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On a General Class of Multiple Eulerian Integrals with Multivariable I-Functions

By Frederic Ayant

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On a General Class of Multiple Eulerian Integrals with Multivariable I-Functions

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Abstract- Recently, Raina and Srivastava and Srivastava and Hussain have provided closed-form expressions for a number of a Eulerian integral involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerian integrals concerning the product of two multivariable I-functions defined by Prathima et al. [6], a class of multivariable polynomials and the spheroidal function. These integrals will serve as a capital formula from which one can deduce numerous integrals.

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I. Introduction and Prerequisites

The well-known Eulerian Beta integral [5]

$$\int_a^b (z-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(Re(\alpha)>0, Re(\beta)>0, b>a) \tag{1.1}$$

Is a basic result of evaluation of numerous other potentially useful integrals involving various special functions and polynomials. The mathematicians Raina and Srivastava [7], Saigo and Saxena [9], Srivastava and Hussain [14], Srivastava and Garg [13] et cetera have established a number of Eulerian integrals involving the various general class of polynomials, Meijer's G-function and Fox's H-function of one and more variables with general arguments. Recently, several Author study some multiple Eulerian integrals, see Bhargava [2], Goyal and Mathur [4] and others. The aim of this paper is to obtain general multiple Eulerian integrals of the product of two multivariable I-functions defined by Prathima et al [7], a general class of multivariable polynomials [12] and the spheroidal functions.

The spheroidal function $\psi_{\alpha n'}(c,\eta)$ of general order $\alpha > -1$ can be expanded as ([10], [18]).

$$\psi_{\alpha n''}(c,\eta) = \frac{i^{n''}\sqrt{2\pi}}{V_{\alpha n''}(c)} \sum_{k=0,or1}^{\infty_*} a_k(c|\alpha n'')(c\eta)^{-\alpha-\frac{1}{2}} J_{k+\alpha+\frac{1}{2}}(c\eta) \tag{1.2}$$

Which represents the expression uniformly on (∞,∞), where the coefficients $a_k(c|\alpha n'')$ satisfy the recursion formula and the asterisk over the summation sign indicate that the sum is taken over only even or odd values of according as n'' is even or odd. As $c \to 0, a_k(c|\alpha n'') \to 0, k \neq n''$

The class of multivariable polynomials defined by Srivastava [12], is given in the following manner:

Ref

2. A. Bhargava, A. Srivastava and O. Mukherjee, On a General Class of Multiple Eulerian Integrals. International Journal of Latest Technology in Engineering, Management & Applied Science (IJLTEMAS), 3(8) (2014), 57-64.

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$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \tag{1.3}$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary real or complex constants.

We shall note $a'_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]$

The I-function of several variables is a generalization of the multivariable H-function studied by Srivastava et Panda [16,17]. The multiple Mellin-Barnes integrals occurring in this paper will be referred to as the multivariables *I*–function of *r*-variables throughout our present study and will refer and represented as follows:

$$\bar{I}(z_1,\cdots,z_r) = \bar{I}_{p,q:p_1,q_1;\cdots;p_r,q_r}^{0,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|c} z_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}: \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n_1},(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{n_1+1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,n_r},(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{n_r+1,p_r} \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};1)_{1,m_1},(d_j^{(1)},\delta_j^{(1)};D_1)_{m_1+1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{1,m_r},(d_j^{(r)},\delta_j^{(r)};D_r)_{m_r+1,q_r} \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.4}$$

Where $\psi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}-\delta_j^{(i)}s_i\right)}$$

For more details, see Prathima et al. [6].
Following the result of Braaksma [3] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{p}A_j\alpha_j^{(i)} - \sum_{j=1}^{q}B_j\beta_j^{(i)} + \sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)} \leqslant 0; i=1,\cdots,r \tag{1.5}$$

The integral (1.4) converges absolutely if

$|arg(z_k)| < \frac{1}{2}\Delta_k\pi, k=1,\cdots,r$ where

$$\Delta_k = -\sum_{j=n+1}^{p}A_j\alpha_j^{(k)} - \sum_{j=1}^{q}B_j\beta_j^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)} > 0 \tag{1.6}$$

