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SELBERGINTEGRALINVOLVINGTHEPRODUCTOFMULTIVARIABLESPECIALFUNCTIONS

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Selberg Integral Involving the Product of Multivariable Special Functions

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Abstract- The Selberg integral was an integral first evaluated by Selberg in 1944. The aim of the present paper is to estimate generalized Selberg integral. It involves the product of the general class of multivariable polynomials, multivariable I-function and modified multivariable H-function. The result is believed to be new and is capable of giving a large number of integrals involving a variety of functions and polynomials as its cases. We shall see several corollaries and particular cases at the end.

Keywords: modified multivariable H-function, selberg integral, multivariable I-function, class of multivariable polynomials, h-function.

I. Introduction and Prerequisites.

The Selberg integral is the following integral first evaluated by Selberg [6] in 1944 :

$$S_n(a,b,c) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{a-1}(1-x_i)^{b-1} \prod_{1\leqslant j<k\leqslant n} |x_j - x_k|^{2c}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

$$= \prod_{j=0}^{n-1} \frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma(1+(j+1)c)}{\Gamma(a+b+(n-1+j)c)\Gamma(1+c)} \tag{1.1}$$

where n is a positive integer, a, b and c are the complex number such that

$$Re(a) > 0, Re(b) > 0, Re(c) > Max\left\{-\frac{1}{n}, -\frac{Re(a)}{n-1}, -\frac{Re(b)}{n-1}\right\}$$

We refer the reader to Forrester and Warnaar's exposition [2] for the history and importance of the Selberg integral.

In this document, we evaluate a generalized Selberg integral involving the product of the multivariable I-function defined by Prasad [4], modified multivariable H-function defined by Prasad and Singh [5] and class of multivariable polynomials defined by Srivastava [7].

The generalized multivariable polynomials defined by Srivastava [7], is given in the following manner:

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1,K_1;\cdots;N_v,K_v] y_1^{K_1} \cdots y_v^{K_v} \tag{1.2}$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are constants Real or complex. On suitably specializing the quantities, $A[N_1,K_1;\cdots;N_v,K_v]$, $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]$ yields a Number of known polynomials, the

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4. Y. N. Prasad, Multivariable I-function, Vijnana Parishad Anusandhan Patrika 29 (1986), 231-237.

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Laguerre polynomials, the Jacobi polynomials, and several other ([10], page. 158-161]. We shall note.

$$a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots ; N_v, K_v] \tag{1.3}$$

The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes types integral:

$$I(z_1, z_2, \cdots, z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}; \cdots ; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2}; \cdots ; \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{a}_{rj}; \alpha^{(1)}_{rj}, \cdots, \alpha^{(r)}_{rj})_{1,p_r} : (a^{(1)}_j, \alpha^{(1)}_j)_{1,p^{(1)}}; \cdots ; (a^{(r)}_j, \alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj}; \beta^{(1)}_{rj}, \cdots, \beta^{(r)}_{rj})_{1,q_r} : (b^{(1)}_j, \beta^{(1)}_j)_{1,q^{(1)}}; \cdots ; (b^{(r)}_j, \beta^{(r)}_j)_{1,q^{(r)}} \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi(s_1, \cdots, s_r) \prod_{i=1}^{r} \phi_i(s_i) z_i^{s_i} \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.4}$$

where

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}} \Gamma(b^{(i)}_j - \beta^{(i)}_j s_i) \prod_{j=1}^{n^{(i)}} \Gamma(1 - a^{(i)}_j + \alpha^{(i)}_j s_i)}{\prod_{j=m^{(i)}+1}^{q^{(i)}} \Gamma(1 - b^{(i)}_j + \beta^{(i)}_j s_i) \prod_{j=n^{(i)}+1}^{p^{(i)}} \Gamma(a^{(i)}_j - \alpha^{(i)}_j s_i)}, \ i = 1, \cdots, r \tag{1.5}$$

