Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 5 Version 1.0 Year 2018
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Monotonic Behaviour of Relative Increments of Pearson Distributions

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GJSFR-F Classification: *FOR Code: MSC 2010: 26A48*

Strictly as per the compliance and regulations of:

Monotonic Behaviour of Relative Increments of Pearson Distributions

Sereko Kooepile-Reikeletseng

Abstract- Theory has been developed in order to classify distributions ac-cording to monotonic behaviour of their relative increment functions. We apply the results to Pearson distributions.

I. Introduction

Let $f(x)$ be the probability density function of a continuous random variable X with open support I. The corresponding distribution function of X, $X(x)$, is defined by

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

The relative increment function, h, of a distribution function, F, is defined by

$$h(x) = \frac{F(x+c) - F(x)}{1 - F(x)}$$

where c is a positive constant.

Lemma 1. Let F be a twice differentiable distribution function with

$$F(x) < 1, \ F'(x) = f(x) > 0$$

for all x in I. We define the function Ψ as follows

$$\Psi(x) = \frac{(F(x) - 1).f'(x)}{f^2(x)}$$

If $\Psi < 1 (\Psi > 1)$, then it has been proven that the function h strictly increases (strictly decreases)[2]

A probability distribution with probability density function $f(x)$ is said to be a Pearson distribution if

$$\frac{f'(x)}{f(x)} = \frac{Q(x)}{q(x)}$$

where $Q(x) = Ax + B$ and $q(x) = ax^2 + bx + c$ and A, B, a, b, c are real constants with

$$a^2 + b^2 + c^2 > 0, \ A^2 + B^2 > 0 \qquad [1]$$

The following four theorems have been proven before and we will use them to formulate results about Pearson distributions.

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Ref

2. Szabo, Z. : Investigation of Relative Increments of Distribution Functions. Publ. Math. De-brecen 49 (1996), pp. 99-112

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Theorem 1.

If the probability density function f has the following properties.

(1.1) $I = (r, s) \subseteq \mathbb{R}$ is the largest finite or infinite open interval in which $f > 0$.
(1.2) There exists m in I at which f' is continuous and $f'(m) = 0$
(1.3) $f' > 0$ in (r, m) and $f' < 0$ in (m, s)
(1.4) f is twice differentiable in (m, s)
(1.5) $\left(\frac{f}{f'}\right)' = \frac{d}{dx}\left(\frac{f}{f'}\right) > 0$ in (m, s),

Then the corresponding continuous relative increment function, h, behaves as follows: If $\Psi(s^-) = \lim_{x \to s^-} \Psi(x)$ exists, then

- h strictly increases in I if $\Psi(s^-) \leq 1$
- h strictly increases in (r, y) and strictly decreases in (y, s) for some y in I if $\Psi(s^-) > 1$.[2]

Theorem 2.

If the probability density function f has the following properties.

(2.1) $I = (r, s) \subseteq \mathbb{R}$ is the largest finite or infinite open interval in which $f > 0$.
(2.2) $m = r$
(2.3) $f' < 0$ in (r, s)
(2.4) f is twice differentiable in (r, s)
(2.5) $\left(\frac{f}{f'}\right)' = \frac{d}{dx}\left(\frac{f}{f'}\right) < 0$ in (r, s), then r is finite.

(i) If $\Psi(r^+) < 1$ or $(\Psi(r^+) = 1$ and $\Psi < 1$ in some right neighbourhood of $r)$, then $\Psi < 1$ in I and the corresponding relative increment function strictly increases in I.
(ii) If $\Psi(r^+) > 1$, then
- If $\Psi(s^-) \geq 1$, then $\Psi > 1$ and the relative increment function strictly decreases in I.
- If $\Psi(s^-) < 1$, then $\Psi > 1$ in (r, y) and $\Psi < 1$ in (y, s) for some $y \in I$, so the relative increment function strictly decreases first and then strictly increases.[2]

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Theorem 3.

If the probability density function f has the following properties.