Ref

3. B. L. J. Braaksma, "Asymptotic expansions and analytic continuations for a class of Barnes integrals," Composition Mathematical, vol. 15, pp. 239–341, 1964.

and if all the poles of (1.6) are simple ,then the integral (1.4) can be evaluated with the help of the residue theorem to give

$$\bar{I}(z_1,\cdots,z_r) = \sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!} \tag{1.7}$$

Where ϕ and ϕ_i are defined by

$$\phi = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}S_k\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}S_k\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}S_k\right)}$$

and

$$\phi_i = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}S_k\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}S_k\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}S_k\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}S_k\right)}, i=1,\cdots,r$$

where

$$S_k = \eta_{G_k,g_k} = \frac{d_{g_k}^{(k)}+G_k}{\delta_{g_k}^{(k)}} \text{ for } k=1,\cdots,r$$

which is valid under the following conditions: $\epsilon_{M_k}^{(k)}[p_j^{(k)}+p_k'] \neq \epsilon_j^{(k)}[p_{M_k}+g_k]$
Consider the second multivariable I-function.

$$I(z_1',\cdots,z_s') = I_{p',q':p_1',q_1';\cdots;p_s',q_s'}^{0,n':m_1',n_1';\cdots;m_s',n_s'}\left(\begin{array}{c|c} \mathrm{z}_1' & (\mathrm{a}'_j;\alpha_j'^{(1)},\cdots,\alpha_j'^{(s)};A_j')_{1,p'}: \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_s' & (\mathrm{b}'_j;\beta_j'^{(1)},\cdots,\beta_j'^{(s)};B_j')_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{c}'^{(1)}_j,\gamma_j'^{(1)};C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(s)},\gamma_j'^{(s)};C_j'^{(s)})_{1,p_s'} \\ \\ (\mathrm{d}'^{(1)}_j,\delta_j'^{(1)};D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},\delta_j'^{(s)};D_j'^{(s)})_{1,q_s'} \end{array}\right) = \frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k'^{t_k}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.8}$$

where $\zeta(t_1,\cdots,t_s)$, $\phi_i(s_i)$, $i=1,\cdots,s$ are given by :

$$\zeta(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{n'}\Gamma^{A_j'}\left(1-a_j'+\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)}{\prod_{j=n'+1}^{p'}\Gamma^{A_j'}\left(a_j'-\sum_{i=1}^{s}\alpha_j'^{(i)}t_j\right)\prod_{j=1}^{q'}\Gamma^{B_j'}\left(1-b_j'+\sum_{i=1}^{s}\beta_j'^{(i)}t_j\right)}$$

$$\phi_i(s_i) = \frac{\prod_{j=1}^{n_i'}\Gamma^{C_j'^{(i)}}\left(1-c_j'^{(i)}+\gamma_j'^{(i)}t_i\right)\prod_{j=1}^{m_i'}\Gamma^{D_j'^{(i)}}\left(d_j'^{(i)}-\delta_j'^{(i)}t_i\right)}{\prod_{j=n_i'+1}^{p_i'}\Gamma^{C_j'^{(i)}}\left(c_j'^{(i)}-\gamma_j'^{(i)}t_i\right)\prod_{j=m_i'+1}^{q_i'}\Gamma^{D_j'^{(i)}}\left(1-d_j'^{(i)}-\delta_j'^{(i)}t_i\right)}$$

For more details, see Prathima et al. [6].
Following the result of Braaksma [3], the I-function of r variables is analytic if:

Ref

6. J. Prathima, V. Nambisan and S.K.Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 2014, 1-12.

$$U'_i = \sum_{j=1}^{p'} A'_j \alpha'^{(i)}_j - \sum_{j=1}^{q'} B'_j \beta'^{(i)}_j + \sum_{j=1}^{p'_i} C'^{(i)}_j \gamma'^{(i)}_j - \sum_{j=1}^{q'_i} D'^{(i)}_j \delta'^{(i)}_j \leqslant 0; i = 1, \cdots, s \tag{1.9}$$