and

$$\phi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{n_2} \Gamma(1 - a_{2j} + \sum_{i=1}^{2} \alpha^{(i)}_{2j} s_i) \prod_{j=1}^{n_3} \Gamma(1 - a_{3j} + \sum_{i=1}^{3} \alpha^{(i)}_{3j} s_i) \cdots}{\prod_{j=n_2+1}^{p_2} \Gamma(a_{2j} - \sum_{i=1}^{2} \alpha^{(i)}_{2j} s_i) \prod_{j=n_3+1}^{p_3} \Gamma(a_{3j} - \sum_{i=1}^{3} \alpha^{(i)}_{3j} s_i) \cdots}$$

$$\frac{\cdots \prod_{j=1}^{n_r} \Gamma(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj} s_i)}{\cdots \prod_{j=n_r+1}^{p_r} \Gamma(a_{rj} - \sum_{i=1}^{r} \alpha^{(i)}_{rj} s_i) \prod_{j=1}^{q_2} \Gamma(1 - b_{2j} - \sum_{i=1}^{2} \beta^{(i)}_{2j} s_i)}$$

$$\times \frac{1}{\prod_{j=1}^{q_3} \Gamma(1 - b_{3j} + \sum_{i=1}^{3} \beta^{(i)}_{3j} s_i) \cdots \prod_{j=1}^{q_r} \Gamma(1 - b_{rj} - \sum_{i=1}^{r} \beta^{(i)}_{rj} s_i)} \tag{1.6}$$

About the above integrals and these existence and convergence conditions, see Prasad [4] for more details. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. We have:

$|arg z_i| < \frac{1}{2}\Omega_i \pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha^{(i)}_k + \sum_{k=1}^{m^{(i)}} \beta^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta^{(i)}_k + \left(\sum_{k=1}^{n_2} \alpha^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2} \alpha^{(i)}_{2k} \right) + \cdots +$$

4. Y. N. Prasad, Multivariable I-function, Vijnana Parishad Anusandhan Patrika 29 (1986), 231-237.

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$$\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right) \tag{1.7}$$

where $i=1,\cdots,r$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function. We may establish the asymptotic expansion in the following convenient form:

$$I(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\to 0$$

$$I(z_1,\cdots,z_r)\;=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\to\infty$$

where $k=1,\cdots,r:\alpha'_k=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m^{(k)}$ and

$$\beta'_k=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n^{(k)}$$

If all the poles of (1.7) are simples, then the integral (1.6) can be evaluated with the help of the residue theorem to give

$$I(z_1,\cdots,z_r)=\sum_{G_i=1}^{m^{(i)}}\sum_{g_i=0}^{\infty}\phi\frac{\prod_{i=1}^r\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r\delta_{G^{(i)}}^{(i)}\prod_{i=1}^r g_i!} \tag{1.8}$$

where

$$\phi=\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\,,\phi_i=\phi_i(\eta_{G_i,g_i}),i=1,\cdots,r \tag{1.9}$$

$$\eta_{G_i,g_i}=\frac{d_{g_i}^{(i)}+G_i}{\delta_{g_i}^{(i)}}\quad\text{for } i=1,\cdots,r\quad\text{and}\quad\sum_{G_i=1}^{m^{(i)}}\sum_{g_i=0}^{\infty}=\sum_{G_1,\cdots,G_r=1}^{m^{(1)},\cdots,m^{(r)}}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

which is valid under the following conditions: $\epsilon_{M_i}^{(i)}[p_j^{(i)}+p'_i]\neq\epsilon_j^{(i)}[p_{M_i}+g_i]$. ϕ_i and ϕ are are given by (1.5) and (1.6) respectively.

