Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Decomposition of Fuzzy Automata based on Lattice-Ordered Monoid

By Anupam K. Singh

Abstract- The purpose of the present work is to introduce decomposable concepts of fuzzy automaton having the membership values in the lattice-ordered monoids. Unlike to the usual fuzzy automata, we show that such concepts for fuzzy automata having the membership values in lattice-ordered monoids by the associated monoid structure.

Keywords: lattice-ordered monoid; fuzzy automaton; primary; f-primary; decomposition.

GJSFR-F Classification: MSC 2010: 06D72

DECOMPOSITIONOFFUZZYAUTOMATABASEDONLATTICEORDEREDMONOID

Strictly as per the compliance and regulations of:

10. D.S. Malik, J.N. Mordeson, M.K. Sen, Submachines of fuzzy finite state machine, *Journal of Fuzzy Mathematics*, 2 1994) 781-792.

Decomposition of Fuzzy Automata based on Lattice-Ordered Monoid

Anupam K. Singh

Abstract- The purpose of the present work is to introduce decomposable concepts of fuzzy automaton having the membership values in the lattice-ordered monoids. Unlike to the usual fuzzy automata, we show that such concepts for fuzzy automata having the membership values in lattice-ordered monoids by the associated monoid structure.

Keywords: *lattice-ordered monoid; fuzzy automaton; primary; f-primary; decomposition.*

I. Introduction And Preliminaries

The study of fuzzy automata was initiated by [16] and [26] in 1960's after the introduction of fuzzy set theory by [27]. Much later, a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) was introduced by [10] (cf. [11], for more details). Somewhat different notions were introduced subsequently by [7, 8, 13]. Recently, Jun [4, 5, 6] generalized the concept of fuzzy finite state machine corresponding to one higher order fuzzy sets, viz., the intuitionistic fuzzy sets, and called it intuitionistic fuzzy finite state machine. In these studies, the membership values in the closed interval [0, 1] were considered. During the recent years, the researchers were initiated to work with fuzzy automata with membership values in complete residuated lattices, lattice ordered monoids and some other kind of lattices (cf., [3, 9, 14, 15]Anu5, Anu6).

In this paper, we study the decomposable properties of fuzzy automata with membership values in lattice ordered monoid via their primaries. We show that several results related to the decomposable properties of fuzzy automata introduced in [19, 20, 21, 22] may not hold well in the case of fuzzy automata with membership values in lattice ordered monoid. This paper is organised as follows: In section 1, we recall some notions related to lattice ordered monoid, monoid with and without zero divisors and some known results which are used in the paper. In Section 2, we introduce and study the concept of a primaries of a fuzzy automata and a compact fuzzy automata and their primary decomposition having the membership values in lattice ordered monoid. In Section 3, we study the primary decomposition and f-primary decomposition for an L-fuzzy automaton.

We recall the following from [2, 19, 23].

Definition 1.1 *An algebra* $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *is called a* **lattice-ordered monoid** *if*

1. $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *is a lattice with the least element 0 and the greatest element 1,*
2. $(L, \bullet, e)$ *is a monoid with identity* $e \in L$ *such that for all* $a, b, c \in L$

   (i) $a \bullet 0 = 0 \bullet a = 0,$

Author: *Janta Koshi College, Biraul A Constituent Unit of Lalit Narayan Mithila University Darbhanga, Bihar-847203, India.*
e-mail: anupam09.bhu@gmail.com

(ii) $a \leq b \Rightarrow \forall x \in L, a \bullet x \leq b \bullet x$ *and* $x \bullet a \leq x \bullet b$,
(iii) $a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$ *and* $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

Definition 1.2 *A monoid* $(L, \bullet, e)$ *is called* **monoid without zero divisors** *if for all* $a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \bullet b \neq 0$.

Definition 1.3 *A monoid* $(L, \bullet, e)$ *is called* **monoid with zero divisors** *if for all* $a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \bullet b = 0$.

Definition 1.4 *Let* L *be an lattice-ordered monoid. An* L-**fuzzy automaton** *is a triple* $M = (Q, X, \delta)$*, where* Q *is a nonempty set (of* **states** *of* M*),* X *is a monoid (the* **input monoid** *of* M*), whose identity shall be denoted as* e_X*, and* $\delta : Q \times X \times Q \to L$ *is a map, such that* $\forall q, p \in Q, \forall x, y \in X$,

$$\delta(q, e_X, p) = \begin{cases} e & \textit{if } q = p \\ 0 & \textit{if } q \neq p \end{cases}$$

$$\textit{and} \quad \delta(q, xy, p) = \vee\{\delta(q, x, r) \bullet \delta(r, y, p) : r \in Q\}.$$

Definition 1.5 *Let* (Q, X, δ) *be an* L*-fuzzy automaton and* $A \subseteq Q$*. The* **source***, the* **successor** *and the* **core** *of* A *are respectively the sets*

$$\sigma_Q(A) = \{q \in Q : \delta(q, x, p) > 0, \textit{ for some } (x, p) \in X \times A\}, \text{ and}$$
$$s_Q(A) = \{p \in Q : \delta(q, x, p) > 0, \textit{ for some } (x, q) \in X \times A\}.$$

We shall frequently write $\sigma_Q(A), s_Q(A)$ as just $\sigma(A), s(A)$and $\sigma(\{q\})$ and $s(\{q\})$ as just $\sigma(q)$ and $s(q)$.