The integral (1.13) converges absolutely if

$|arg(z'_k)| < \frac{1}{2}\Delta'_k \pi; k = 1, \cdots, s$. Where

$$\Delta'_k = -\sum_{j=n'+1}^{p'} A'_j \alpha'^{(k)}_j - \sum_{j=1}^{q'} B'_j \beta'^{(k)}_j + \sum_{j=1}^{m'_k} D'^{(k)}_j \delta'^{(k)}_j - \sum_{j=m'_k+1}^{q'_k} D'^{(k)}_j \delta'^{(k)}_j + \sum_{j=1}^{n'_k} C'^{(k)}_j \gamma'^{(k)}_j - \sum_{j=n'_k+1}^{p'_k} C'^{(k)}_j \gamma'^{(k)}_j > 0 \tag{1.10}$$

II. Integral Representation of Generalized Hypergeometric Function

The following generalized hypergeometric function regarding multiple integrals contour is also required [15, page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)}\, {}_PF_Q\left[(A_P); (B_Q); -(x_1 + \cdots + x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \frac{\prod_{j=1}^{P}\Gamma(A_j + s_1 + \cdots + s_r)}{\prod_{j=1}^{Q}\Gamma(B_j + s_1 + \cdots + s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r) x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j + s_1 + \cdots + s_r)$ are separated from those of $\Gamma(-s_j), j = 1, \cdots, r$. The above result (2.1) is easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j = 1, \cdots, r$

The equivalent form of Eulerian beta integral is given by (1.1):

III. Main Integral

We shall note:

$$X = m'_1, n'_1; \cdots; m'_s, n'_s; 1, 0; \cdots; 1, 0; 1, 0; \cdots; 1, 0$$

$$Y = p'_1, q'_1; \cdots; p'_s, q'_s; 0, 1; \cdots; 0, 1; 0, 1; \cdots; 0, 1$$

$$\mathbb{A} = [1 + \sigma^{(1)}_i - \sum_{k'=1}^{v} K_{k'}\rho''^{(1,k')}_i - \sum_{k=1}^{r}\eta_{G_k,g_k}\rho^{(1,k)}_i - \theta^{(1)}_i(2R + k'); \rho'^{(1,1)}_i, \cdots, \rho'^{(1,s)}_i, \tau^{(1,1)}_i, \cdots, \tau^{(1,l)}_i, 1, 0, \cdots, 0; 1]_{1,s}$$

$$, \cdots, [1 + \sigma^{(T)}_i - \sum_{k'=1}^{v} K_{k'}\rho''^{(T,k')}_i - \sum_{k=1}^{r}\eta_{G_k,g_k}\rho^{(T,k)}_i - \theta^{(T)}_i(2R + k'); \rho'^{(T,1)}_i, \cdots, \rho'^{(T,s)}_i, \tau^{(T,1)}_i, \cdots, \tau^{(T,l)}_i, 1, 0, \cdots, 0; 1]_{1,s},$$

$$[1 - A_j; 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0; 1]_{1,P},$$

$$[1 - \alpha_i - \sum_{k'=1}^{v} K_{k'}\delta''^{(k')}_i - \sum_{k=1}^{r}\eta_{G_k,g_k}\delta^{(k)}_i - (2R + k')\zeta_i; \delta'^{(1)}_i, \cdots, \delta'^{(s)}_i, \mu^{(1)}_i, \cdots, \mu^{(l)}_i, 1, \cdots, 1, 0, \cdots, 0; 1]_{1,s},$$

Ref

15. H.M. Srivastava and P.W. Karlsson, Multiple Gaussian Hypergeometric series. Ellis. Horwood. Limited. New-York, Chichester. Brisbane. Toronto, 1985.

$$[1-\beta_i-\sum_{k'=1}^{v}K_{k'}\eta_i^{\prime\prime(k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\eta_i^{(k)}-(2R+k')\lambda_i;\eta_i^{\prime(1)},\cdots,\eta_i^{\prime(s)},\theta_i^{(1)},\cdots,\theta_i^{(l)},1,\cdots,1,0,\cdots,0;1]_{1,s}$$

$$\mathbf{A}=(a_j';\alpha_j'^{(1)},\cdots,\alpha_j'^{(s)},0,\cdots,0,0,\cdots,0;A_j')_{1,p'}:(c_j'^{(1)},\gamma_j'^{(1)};C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(r)},\gamma_j^{(s)};C_j'^{(s)})_{1,p_s'};$$