The modified H-function studied by Prasad and Singh [5] generalizes the multivariable H-function defined by Srivastava and Panda [8,9]. It is defined in term of multiple Mellin-Barnes types integral:

$$H(z'_1,\cdots,z'_s)=H^{\mathbf{m},\mathbf{n}:|\mathbf{R}:m_1,n_1;\cdots;m_s,n_s}_{\mathbf{p},\mathbf{q}:|R:p_1,q_1;\cdots,p_s,q_s}\left(\begin{array}{c|l}\mathrm{z}'_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(s)})_{1,\mathbf{p}}:\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \\ \mathrm{z}'_s & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(s)})_{1,\mathbf{q}}:\end{array}\right.$$

$$\left.\begin{array}{l}(\mathrm{e}_j;u_j^{(1)}g_j^{(1)},\cdots,u_j^{(s)}g_j^{(s)})_{1,\mathbf{R}}:(c_j^{(1)};\gamma_j^{(1)})_{1,p_1},\cdots,(c_j^{(s)};\gamma_j^{(s)})_{1,p_s}\\ \cdot\\ \cdot\\ (\mathrm{l}_j;U_j^{(1)}f_j^{(1)},\cdots,U_j^{(s)}f_j^{(r)})_{1,R}:(d_j^{(1)};\delta_j^{(1)})_{1,q_1},\cdots,(d_j^{(s)};\delta_j^{(s)})_{1,q_s}\end{array}\right) \tag{1.10}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\theta(t_1,\cdots,t_s)\prod_{i=1}^s\theta_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.11}$$

Ref

5. Y. N. Prasad and A. K. Singh, Basic properties of the transform involving and H-function of r-variables as kernel, Indian Acad Math, (2) (1982), 109-115.

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where $\theta(t_1,\cdots,t_s), \theta_i(t_i), i=1,\cdots,\ s$ are given by:

$$\theta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{\mathbf{m}}\Gamma(b_j-\sum_{i=1}^{s}\beta_j^{(i)}t_i)\prod_{j=1}^{\mathbf{n}}\Gamma\left(1-a_j+\sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)}{\prod_{j=\mathbf{n}+1}^{\mathbf{p}}\Gamma\left(a_j-\sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)\prod_{j=\mathbf{m}+1}^{\mathbf{q}}\Gamma\left(1-b_j+\sum_{i=1}^{s}\beta_j^{(i)}t_j\right)}$$

$$\frac{\prod_{j=1}^{\mathbf{R}}\Gamma(e_j+\sum_{i=1}^{s}u_j^{(i)}g_j^{(i)}t_i)}{\prod_{j=1}^{R}\Gamma(l_j+\sum_{i=1}^{s}U_j^{(i)}f_j^{(i)}t_i)} \tag{1.12}$$

$$\theta_i(t_i)=\frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}t_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}t_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}t_i\right)\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}t_i\right)} \tag{1.13}$$

The integrals (1.14) converges absolutely if

$$|argz_i'|<\frac{1}{2}U_i\pi \quad (i=1,\cdots,s) \tag{1.14}$$

with $$U_i=\sum_{j=1}^{\mathbf{m}}\beta_j^{(i)}-\sum_{j=\mathbf{m}+1}^{\mathbf{q}}\beta_j^{(i)}+\sum_{j=1}^{\mathbf{n}}\alpha_j^{(i)}-\sum_{j=\mathbf{n}+1}^{\mathbf{p}}\alpha_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=1+m_i}^{q_i}\delta_j^{(i)}$$

$$+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}+\sum_{j=1}^{\mathbf{R}}g_j^{(i)}-\sum_{j=1}^{R}f_j^{(i)}>0(i=1,\cdots,s) \tag{1.15}$$

For more details, see Prasad and Singh [5].

II. Required Integral

We have the following integrals, see Andrew and R. Askey for more details ([1], p. 402).
Lemma.

$$\int_0^1\cdots\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n=\prod_{i=1}^{k}\frac{(a+(n-i)c)}{(a+b+(2n-i-1)c)}S_n(a,b,c) \tag{2.1}$$

Where $Re(a)>0, Re(b)>0, Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$ and $k\leqslant n$. $S_n(a,b,c)$ is defined by (1.1).

