Decomposition of Fuzzy Automata based on Lattice-Ordered Monoid

By Anupam K. Singh

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GJSFR-F Classification: MSC 2010: 06D72
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I. INTRODUCTION AND PRELIMINARIES

The study of fuzzy automata was initiated by [16] and [26] in 1960’s after the introduction of fuzzy set theory by [27]. Much later, a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) was introduced by [10] (cf. [11], for more details). Somewhat different notions were introduced subsequently by [7, 8, 13]. Recently, Jun [4, 5, 6] generalized the concept of fuzzy finite state machine corresponding to one higher order fuzzy sets, viz., the intuitionistic fuzzy sets, and called it intuitionistic fuzzy finite state machine. In these studies, the membership values in the closed interval [0, 1] were considered. During the recent years, the researchers were initiated to work with fuzzy automata with membership values in complete residuated lattices, lattice ordered monoids and some other kind of lattices (cf., [3, 9, 14, 15, Anu5, Anu6]).

In this paper, we study the decomposable properties of fuzzy automata with membership values in lattice ordered monoid via their primaries. We show that several results related to the decomposable properties of fuzzy automata introduced in [19, 20, 21, 22] may not hold well in the case of fuzzy automata with membership values in lattice ordered monoid. This paper is organised as follows: In section 1, we recall some notions related to lattice ordered monoid, monoid with and without zero divisors and some known results which are used in the paper. In Section 2, we introduce and study the concept of a primaries of a fuzzy automata and a compact fuzzy automata and their primary decomposition having the membership values in lattice ordered monoid. In Section 3, we study the primary decomposition and f-primary decomposition for an L-fuzzy automaton.

We recall the following from [2, 19, 23].

Definition 1.1 An algebra $L = (L, \leq, \wedge, \vee, \cdot, 0, 1)$ is called a lattice-ordered monoid if

1. $L = (L, \leq, \wedge, \vee, \cdot, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
2. $(L, \cdot, e)$ is a monoid with identity $e \in L$ such that for all $a, b, c \in L$

(i) $a \cdot 0 = 0 \cdot a = 0$, 

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(ii) \( a \leq b \Rightarrow \forall x \in L, a \circ x \leq b \circ x \) and \( x \circ a \leq x \circ b \).

(iii) \( a \circ (b \lor c) = (a \circ b) \lor (a \circ c) \) and \( (b \lor c) \circ a = (b \circ a) \lor (c \circ a) \).

Definition 1.2 A monoid \((L, \cdot, e)\) is called **monoid without zero divisors** if for all \( a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \cdot b \neq 0 \).

Definition 1.3 A monoid \((L, \cdot, e)\) is called **monoid with zero divisors** if for all \( a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \cdot b = 0 \).

Definition 1.4 Let \( L \) be an lattice-ordered monoid. An **\( L \)**-fuzzy automaton is a triple \( M = (Q, X, \delta) \), where \( Q \) is a nonempty set (of states of \( M \)), \( X \) is a monoid (the input monoid of \( M \)), whose identity shall be denoted as \( e_X \), and \( \delta : Q \times X \times Q \rightarrow L \) is a map, such that \( \forall q, p \in Q, \forall x, y \in X \),

\[
\delta(q, e_X, p) = \begin{cases} 
  e & \text{if } q = p \\
  0 & \text{if } q \neq p
\end{cases}
\]

and \( \delta(q, xy, p) = \lor \{ \delta(q, x, r) \cdot \delta(r, y, p) : r \in Q \} \).

Definition 1.5 Let \((Q, X, \delta)\) be an \( L \)-fuzzy automaton and \( A \subseteq Q \). The **source**, the **successor** and the **core** of \( A \) respectively the sets

\[
\sigma_Q(A) = \{ q \in Q : \delta(q, x, p) > 0, \text{ for some } (x, p) \in X \times A \}, \\
\sigma_Q(A) = \{ p \in Q : \delta(q, x, p) > 0, \text{ for some } (x, q) \in X \times A \}.
\]

We shall frequently write \( \sigma_Q(A), s_Q(A) \) as just \( \sigma(A), s(A) \) and \( s\{q\} \) as just \( s(q) \) and \( s(q) \).

Definition 1.6 The **core** of any subset \( R \) of the state-set \( Q \) of an \( L \)-fuzzy automaton is the set

\[
\mu(R) = \{ q \in Q : \sigma(q) \subseteq R \}.
\]

We shall frequently write \( \mu(\{q\}) \) as just \( \mu(q) \).