(3.1) $I = (r, s) \subseteq \mathbb{R}$ is the largest finite or infinite open interval in which $f > 0$.
(3.2) $m = r$
(3.3) $f' < 0$ in (r, s)
(3.4) f is twice differentiable in (r, s)
(3.5) $\left(\frac{f}{f'}\right)' = \frac{d}{dx}\left(\frac{f}{f'}\right) > 0$ in (r, s), then r is finite

(i) If $\Psi(m^+) > 1$ or $(\Psi(m^+) = 1$ and $\Psi > 1$ in some right neighbourhood of $m)$, then $\Psi > 1$ in I and the corresponding relative increment function strictly decreases in I.
(ii) If $\Psi(m^+) < 1$, then
- If $\Psi(s^-) < 1$, then $\Psi < 1$ and the relative increment function strictly increases in I.
- If $\Psi(s^-) > 1$, then $\Psi < 1$ in (m, x) and $\Psi > 1$ in (x, s) for some $x \in I$, so the relative increment function strictly increases first and then strictly decreases.[4]

Theorem 4.

Let $f(x)$ be a probability density function of U type distribution where (1.1) – (1.4) are fulfilled. Suppose $(\frac{f}{f'})' > 0$ in (m, y) and $(\frac{f}{f'})' < 0$ in (y, s) for some $y \in (m, s)$. Then the corresponding continuous relative increment function, h, behaves as follows:

- If $\Psi(y) < 1$ and $\Psi(s^-) < 1$ then strictly increases in I or

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2. Szabo, Z. : Investigation of Relative Increments of Distribution Functions. Publ. Math. De-brecen 49 (1996), pp. 99-112

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- If $\Psi(y) \geq 1$ then
  - if $\Psi(s^-) > 1$ h strictly increases in (r, x_0) for some $x_0 \in (m, y)$ and strictly decreases in (x_0, s).
  - if $\Psi(s^-) <$ then h strictly increases in (r, x_1) for some $x_1 \in (m, y)$ and strictly decreases in (x_1, x_2) for some $x_2 \in (y, s)$ then finally strictly increases in (x_2, s).[4]

Theorem 5.

Let $f(x)$ be a probability density function of J type distribution. Assume $m = r$, (1.1), (1.3), (1.4) are fulfilled. Suppose $(\frac{f}{f'})' < 0$ in (m, Y) and $(\frac{f}{f'})' > 0$ in (Y, s) for some $Y \in (m, s)$. Then the corresponding continuous relative increment function, h, behaves as follows: if $\Psi(m^+) = \lim_{x \to m^+} \Psi(x)$ exists, then

- If $\Psi(m^+) < 1$ and
  - $\Psi(s^-) < 1$ then h strictly increases in I or
  - $\Psi(s^-) > 1$ then h strictly increases in (m, x_3) for some $x_3 \in (y, s)$ and strictly decreases in (x_3, s)
- If $\Psi(m^+) \geq 1$ and
  - $\Psi(y) > 1$ then h strictly decreases in I
  - $\Psi(y) < 1$ then if
    - ∗ if $\Psi(s^-) < 1$ then h strictly decreases in (m, x_4) for some $x_4 \in (m, y)$ and strictly increases in (x_4, s)

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- ∗ if $\Psi(s^-) > 1$ then h strictly decreases in (m, x_5) for some $x_5 \in (m, y)$ and strictly increases in (x_5, x_6) for some $x_6 \in (y, s)$ then finally strictly decreases in (x_6, s) [4]

V. Main Results

For the next two theorems, we use the notation used in [3] **Theorem 6.** Let $f(x)$ be a density function of a Pearson distribution where (1.1) – (1.4) are fulfilled Let $M = b.B - A.c$, $L = a.B^2 - A.M$, $D = a.L$, $D_1 = \sqrt{D}$ and $b_1 = \frac{b}{2a}$ for $a \neq 0$.

(1) If $A = 0$ and $a < 0$ then $s + b_1 \geq 0$.