III. Main Integral

Let

$$X_{u,v,w}(x_1,\cdots,x_n)=\prod_{i=1}^{n}x_i^u(1-x_i)^v\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2w} \tag{3.1}$$

$$X=m_1,n_1;\cdots;m_s,n_s \tag{3.2}$$

$$Y=p_1,q_1;\cdots;p_s,q_s \tag{3.3}$$

Ref

1. G. G Andrew and R. Askey, Special function. Cambridge. University. Press 1999.

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$$\mathbb{A} = (a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(s)})_{1,\mathbf{p}};\,(\mathrm{e}_j;u_j^{(1)}g_j^{(1)},\cdots,u_j^{(s)}g_j^{(s)})_{1,\mathbf{R}}:\mathrm{C} = (\mathrm{c}_j^{(1)};\gamma_j^{(1)})_{1,p_1};\cdots,(c_j^{(s)};\gamma_j^{(s)})_{1,p_s} \tag{3.4}$$

$$\mathbb{B} = (b_j;\beta_j^{(1)},\cdots,\beta_j^{(s)})_{1,\mathbf{q}};\quad (\mathrm{l}_j;U_j^{(1)}f_j^{(1)},\cdots,U_j^{(s)}f_j^{(s)})_{1,R}:\,D = (d_j^{(1)};\delta_j^{(1)})_{1,q_1};\cdots;(d_j^{(s)};\delta_j^{(s)})_{1,q_s} \tag{3.5}$$

In this section, we establish the general Selberg integral about the product of a class of multivariable polynomials, multivariable I-function and modified H-function of several variables.

Notes

Theorem.

$$\int_0^1\cdots\int_0^1\prod_{i=1}^k x_i\prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{y}_1X_{\alpha_1,\beta_1,\gamma_1}(x_1,\cdots,x_n)\\ \cdot\\ \cdot\\ \mathrm{y}_vX_{\alpha_v,\beta_v,\gamma_v}(x_1,\cdots,x_n)\end{pmatrix}$$

$$I\begin{pmatrix}\mathrm{z}_1X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n)\\ \cdot\\ \cdot\\ \mathrm{z}_rX_{\delta_r,\psi_r,\phi_r}(x_1,\cdots,x_n)\end{pmatrix}H_{\mathbf{p},\mathbf{q}:|R:Y}^{\mathbf{m},\mathbf{n}:|\mathbf{R}:X}\begin{pmatrix}\mathrm{z}_1'X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n)\\ \cdot\\ \cdot\\ \mathrm{z}_s'X_{\epsilon_s,\eta_s,\zeta_s}(x_1,\cdots,x_n)\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_n =$$

$$\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_i=1}^{m^{(i)}}\sum_{g_i=0}^{\infty}\phi\frac{\prod_{i=1}^r\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r\delta_{G^{(i)}}^{(i)}\prod_{i=1}^r g_i!}\quad a_v y_1^{R_1}\cdots y_v^{R_v}$$

$$H_{\mathbf{p}+3n+2k,\mathbf{q}+2n+2k:|R:Y}^{\mathbf{m},\mathbf{n}+3n+2k:|\mathbf{R}:X}\left(\begin{array}{c|c}\mathrm{z}_1' & \mathrm{A}_1,A_2,A_3,A_4,A_5,\mathbb{A}:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \\ \mathrm{z}_s' & \mathbb{B},\mathrm{B}_1,B_2,B_3,B_4:D\end{array}\right) \tag{3.6}$$

where

$$A_1 = \left[1-a-\sum_{i=1}^v K_i\alpha_i-\sum_{i=1}^r\eta_{G_i,g_i}\delta_i-j(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\epsilon_1+j\zeta_1,\cdots,\epsilon_s+j\zeta_s\right]_{0,n-1} \tag{3.7}$$

$$A_2 = \left[1-b-\sum_{i=1}^v K_i\beta_i-\sum_{i=1}^r\eta_{G_i,g_i}\psi_i-j(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\eta_1+j\zeta_1,\cdots,\eta_s+j\zeta_s\right]_{0,n-1} \tag{3.8}$$

$$A_3 = \left[-j(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});j\zeta_1,\cdots,j\zeta_s)\right]_{1,n} \tag{3.9}$$

$$A_4 = \Bigg[-a-\sum_{i=1}^v K_i\alpha_i-\sum_{i=1}^r\eta_{G_i,g_i}\delta_i-(n-j)(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\Big]_{1,k} \tag{3.10}$$

$$A_5 = \Bigg[1-a-b-\sum_{i=1}^v K_i(\alpha_i+\beta_i)-\sum_{j=1}^r\eta_{G_j,g_j}(\delta_j+\psi_j)-(2n-j-1)(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});$$