Definition 1.6 *The* **core** *of any subset* R *of the state-set* Q *of an* L*-fuzzy automaton is the set*

$$\mu(R) = \{q \in Q : \sigma(q) \subseteq R\}.$$

We shall frequently write $\mu(\{q\})$ as just $\mu(q)$.

Proposition 1.1 *Let* $(L, \bullet, e)$ *be a monoid without zero divisors and* (Q, X, δ) *be an* L*-fuzzy automaton. Then for all* $A \subseteq Q, s(s(A)) = s(A)$ *and hence* $\sigma(\sigma(A)) = \sigma(A)$.

Remark 1.1 *Let* $M = (Q, X, \delta)$ *be an* L*-fuzzy automaton and* $p, q, r \in Q$*. Then* $p \in \sigma(q), q \in \sigma(r) \not\Rightarrow p \in \sigma(r)$.

Proposition 1.2 *Let* $(L, \bullet, e)$ *be a monoid without zero divisors and* (Q, X, δ) *be an* L*-fuzzy automaton. Then for all* $p, q, r \in Q$*. Then* $p \in \sigma(q), q \in \sigma(r) \Rightarrow p \in \sigma(r)$.

Definition 1.7 *Let* $M = (Q, X, \delta)$ *be an* L*-fuzzy automaton.* $R \in L^Q$ *is called an* L*-fuzzy subautomaton of* M *if* $s(R) \leq R$ *and* $\lambda = \delta|_{R \times X \times R}$.

A fuzzy automaton $M = (Q, X, \delta)$ is retrievable if $\forall p, q \in Q, q \in \sigma(p) \Rightarrow p \in \sigma(q)$ and strongly connected if $\forall q, p \in Q, q \in s(p)$.

II. Compact L-Fuzzy Automata and Their Primary Decomposition

In this section, we introduce the concept of a primary of an L-fuzzy automaton $M = (Q, X, \delta)$ and provide its topological interpretation through the concept of a *regular closed set* in topology, as introduced in [19, 20]. Also, we introduce the concept of a primary decomposition of an L-fuzzy automaton $M = (Q, X, \delta)$ and provide state-set topologies are compact (cf. [17] for a similar observation).

Definition 2.1 *A closed subset of topological space is called* **regular closed** *if it is equal to the closure of its interior*

Definition 2.2 *A subset* $R \subseteq Q$ *is called*

(i) **genetic** *if* $\sigma(R) \subseteq s(R)$,

(ii) **genetically closed** *if* $\exists P \subseteq R$ *such that* $\sigma(P) \subseteq s(P)$ *and* $s(P) = R$,

Ref

17. A.K. Srivastava, W. Shukla, A topology for automata , *Information and control* A note, 32 (1976) 163-168.

(iii) $\mathbf{Gen}(Q)$ *is defined as,* $\{R : R \subseteq Q$ *and* $s(R) = Q\}$,

(iv) *a* **primary** *subset of* Q *if* R *is a nonempty minimal genetically closed subset of* Q.

Proposition 2.1 *Let* (R, X, λ) *be a primary subautomaton of an* L*-fuzzy automaton* (Q, X, δ). *Then* $s(\sigma(p)) = R, \forall p \in \mu(R)$.

Proof : The proof is similar, as given in [23].

Proposition 2.2 *If* $(L, \bullet, e)$ *be a monoid without zero divisors. Then for each* $q \in Q, s(\mu(q))$ *is a primary of* Q, *if* $\mu(q) \neq \phi$.

Proof: The proof is similar, as given in [23].

Remark 2.1 *If* $(L, \bullet, e)$ *be a monoid with zero divisors. Then* $s(q)$ *is not a primary of* Q. *Hence for each* $q \in Q, s(\mu(q))$ *is not a primary of* Q.

Remark 2.2 *From the above proposition it is clear that whenever* $s(p)$ *is regular closed, it must be minimal regular closed.*

Proposition 2.3 *Let* $M = (Q, X, \delta)$ *be an* L*-fuzzy automaton and* $p \in Q$. *Then as* $(L, \bullet, e)$ *be a monoid without zero divisors. The following statements are equivalent.*

(i) $s(p)$ *is a primary of* Q:

(ii) $s(p)$ *is a regular closed subset of* Q:

(iii) $\{p\}$ *is not a nowhere dense subset of* Q:

(iv) $s(p)$ *is a minimal regular closed subset of* Q:

(v) $p \in \mu(s(p))$.

Proof: The proof is similar, as given in [19].

Proposition 2.4 *Let* $R \subseteq Q$ *and* $(L, \bullet, e)$ *be a monoid without zero divisors. Then* $s(\sigma(R))$ *is the union of all primaries of* Q *which contains at least one member of* R.