Proposition 1.1 Let \((L, \cdot, e)\) be a monoid without zero divisors and \((Q, X, \delta)\) be an \( L \)-fuzzy automaton. Then for all \( A \subseteq Q, s(s(A)) = s(A) \) and hence \( \sigma(\sigma(A)) = \sigma(A) \).

Remark 1.1 Let \( M = (Q, X, \delta) \) be an \( L \)-fuzzy automaton and \( p, q, r \in Q \). Then \( p \in \sigma(q), q \in \sigma(r) \Rightarrow p \in \sigma(r) \).

Proposition 1.2 Let \((L, \cdot, e)\) be a monoid without zero divisors and \((Q, X, \delta)\) be an \( L \)-fuzzy automaton. Then for all \( p, q, r \in Q \). Then \( p \in \sigma(q), q \in \sigma(r) \Rightarrow p \in \sigma(r) \).

Definition 1.7 Let \( M = (Q, X, \delta) \) be an \( L \)-fuzzy automaton. \( R \in L^Q \) is called an **\( L \)**-fuzzy subautomaton of \( M \) if \( s(R) \leq R \) and \( \lambda = \delta|_{R \times X \times R} \).

A fuzzy automaton \( M = (Q, X, \delta) \) is retrievable if \( \forall p, q \in Q, q \in \sigma(p) \Rightarrow p \in \sigma(q) \) and strongly connected if \( \forall q, p \in Q, q \in s(p) \).

II. **COMPACT L-FUZZY AUTOMATA AND THEIR PRIMARY DECOMPOSITION**

In this section, we introduce the concept of a primary of an \( L \)-fuzzy automaton \( M = (Q, X, \delta) \) and provide its topological interpretation through the concept of a **regular closed set** in topology, as introduced in [19, 20].

Also, we introduce the concept of a primary decomposition of an \( L \)-fuzzy automaton \( M = (Q, X, \delta) \) and provide state-set topologies are compact (cf. [17] for a similar observation).

Definition 2.1 A closed subset of topological space is called **regular closed** if it is equal to the closure of its interior

**Definition 2.2** A subset \( R \subseteq Q \) is called

(i) **genetic** if \( \sigma(R) \subseteq s(R) \),

(ii) **genetically closed** if \( \exists P \subseteq R \) such that \( \sigma(P) \subseteq s(P) \) and \( s(P) = R \).
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(iii) $\text{Gen}(Q)$ is defined as, \{ $R : R \subseteq Q$ and $s(R) = Q$ \},

(iii) a primary subset of $Q$ if $R$ is a nonempty minimal genetically closed subset of $Q$.

Proposition 2.1 Let $(R, X, \lambda)$ be a primary subautomaton of an $L$-fuzzy automaton $(Q, X, \delta)$. Then $s(\sigma(p)) = R, \forall p \in \mu(R)$.

Proof: The proof is similar, as given in [23].

Proposition 2.2 If $(L, \cdot, e)$ be a monoid without zero divisors. Then for each $q \in Q, s(\mu(q))$ is a primary of $Q$, if $\mu(q) \neq \phi$.

Proof: The proof is similar, as given in [23].

Remark 2.1 If $(L, \cdot, e)$ be a monoid with zero divisors. Then $s(q)$ is not a primary of $Q$. Hence for each $q \in Q, s(\mu(q))$ is not a primary of $Q$.

Remark 2.2 From the above proposition it is clear that whenever $s(p)$ is regular closed, it must be minimal regular closed.

Proposition 2.3 Let $M = (Q, X, \delta)$ be an $L$-fuzzy automaton and $p \in Q$. Then as $(L, \cdot, e)$ be a monoid without zero divisors. The following statements are equivalent.

(i) $s(p)$ is a primary of $Q$;
(ii) $s(p)$ is a regular closed subset of $Q$;
(iii) $\{ p \}$ is not a nowhere dense subset of $Q$;
(iv) $p \in \mu(s(p))$.

Proof: The proof is similar, as given in [19].

Proposition 2.4 Let $R \subseteq Q$ and $(L, \cdot, e)$ be a monoid without zero divisors. Then $s(\sigma(R))$ is the union of all primaries of $Q$ which contains at least one member of $R$.

Proof: The proof is similar, as given in [12].

Lemma 2.1 Let $Q = s(q)$ and $(L, \cdot, e)$ be a monoid without zero divisors. Then $\sigma(q) = g_1(Q)$, where $g_1(Q) = \{ p : \{ p \} \in \text{Gen}(Q) \}$.