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$$\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots\epsilon_s+\eta_s+(2n-j-1)\zeta_s\big]_{1,k} \tag{3.11}$$

$$B_1=\left(-c-\sum_{i=1}^{v}K_i\gamma_i-\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s\right),\cdots,\left(-c-\sum_{i=1}^{v}\gamma_iR_i-\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s\right) \tag{3.12}$$

$$B_2=\Bigg[1-a-\sum_{i=1}^{v}K_i\alpha_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\delta_i-(n-j)(c+\sum_{i=1}^{v}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\big]_{1,k} \tag{3.13}$$

$$B_3=\Bigg[-a-b-\sum_{i=1}^{v}K_i(\alpha_i+\beta_i)-\sum_{i=1}^{r}\eta_{G_i,g_i}(\delta_i+\psi_i)-(2n-j-1)(c+\sum_{i=1}^{v}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots\epsilon_s+\eta_s+(2n-j-1)\zeta_s\big]_{1,k} \tag{3.14}$$

$$B_4=\Bigg[1-a-b-\sum_{i=1}^{v}K_i(\alpha_i+\beta_i)-\sum_{i=1}^{r}\eta_{G_i,g_i}(\delta_i+\psi_j)-(n+j-1)(c+\sum_{i=1}^{v}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+\eta_1+(n+j-1)\zeta_1,\cdots\epsilon_s+\eta_s+(n+j-1)\zeta_s\big]_{0,n-1} \tag{3.15}$$

Provided

$min\{\alpha_i,\beta_i,\gamma_i,\delta_j,\psi_j,\phi_j,\epsilon_l,\eta_l,\zeta_l\}>0, i=1,\cdots,v, j=1,\cdots,r, l=1,\cdots,s\,; a,b,c\in\mathbb{C}$

$$A=Re\left(a+\sum_{i=1}^{r}\delta_i\eta_{G_i,g_i}\right)+\sum_{i=1}^{s}\epsilon_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

$$B=Re\left(b+\sum_{i=1}^{r}\psi_i\eta_{G_i,g_i}\right)+\sum_{i=1}^{s}\eta_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

$$C=Re\left(c+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i}\right)+\sum_{i=1}^{s}\zeta_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>Max\left\{-\frac{1}{n},-\frac{A}{n-1},-\frac{B}{n-1}\right\}\text{ and } k\leqslant n.$$

$|arg(z_iX_{\delta_i,\psi_i,\phi_i}(x_1,\cdots,x_n))|<\frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.7) for $i=1,\cdots,r$

$|arg(z_i'X_{\epsilon_i,\eta_i,\zeta_i}(x_1,\cdots,x_n))|<\frac{1}{2}U_i\pi$, where U_i is defined by (1.15) for $i=1,\cdots,s$

the multiple series on the left-hand side of (3.6) converges absolutely.

Proof

To evaluate the integrals (3.6), first, we replace the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ and multivariable I-function occurring on the left-hand side of your integral in term of series with the help of (1.2) and (1.8) Respectively. Now we express the modified multivariable H-function in Mellin-Barnes contour integrals by using (1.11). Nest, we change the order of the $(t_1,\cdots,t_s)$-integrals and $(x_1,\cdots,x_n)$-integrals, (which is justified under the conditions stated), we obtain the following result (say L.H.S.):

Notes

Global Journal of Science Frontier Research (F) Volume XVIII Issue IV Version I 6 Year 2018