Notes

$$(1,0;1);\cdots;(1,0;1);(1,0;1);\cdots;(1,0;1)$$

$$\mathbb{B}=[1+\sigma_i^{(1)}-\sum_{k'=1}^{v}K_{k'}\rho_i^{\prime\prime(1,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(1,k)}-\theta_i^{(1)}(2R+k');\rho_i^{\prime(1,1)},\cdots,\rho_i^{\prime(1,s)},\tau_i^{(1,1)},\cdots,\tau_i^{(1,l)},0,\cdots,0;1]_{1,s}$$

$$,\cdots,[1+\sigma_i^{(T)}-\sum_{k'=1}^{v}K_{k'}\rho_i^{\prime\prime(T,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(T,k)}-\theta_i^{(T)}(2R+k');\rho_i^{\prime(T,1)},\cdots,\rho_i^{\prime(T,s)},\tau_i^{(T,1)},\cdots,\tau_i^{(T,l)},0,\cdots,0;1]_{1,s},$$

$$[1-B_j;0,\cdots,0,1,\cdots,1,0,\cdots,0]_{1,Q},$$

$$[1-\alpha_i-\beta_i-\sum_{k'=1}^{v}(\delta_i^{\prime\prime(k')}+\eta_i^{\prime\prime(k')})K_{k'}-\sum_{k=1}^{r}(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}-(\zeta_i+\lambda_i)(2R+k');$$

$$(\delta_i^{\prime(1)}+\eta_i^{\prime(1)}),\cdots,(\delta_i^{\prime(s)}+\eta_i^{\prime(s)}),(\mu_i^{(1)}+\theta_i^{(1)}),\cdots,(\mu_i^{(l)}+\theta_i^{(l)}),1,\cdots,1;1]_{1,s}$$

$$\mathbf{B}=(b_j';\beta_j'^{(1)},\cdots,\beta_j'^{(s)},0,\cdots,0,0,\cdots,0;B_j')_{1,q'}:(\mathrm{d}_j'^{(1)},\delta_j'^{(1)};D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},\delta_j'^{(s)};D_j'^{(s)})_{1,q_s'};$$

$$(0,1;1);\cdots;(0,1;1);(0,1;1);\cdots;(0,1;1)$$

We have the following multiple Eulerian integrals, and we obtain the I-function of variables.

Theorem

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$I\begin{pmatrix}\mathrm{z}_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{pmatrix} I\begin{pmatrix}\mathrm{z}_1'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{\prime(1)}}(v_i-x_i)^{\eta_i^{\prime(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{\prime(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{\prime(s)}}(v_i-x_i)^{\eta_i^{\prime(s)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{\prime(j,s)}}}\right]\end{pmatrix}$$

$$S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}_{N_1,\cdots,N_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right]\end{array}\right)\psi_{\alpha n''}\left[c^\sigma,\prod_{j=1}^t\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^l g_k\prod_{i=1}^t\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^t\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^W\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^T\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{k''=0,or1}^{\infty_*}\sum_{R=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^r\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^r g_k}}{\prod_{k=1}^r\delta_{G^{(k)}}^{(k)}\prod_{k=1}^r g_k!}a'_v z_1''^{K_1}\cdots z_v''^{K_v}\frac{(-)^R a_{k''}(c^\sigma|\alpha n'')}{R!\Gamma(R+k''+\alpha+\frac{3}{2})}E_{ij}$$

$$I^{0,sT+P+2s+n':X}_{sT+P+2s+p',sT+Q+s+q':Y}\left(\begin{array}{c|c} \mathrm{z}'_1w_1 & \mathbb{A},\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}'_1w_s & \cdot\\ \mathrm{g}_1W_1 & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{g}_lW_l & \cdot\\ \mathrm{G}_1 & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{G}_T & \mathbb{B},\ \mathbf{B}\end{array}\right)\tag{3.1}$$