Proof: The proof is similar, as given in [12].

Lemma 2.1 *Let* $Q = s(q)$ *and* $(L, \bullet, e)$ *is a monoid without zero divisors. Then* $\sigma(q) = g_1(Q)$, *where* $g_1(Q) = \{p : \{p\} \in Gen(Q)\}$.

Proof: Let $p \in \sigma(q)$, then $q \in s(p)$ and $(L, \bullet, e)$ is a monoid without zero divisors. Then $Q = s(q) \subseteq s(s(p)) \subseteq s(p) \subseteq Q$. Hence $Q = s(p)$ and so $\{p\} \in Gen(Q)$. Thus $\sigma(q) \subseteq g_1(Q)$. On the other hand, if $p \in g_1(Q)$, then $q \in s(p)$. Hence $p \in \sigma(q)$. Thus $g_1(Q) \subseteq \sigma(q)$.

Remark 2.3 *Let* $Q = s(q)$ *and* $(L, \bullet, e)$ *be a monoid with zero divisors. Then* $\sigma(q) \neq g_1(Q)$, *where* $g_1(Q) = \{p : \{p\} \in Gen(Q)\}$, *as the following counter-example shows.*

counter-example 2.1 *For the lattice-ordered monoid* L, *consider the monoid* $(L, \bullet, e)$, *where* $L = [0, 1], a \bullet b = max(0, a + b - 1), \forall a, b \in L$ *and* $e = 1$. *Consider a* L*-fuzzy automaton* $M = (Q, X, \delta)$, *where* Q *is the set of integers,* $X = \{0, 1, 2, ...\}$, *and* $\delta : Q \times X \times Q \to L$ *is given by*

$$\delta(m, 0, n) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n, \end{cases}$$

$\forall m, n \in Q$, *and* $\delta(m_0, x_0, n_0) = 1/3, \delta(n_0, x_0, k_0) = 1/3, \delta(k_0, x_0, l_0) = 1/3$, *for fixed* $m_0, n_0, k_0, l_0 \in Q$ *and for fixed* $x_0 \in X (x_0 \neq 0)$. *For other* $m, n \in Q$ *and* $x \in X, \delta(m, x, n) = 0$. *Here, Let* $s(\{m_0\}) = \{n_0\} = Q$ *then* $\sigma(\{m_0\}) = \phi$. *Now if* $\{m_0\} \in g_1(Q)$. *Then* $\{n_0\} \in s(\{m_0\}) \Rightarrow \sigma(\{n_0\}) = \{m_0\} \neq \phi$. *Thus* $\sigma(\{m_0\}) \neq g_1(Q)$.

Proposition 2.5 *Let* R *be a non-empty genetic subset of* Q *and* $(L, \bullet, e)$ *be a monoid without zero divisors. Then* $s(R)$ *is the union of those primaries of* Q *which contains at least one member of* R.

Ref

23. S.P. Tiwari and Shambhu Sharan, Fuzzy automata based on latticeordered monoid with algebraic and topological aspects, *International Journal of Fuzzy and Information Engineering, Communicated.*

Proof: Since R is genetic, $\sigma(R) \subseteq s(R)$ and $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(\sigma(R)) \subseteq s(s(R)) = s(R)$. Since $R \subseteq \sigma(R), s(R) \subseteq s(\sigma(R))$. Hence $s(\sigma(R)) = s(R)$. The result follows from Proposition 2.4.

Remark 2.4 Let R be a non-empty genetic subset of Q and $(L, \bullet, e)$ be a monoid with zero divisors. Then $s(R)$ is not the union of those primaries of Q which contains at least one member of R, as the following counter-examples shows.

counter-example 2.2 Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{n_o\}$. Then $s(A) = \{n_o, k_o\}, \sigma(A) = \{m_o, n_o\}$, $\sigma(\sigma(A)) = \{m_o, n_o\}$ and $s(\sigma(A)) = \{m_o, n_o, k_o\}$. Then $\sigma(\sigma(A)) \subseteq s(\sigma(A))$. Thus $\sigma(A)$ is genetic and hence A be genetic subset of Q. But $s(\sigma(A)) \neq s(A)$.

Lemma 2.2 Let (R, X, λ) be a primary subautomaton of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ be a monoid without zero divisors. Then for every finite subset T of R, $T \subseteq s(r)$, for some $r \in R$.