(2) If $a.A > 0$ and $q(\frac{-B}{A}) \neq 0$ and $D \geq 0$ then $Y(m) \leq 0$ and $0 \leq Y(s) \leq 2D_1$

(3) If $a.A < 0$ and $q(\frac{-B}{A}) \neq 0$ and $D \geq 0$ then $0 \leq Y(m) \leq 2D_1$ and $Y(s) \leq 0$

where $Y(v) = a(Av + B) + D_1$.

Then all the assertions of Theorem 4 hold.

Proof. By theorem 4, it is sufficient to prove that

$$(\tfrac{f}{f'})' > 0 \text{ in } (m, y) \text{ and } (\tfrac{f}{f'})' < 0 \text{ in } (y, s)$$

for some y in (m, s) We have

$$\left(\tfrac{f}{f'}\right)' = \left(\tfrac{q}{Q}\right)' = \tfrac{p(x)}{Q^2}$$

where

$$p(x) = aAx^2 + 2aBx + M$$

and

$$\left(\tfrac{f}{f'}\right)' > 0 \text{ if } p(x) > 0 \text{ and } \left(\tfrac{f}{f'}\right)' < 0 \text{ if } p(x) < 0$$

If $a \neq 0$ and $D \geq 0$ then the roots of $p(x)$ are

$$x_1 = \tfrac{(-aB - D_1)}{(aA)} \text{ and } x_2 = \tfrac{(-aB + D_1)}{(aA)}$$

where $D_1 = D^{\frac{1}{2}}$.

Ref

4. Szabo Z. : On Monotonic Behaviour of Relative Increments of Unimodal Distributions.

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Case 1: $A \neq 0$, $a \neq 0$ and $q(\frac{-B}{A}) \neq 0$, so q and Q have no common zero.

1.1. If $a.A > 0$, then $p(x)$ is convex. If $D \geq 0$ then there will be two real roots x_1 and x_2 where $x_1 \leq x_2$.

$$p(x) > 0 \text{ in } (m, y) \text{ and } p(x) < 0 \text{ in } (y, s)$$

if and only if

$$m \leq x_1,\ x_1 = y \text{ and } x_1 < s < x_2$$

which means that

$$Y(m) < 0 \text{ and } 0 \leq Y(s) \leq 2D_1$$

1.2. If $a.A < 0$, then $p(x)$ is concave.

$$p(x) > 0 \text{ in } (m, y) \text{ and } p(x) < 0 \text{ in } (y, s)$$

if and only if

$$x_1 \leq m \leq x_2 \text{ and } s \geq x_2 = y$$

which means that

$$0 \leq Y(m) \leq 2D_1 \text{ and } Y(s) \leq 0$$

Case 2 $A = 0$ and $a \neq 0$ therefore

$$\frac{q}{Q} = \frac{ax^2 + bx + c}{B}$$

and

$$(\frac{q}{Q})' = \frac{2ax + b}{B}$$

If $B > 0$ then $(\frac{q}{Q})' > 0$ if $x > \frac{-b}{a}$ and $(\frac{q}{Q})' < 0$ if $x < \frac{-b}{a}$. If $a < 0$ then $s > \frac{-b}{2a}$

If $B < 0$ then $(\frac{q}{Q})' > 0$ if $x < \frac{-b}{a}$ and $(\frac{q}{Q})' < 0$ if $x > \frac{-b}{a}$. If $a > 0$ then $s > \frac{-b}{2a}$

Case 3 If $a = 0$, then $p(x) = \frac{b}{B}$. If $\frac{b}{B} > 0$ then Theorem 1 applies. If $\frac{b}{B} < 0$ then there is no case like it as seen in remark 2.1 in [3].

Theorem 7

Let $f(x)$ be the density function of a Pearson distribution with $m = r$, (1.1), (1.3), (1.4) are fulfilled and M, L, D defined as in theorem 5.