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$$L.H.S = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{G_i=1}^{m^{(i)}} \sum_{g_i=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i z_i^{\eta_{G_i,g_i}} (-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r \delta_{G^{(i)}}^{(i)} \prod_{i=1}^r g_i!}$$

$$a_v y_1^{K_1} \cdots y_v^{K_v} \frac{1}{(2\pi\omega)^s} \int_{L_1'} \cdots \int_{L_s'} \theta(t_1, \cdots, t_s) \prod_{l'=1}^s \theta_{l'}(t_{l'}) z_{l'}^{\prime t_{l'}} \left[\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \right.$$

$$\prod_{i=1}^n x_i^{a-1+\sum_{l=1}^v K_l\alpha_l + \sum_{l=1}^r \eta_{G_l,g_l}\delta_l + \sum_{l=1}^s \epsilon_l t_l} (1-x_i)^{b-1+\sum_{l=1}^v K_l\beta_l + \sum_{l=1}^r \eta_{G_l,g_l}\psi_l + \sum_{l=1}^s \eta_l t_l}$$

$$\left. \prod_{1 \leqslant j<k \leqslant n} |x_j - x_k|^{2(c+\sum_{l=1}^v K_l\gamma_l + \sum_{l=1}^r \eta_{G_l,g_l}\phi_l + \sum_{l=1}^s \zeta_l t_l)} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \right] \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{3.16}$$

Now we evaluate the inner $(x_1, \cdots, x_n)$ - integrals with the help of the lemma and reinterpreting the Mellin-Barnes contour integrals thus obtained regarding modified H-function of s-variables, we arrive at the required result after algebraic manipulations.

IV. Special Cases

The multivariable polynomial vanishes and the multivariable I-function reduces to H-function defined by Fox [3], we obtain:

Corollary 1.

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1} \prod_{1 \leqslant j<k \leqslant n} |x_j - x_k|^{2c} H\left(\ z_1 X_{\delta_1,\psi_1,\phi_1}(x_1, \cdots, x_n) \ \right)$$

$$H_{\mathbf{p},\mathbf{q}:|R:Y}^{\mathbf{m},\mathbf{n}:|\mathbf{R}:X} \begin{pmatrix} z_1' X_{\epsilon_1,\eta_1,\zeta_1}(x_1, \cdots, x_n) \\ \cdot \\ \cdot \\ z_s' X_{\epsilon_s,\eta_s,\zeta_s}(x_1, \cdots, x_n) \end{pmatrix} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \sum_{G=1}^{m^{(1)}} \sum_{g=0}^{\infty} \phi_1 \frac{z_1^{\eta_{G,g}}(-)^g}{\delta_G g!}$$

$$H_{\mathbf{p}+3n+2k,\mathbf{q}+2n+2k:|R:Y}^{\mathbf{m},\mathbf{n}+3n+2k:|\mathbf{R}:X} \left(\begin{array}{c|c} z_1' & A_1', A_2', A_3', A_4', A_5', \mathbb{A} : C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathbb{B}, B_1', B_2', B_3', B_4' : D \\ z_s' & \end{array} \right) \tag{4.1}$$

where

$$A_1' = [1 - a - \eta_{G,g}\delta_1 - j(c + \phi_1\eta_{G,g}); \epsilon_1 + j\zeta_1, \cdots, \epsilon_s + j\zeta_s]_{0,n-1} \tag{4.2}$$

$$A_2' = [1 - b - \eta_{G,g}\psi_1 - j(c + \phi_1\eta_{G,g}); \eta_1 + j\zeta_1, \cdots, \eta_s + j\zeta_s]_{0,n-1} \tag{4.3}$$

$$A_3' = [-j(c + \phi_1\eta_{G,g}); j\zeta_1, \cdots, j\zeta_s)]_{1,n} \tag{4.4}$$

$$A_4' = \big[-a - \eta_{G,g}\delta_1 - (n-j)(c + \phi_1\eta_{G,g}); \epsilon_1 + (n-j)\zeta_1, \cdots, \epsilon_s + (n-j)\zeta_s \big]_{1,k} \tag{4.5}$$