Proof: We prove this Lemma by induction. Let $T = \{p_1, p_2 ...... p_n\}$ be any finite subset of R. Then the result is obvious for $n = 1$. Now assume the result is true for $n = k - 1$; in particular for $T_{k-1} = \{p_1, p_2 ..... p_{k-1}\}$. Then $\exists q \in R$ such that $T_{k-1} \subseteq s(q)$. Thus $T_k = \{p_1, p_2 ...... p_k\} \subseteq s(q) \cup \{p_k\} \subseteq s(\{q, p_k\})$. Put $S = \{q, p_k\}$ and let $m \in \mu(R)$. Then by Proposition 2.1. $R = s(\sigma(m))$. Note that $q \in R \Rightarrow q \in s(\sigma(m)) \Rightarrow \delta(m', x, q) > 0$, for some $(m', x) \in \sigma(m) \times X$ now as $(L, \bullet, e)$ is a monoid without zero divisors. Then $m' \in \sigma(m) \Rightarrow \sigma(m') \subseteq \sigma(\sigma(m)) = \sigma(m) \subseteq R$ (as $m \in \mu(R)$) $\Rightarrow m' \in \mu(R) \Rightarrow s(\sigma(m')) = R$ (by Proposition 2.1). Consequently, $p_k \in R \Rightarrow p_k \in s(\sigma(m')) \Rightarrow \delta(r, y, p_k) > 0$, for some $(r, y) \in \sigma(m') \times X$. From $r \in \sigma(m')$, we get $m' \in s(r)$, whereby $s(m') \subseteq s(s(r)) = s(r)$. This together with the facts that $q \in s(m')$ and $p_k \in s(r)$, gives $q, p_k \in s(r)$. Hence $S = \{q, p_k\} \subseteq s(r)$. Hence $T_k \subseteq s(r)$.

Remark 2.5 Let (R, X, λ) be a primary subautomaton of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ be a monoid with zero divisors. Then for every finite subset T of R, $T \nsubseteq s(r)$, for some $r \in R$, as the following counter-example shows.

counter-example 2.3 Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{m_o, n_o, k_o\}$. Then $\sigma(\{n_o, k_o\}) = \{m_o, n_o\}$, $s(\sigma(\{n_o, k_o\})) = \{m_o, n_o, k_o\}$. Thus $s(\sigma(\{n_o, k_o\})) = A, \forall \{n_o, k_o\} \in \mu(A)$, which shows that A be a primary of an L-fuzzy automaton Q. But for every finite subset $T = \{n_o\} \subseteq A$, $T = \{n_o\} \nsubseteq s(\{k_0\}) = \{l_0\}$, for some $\{k_0\} \in A$.

Proposition 2.6 If $(L, \bullet, e)$ be a monoid without zero divisors. Then a primary of a compact L-fuzzy automaton is a maximal singly generated subautomaton.

Proof: Let N be a primary of a compact L-fuzzy automaton $M = (Q, X, \delta)$ and let $p \in R$. Then $p \in s(\mu(R))$. So $\exists q \in \mu(R)$ with $p \in s(q)$. As $q \in \mu(R), \sigma(q) \subseteq R$. Now $\sigma(q)$ is finite owing to the compactness of (Q, X, δ), so $\exists q' \in R$ such that $\sigma(q) \subseteq s(q')$ (by Lemma 3.1) and $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(\sigma(q)) \subseteq s(s(q')) = s(q')$. Also, $q \in s(q') \Rightarrow q' \in \sigma(q) \Rightarrow s(q') \subseteq s(\sigma(q'))$. Thus $s(\sigma(q')) = s(q')$, whereby $s(q')$ is a genetically closed subset of R (as $q' \in R$). So by the minimality of R, say $s(q') = R$. Hence the primary (R, X, λ) is singly generated. Let $S = s(t)$ be the state-set of another singly generated subautomaton of M such that $R = s(q') \subseteq S$. To prove that $R = S$. It is enough to show that $t \in s(q')$. Now $s(q') \subseteq S = s(t) \Rightarrow q' \in s(t) \Rightarrow t \in \sigma(q')$, so that $t \in s(q')$ (as $s(q')$ is a primary: cf. Proposition 2.5).

Remark 2.6 If $(L, \bullet, e)$ be a monoid with zero divisors. Then a primary of a compact L-fuzzy automaton is not a maximal singly generated subautomaton.

Notes

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version I Year 2018

Proposition 2.7 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $R \subseteq Q$. Then $s(\sigma(R))$ can be written as the union of those primaries of Q, which have nonempty intersection with R.*

Proof: The proof is similar, as given in [19].

Proposition 2.8 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $R \subseteq Q$ be genetic with $(L, \bullet, e)$ be a monoid without zero divisors. Then $s(R)$ is the union of primaries of Q having nonempty intersection with R.*

Proof: As R be genetic, $\sigma(R) \subseteq s(R)$, implying that $s(\sigma(R)) \subseteq s(s(R)) = s(R)$, as $(L, \bullet, e)$ is a monoid without zero divisors. On the other hand , as $R \subseteq \sigma(R) \subseteq s(\sigma(R))$, we have $s(R) \subseteq s(\sigma(R))$. Thus $s(R) = s(\sigma(R))$. Hence by the Proposition 3.2. $s(R)$ is the union of primaries of Q having nonempty intersection with R.

