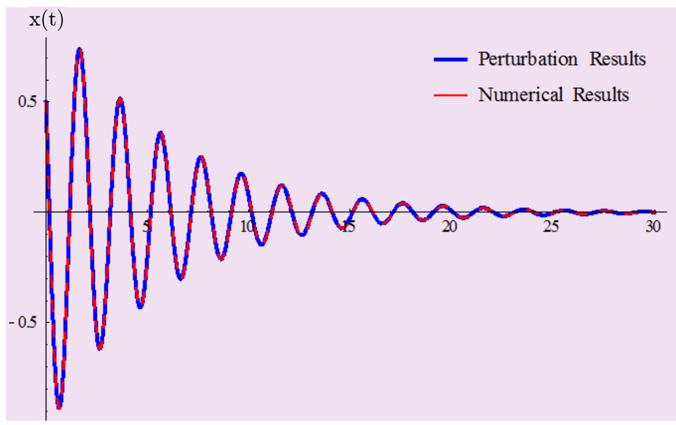

Figure 7: Comparison between perturbation and numerical results for $R = 2.5\ m,\ k = 0.18\ s^{-1},\ g = 9.8\ ms^{-2}$ with the initial conditions $c_0 = 1.00\ m,\ \phi_0 = 60°$

VII. Conclusions

In this article, we have successfully applied the modified method to the half-submerged cylinder for oscillatory and damped oscillatory nonlinear systems. The system is oscillating in a liquid due to the gravitational force and upward pressure. A second order nonlinear equation has been derived from a horizontally half submerged cylinder, which is floating in a liquid for both oscillatory and damped oscillatory motion. Based on the modified KBM method transient responses of nonlinear differential systems have been investigated. For different sets of initial conditions, the modified KBM method provides solutions which show well agreement with the numerical solutions. In the KBM method, much error occurs in the case of rapid changes of x with respect to time, t. But it is noteworthy to observe from all figures, x changes rapidly in the time period $t = 0$ to $t = 30$, the results obtained via the modified KBM method show good coincidence with those obtained by the numerical method.

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A Note on the Representation and Definition of Dual Split Semi-Quaternions Algebra

By Mehdi Jafari
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Abstract- In this paper, dual split semi-quaternions algebra, $\tilde{\mathrm{H}}_{ss}$, is defined for the first time, and some fundamental algebraic properties of its is studied. The set of all unit dual split semi-quaternions is a subgroup of $\tilde{\mathrm{H}}_{ss}$. Fortheremore, by De-Moivre's formula, any powers of these quaternions are obtained.

Keywords: *dual split semi-quaternion, de-moivre's theorem, subgroup.*

GJSFR-F Classification: *MSC 2010: 11R52*

ANOTEONTHEREPRESENTATIONANDDEFINITIONOFDUALSPLITSEMIQUATERNIONSALGEBRA

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A Note on the Representation and Definition of Dual Split Semi-Quaternions Algebra

Mehdi Jafari

Abstract- In this paper, dual split semi-quaternions algebra, $\tilde{H}_{ss}$, is defined for the first time, and some fundamental algebraic properties of its is studied. The set of all unit dual split semi-quaternions is a subgroup of $\tilde{H}_{ss}$. Fortheremore, by De-Moivre's formula, any powers of these quaternions are obtained.

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I. Introduction

The quaternion number system was discovered by Hamilton, who was looking for an extension of the complex number system to use in various areas of mathematics. The different type of quaternions are suitable algebraic instructure for expressing important space-time transformations as well as description of the classical and quantum filds.

Dual numbers and dual quaternions were introduced in the 19th century by W.K. Clifford, as a tool for his geometrical investigation.

In our previos work, we have studied the split semi-quaternions, and have presented some of their algebric properties. De Moivre's and Euler's formula for these quaternions are given (Jafari, 2015).We have shown that the set of all unit split semi-quaternions with the group operation of quaternion multiplication is a Lie group of 3-dimension and find its Lie algebra and Killing bilinear form (Jafari, 2016).