Proof: Let $p \in \sigma(q)$, then $q \in s(p)$ and $(L, \cdot, e)$ is a monoid without zero divisors. Then $Q = s(q) \subseteq s(s(p)) \subseteq s(p) \subseteq Q$. Hence $Q = s(p)$ and so $\{ p \} \in \text{Gen}(Q)$. Thus $\sigma(q) \subseteq g_1(Q)$. On the other hand, if $p \in g_1(Q)$, then $q \in s(p)$. Hence $p \in \sigma(q)$. Thus $g_1(Q) \subseteq \sigma(q)$.

Remark 2.3 Let $Q = s(q)$ and $(L, \cdot, e)$ be a monoid with zero divisors. Then $\sigma(q) \neq g_1(Q)$, where $g_1(Q) = \{ p : \{ p \} \in \text{Gen}(Q) \}$, as the following counter-example shows.

counter-example 2.1 For the lattice-ordered monoid $L$, consider the monoid $(L, \cdot, e)$, where $L = \{ 0, 1 \}, a \cdot b = \max(0, a + b - 1), \forall a, b \in L$ and $e = 1$. Consider a $L$-fuzzy automaton $M = (Q, X, \delta)$, where $Q$ is the set of integers, $X = \{ 0, 1, 2, \ldots \}$, and $\delta : Q \times X \times Q \rightarrow L$ is given by

$$
\delta(m, 0, n) = \begin{cases} 
1 & \text{if } m = n \\
0 & \text{if } m \neq n,
\end{cases}
$$

$\forall m, n \in Q$, and $\delta(m_0, x_0, n_0) = 1/3, \delta(n_0, x_0, k_0) = 1/3, \delta(k_0, x_0, l_0) = 1/3$, for fixed $m_0, n_0, k_0, l_0 \in Q$ and for fixed $x_0 \in X(x_0 \neq 0)$. For other $m, n \in Q$ and $x \in X, \delta(m, x, n) = 0$. Here, Let $s(\{ m_0 \}) = \{ m_0 \} = Q$ then $\sigma(\{ m_0 \}) = \phi$. Now if $\{ m_0 \} \in g_1(Q)$. Then $\{ m_0 \} \in s(\{ m_0 \}) \Rightarrow \sigma(\{ m_0 \}) = \{ m_0 \} \neq \phi$. Thus $\sigma(\{ m_0 \}) \neq g_1(Q)$.

Proposition 2.5 Let $R$ be a non-empty genetic subset of $Q$ and $(L, \cdot, e)$ be a monoid without zero divisors. Then $s(R)$ is the union of those primaries of $Q$ which contains at least one member of $R$. 
Proof: Since $R$ is genetic, $\sigma(R) \subseteq s(R)$ and $(L \odot, e)$ is a monoid without zero divisors. Then $s(\sigma(R)) \subseteq s(s(R)) = s(R)$. Since $R \subseteq s(R), s(R) \subseteq s(\sigma(R))$. Hence $s(\sigma(R)) = s(R)$. The result follows from Proposition 2.4.

Remark 2.4 Let $R$ be a non-empty genetic subset of $Q$ and $(L \odot, e)$ be a monoid without zero divisors. Then $s(R)$ is not the union of those primaries of $Q$ which contains at least one member of $R$, as the following counterexamples shows.

counter-example 2.2 Consider the $L$-fuzzy automaton given in counterexample 2.1. Let $A = \{n_0\}$. Then $s(A) = \{n_0, k_0\}, \sigma(A) = \{m_0, n_0\}, \sigma(\sigma(A)) = \{m_0, n_0, k_0\}$. Then $s(\sigma(A)) \subseteq s(\sigma(A))$. Thus $\sigma(A)$ is genetic and hence $A$ be genetic subset of $Q$. But $s(\sigma(A)) \neq s(A).

Lemma 2.2 Let $(R, X, \lambda)$ be a primary subautomaton of an $L$-fuzzy automaton $(Q, X, \delta)$ and $(L \odot, e)$ be a monoid without zero divisors. Then for every finite subset $T$ of $R, T \subseteq s(t)$, for some $t \in R$.