(1) If $A = 0$ and $a < 0$ then $m + b_1 \leq 0$.
(2) If $a.A > 0$ and $q(\frac{-B}{A}) \neq 0$ and $D \geq 0$ and $Y(m) \leq 0$ and $0 \leq Y(s) \leq 2D_1$
(3) If $a.A < 0$ and $q(\frac{-B}{A}) \neq 0$ and $D \geq 0$ and $0 \leq Y_1(m) \leq 2D_1$ and $Y_1(s) \leq 0$

where $Y_1(v) = a(Av + B) - D_1$.

Then all the assertions of Theorem 5 hold.

Proof. By theorem 5, it is sufficient to prove that

$$(\frac{f}{f'})' < 0 \text{ in } (m, y) \text{ and } (\frac{f}{f'})' > 0 \text{ in } (y, s)$$

for some y in (m, s).

Case 1: $A \neq 0$ and $q(\frac{-B}{A}) \neq 0$, so q and Q have no common zero.

1.1. If $a.A > 0$, then $p(x)$ is convex. If $D \geq 0$ then there will be two real roots x_1 and x_2 where $x_1 < x_2$.

$$p(x) < 0 \in (m, y) \text{ and } p(x) > 0 \in (y, s)$$

if and only if

$$m = x_1,\ x_2 = y \text{ and } s > x_2$$

Ref

3. Szabo, Z. : Relative Increments of Pearson Distributions. Acta Math. Aca. Paed. Nyiregyh. 15 (1999), pp. 45-54. [Electronic Journal, websites www.bgytf.hu/amapn, www.emis.de/journals]

which means that

$$Y_1(s) > 0$$

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1.2. If $a.A < 0$, then $p(x)$ is concave.

$$p(x) < 0 \in (m, y) \text{ and } p(x) > 0 \in (y, s)$$

if and only if

$$m < x_1 \text{ which means that } Y_1(m) > 0,\ y = x_1 \text{ and } s = x_2$$

Case 2 $A = 0$ therefore

$$\frac{q}{Q} = \frac{ax^2 + bx + c}{B}$$

and

$$(\frac{q}{Q})' = \frac{2ax + b}{B}$$

If $B > 0$ then $(\frac{q}{Q})' < 0$ if $x < \frac{-b}{a}$ and $(\frac{q}{Q})' > 0$ if $x > \frac{-b}{a}$. If $a > 0$ then $m < \frac{-b}{2a}$

If $B < 0$ then $(\frac{q}{Q})' < 0$ if $x > \frac{-b}{a}$ and $(\frac{q}{Q})' > 0$ if $x < \frac{-b}{a}$. If $a < 0$ then $m < \frac{-b}{2a}$

Case 3 If $a = 0$, then $p(x) = \frac{b}{B}$. If $\frac{b}{B} < 0$ then Theorem 2 applies.

If $\frac{b}{B} > 0$ then Theorem 3 applies.

We state the following Lemma as it helps in classifying distributions. These are outlined and proven in [3].

Lemma 2. Let $f(x)$ be the density function of a Pearson distribution, then then:

- I Let $s = \infty$ and $f_\infty = lim_{x\to\infty} x.f(x)$
  - I.1 If $A = a = f_\infty = 0$ and $b + B \neq 0$ then $\Psi(\infty) = \frac{B}{(b+B)}$
  - I.2 If $A \neq 0$ and $a = 0$ then $\Psi(\infty) = 1$
  - I.3 If $A.a.(a + A) \neq 0$ and $f_\infty = 0$ then $\Psi(\infty) = \frac{A}{(a+A)}$
  - I.4 If $(a = A = 0\ ,\ b.f_\infty = 0)$ or $(A = 0\ ,\ a \neq 0)$ or $(a.A.f_\infty \neq 0)$ then $\Psi(\infty) = 0$
- II Let s be a finite number and let $lim_{x\to s^-} f(x) = 0$
  - II.1 If $[A.Q(s).q(s) \neq 0]$ or $[A.q'(s) \neq 0, Q(s) = q(s) = 0]$ or $[A = 0, q(s) \neq 0]$ or $[AA.Q(s) \neq 0, q(s) = q'(s) = 0]$, then $\Psi(s^-) = 1$
  - II.2 If $A \neq 0, q(-\frac{B}{A}).Q(s).[Q(s) + q'(s)] \neq 0\ ,\ q(s) = 0$ then $\Psi(s^-) = \frac{Q(s)}{[Q(s)+q'(s)]}$
  - II.3 If $A \neq 0\ ,\ q(-\frac{B}{A}).q(s) \neq 0\ ,\ Q(s) = 0$ then $\Psi(s^- = 0)$
  - II.4 If $A \neq 0\ ,\ q(-\frac{B}{A}) = q(s) = 0\ ,\ a + A \neq 0$ and either $Q(s) = q'(s) = 0$ or $Q(s) \neq 0$ then $\Psi(s^-) = \frac{A}{(a+A)}$
  - II.5 If $A = a = q(s) = 0\ ,\ q'(s).(b + B) \neq 0$ then $\Psi(s^-) = \frac{B}{b+B}$