Where

$$E_{ij}=\frac{\frac{i^{n''}\sqrt{2\pi}}{V_{an''(c)}}}{\prod_{i=1}^t\prod_{j=1}^W(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^v\rho_i''^{(j,k')}K_{k'}+\sum_{k=1}^r\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}(2R+k'')}}$$

$$\times\frac{\prod_{i=1}^t(v_i-u_i)^{\sum_{k'=1}^v(\delta_i''^{(k')}+\eta_i''^{(k')})K_{k'}+\sum_{k=1}^r(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}+(\zeta_i+\lambda_i)(2R+k'')}}{\prod_{i=1}^t\prod_{j=W+1}^T(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^v\rho_i''^{(j,k')}K_{k'}+\sum_{k=1}^r\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}(2R+k'')}}$$

$$w_m=\prod_{i=1}^t\left[(v_i-u_i)^{\delta_i'^{(m)}+\eta_i'^{(m)}}\prod_{j=1}^W\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i'^{(j,m)}}\prod_{j=W+1}^T\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\rho_i'^{(j,m)}}\right],m=1,\cdots,s$$

Notes

$$W_k=\prod_{i=1}^{t}\left[(v_i-u_i)^{\mu_i^{(k)}+\theta_i^{(k)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\right],k=1,\cdots,l$$

$$G_j=\prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right],j=1,\cdots,W$$

Notes

$$G_j=-\prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right],j=W+1,\cdots,T$$

$$\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}=\sum_{G_1,\cdots,G_r=1}^{m_1,\cdots,m_r}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

Provided that:

(A) $W\in[0,T];u_i,v_i\in\mathbb{R};i=1,\cdots,t$

(B) $min\{\delta_i^{(g)},\eta_i^{(g)},\delta_i'^{(h)},\eta_i'^{(h)},\delta_i''^{(k)},\eta_i''^{(k)},\zeta_i,\eta_i\}\geqslant 0;g=1,\cdots,r;i=1,\cdots,t;h=1,\cdots,s;k=1,\cdots,v$

$min\{\rho_i^{(j,g)},\rho_i'^{(j,h)},\rho_i''^{(j,k')},\theta_i^{(j)},\tau_i^{(j,k)}\}\geqslant 0;j=1,\cdots,T;i=1,\cdots,t;g=1,\cdots,r;h=1,\cdots,s;k'=1,\cdots,v,k=1,\cdots,l$

(C) $\sigma_i^{(j)}\in\mathbb{R},U_i^{(j)},V_i^{(j)}\in\mathbb{C},z_{i'},z_{j'},z''_{k'},g_k,G_j\in\mathbb{C};i=1,\cdots,t;j=1,\cdots,T;i'=1,\cdots,r;$

$j'=1,\cdots,s;k'=1,\cdots,v;k=1,\cdots,l$

(D) $max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right]<1,i=1,\cdots,s;j=1,\cdots,W$ and

$$max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right]<1,i=1,\cdots,s;j=W+1,\cdots,T$$

(E) $\left|arg\left(z_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i^{(j,k)}}\right)\right|<\frac{1}{2}\Delta_k\pi,k=1,\cdots,r,$ where

$$\Delta_k=-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}$$

$$-\delta_i^{(k)}-\eta_i^{(k)}-\sum_{j=1}^{T}\rho_i^{(j,k)}>0$$

$\left|arg\left(z_i'\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i'^{(j,k)}}\right)\right|\frac{1}{2}\Delta_k'\pi,k=1,\cdots,s,$ where

$$\Delta_k'=-\sum_{j=n'+1}^{p'}A_j'\alpha_j'^{(k)}-\sum_{j=1}^{q'}B_j'\beta_j'^{(k)}+\sum_{j=1}^{m_k'}D_j'^{(k)}\delta_j'^{(k)}-\sum_{j=m_k'+1}^{q_k'}D_j'^{(k)}\delta_j'^{(k)}+\sum_{j=1}^{n_k'}C_j'^{(k)}\gamma_j'^{(k)}-\sum_{j=n_k'+1}^{p_k'}C_j'^{(k)}\gamma_j'^{(k)}$$

$$-\delta_i'^{(k)}-\eta_i'^{(k)}-\sum_{j=1}^{T}\rho_i'^{(j,k)}>0$$

(F) $Re\left(\alpha_i+\zeta_i(2R+k'')+\sum_{j=1}^{r}\delta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\delta_i'^{(k)}\min_{1\leqslant j\leqslant m_k'}Re\left(\frac{d_j'^{(k)}}{\delta_j'^{(k)}}\right)>0$ and

$$Re\left(\beta_i+\lambda_i(2R+k'')+\sum_{j=1}^{r}\eta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\eta_i'^{(k)}\min_{1\leqslant j\leqslant m_k'}Re\left(\frac{d_j'^{(k)}}{\delta_j'^{(k)}}\right)>0 \text{ for } i=1,\cdots,t$$