Proof: We prove this Lemma by induction. Let $T = \{p_1, p_2, \ldots, p_n\}$ be any finite subset of $R$. Then the result is obvious for $n = 1$. Now assume the result is true for $n = k - 1$; in particular for $T_{k-1} = \{p_1, p_2, \ldots, p_{k-1}\}$. Then $\exists q \in R$ such that $T_{k-1} \subseteq s(q)$. Thus $T_k = \{p_1, p_2, \ldots, p_n\} \subseteq s(q) \cup \{p_k\} \subseteq s(s(q))$. Let $S = \{q, p_k\}$ and let $m \in \mu(R)$. Then by Proposition 2.1. $R = s(\sigma(m))$. Note that $q \in R \Rightarrow q \in s(\sigma(m)) \Rightarrow \delta(m', x, q) > 0$, for some $(m', x) \in \sigma(m) \times X$ now as $(L \odot, e)$ is a monoid without zero divisors. Then $m' \in \sigma(m) \Rightarrow m' \subseteq s(\sigma(m)) = \sigma(m) \subseteq R \Rightarrow s(s(m')) = R$ by Proposition 2.1. Consequently, $p_k \in R \Rightarrow p_k \in s(\sigma(m')) \Rightarrow \delta(r, y, p_k) > 0$, for some $(r, y) \in \sigma(m') \times X$. From $r \in \sigma(m')$, we get $m' \in s(r)$, whereby $s(m') \subseteq s(s(r)) = s(r)$. This together with the facts that $q \in s(m')$ and $p_k \in s(r)$, gives $q, p_k \in s(r)$. Hence $S = \{q, p_k\} \subseteq s(r)$. Hence $T_k \subseteq s(r).

Remark 2.5 Let $(R, X, \lambda)$ be a primary subautomaton of an $L$-fuzzy automaton $(Q, X, \delta)$ and $(L \odot, e)$ be a monoid with zero divisors. Then for every finite subset $T$ of $R, T \not\subseteq s(t)$, for some $t \in R$, as the following counter-example shows.

counter-example 2.3 Consider the $L$-fuzzy automaton given in counterexample 2.1. Let $A = \{m_0, n_0, k_0\}$. Then $s(A) = \{m_0, n_0, k_0\}, s(\sigma(A)) = \{m_0, n_0, k_0\}$. Thus $s(\sigma(A)) = A, s(\sigma(A)) \subseteq \mu(A)$, which shows that $A$ be a primary of an $L$-fuzzy automaton $Q$. But for every finite subset $T = \{n_0\} \subseteq A, T = \{n_0\} \not\subseteq s(\{k_0\}) = \{l_0\}$, for some $\{k_0\} \in A$.

Proposition 2.6 If $(L \odot, e)$ be a monoid without zero divisors. Then a primary of a compact $L$-fuzzy automaton is a maximal singly generated subautomaton.

Proof: Let $N$ be a primary of a compact $L$-fuzzy automaton $M = (Q, X, \delta)$ and let $p \in R$. Then $p \in s(\mu(R))$. So $\exists q \in \mu(R)$ with $p \in s(q)$. As $q \in \mu(R), \sigma(q) \subseteq R$. Now $\sigma(q)$ is finite owing to the compactness of $(Q, X, \delta)$, so $\exists q' \in R$ such that $\sigma(q) \subseteq s(q')$ (by Lemma 3.1) and $(L \odot, e)$ is a monoid without zero divisors. Then $s(\sigma(q)) \subseteq s(s(q')) = s(q')$. Also, $q \in s(q) \Rightarrow q' \in s(q) \Rightarrow s(q) \subseteq s(q')$. Thus $s(\sigma(q')) = s(q')$, whereby $s(q')$ is a genetically closed subset of $R$ (as $q' \in R$). So by the minimality of $R$, say $s(q') = R$. Hence the primary $(R, X, \lambda)$ is singly generated. Let $S = s(t)$ be the state-set of another singly generated subautomaton of $M$ such that $R = s(q') \subseteq S$. To prove that $R = S$. It is enough to show that $t \in s(q')$. Now $s(q') \subseteq S \Rightarrow s(t) \Rightarrow q' \in s(t) \Rightarrow t \in s(q')$, so that $t \in s(q')$ (as $s(q')$ is a primary: cf. Proposition 2.5).

Remark 2.6 If $(L \odot, e)$ be a monoid with zero divisors. Then a primary of a compact $L$-fuzzy automaton is not a maximal singly generated subautomaton.
Proposition 2.7 Let \( M = (Q, X, \delta) \) be a compact \( L \)-fuzzy automaton and \( R \subseteq Q \). Then \( s(\sigma(R)) \) can be written as the union of those primaries of \( Q \), which have nonempty intersection with \( R \).

Proof: The proof is similar, as given in [19].

Proposition 2.8 Let \( M = (Q, X, \delta) \) be a compact \( L \)-fuzzy automaton and \( R \subseteq Q \) be genetic with \((L, \cdot, e)\) be a monoid without zero divisors. Then \( s(R) \) is the union of primaries of \( Q \) having nonempty intersection with \( R \).