Remark 2.7 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $R \subseteq Q$ be genetic with $(L, \bullet, e)$ be a monoid with zero divisors. Then $s(R)$ is not the union of primaries of Q having nonempty intersection with R, as the following counter-examples shows.*

counter-example 2.2 *Similar to counter-example 2.2.*

III. Primary Decomposition of an L-Fuzzy Automata

Finite state automata admit a primary decomposition (cf. e.g., [1]). Even an infinite state automaton can admit a primary decomposition, for example, when that automaton is compact (cf. [18]). In [19] we extended this for L-fuzzy automaton provided that $(L, \bullet, e)$ is a monoid without zero divisors. Also, we introduce the concept of a source-splitting subautomaton, including their characterization and a topological description with $(L, \bullet, e)$ is a monoid without zero divisors. lastly, we introduce the concept of f-primaries of an L-fuzzy automaton $M = (Q, X, \delta)$ and provided $(L, \bullet, e)$ is a monoid without zero divisors.

Proposition 3.1 *An L-fuzzy auotmata M is strongly connected if and only if M has no proper subautomaton but if $(L, \bullet, e)$ be a monoid without zero divisors then the convere is true*

Proof: The proof is similar, as given in [23].

Lemma 3.1 *Let (Q, X, δ) be an L-fuzzy automaton and $q \in Q$ such that $R \subseteq s(\sigma(q))$ is a non-empty regular closed (genetically closed) subset of Q. Then $q \in R$ and $(L, \bullet, e)$ be a monoid without zero divisors.*

Proof: Let $p \in \mu(R) \subseteq R \subseteq s(\sigma(q))$. Then $\sigma(p) \subseteq R$ and $p \in s(\sigma(q))$. Now $\sigma(p) \subseteq R \Rightarrow s(\sigma(p)) \subseteq s(R) = R$. Also, $p \in s(\sigma(q)) \Rightarrow p \in s(t)$, for some $t \in \sigma(q) \Rightarrow t \in \sigma(p)$, for some t such that $q \in s(t)$. Again, $t \in \sigma(p) \Rightarrow \sigma(t) \subseteq \sigma(\sigma(p)) = \sigma(p)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Now $q \in s(t) \Rightarrow q \in s(\sigma(t)) \subseteq s(\sigma(p)) \subseteq R$. Thus, $q \in R$.

Remark 3.1 *Let (Q, X, δ) be an L-fuzzy automaton and $q \in Q$ such that $R \subseteq s(\sigma(q))$ is a non-empty regular closed (genetically closed) subset of Q and $(L, \bullet, e)$ be a monoid with zero divisors. Then $q \notin R$, as the following counter-example shows.*

counter-example 3.1 *Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{n_0\}$ and $\{k_0\} \in Q$. Then $\sigma\{k_0\} = \{n_0\}, s(\sigma\{k_0\}) = \{n_0, k_0\}$. Then $A \subseteq s(\sigma\{k_0\})$ is not a non-empty regular closed subset of Q, with $\{k_0\} \notin A$.*

Proposition 3.2 *[20] (Primary Decomposition Theorem) Let the L-fuzzy automaton $M = (Q, X, \delta)$ be compact (having possibly an infinite state-set Q). Then*

(i) $M = \cup_{i=1}^{n} P_i$, and

(ii) for any $j, 1 \leq j \leq n, M = \cup_{i=1, i \neq j}^{n} P_i$,

where $P_1, P_2, P_3, ..., P_n$ are all the distinct primaries of M.

Ref

19. A.K. Srivastava and S.P. Tiwari, A topology for fuzzy automata, in: Proc. 2002 AFSS Internat. Conf. on Fuzzy Systems, *Lecture Notes in Artificial Intelligence*, *Springer-verlag*, 2275 (2002) 485-490.

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version 1 Year 2018

Proposition 3.3 *A L-fuzzy automaton $M = (Q, X, \delta)$ is retrievable if and only if M is decomposable and the primaries of M are strongly connected.*

Proof: The proof is similar, as given in [20].

Remark 3.2 *The converse of the above proposition is not true, if $(L, \bullet, e)$ be a monoid with zero divisors.*

Proposition 3.4 *Let $(L, \bullet, e)$ be a monoid without zero divisors M is decomposable and the primaries of M be strongly connected. Then M is retrievable.*

Proof: Let M be decomposable and the primaries of M be strongly connected. Also, let $p, q \in Q$ be such that $p \in \sigma(q)$ and as $(L, \bullet, e)$ be a monoid without zero divisors. Then $\sigma(p) \subseteq \sigma(\sigma(q)) = \sigma(q)$ and so $s(\sigma(p)) \subseteq s(\sigma(q))$. Since M is decomposable and $s(\sigma(p))$ is a regular closed subset of Q, $s(\sigma(p))$ should contain a primary subset of Q, say R. Now, $R \subseteq s(\sigma(p)) \subseteq s(\sigma(q)) \Rightarrow p, q \in R$(cf. Lemma 4.1). But as R is strongly connected, $p \in s(q)$, whereby $q \in \sigma(p)$. Thus, M is retrievable.

Definition 3.1 [20] *A subautomaton $N = (R, X, \lambda)$ of an L-fuzzy automaton $M = (Q, X, \delta)$ is called source-splitting in M if*

(i) *R is genetically closed subset of Q, and*

(ii) $\forall r \in R, \exists r_1, r_2 \in \sigma(r)$ *such that* $\sigma(r_1) \cap \sigma(r_2) = \phi$.