Ref

3. C. Fox, The G and H-functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961), 395-429.

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$$A'_5 = \left[1 - a - b - \eta_{G,g}(\delta_1+\psi_1) - (2n-j-1)(c+\phi_1\eta_{G,g}); \epsilon_1+\eta_1+(2n-j-1)\zeta_1, \cdots \right.$$

$$\left. \epsilon_s+\eta_s+(2n-j-1)\zeta_s\right]_{1,k} \tag{4.6}$$

$$B'_1 = (-c-\phi_1\eta_{G,g};\zeta_1,\cdots,\zeta_s),\cdots,(-c-\phi_1\eta_{G,g};\zeta_1,\cdots,\zeta_s) \tag{4.7}$$

$$B'_2 = \left[1-a-\eta_{G,g}\delta_1-(n-j)(c+\phi_1\eta_{G,g});\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\right]_{1,k} \tag{4.8}$$

$$B'_3 = \left[-a-b-\eta_{G,g}(\delta_1+\psi_1)-(2n-j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots \right.$$

$$\left. \epsilon_s+\eta_s+(2n-j-1)\zeta_s\right]_{1,k} \tag{4.9}$$

$$B'_4 = \left[1-a-b-\eta_{G,g}(\delta_1+\psi_1)-(n+j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(n+j-1)\zeta_1,\cdots \right.$$

$$\left. \epsilon_s+\eta_s+(n+j-1)\zeta_s\right]_{0,n-1} \tag{4.10}$$

Provided

$$min\{\delta_1,\psi_1,\phi_1,\epsilon_l,\eta_l,\zeta_l\} > 0; l=1,\cdots,s; a,b,c \in \mathbb{C}$$

$$A' = Re\,(a+\delta_1\eta_{G,g}) + \sum_{i=1}^{s}\epsilon_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

$$B' = Re\,(b+\psi_1\eta_{G,g}) + \sum_{i=1}^{s}\eta_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

$$C' = Re\,(c+\phi_1\eta_{G,g}) + \sum_{i=1}^{s}\zeta_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > Max\left\{-\frac{1}{n},-\frac{A'}{n-1},-\frac{B'}{n-1}\right\} \text{ and } k \leqslant n.$$

$$|arg(z_1X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n))| < \frac{1}{2}\Omega_1\pi\,, \text{ where } \Omega_1 = \sum_{k=1}^{n^{(1)}}\alpha_k^{(1)} - \sum_{k=n^{(1)}+1}^{p^{(1)}}\alpha_k^{(1)} + \sum_{k=1}^{m^{(1)}}\beta_k^{(1)} - \sum_{k=m^{(1)}+1}^{q^{(1)}}\beta_k^{(1)}$$

$|arg(z'_iX_{\epsilon_i,\eta_i,\zeta_i}(x_1,\cdots,x_n))| < \frac{1}{2}U_i\pi$, where U_i is defined by (1.15) for $i=1,\cdots,s$

the series on the left-hand side of (4.1) converges absolutely.

Consider the above corollary, now the modified multivariable H-function reduces to H-function of one variable defined by Fox [3], we have.

Corollary 2.

$$\int_0^1\cdots\int_0^1 \prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,H\left(\ z_1X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n)\ \right)$$

3. C. Fox, The G and H-functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961), 395-429.

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$$H_{p_1,q_1}^{m_1,n_1}\left(z_1' X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n) \right)\mathrm{d}x_1\cdots\mathrm{d}x_n = \sum_{G=1}^{m^{(1)}}\sum_{g=0}^{\infty}\phi_1\frac{z_1^{\eta_{G,g}}(-)^g}{\delta_G g!}$$

$$H_{p_1+3n+2k,q_1+2n+2k}^{m_1,n_1+3n+2k}\left(z_1' \left| \begin{array}{c} A_1'', A_2'', A_3'', A_4'', A_5'', (c_j^{(1)},\gamma_j^{(1)})_{1,p_1} \\ \cdot \\ \cdot \\ (d_j^{(1)},\delta_j^{(1)})_{1,q_1}, B_1'', B_2'', B_3'', B_4'' \end{array} \right. \right) \tag{4.11}$$