Proof: As \( R \) be genetic, \( \sigma(R) \subseteq s(R) \), implying that \( s(\sigma(R)) \subseteq s(s(R)) = s(R) \), as \((L, \cdot, e)\) is a monoid without zero divisors. On the other hand, as \( R \subseteq \sigma(R) \subseteq s(\sigma(R)) \), we have \( s(R) \subseteq s(R) \). Thus \( s(R) = s(\sigma(R)) \). Hence by the Proposition 3.2. \( s(R) \) is the union of primaries of \( Q \) having nonempty intersection with \( R \).

Remark 2.7 Let \( M = (Q, X, \delta) \) be a compact \( L \)-fuzzy automaton and \( R \subseteq Q \) be genetic with \((L, \cdot, e)\) be a monoid with zero divisors. Then \( s(R) \) is not the union of primaries of \( Q \) having nonempty intersection with \( R \), as the following counter-examples shows.

counter-example 2.2 Similar to counter-example 2.2.

III. Primary Decomposition of an \( L \)-Fuzzy Automata

Finite state automata admit a primary decomposition (cf. e.g., [1]). Even an infinite state automaton can admit a primary decomposition, for example, when that automaton is compact (cf. [18]). In [19] we extended this for \( L \)-fuzzy automaton provided that \((L, \cdot, e)\) is a monoid without zero divisors. Also, we introduce the concept of a source-splitting sub-automaton, including their characterization and a topological description with \((L, \cdot, e)\) is a monoid without zero divisors. Lastly, we introduce the concept of \( f \)-primaries of an \( L \)-fuzzy automaton \( M = (Q, X, \delta) \) and provided \((L, \cdot, e)\) is a monoid without zero divisors.

Proposition 3.1 An \( L \)-fuzzy automata \( M \) is strongly connected if and only if \( M \) has no proper subautomaton but if \((L, \cdot, e)\) be a monoid without zero divisors then the converse is true.

Proof: The proof is similar, as given in [23].

Lemma 3.1 Let \( (Q, X, \delta) \) be an \( L \)-fuzzy automaton and \( q \in Q \) such that \( R \subseteq s(\sigma(q)) \) is a non-empty regular closed (genetically closed) subset of \( Q \). Then \( q \in R \) and \((L, \cdot, e)\) be a monoid without zero divisors.

Proof: Let \( p \in \mu(R) \subseteq R \subseteq s(\sigma(q)) \). Then \( \sigma(p) \subseteq R \) and \( p \in s(\sigma(q)) \). Now \( \sigma(p) \subseteq R \Rightarrow s(\sigma(p)) \subseteq s(R) = R \). Also, \( p \in s(\sigma(q)) \Rightarrow p \in s(t) \), for some \( t \in \sigma(q) \Rightarrow t \in \sigma(p) \), for some \( t \) such that \( q \in t(t) \). Again, \( t \in \sigma(p) \Rightarrow \sigma(t) \subseteq s(\sigma(p)) = \sigma(p) \), as \((L, \cdot, e)\) is a monoid without zero divisors. Now \( q \in s(t) \Rightarrow q \in s(\sigma(t)) \subseteq s(\sigma(p)) \subseteq R \). Thus, \( q \in R \).

Remark 3.1 Let \( (Q, X, \delta) \) be an \( L \)-fuzzy automaton and \( q \in Q \) such that \( R \subseteq s(\sigma(q)) \) is a non-empty regular closed (genetically closed) subset of \( Q \) and \((L, \cdot, e)\) be a monoid with zero divisors. Then \( q \notin R \), as the following counter-example shows.

counter-example 3.1 Consider the \( L \)-fuzzy automaton given in counterexample 2.1. Let \( A = \{ n_0 \} \) and \( \{ k_0 \} \in Q \). Then \( \sigma(k_0) = \{ n_0 \}, s(\sigma(k_0)) = \{ n_0, k_0 \} \). Then \( A \subseteq s(\sigma(k_0)) \) is not a non-empty regular closed subset of \( Q \), with \( \{ k_0 \} \notin A \).

Proposition 3.2 [20] (Primary Decomposition Theorem) Let the \( L \)-fuzzy automaton \( M = (Q, X, \delta) \) be compact (having possibly an infinite state-set \( Q \) ). Then

(i) \( M = \bigcup_{i=1}^{n} P_i \), and

(ii) for any \( j, 1 \leq j \leq n, M = \bigcup_{i=1, i \neq j}^{n} P_i \), where \( P_1, P_2, P_3, \ldots, P_n \) are all the distinct primaries of \( M \).
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**Proposition 3.3** A L-fuzzy automaton $M = (Q, X, \delta)$ is retrievable if and only if $M$ is decomposable and the primaries of $M$ are strongly connected.