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Example 1

$$f(x) = \frac{1}{\Gamma(n-1)} x^{-n} \exp(\frac{-k}{x})\ ,\ n = 2, 3, 4, \dots\ ,\ I = (0, \infty)$$

We have

$$\frac{f'}{f} = \frac{-nx+k}{x^2} \text{ so } A = -n\ ,\ B = k\ ,\ a = 1\ ,\ b = 0\ ,\ c = 0$$

Ref

3. Szabo, Z. : Relative Increments of Pearson Distributions. Acta Math. Aca. Paed. Nyiregyh. 15 (1999), pp. 45-54. [Electronic Journal, websites www.bgytf.hu/amapn, www.emis.de/journals]

Theorem 6 part 3 applies since $q(\frac{-B}{A}) = \frac{k^2}{n^2} \neq 0$ and $a.A = -n < 0$.
$mode = \frac{k}{n} < y = \frac{2k}{n}$, so it's a U-type distribution with
$p(x) = x(2k - nx) > 0$ for $x < \frac{2k}{n}$ and $p(x) < 0$ for $x > \frac{2k}{n}$

$$\lim_{x\to\infty} xf(x) = \lim_{x\to\infty} \frac{1}{\Gamma(n-1)} x^{(-n+1)} \exp(-k/x) = 0$$

so $I.3$ in Lemma 2 applies giving

$$\psi(\infty) = \frac{A}{a+A} = \frac{-n}{-n+1} > 1$$

This means that $\psi(y) > 1$ since if it was less than 1 then according to theory $\psi(x)$ would decrease and remain less than 1 in (y, s). So the relative increment function increases and then decreases.

Example 2

$f(x) = K(1+x^2)^{-n} \exp(-\arctan(x))$, $n > \frac{1}{2}$, K is a constant and $I = (-\infty, \infty)$

We have

$$\frac{f'}{f} = \frac{2nx \quad 1}{x^2+1} \text{ so } A = -2n \ , B = -1 \ , a = 1 \ , b = 0 \ , c = 1$$

Theorem 6 part 3 applies since $q(\frac{-B}{A}) \neq 0$ and $a.A = -2n < 0$.
$mode = \frac{-1}{2n} < y = \frac{-1+\sqrt{4n^2+1}}{2n}$, so it's a U-type distribution with
$p(x) = -2(-nx^2 + x - n) > 0$ for

$$x < \frac{-1+\sqrt{4n^2+1}}{2n}$$

and

$$p(x) < 0 \text{ for } x > \frac{-1+\sqrt{4n^2+1}}{2n}$$

$$\lim_{x\to\infty} xf(x) = 0$$

so I.3 in Lemma 2 applies giving

$$\psi(\infty) = \frac{A}{a+A} = \frac{-2n}{-2n+1} > 1$$

This means that $\psi(y) > 1$ since if it was less than 1 then according to theory $\psi(x)$ would decrease and remain less than 1 in (y, s). So the relative increment function increases and then decreases.

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Notes