In this paper, we study the dual split semi-quaternions algebra and give some of their basic properties. We express De Moivre's and Euler's formulas for dual split semi-quaternions and find roots of a quaternion using these formulas. Finally, we give some examples for more clarification.We hope that these results will contribute to the study of physical science.

a) Split Semi-quaternions Algebra

A split semi-quaternion q has an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^{\,2} = 1,\ \vec{j}^{\,2} = \vec{k}^{\,2} = 0,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i},\ \ \vec{j}\vec{k} = 0 = -\vec{k}\vec{j},$$

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and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

The set of all split semi-quaternions is denoted by H_{ss}. For detailed information about this concept, we refer the reader to [3,4,6,9].

b) Dual Numbers Algebra

Let a and a^* be two real numbers, the combination

$$A = a + \varepsilon a^*,$$

is called a dual number. Here ε is the dual unit. Dual numbers are considered as polynomials in ε, subject to the rules

$$\varepsilon \neq 0,\ \varepsilon^2 = 0,\ \varepsilon.r = r.\varepsilon = \varepsilon, \text{ for all } r \in \mathbb{R}.$$

The set of dual numbers, D, forms a commutative ring having the εa^* (a^* real) as divisors of zero, not field. Some properties of dual numbers are

$$\sin(a+\varepsilon a^*) = \sin a + \varepsilon a^* \cos a,$$

$$\cos(a+\varepsilon a^*) = \cos a - \varepsilon a^* \sin a,$$

$$\sqrt{a+\varepsilon a^*} = \sqrt{a} + \varepsilon \frac{a^*}{2\sqrt{a}} \quad \mathit{for}\ a > 0.$$

For detailed information about dual numbers algebra, we refer the reader to (Keler, 2000).

c) Generalized Dual Quaternions Algebra

A generalized dual quaternion Q has an expression of form

$$Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

where A_0, A_1, A_2 and A_3 are dual numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units which satisfy the equalities

$$\vec{i}^2 = -\alpha,\quad \vec{j}^2 = -\beta,\quad \vec{k}^2 = -\alpha\beta,$$

$$\vec{i}\vec{j} = \vec{\mathrm{k}} = -\vec{j}\vec{i},\quad \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \alpha\vec{j} = -\vec{i}\vec{k},\quad \alpha, \beta \in \mathrm{R}.$$

The set of all generalized dual quaternions (abbreviated *GDQ*) are denoted by $\tilde{\mathrm{H}}_{\alpha\beta}$. A generalized dual quaternion Q is a sum of a scalar and a vector, called scalar part, $S_Q = A_0$, and vector part $\vec{V}_Q = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ (Jafari,2015).

If $S_Q = 0$, then Q is called pure generalized dual quaternion, we may be called its generalized dual vector. The set of all generalized dual vectors denoted by $D^3_{\alpha\beta}$.

Special cases:

1. $\alpha = \beta = 1$, is considered, then $\tilde{\mathrm{H}}_{\alpha\beta}$ is the algebra of dual quaternions.

Ref

3. Jafari M., *Some results on the matrices of Split Semi-quaternions*, 2016, submitted.

2. $\alpha=1, \beta=-1,$ is considered, then $\tilde{\mathrm{H}}_{\alpha\beta}$ is the algebra of split dual quaternions.
3. $\alpha=1, \beta=0,$ is considered, then $\tilde{\mathrm{H}}_{\alpha\beta}$ is the algebra of dual semi-quaternions.
4. $\alpha=-1, \beta=0,$ is considered, then $\tilde{\mathrm{H}}_{\alpha\beta}$ is the algebra of dual split semi-quaternions.
5. $\alpha=0, \beta=0,$ is considered, then $\tilde{\mathrm{H}}_{\alpha\beta}$ is the algebra of dual quasi-quaternions (Jafari, 2016).