Lemma 3.2 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and let $\{P_i : i \in I\}$ be a family of all distinct primaries of M, with respective state-sets R_i. Then $s(Q - \cup_{i \in I} R_i)$ is a source-splitting in M.*

Proof: The proof is similar, as given in [20].

Lemma 3.3 *Let $N = (R, X, \lambda)$ be a source-splitting subautomaton of an L-fuzzy automaton $M = (Q, X, \delta)$ and $(L, \bullet, e)$ be a monoid without zero divisors . Then $\mu(R) \subseteq Q - \cup_{i \in I} R_i$, where R_i's are in Lemma 4.2.*

Proof: To show that $\mu(R) \subseteq Q - \cup_{i \in I} R_i$, we show that $\mu(R) \cap R_i = \phi, \forall i \in I$. If possible, let $\mu(R) \cap R_i \neq \phi$, for some $i \in I$. Then $\mu(R) \cap s(\mu(R_i)) \neq \phi$ (since $s(\mu(R_i)) = R_i, \forall i \in I$). Now $q \in \mu(R) \cap s(\mu(R_i)) \Rightarrow \sigma(q) \subseteq R$ and $q \in s(t)$, for some $t \in \mu(R_i) \Rightarrow s(\sigma(q)) \subseteq s(R) = R$ and $q \in s(t)$, for some t with $\sigma(t) \subseteq R_i$. Also $\sigma(t) \subseteq R_i \Rightarrow s(\sigma(t)) \subseteq s(R_i) = R_i$. So, as $s(\sigma(t))$ is regular closed and R_i, being a primary subset, is minimal regular closed, we have $s(\sigma(t)) = R_i$. As $(L, \bullet, e)$ is a monoid without zero divisors. Then $q \in s(t) \Rightarrow t \in \sigma(q) \Rightarrow \sigma(t) \subseteq \sigma(\sigma(q)) = \sigma(q) \Rightarrow s(\sigma(t)) \subseteq s(\sigma(q)) \Rightarrow R_i \subseteq R$. This shows that R contains a primary subset, a contradiction, as N, being source-splitting in M, cannot contain any primary. Hence, $\mu(R) \subseteq Q - \cup_{i \in I} R_i$.

Remark 3.3 *Let $N = (R, X, \lambda)$ be a source-splitting subautomaton of an L-fuzzy automaton $M = (Q, X, \delta)$ and $(L, \bullet, e)$ be a monoid with zero divisors . Then $\mu(R) \nsubseteq Q - \cup_{i \in I} R_i$, where R_i's are in Lemma 4.2, as the following counter-example shows.*

counter-example 3.2 *Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{m_0, n_0, k_0\}, A_1 = \{n_0, k_0\}, A_2 = \{k_0, l_0\}$. Then $\mu(A) = \{n_0, k_0\}$, now $\mu(A) \cap A_i = \{n_0, k_0\} \cap \{n_0, k_0\} \cap \{k_0, l_0\} = \{k_0\} \neq \phi, \forall i = 1, 2$. Thus $\mu(R) \nsubseteq Q - \cup_{i \in I} R_i$.*

Proposition 3.5 *An L-fuzzy automaton (Q, X, δ) is compact if and only if their exists a finite subset Q' of Q such that $s(Q') = Q$, or equivalently, if and only if $\sigma(Q)$ is finite.*

Proof: The proof is similar, as given in [21].

Lemma 3.4 *Let (R, X, λ) be a primary of an L-fuzzy automaton (Q, X, δ). Then $s(\sigma(p)) = R, \forall p \in \mu(R)$.*

Proof: The proof is similar, as given in [21].

Ref

20. S.P. Tiwari and A.K. Srivastava, On a decomposition of fuzzy automata, *Fuzzy Sets and Systems* 151 (2005) 503-511.

Lemma 3.5 *Let (R, X, λ) be a primary of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ be a monoid without zero divisors. Then for every finite subset T of R, $T \subseteq s(r)$ for some $r \in R$.*

Proof: We prove this Lemma by induction on the number of elements in T. The results is obvious if $|T| = 1$. Now assume the result to be true for all T having $k-1$ elements. Consider some $T \subseteq R$ having k elements. Pick any $p \in T$. By induction hypothesis, $\exists q \in R$ such that $T - p \subseteq s(q)$. Thus $T \subseteq s(q) \cup p \subseteq s(\{q, p\})$. Put $S = \{q, p\}$ and let $m \in \mu(R)$. Then by Lemma 4.4, $R = s(\sigma(m))$. Note that $q \in R \Rightarrow q \in s(\sigma(m)) \Rightarrow \delta(m', x, q) > 0$ for some $(m', x) \in \sigma(m) \times X$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $m' \in \sigma(m) \Rightarrow \sigma(\sigma(m')) = \sigma(m') \subseteq \sigma(m) \subseteq R$ (as $m \in \mu(R)) \Rightarrow m' \in \mu(R) \Rightarrow s(\sigma(m')) = R$ (by Lemma 4.4). Consequently, $p \in R \Rightarrow p \in s(\sigma(m')) \Rightarrow \delta(r, y, p) > 0$ for some $(r, y) \in \sigma(m') \times X$. From $r \in \sigma(m')$, we get $m' \in s(r)$, whereby $s(m') \subseteq s(s(r)) = s(r)$. This, together with the facts that $q \in s(m')$ and $p \in s(r)$. Hence $S = \{q, p\} \subseteq s(r)$. Thus $s(\{q, p\}) \subseteq s(s(r))$, i.e., $s(\{q, p\}) \subseteq s(r)$. Hence $T \subseteq s(r)$.