(G) $P\leqslant Q+1$. The equality holds, when , in addition,

either $P>Q$ and $\sum_{k=1}^{l}\left|g_k\left(\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\tau_i^{(j,k)}}\right)\right|^{\frac{1}{Q-P}}<1 \qquad (u_i\leqslant x_i\leqslant v_i; i=1,\cdots,t)$

or $P\leqslant Q$ and $\max_{1\leqslant k\leqslant l}\left[\left|\left(g_k\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{-\tau_i^{(j,k)}}\right)\right|\right]<1 \qquad (u_i\leqslant x_i\leqslant v_i; i=1,\cdots,t)$

Proof

To establish the formula (4.7), we first express the spheroidal function, the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ and the multivariable I-function $I(z_1,\cdots,z_r)$ in series with the help of (1,2), (1,3) and (1.7) respectively, use integral contour representation with the help of (1.8) for the multivariable I-function $I(z_1',\cdots,z_s')$ occurring in its left-hand side and use the integral contour representation with the help of (2.1) for the Generalized hypergeometric function ${}_PF_Q(.)$. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now we write:

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} \tag{3.2}$$

where $K_i^{(j)}=\upsilon_i^{(j)}-\theta_i^{(j)}(2R+k'')-\sum_{l=1}^{r}\rho_i^{(j,l)}\eta_{G_l,g_l}-\sum_{l=1}^{s}\rho_i'^{(j,l)}\psi_l-\sum_{l=1}^{v}\rho_i''^{(j,v)}K_l$ where $i=1,\cdots,t; j=1,\cdots,T$

and express the factors occurring in R.H.S. Of (3.1) in terms of following Mellin-Barnes integrals contour, we obtain:

$$\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^W}\int_{L_1'}\cdots\int_{L_W'}\prod_{j=1}^{W}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')$$

$$\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i-u_i)}{(u_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\Bigg]\mathrm{d}\zeta_1'\cdots\mathrm{d}\zeta_W' \tag{3.3}$$

$$\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^{T-W}}\int_{L_{W+1}'}\cdots\int_{L_T'}\prod_{j=W+1}^{T}\Big[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')$$

$$\prod_{j=W+1}^{T}\left[-\frac{(U_i^{(j)}(v_i-x_i)}{(v_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\Bigg]\mathrm{d}\zeta_{W+1}'\cdots\mathrm{d}\zeta_T' \tag{3.4}$$

Notes

We apply the Fubini's theorem for multiple integrals. Finally evaluating the innermost **x**-integral with the help of (1.1) and reinterpreting the multiple Mellin-Barnes integrals contour in terms of multivariable I-function of $(r+l+T)$ variables, we obtain the formula (3.1).

IV. Particular Cases

a) I-functions of two variables

Corollary 1

If $r=s=2$, then the multivariable I-functions reduce to I-functions of two variables defined by Rathie et al. [9]. We have.

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$\bar{I}\begin{pmatrix} \mathrm{z}_1\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i^{(2)}}(v_i-x_i)^{\eta_i^{(2)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,2)}}}\right]\end{pmatrix} I\begin{pmatrix} \mathrm{z}'_1\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i'^{(2)}}(v_i-x_i)^{\eta_i'^{(2)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,2)}}}\right]\end{pmatrix}$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix} \mathrm{z}''_1\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^t\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right]\end{pmatrix}\psi_{\alpha n''}\left[c^\sigma,\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^T\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{k''=0,or1}^{\infty_*}\sum_{R=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{G_2=1}^{m_2}\sum_{g_1,g_2=0}^{\infty}\phi_2\frac{\prod_{k=1}^2\phi_{2k}z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^2 g_k}}{\prod_{k=1}^2\delta_{G^{(k)}}^{(k)}\prod_{k=1}^2 g_k!}a'_v z_1''^{K_1}\cdots z_v''^{K_v}\frac{(-)^R a_{k''}(c^\sigma|\alpha n'')}{R!\Gamma(R+k''+\alpha+\frac{3}{2})}$$