**Proof**: The proof is similar, as given in [20].

**Remark 3.2** The converse of the above proposition is not true, if $(L, \cdot, e)$ be a monoid with zero divisors.

**Proposition 3.4** Let $(L, \cdot, e)$ be a monoid without zero divisors $M$ is decomposable and the primaries of $M$ are strongly connected. Then $M$ is retrievable.

**Proof**: Let $M$ be decomposable and the primaries of $M$ be strongly connected. Also, let $p, q \in Q$ be such that $p \in \sigma(q)$ and as $(L, \cdot, e)$ be a monoid without zero divisors. Then $\sigma(p) \subseteq \sigma(q) = \sigma(p)$ and so $s(\sigma(p)) \subseteq s(\sigma(q))$. Since $M$ is decomposable and $s(\sigma(p))$ is a regular closed subset of $Q$, $s(\sigma(p))$ should contain a primary subset of $Q$, say $R$. Now, $R \subseteq s(\sigma(p)) \subseteq s(\sigma(q)) \Rightarrow p, q \in R$(cf. Lemma 4.1). But as $R$ is strongly connected, $p \in s(q)$, thereby $q \in \sigma(p)$. Thus, $M$ is retrievable.

**Definition 3.1** [20] A subautomaton $N = (R, X, \lambda)$ of an L-fuzzy automaton $M = (Q, X, \delta)$ is called source-splitting in $M$ if

(i) $R$ is genetically closed subset of $Q$, and

(ii) $\forall r \in R, \exists r_1, r_2 \in \sigma(r)$ such that $\sigma(r_1) \cap \sigma(r_2) = \phi$.

**Lemma 3.2** Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and let $\{P_i : i \in I\}$ be a family of all distinct primaries of $M$, with respective state-sets $R_i$. Then $s(Q - \cup_{i \in I} R_i)$ is a source-splitting in $M$.

**Proof**: The proof is similar, as given in [20].

**Lemma 3.3** Let $N = (R, X, \lambda)$ be a source-splitting subautomaton of an L-fuzzy automaton $M = (Q, X, \delta)$ and $(L, \cdot, e)$ be a monoid without zero divisors. Then $\mu(R) \subseteq Q - \cup_{i \in I} R_i$, where $R_i$s are in Lemma 4.2.

**Proof**: To show that $\mu(R) \subseteq Q - \cup_{i \in I} R_i$, we show that $\mu(R) \cap R_i = \phi, \forall i \in I$. If possible, let $\mu(R) \cap R_i \neq \phi$, for some $i \in I$. Then $\mu(R) \cap s(\mu(R_i)) \neq \phi$ (since $s(\mu(R_i)) = R_i, \forall i \in I$). Now $q \in \mu(R) \cap s(\mu(R_i)) \Rightarrow s(q) \subseteq R$ and $q \in s(t)$, for some $t \in \mu(R_i)$, $\Rightarrow s(\sigma(q)) \subseteq s(R) = R$ and $q \in s(t)$, for some $t$ with $s(t) \subseteq R_i$. Also $s(t) \subseteq R_i$ $\Rightarrow s(\sigma(t)) \subseteq s(R_i) = R_i$. So, as $s(\sigma(t))$ is regular closed and $R_i$, being a primary subset, is minimal regular closed, we have $s(\sigma(t)) = R_i$. As $(L, \cdot, e)$ is a monoid without zero divisors. Then $q \in s(t) \Rightarrow t \in \sigma(q)$ $\Rightarrow \sigma(t) \subseteq s(\sigma(q)) = \sigma(q) \Rightarrow s(\sigma(t)) \subseteq s(\sigma(q)) \Rightarrow R_i \subseteq R$. This shows that $R$ contains a primary subset, a contradiction, as $N$, being source-splitting in $M$, cannot contain any primary. Hence, $\mu(R) \subseteq Q - \cup_{i \in I} R_i$.

**Remark 3.3** Let $N = (R, X, \lambda)$ be a source-splitting subautomaton of an L-fuzzy automaton $M = (Q, X, \delta)$ and $(L, \cdot, e)$ be a monoid with zero divisors. Then $\mu(R) \subseteq Q - \cup_{i \in I} R_i$, where $R_i$s are in Lemma 4.2, as the following counter-example shows.