Remark 3.4 *Let (R, X, λ) be a primary of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ is a monoid with zero divisors. Then for every finite subset T of R, $T \nsubseteq s(r)$ for some $r \in R$, as the following counter-example shows.*

counter-example 3.3 *Similar to counter-example 3.1.*

Proposition 3.6 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $(L, \bullet, e)$ be a monoid without zero divisors. Then each maximal finitely closed subset of Q is $T(Q)$-open.*

Proof: Let R be a maximal finitely closed subset of Q. We have to show that $s(R) = R$. Let T be a finite subset of $s(R)$. For each $q \in T$, we can find some $p \in R$ with $q \in s(p)$. The set S of all such p's must be finite. Also, $S \subseteq R$. Obviously, $T \subseteq s(S)$. Now as R is finitely closed, $\exists t \in Q$ such that $S \subseteq s(t)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(S) \subseteq s(t)$ (since $s(s(t)) = s(t)$), whereby $T \subseteq s(t)$. Thus for each finite subset T of $s(R), \exists t \in Q$ such that $T \subseteq s(t)$. Hence $s(R)$ is a finitely closed subset of Q. But as R is a maximal finitely closed set and $R \subseteq s(R)$, we find that $s(R) = R$.

Remark 3.5 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $(L, \bullet, e)$ be a monoid with zero divisors. Then each maximal finitely closed subset of Q is not $T(Q)$-open.*

Proposition 3.7 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton. Then every primary of M is an f-primary of M.*

Proof: The proof is similar, as given in [21].

Remark 3.6 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $p \in Q$, if $s(p)$ is a primary of M. Then $s(p)$ is an f-primary of M.*

Proof: In view of Proposition 4.7, $s(p)$ is an f-primary of M.

Remark 3.7 *$s(p)$ is a primary of M if $(L, \bullet, e)$ be a monoid without zero divisors. So, converse is true only if $(L, \bullet, e)$ be a monoid without zero divisors.*

Proposition 3.8 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $p \in Q$, if $s(p)$ is an f-primary of M. Then it is a primary of M and $(L, \bullet, e)$ be a monoid without zero divisors.*

Proof: We only need to prove that if $s(p)$ is an f-primary then it is a primary. So let $s(p)$ be an f-primary of M. In view of Proposition 2.2, it suffices to show that $p \in \mu(s(p))$, or that $\sigma(p) \subseteq s(p)$. Let $q \in \sigma(p)$. Then $p \in s(q)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(p) \subseteq s(s(q)) = s(q)$. As $s(q)$ is finitely closed and $s(p)$, being an f-primary, is maximal finitely closed, $s(q) = s(p)$. Thus $q \in s(p)$, showing that $\sigma(p) \subseteq s(p)$.

Ref

21. S.P. Tiwari and A.K. Srivastava, On another decomposition of fuzzy automata, *Journal of Uncertain Systems*, 5 (2011) 33-37.

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version I Year 2018

Proposition 3.9 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $N = (R, X, \lambda)$ be its subautomaton, if N is a primary of M. Then it is an f-primary of M.*

Proof : Once again, in view of Proposition 4.7, N is an f-primary of M.

Remark 3.8 *If N is a primary of M and $(L, \bullet, e)$ be a monoid with zero divisors. Then it is not an f-primary of M.*

Proposition 3.10 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $N = (R, X, \lambda)$ be its subautomaton, if N is an f-primary of M. Then it is a primary of M and $(L, \bullet, e)$ be a monoid without zero divisors.*

Proof: Once again, in view of Proposition 4.7, we only need to prove that if N is an f-primary of M then it is a primary. Now, as M is compact, there exists a finite subset R' of R such that $s(R') = R$ (cf. Proposition 4.5). Also, as R is finitely closed, $\exists q \in Q$ such that $R' \subseteq s(q)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(R') \subseteq s(s(q)) = s(q)$. Thus $R \subseteq s(q)$. But as $s(q)$ is finitely closed and R is maximal finitely closed (as N is an f-primary), $s(q) = R$. Hence N is a primary of M (cf. Proposition 3.1).

Remark 3.9 *If N is an f-primary of M and $(L, \bullet, e)$ be a monoid with zero divisors. Then it is not a primary of M.*

IV. Conclusion

In this paper, we have introduced and studied here the decomposition of fuzzy automata via their primary and f-primary based on lattice ordered monoids. Interestingly, we found that decomposition concepts for fuzzy automata based on lattice-ordered monoids depends on the associated monoid structure. The obtained results generalize the observations made in [19, 20, 21].