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Theorem 2. Every unit generalized dual quaternion is a screw operator.

d) Dual split semi-quaternions Algebra

A dual split semi-quaternion Q is defined as

$$Q=A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$$

where A_0, A_1, A_2 and A_3 are dual numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^{\,2}=1, \quad \vec{j}^{\,2}=\vec{k}^{\,2}=0,$$
$$\vec{i}\vec{j}=\vec{k}=-\vec{j}\vec{i} \quad , \quad \vec{j}\vec{k}=0=\vec{k}\vec{j}$$

and

$$\vec{k}\vec{i}=j=-\vec{i}\vec{k}.$$

In other words, this may also be given as $Q=q+\varepsilon q^*$, where q, q^* are split semi-quaternions. The set of all dual split semi-quaternions(abbreviated dual *SSQ*) is denoted by $\tilde{\mathrm{H}}_{ss}$. We express the basic operations in terms of $\vec{i}, \vec{j}, \vec{k}$.

Given $Q=A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$, A_0 is called the *scalar part* of Q, denoted by

$$S_Q=A_0,$$

and $A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ is called the *vector part* of Q, denoted by

$$\vec{V}(Q)=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}.$$

If $S_Q=0$, then Q is called pure dual *SSQ*.

The addition becomes as

$$\begin{aligned}Q+P&=(A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k})+(B_0+B_1\vec{i}+B_2\vec{j}+B_3\vec{k})\\&=(A_0+B_0)+(A_1+B_1)\vec{i}+(A_2+B_2)\vec{j}+(A_3+B_3)\vec{k}\end{aligned}$$

This rule preserves the associativity and commutativity properties of addition.

The multiplication as

$$\begin{aligned}QP&=(A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k})(B_0+B_1\vec{i}+B_2\vec{j}+B_3\vec{k})\\&=(A_0B_0+A_1B_1)+(A_1B_0+A_0B_1)\vec{i}+\\&+(A_2B_0-A_3B_1+A_0B_2+A_1B_3)\vec{j}+(A_3B_0-A_2B_1+A_1B_2+A_0B_3)\vec{k}\end{aligned}$$

Also, this can be written as

$$QP=\begin{bmatrix} A_0 & A_1 & 0 & 0 \\ A_1 & A_0 & 0 & 0 \\ A_2 & -A_3 & A_0 & A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

Obviously, the quaternion multiplication is associative and distributive with respect to addition and subtraction, but the commutativity law does not hold in general.

Corollary 1. $\tilde{H}_{ss}$ with addition and multiplication has all the properties of a number field expect commutativity of the multiplication. It is therefore called the skew field of quaternions.

e) Some Properties of Dual Semi-Quaternions

1) The Hamilton *conjugate* of $Q=A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ is

$$\overline{Q}=A_0-A_1\vec{i}-A_2\vec{j}-A_3\vec{k}.$$

The dual *conjugate* of $Q=A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ is

$$\begin{aligned} Q^* &= \overline{A}_0+\overline{A}_1\vec{i}+\overline{A}_2\vec{j}+\overline{A}_3\vec{k} \\ &= (a_0-\varepsilon a_0^*)+(a_1-\varepsilon a_1^*)\vec{i}+(a_2-\varepsilon a_2^*)\vec{j}+(a_3-\varepsilon a_3^*)\vec{k}. \end{aligned}$$

The Hermitian *conjugate* of $Q=A_0+A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ is

$$\begin{aligned} Q^{\dagger} &= \overline{A}_0-\overline{A}_1\vec{i}-\overline{A}_2\vec{j}-\overline{A}_3\vec{k} \\ &= (a_0-\varepsilon a_0^*)-(a_1-\varepsilon a_1^*)\vec{i}-(a_2-\varepsilon a_2^*)\vec{j}-(a_3-\varepsilon a_3^*)\vec{k}. \end{aligned}$$

2) The *norm* of Q is

$$N_Q=Q\overline{Q}=\overline{Q}Q=A_0^2-A_1^2$$

The norm Q canbe dual number, real number, or zero. If $N_Q=1$, then Q is called a unit dual *SSQ*. We will use $\tilde{H}_{ss}^1$ to denote the set of all the unit dual *SSQ*.