**counter-example 3.2** Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{n_0, n_0, k_0\}, A_1 = \{n_0, k_0\}, A_2 = \{k_0, l_0\}$. Then $\mu(A) = \{n_0, k_0\}$, now $\mu(A) \cap A_1 = \{n_0, k_0\} \cap \{n_0, k_0\} \cap \{k_0, l_0\} = \{k_0\} \neq \phi, \forall i = 1, 2$. Thus $\mu(R) \subseteq Q - \cup_{i \in I} R_i$.

**Proposition 3.5** An L-fuzzy automaton $M = (Q, X, \delta)$ is compact if and only if there exists a finite subset $Q'$ of $Q$ such that $s(Q') = Q$, or equivalently, if and only if $\sigma(Q')$ is finite.

**Proof**: The proof is similar, as given in [21].

**Lemma 3.4** Let $(R, X, \lambda)$ be a primary of an L-fuzzy automaton $(Q, X, \delta)$. Then $s(\sigma(p)) = R, \forall p \in \mu(R)$.

**Proof**: The proof is similar, as given in [21].
Lemma 3.5 Let \((R, X, \lambda)\) be a primary of an \(L\)-fuzzy automaton \((Q, X, \delta)\) and \((L, \bullet, e)\) be a monoid without zero divisors. Then for every finite subset \(T\) of \(R\), \(T \subseteq s(r)\) for some \(r \in R\).

**Proof:** We prove this Lemma by induction on the number of elements in \(T\). The results is obvious if \(|T| = 1\). Now assume the result to be true for all \(T\) having \(k - 1\) elements. Consider some \(T \subseteq R\) having \(k\) elements. Pick any \(p \in T\). By induction hypothesis, \(\exists q \in R\) such that \(T = p \subseteq s(q)\). Thus \(T \subseteq s(q) \cup p \subseteq s(q, p)\). Put \(S = \{q, p\}\) and let \(m \in \mu(R)\). Then by Lemma 4.4, \(R = s(\sigma(m))\). Note that \(q \in R \Rightarrow q \in s(\sigma(m)) \Rightarrow \delta(m, x, q) > 0\) for some \((m, x) \in \sigma(m) \times X\), as \((L, \bullet, e)\) is a monoid without zero divisors. Then \(m' \in \sigma(m) \Rightarrow \sigma(\sigma(m)) = \sigma(m') \subseteq \sigma(m) \subseteq R\) (as \(m \in \mu(R)\) \(\Rightarrow m' \in \mu(R) \Rightarrow s(\sigma(m)) = R\) by Lemma 4.4). Consequently, \(p \in R \Rightarrow p \in s(\sigma(m)) \Rightarrow \delta(r, y, p) > 0\) for some \((r, y) \in \sigma(m') \times X\). From \(r \in \sigma(m')\), we get \(m' \subseteq s(r)\), whereby \(s(m') \subseteq s(r)\). This, together with the facts that \(q \in s(m')\) and \(p \in s(r)\). Hence \(S = \{q, p\} \subseteq s(r)\). Thus \(s(q, p) \subseteq s(r)\), i.e., \(s(q, p) \subseteq s(r)\).

Remark 3.4 Let \((R, X, \lambda)\) be a primary of an \(L\)-fuzzy automaton \((Q, X, \delta)\) and \((L, \bullet, e)\) be a monoid with zero divisors. Then for every finite subset \(T\) of \(R\), \(T \not\subseteq s(r)\) for some \(r \in R\), as the following counter-example shows.

counter-example 3.3 Similar to counter-example 3.1.

Proposition 3.6 Let \(M = (Q, X, \delta)\) be an \(L\)-fuzzy automaton and \((L, \bullet, e)\) be a monoid without zero divisors. Then each maximal finitely closed subset of \(Q\) is \(T(Q)\)-open.

**Proof:** Let \(R\) be a maximal finitely closed subset of \(Q\). We have to show that \(s(R) = R\). Let \(T\) be a finite subset of \(s(R)\). For each \(q \in T\), we can find some \(p \in R\) with \(q \in s(p)\). The set \(S\) of all such \(p\)’s must be finite. Also, \(S \subseteq R\). Obviously, \(T \subseteq s(S)\). Now as \(R\) is finitely closed, \(\exists t \in Q\) such that \(S \subseteq s(t)\), as \((L, \bullet, e)\) is a monoid without zero divisors. Then \(s(S) \subseteq s(t)\) (since \(s(s(t)) = s(t)\)), whereby \(T \subseteq s(t)\). Thus for each finite subset \(T\) of \(s(R)\), \(\exists t \in Q\) such that \(T \subseteq s(t)\). Hence \(s(R)\) is a finitely closed subset of \(Q\). But as \(R\) is a maximal finitely closed set and \(R \subseteq s(R)\), we find that \(s(R) = R\).