References Références Referencias

1. Z. Bavel, Structure and transition preserving functions of finite automata, *J. Assoc. Comput. Mach.*, 15 (1968) 135-158.
2. X. Guo, Grammar theory based on lattice ordered monoid, *Fuzzy Sets and Systems*, 160 (2009) 1152-1161.
3. J. Ignjatovijc, M. J Ciric, S. Bogdanovijc, Determinization of fuzzy automata with membership values in complete residuated lattices, *In-formation Sciences*, 178 (2008) 164-180.
4. Y.B. Jun, Intuitionistic fuzzy finite state machines, *Journal of Ap-plied Mathematics and Computing*, 17 (2005) 109-120.
5. Y.B. Jun, Intuitionistic fuzzy finite switchboard state machines, *Journal of Applied Mathematics and Computing*, 20 (2006) 315-325.
6. Y.B. Jun, Quotient structures of intuitionistic fuzzy finite state machines, *Information Sciences*, 177 (2007) 4977-4986.
7. Y.H. Kim, J.G. Kim and S.J. Cho, Products of *T*-generalized state machines and *T*-generalized transformation semigroups, *Fuzzy Sets and Systems*, 93 (1998) 87-97.
8. H.V. Kumbhojkar, S.R. Chaudhri, On proper fuzzification of fuzzy finite state machines, *International Journal of Fuzzy Mathematics*, 4 (2008) 1019-1027.
9. Y. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-orderd monoids, *Fuzzy Sets and Systems*, 156 (2005) 68-92.
10. D.S. Malik, J.N. Mordeson, M.K. Sen, Submachines of fuzzy finite state machine, *Journal of Fuzzy Mathematics*, 2 (1994) 781-792.
11. J.N. Mordeson, D.S. Malik, Fuzzy Automata and Languages: Theory and Applications. London/Boca Raton: Chapman and Hall/CRC (2002).

Ref

19. A.K. Srivastava and S.P. Tiwari, A topology for fuzzy automata, in: Proc. 2002 *AFSS Internat. Conf. on Fuzzy Systems, Lecture Notes in Artificial Intelligence, Springer-verlag*, 2275 (2002) 485-490.

12. J.N. Mordeson and P.S. Nair, Retrievability and Connectedness in Fuzzy Finite State Machines, in: Proc. 1996 *Internat. Conf. on IEEE*, 3 1996 1586-1590.
13. D. Qiu, Characterizations of fuzzy finite automata, *Fuzzy Sets and Systems*, 141 (2004) 391-414.
14. D. Qiu, Automata theory based on complete residuated lattice-valued logic(I), *Sci. China Ser. (F)*, 44 (2001) 419-429.
15. D. Qiu, Automata theory based on complete residuated lattice-valued logic(II), *Sci. China Ser. (F)*, 45 (2002) 442-452.
16. E. S. Santos, Maximin automata, *Information and Control*, 12 (1968) 367-377.
17. A.K. Srivastava, W. Shukla, A topology for automata , *Information and control* A note, 32 (1976) 163-168.
18. A.K. Srivastava, W. Shukla, A topology for automata II, *Internat. J. Math. Math. Sci.*, 9 (1986) 425-428.
19. A.K. Srivastava and S.P. Tiwari, A topology for fuzzy automata, in: Proc. 2002 *AFSS Internat. Conf. on Fuzzy Systems, Lecture Notes in Artificial Intelligance, Springer-verlag*, 2275 (2002) 485-490.
20. S.P. Tiwari and A.K. Srivastava, On a decomposition of fuzzy automata, *Fuzzy Sets and Systems*, 151 (2005) 503-511.
21. S.P. Tiwari and A.K. Srivastava, On another decomposition of fuzzy automata, *Journal of Uncertain Systems*, 5 (2011) 33-37.
22. S.P. Tiwari, Anupam K. Singh, Shambhu Sharan, Fuzzy automata based on lattice-ordered monoid and associated topology, *Journal of Uncertain Systems*, 6 (2012) 51-55.
23. S.P. Tiwari and Shambhu Sharan, Fuzzy automata based on latticeordered monoid with algebraic and topological aspects, *Internation Journal of Fuzzy and Information Engineering*, *Communicated*.
24. S.P. Tiwari, Anupam K. Singh, Fuzzy preorder, fuzzy topology and fuzzy transition system, in: Proc. ICLA 2013, *Lecture Notes in Com-puter Science, Springer, Berlin* 7750 (2013), 210-219.
25. Anupam K. Singh, Fuzzy preordered set, fuzzy topology and fuzzy automaton based on generalized residuated lattice, *Iranian Journal of Fuzzy Systems*,14 (2017) 55-66.
26. W.G. Wee, On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification, Ph. D. Thesis, Purdue University, Lafayette, IN 1967.
27. L.A. Zadeh, Fuzzy Sets, *Information and control*, 8 (1965) 338-353.

Notes