If $N_Q=0$, then Q is called a null dual *SSQ*. A dual split semi-quaternion Q for which $N_Q=0$ has form $Q=A_2\vec{j}+A_3\vec{k},(A_0=A_1=0)$ and it is a zero divisor.

3) The *inverse* of Q with $N_Q\neq 0$, is

$$Q^{-1}=\frac{1}{N_Q}\overline{Q}.$$

Clearly $QQ^{-1}=1+0\vec{i}+0\vec{j}+0\vec{k}$. Note also that $\overline{QP}=\overline{P}\,\overline{Q}$ and $(QP)^{-1}=P^{-1}Q^{-1}$.

Theorem 1. The set $\tilde{H}_{ss}^1$ of unit dual *SSQ* is a subgroup of the group $\tilde{H}_{ss}^0$ where $\tilde{H}_{ss}^0$ is the set of all non-zero dual split semi-quaternions.

Proof: Let $Q,P\in\tilde{H}_{ss}^1$. We have $N_{QP}=1$, *i.e.* $QP\in\tilde{H}_{ss}^1$ and thus the first subgroup requirement is satisfied. Also, by the property

$$N_Q=N_{\overline{Q}}=N_{Q^{-1}}=1,$$

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the second subgroup requirement $Q^{-1} \in \tilde{H}_{ss}^{1}$.

Example 4. Consider the dual split semi-quaternions

$$Q_1 = 1 + (1+\varepsilon)\vec{i} - \varepsilon\vec{j} + (\varepsilon - 1)\vec{k}\,,\; Q_2 = 2\varepsilon + \vec{i} - (1-\varepsilon)\vec{j} + (-1+2\varepsilon)\vec{k}\,,$$

and

$$Q_3 = (1-\varepsilon) + (1-\varepsilon)\vec{i} + (1-3\varepsilon)\vec{j} + \vec{k}\,,\; Q_4 = \sqrt{2} + 2\varepsilon\vec{i} + \vec{j} - \varepsilon\vec{k}\,,$$

1. The vector parts of Q_1, Q_2 are

$$\vec{V}_{Q_1} = (1+\varepsilon)\vec{i} - \varepsilon\vec{j} + (-1+2\varepsilon)k,\; \vec{V}_{Q_2} = \vec{i} - (1-\varepsilon)\vec{j} + (-1+2\varepsilon)k.$$

2. The Hamilton conjugates of Q_2, Q_3 are

$$\overline{Q_2} = 2\varepsilon - \vec{i} + (1-\varepsilon)\vec{j} - (-1+2\varepsilon)\vec{k}\,,\; \overline{Q_3} = (1-\varepsilon) - (1-\varepsilon)\vec{i} - (1-3\varepsilon)\vec{j} - \vec{k}\,,$$

3. The dual conjugates of Q_2, Q_3 are

$$Q_2^* = -2\varepsilon + \vec{i} + (1+\varepsilon)\vec{j} - (-1-2\varepsilon)\vec{k}\,,\; Q_3^* = (1+\varepsilon) + (1+\varepsilon)\vec{i} + (1+3\varepsilon)\vec{j} + \vec{k}\,,$$

4. The Hermitian conjugate of Q_1, Q_4 are

$$Q_1^\dagger = 1 - (1-\varepsilon)\vec{i} - \varepsilon\vec{j} + (1+\varepsilon)\vec{k}\,,\; Q_4^\dagger = \sqrt{2} + 2\varepsilon\vec{i} - \vec{j} - \varepsilon\vec{k}\,,$$

5. The norms are given by

$$N_{Q_1} = -2\varepsilon,\; N_{Q_2} = -1,\; N_{Q_3} = 0, N_{Q_4} = 2$$

6. The inverses of Q_1, Q_2 are

$$Q_1^{-1} = \frac{1}{2\varepsilon}[1 - (1+\varepsilon)\vec{i} + \varepsilon\vec{j} - (\varepsilon-1)\vec{k}]\;,\; Q_2^{-1} = -[2\varepsilon - \vec{i} + (1-\varepsilon)\vec{j} - (-1+2\varepsilon)\vec{k}],$$

and Q_3 not invertible.