Remark 3.5 Let \(M = (Q, X, \delta)\) be an \(L\)-fuzzy automaton and \((L, \bullet, e)\) be a monoid with zero divisors. Then each maximal finitely closed subset of \(Q\) is not \(T(Q)\)-open.

Proposition 3.7 Let \(M = (Q, X, \delta)\) be an \(L\)-fuzzy automaton. Then every primary of \(M\) is an \(f\)-primary of \(M\).

**Proof:** The proof is similar, as given in [21].

Remark 3.6 Let \(M = (Q, X, \delta)\) be an \(L\)-fuzzy automaton and \(p \in Q\), if \(s(p)\) is a primary of \(M\). Then \(s(p)\) is an \(f\)-primary of \(M\).

**Proof:** In view of Proposition 4.7, \(s(p)\) is an \(f\)-primary of \(M\).

Remark 3.7 \(s(p)\) is a primary of \(M\) if \((L, \bullet, e)\) be a monoid without zero divisors. So, converse is true only if \((L, \bullet, e)\) be a monoid without zero divisors.

Proposition 3.8 Let \(M = (Q, X, \delta)\) be an \(L\)-fuzzy automaton and \(p \in Q\), if \(s(p)\) is an \(f\)-primary of \(M\). Then it is a primary of \(M\) and \((L, \bullet, e)\) be a monoid without zero divisors.

**Proof:** We only need to prove that if \(s(p)\) is an \(f\)-primary then it is a primary. So let \(s(p)\) be an \(f\)-primary of \(M\). In view of Proposition 2.2, it suffices to show that \(p \in \mu(s(p))\), or that \(\sigma(p) \subseteq s(p)\). Let \(q \in \sigma(p)\). Then \(p \in s(q)\), as \((L, \bullet, e)\) is a monoid without zero divisors. Then \(s(p) \subseteq s(s(q)) = s(q)\). As \(s(q)\) is finitely closed and \(s(p)\), being an \(f\)-primary, is maximal finitely closed, \(s(q) = s(p)\). Thus \(q \in s(p)\), showing that \(\sigma(p) \subseteq s(p)\).
Proposition 3.9 Let $M = (Q, X, \delta)$ be a compact $L$-fuzzy automaton and $N = (R, X, \lambda)$ be its subautomaton, if $N$ is a primary of $M$. Then it is an $f$-primary of $M$.

Proof: Once again, in view of Proposition 4.7, $N$ is an $f$-primary of $M$.

Remark 3.8 If $N$ is a primary of $M$ and $(L, \cdot, e)$ be a monoid with zero divisors. Then it is not an $f$-primary of $M$.

Proposition 3.10 Let $M = (Q, X, \delta)$ be a compact $L$-fuzzy automaton and $N = (R, X, \lambda)$ be its subautomaton, if $N$ is an $f$-primary of $M$. Then it is a primary of $M$ and $(L, \cdot, e)$ be a monoid without zero divisors.

Proof: Once again, in view of Proposition 4.7, we only need to prove that if $N$ is an $f$-primary of $M$ then it is a primary. Now, as $M$ is compact, there exists a finite subset $R$ of $R$ such that $s(R) = R$ (cf. Proposition 4.5). Also, as $R$ is finitely closed, $\exists q \in Q$ such that $R \subseteq s(q)$, as $(L, \cdot, e)$ is a monoid without zero divisors. Then \(s(R) \subseteq s(s(q)) = s(q)\). Thus $R \subseteq s(q)$. But as $s(q)$ is finitely closed and $R$ is maximal finitely closed (as $N$ is an $f$-primary), $s(q) = R$. Hence $N$ is a primary of $M$ (cf. Proposition 3.1).

Remark 3.9 If $N$ is an $f$-primary of $M$ and $(L, \cdot, e)$ be a monoid with zero divisors. Then it is not a primary of $M$.

IV. Conclusion

In this paper, we have introduced and studied here the decomposition of fuzzy automata via their primary and $f$-primary based on lattice ordered monoids. Interestingly, we found that decomposition concepts for fuzzy automata based on lattice-ordered monoids depends on the associated monoid structure. The obtained results generalize the observations made in [19, 20, 21].

References Références Referencias

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