Ref

9. A.K. Rathie, K.S. Kumari and T.M. Vasudevan Nambisan, A study of I-functions of two variables, Le Matematiche 69(1) (2014), 285-305.

$$E_{ij} I^{0,sT+P+2s+n':X_2}_{sT+P+2s+p',sT+Q+s+q':Y_2}\left(\begin{array}{c|c} \mathrm{z}'_1 w_1 & \mathbb{A}_2,\ \mathbf{A}_2 \\ . & . \\ . & . \\ \mathrm{z}'_2 w_2 & . \\ \mathrm{g}_1 W_1 & ; \\ . & . \\ . & . \\ \mathrm{g}_l W_l & . \\ \mathrm{G}_1 & . \\ . & . \\ . & . \\ \mathrm{G}_T & \mathbb{B}_2,\ \mathbf{B}_2 \end{array}\right) \tag{4.1}$$

The validity conditions are the same that (3.1) with $r=s=2$. The quantities $\phi_2, \phi_{2k}, V_2, W_2, \mathbb{A}_2, \mathbb{B}_2, \mathbf{A}_2, \mathbf{B}_2$ are equal to $\phi, \phi_k, V, W, \mathbb{A}, \mathbb{B}, \mathbf{A}, \mathbf{B}$ respectively for $r=s=2$.

b) I-function of one variable

Corollary 2

If $r=s=1$, the multivariable I- functions reduce to I-functions of one variable defined by Rathie [8]. We have

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$I\left(\mathrm{z}\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\right) I\left(\mathrm{z}'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\right)$$

$$S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}_{N_1,\cdots,N_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right] \\ . \\ . \\ . \\ \mathrm{z}''_v\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right] \end{array}\right)\psi_{\alpha n''}\left[\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{k''=0,or1}^{\infty_*}\sum_{R=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{g_1=0}^{\infty}\phi_1\frac{z_k^{\eta_{G_1,g_1}}(-)^{g_1}}{\delta^{(1)}_{G^{(1)}}g_1!}a'_v z_1''^{K_1}\cdots z_v''^{K_v}\frac{(-)^R a_{k''}(c^\sigma|\alpha n'')}{R!\Gamma(R+k''+\alpha+\frac{3}{2})}E_{ij}$$

Ref

8. A.K. Rathie, A new generalization of generalized hypergeometric functions, Le Matematiche,52(2) (1997), 297-310.

$$I^{0,sT+P+2s+n':X_1}_{sT+P+2s+p',sT+Q+s+q':Y_1}\left(\begin{array}{c|c} \text{z'w}_1 & \mathbb{A}_1,\ \mathbf{A}_1 \\ \text{g}_1 W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \text{g}_l W_l & \cdot \\ \text{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \mathbb{B}_1,\ \mathbf{B}_1 \\ \text{G}_T & \end{array}\right) \tag{4.2}$$

The validity conditions are the same that (3.1) with $r = s = 1$. The quantities $\phi_1, V_1, W_1, \mathbb{A}_1, \mathbb{B}_1, \mathbf{A}_1, \mathbf{B}_1$ are equal to $\phi_k, V, W, \mathbb{A}, \mathbb{B}, \mathbf{A}, \mathbf{B}$ respectively for $r = s = 1$.

Remark: By the similar procedure, the results of this document can be extended to the product of any finite number of multivariable I-functions and class of multivariable polynomials defined by Srivastava [12].

V. Conclusion

Our main integral formula is unified in nature and possesses manifold generality. It acts a fundamental expression and using various particular cases of the multivariable I-function, the class of multivariable polynomials and a general spheroidal functions, one can obtain a large number of other integrals involving simpler special functions and polynomials of one and several variables.

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Notes