7. One can realize the following operations

$$Q_1 + Q_2 = (1+2\varepsilon) + (2+\varepsilon)\vec{i} - (1+2\varepsilon)\vec{j} + (-2+3\varepsilon)\vec{k}$$

$$Q_2 - Q_3 = (-1+3\varepsilon) + \varepsilon\vec{i} + (-2+2\varepsilon)\vec{j} + (-2+2\varepsilon)k$$

$$Q_1 Q_4 = (\sqrt{2} + 2\varepsilon) + [\sqrt{2} + (2+\sqrt{2})\varepsilon]\vec{i} +$$

$$+ [1 + (1-\sqrt{2})\varepsilon]\vec{j} - [(1-\sqrt{2}) + \sqrt{2}\varepsilon]\vec{k}.$$

f) Trigonometric Form and De Moivre's Theorem

In this section, we express De-Moivre's formula for dual *SSQ*. For this, we can cosider two different cases:

Case 1. Let the norm of dual *SSQ* be positive.

The trigonometric (polar) form of a non-null dual *SSQ*

$$Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

is

$$Q = R(\cosh\phi + \vec{W}\sinh\phi)$$

where $R = \sqrt{N_Q}$, and

$$\cosh\phi = \frac{|A_0|}{R}\ ,\ \sinh\phi = \frac{\sqrt{A_1^2}}{R} = \frac{|A_1|}{\sqrt{A_1^2 - A_2^2}}.$$

$\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit dual vector $\vec{W}$ is given by

$$\vec{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{A_1^2}}[A_1\vec{i} + A_2\vec{j} + A_3\vec{k}\ ] = \frac{1}{|A_1|}(A_1, A_2, A_3).$$

This is similar to polar coordinate expression of asplit quaternion [7], split semi-quaternion [3].

Example 1.5. The trigonometric forms of the dual split semi-quaternions

$$Q_1 = \sqrt{2} + \vec{i} + (1+\varepsilon)\vec{j} + 2\varepsilon\vec{k}, \text{is } Q_1 = \cosh\phi_1 + \vec{W}_1\sinh\phi_1,$$

$$Q_2 = (1+\varepsilon) + \vec{i} + (1-\varepsilon)\vec{j} - \vec{k}, \text{is } Q_2 = \sqrt{2\varepsilon}\ [\cosh\phi_2 + \vec{W}_2\sinh\phi_2]$$

where

$$\cosh\phi_1 = \sqrt{2},\ \sin\phi_1 = 1,\ \vec{W}_1 = (1, 1+\varepsilon, 2\varepsilon)$$

$$\cosh\phi_2 = \frac{1+\varepsilon}{\sqrt{2\varepsilon}},\ \sinh\phi_2 = \frac{1}{\sqrt{2\varepsilon}},\ \vec{W}_2 = (1, 1-\varepsilon, -1),$$

and $N_{\vec{W}_1} = N_{\vec{W}_2} = -1.$

Theorem 1.5. (De Moivre's Theorem) If $Q = R(\cosh\phi + \vec{W}\sinh\phi)$ be a dual *SSQ* and *n* is any positive integer, then

$$Q^n = R^n(\cosh n\phi + \vec{W}\sinh n\phi)$$

Proof: The proof is easily followed by induction on *n*.■

The Theorem holds for all integers *n*, since

$$Q^{-1} = R^{-1}(\cosh\phi - \vec{W}\sinh\phi),$$

$$Q^{-n} = R^{-n}[\cosh(-n\phi) + \vec{W}\sinh(-n\phi)]$$

$$= R^{-n}[\cosh n\phi - \vec{W}\sinh n\phi]$$

Example 2.5. Let $Q = 2 + \vec{i} + (1+\varepsilon)\vec{j} - 2\varepsilon\vec{k}$. Find Q^{10} and Q^{-45}.