Notes

where

$$A_1'' = [1-a-\eta_{G,g}\delta_1 - j(c+\phi_1\eta_{G,g});\epsilon_1+j\zeta_1]_{0,n-1} \tag{4.12}$$

$$A_2'' = [1-b-\eta_{G,g}\psi_1 - j(c+\phi_1\eta_{G,g});\eta_1+j\zeta_1]_{0,n-1} \tag{4.13}$$

$$A_3'' = [-j(c+\phi_1\eta_{G,g});j\zeta_1)]_{1,n} \tag{4.14}$$

$$A_4'' = \left[-a-\eta_{G,g}\delta_1-(n-j)(c+\phi_1\eta_{G,g});\epsilon_1+(n-j)\zeta_1\right]_{1,k} \tag{4.15}$$

$$A_5'' = \left[1-a-b-\eta_{G,g}(\delta_1+\psi_1)-(2n-j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(2n-j-1)\zeta_1\right]_{1,k} \tag{4.16}$$

$$B_1'' = (-c-\phi_1\eta_{G,g};\zeta_1),\cdots,(-c-\phi_1\eta_{G,g};\zeta_1) \tag{4.17}$$

$$B_2'' = \left[1-a-\eta_{G,g}\delta_1-(n-j)(c+\phi_1\eta_{G,g});\epsilon_1+(n-j)\zeta_1\right]_{1,k} \tag{4.18}$$

$$B_3'' = \left[-a-b-\eta_{G,g}(\delta_1+\psi_1)-(2n-j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(2n-j-1)\zeta_1\right]_{1,k} \tag{4.19}$$

$$B_4'' = \left[1-a-b-\eta_{G,g}(\delta_1+\psi_1)-(n+j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(n+j-1)\zeta_1\right]_{0,n-1} \tag{4.20}$$

Provided

$$min\{\delta_1,\psi_1,\phi_1,\epsilon_1,\eta_1,\zeta_1\}>0; a,b,c\in\mathbb{C}$$

$$A'' = Re\,(a+\delta_1\eta_{G,g})+\epsilon_1\min_{1\leqslant j\leqslant m_1} Re\left(\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)>0$$

$$B'' = Re\,(b+\psi_1\eta_{G,g})+\eta_1\min_{1\leqslant j\leqslant m_1} Re\left(\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)>0$$

$$C'' = Re\,(c+\phi_1\eta_{G,g})+\zeta_1\min_{1\leqslant j\leqslant m_1} Re\left(\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)> Max\left\{-\frac{1}{n},-\frac{A''}{n-1},-\frac{B''}{n-1}\right\} \text{ and } k\leqslant n.$$

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$$|arg(z_1 X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n))| < \frac{1}{2}\Omega_1\pi\ , \quad \text{where } \Omega_1 = \sum_{k=1}^{n^{(1)}} \alpha_k^{(1)} - \sum_{k=n^{(1)}+1}^{p^{(1)}} \alpha_k^{(1)} + \sum_{k=1}^{m^{(1)}} \beta_k^{(1)} - \sum_{k=m^{(1)}+1}^{q^{(1)}} \beta_k^{(1)}$$

$$|arg(z_1' X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n))| < \frac{1}{2}U_1\pi, \text{where} \quad U_1 = \sum_{j=1}^{m_1} \delta_j^{(1)} - \sum_{j=1+m_1}^{q_1} \delta_j^{(1)} + \sum_{j=1}^{n_1} \gamma_j^{(1)} - \sum_{j=n_1+1}^{p_1} \gamma_j^{(1)}$$

the series on the left-hand side of (4.11) converges absolutely.

V. Conclusion

In this paper, we have evaluated a general Selberg integral involving the product of an expansion of multivariable I function defined by Prasad [5], modified multivariable H-function defined by Prasad and Singh [6] and class of multivariable polynomials defined by Srivastava [9] with general arguments. The formulae evaluated in this paper are very general nature. Thus, the results established in this research work would serve as a formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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