Solution: First write Q in trigonometric form.

$$Q = \sqrt{3}(\cosh\phi + \vec{W}\sinh\phi),$$

where $\phi = \ln\sqrt{3}$, $\vec{W} = (1, 1+\varepsilon, -2\varepsilon)$

Applying de Moivre's Theorem gives:

7. Rosenfeld B., *Geometry of Lie groups*, Kluwer Academic Publishers, Netherlands, (1997).

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$$Q^{10} = 3^5(\cosh 10\phi + \vec{W}\sinh 10\phi) = 3^5(\frac{3^5+3^{-5}}{2} + \vec{W}\frac{3^5-3^{-5}}{2})$$

$$Q^{-45} = (\sqrt{3})^{-45}(\cosh 45\phi - \vec{W}\sinh 45\phi)$$

Corollary 1.5. The equation $Q^n = 1$, does not have solution for a unit dual split semi-quaternion.

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Example 3.5. Let $Q = 1 + 3\varepsilon i + (2+\varepsilon) j + \varepsilon k$, be a dual split semi-quaternion. There is no n (n¿ 0) such that $Q^n = 1$.

Case 2. Let the norm of dual *SSQ* be negative, *i.e.* $N_Q = A_0^2 - A_1^2 < 0$.
The polar form of a non-null dual *SSQ*

$$Q = A_0 + A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$$

is

$$Q = R(\sinh\psi + \vec{W}\cosh\psi)$$

where $R = \sqrt{|N_Q|}$, and

$$\sinh\psi = \frac{|A_0|}{R}\ ,\ \cosh\phi = \frac{\sqrt{A_1^2}}{R} = \frac{|A_1|}{\sqrt{A_1^2 - A_2^2}}.$$

$\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit dual vector $\vec{W}$ is given by

$$\vec{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{A_1^2}}[A_1\vec{i} + A_2\vec{j} + A_3\vec{k}\ ] = \frac{1}{|A_1|}(A_1, A_2, A_3).$$

Futher Work

By the Hamilton operators, dual split semi-quaternions have been expressed in terms of 4×4 matrices. With the aid of the De-Moivre's formula, we will obtain any power of these matrices.

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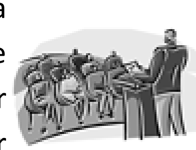

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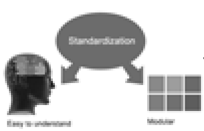

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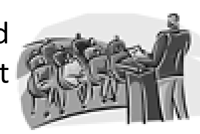

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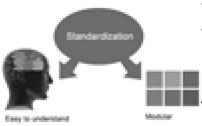

After nomination of your institution as “Institutional Fellow” and constantly functioning successfully for one year, we can consider giving recognition to your institute to function as Regional/Zonal office on our behalf.

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12. Make all efforts: Make all efforts to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in introduction, that what is the need of a particular research paper. Polish your work by good skill of writing and always give an evaluator, what he wants.

13. Have backups: When you are going to do any important thing like making research paper, you should always have backup copies of it either in your computer or in paper. This will help you to not to lose any of your important.

14. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several and unnecessary diagrams will degrade the quality of your paper by creating "hotchpotch." So always, try to make and include those diagrams, which are made by your own to improve readability and understandability of your paper.

15. Use of direct quotes: When you do research relevant to literature, history or current affairs then use of quotes become essential but if study is relevant to science then use of quotes is not preferable.

16. Use proper verb tense: Use proper verb tenses in your paper. Use past tense, to present those events that happened. Use present tense to indicate events that are going on. Use future tense to indicate future happening events. Use of improper and wrong tenses will confuse the evaluator. Avoid the sentences that are incomplete.

17. Never use online paper: If you are getting any paper on Internet, then never use it as your research paper because it might be possible that evaluator has already seen it or maybe it is outdated version.

18. Pick a good study spot: To do your research studies always try to pick a spot, which is quiet. Every spot is not for studies. Spot that suits you choose it and proceed further.

19. Know what you know: Always try to know, what you know by making objectives. Else, you will be confused and cannot achieve your target.

20. Use good quality grammar: Always use a good quality grammar and use words that will throw positive impact on evaluator. Use of good quality grammar does not mean to use tough words, that for each word the evaluator has to go through dictionary. Do not start sentence with a conjunction. Do not fragment sentences. Eliminate one-word sentences. Ignore passive voice. Do not ever use a big word when a diminutive one would suffice. Verbs have to be in agreement with their subjects. Prepositions are not expressions to finish sentences with. It is incorrect to ever divide an infinitive. Avoid clichés like the disease. Also, always shun irritating alliteration. Use language that is simple and straight forward. put together a neat summary.

21. Arrangement of information: Each section of the main body should start with an opening sentence and there should be a changeover at the end of the section. Give only valid and powerful arguments to your topic. You may also maintain your arguments with records.

22. Never start in last minute: Always start at right time and give enough time to research work. Leaving everything to the last minute will degrade your paper and spoil your work.

23. Multitasking in research is not good: Doing several things at the same time proves bad habit in case of research activity. Research is an area, where everything has a particular time slot. Divide your research work in parts and do particular part in particular time slot.

24. Never copy others' work: Never copy others' work and give it your name because if evaluator has seen it anywhere you will be in trouble.

25. Take proper rest and food: No matter how many hours you spend for your research activity, if you are not taking care of your health then all your efforts will be in vain. For a quality research, study is must, and this can be done by taking proper rest and food.

26. Go for seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

27. Refresh your mind after intervals: Try to give rest to your mind by listening to soft music or by sleeping in intervals. This will also improve your memory.

28. Make colleagues: Always try to make colleagues. No matter how sharper or intelligent you are, if you make colleagues you can have several ideas, which will be helpful for your research.

29. Think technically: Always think technically. If anything happens, then search its reasons, its benefits, and demerits.

30. Think and then print: When you will go to print your paper, notice that tables are not be split, headings are not detached from their descriptions, and page sequence is maintained.

31. Adding unnecessary information: Do not add unnecessary information, like, I have used MS Excel to draw graph. Do not add irrelevant and inappropriate material. These all will create superfluous. Foreign terminology and phrases are not apropos. One should NEVER take a broad view. Analogy in script is like feathers on a snake. Not at all use a large word when a very small one would be sufficient. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Amplification is a billion times of inferior quality than sarcasm.

32. Never oversimplify everything: To add material in your research paper, never go for oversimplification. This will definitely irritate the evaluator. Be more or less specific. Also too, by no means, ever use rhythmic redundancies. Contractions aren't essential and shouldn't be there used. Comparisons are as terrible as clichés. Give up ampersands and abbreviations, and so on. Remove commas, that are, not necessary. Parenthetical words however should be together with this in commas. Understatement is all the time the complete best way to put onward earth-shaking thoughts. Give a detailed literary review.

33. Report concluded results: Use concluded results. From raw data, filter the results and then conclude your studies based on measurements and observations taken. Significant figures and appropriate number of decimal places should be used. Parenthetical remarks are prohibitive. Proofread carefully at final stage. In the end give outline to your arguments. Spot out perspectives of further study of this subject. Justify your conclusion by at the bottom of them with sufficient justifications and examples.

34. After conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print to the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects in your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form, which is presented in the guidelines using the template.
- Please note the criterion for grading the final paper by peer-reviewers.

Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.

- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.
- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all, take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described. Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."
- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals Inc. (US):

Segment Draft and Final Research Paper: You have to strictly follow the template of research paper. If it is not done your paper may get rejected.

- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.
- Do not give permission to anyone else to "PROOFREAD" your manuscript.
- Methods to avoid Plagiarism is applied by us on every paper, if found guilty, you will be blacklisted by all of our collaborated research groups, your institution will be informed for this and strict legal actions will be taken immediately.)
- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

Criterion for Grading a Research Paper (Compilation) by Global Journals Inc. (US)

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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