Selberg Integral Involving the Product of Multivariable Special Functions

By FY. AY. Ant

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Keywords: modified multivariable H-function, selberg integral, multivariable I-function, class of multivariable polynomials, h-function.

GJSFR-F Classification: FOR Code: MSC 2010: 33C60, 82C31
Selberg Integral Involving the Product of Multivariable Special Functions

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I. Introduction and Prerequisites.

The Selberg integral is the following integral first evaluated by Selberg [6] in 1944:

\[ S_n(a, b, c) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} x_i^{a_i-1} (1-x_i)^{b_i-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} \, dx_1 \cdots dx_n \]

\[ = \prod_{j=0}^{n-1} \frac{\Gamma(a + jc) \Gamma(b + jc) \Gamma(1 + (j + 1)c)}{\Gamma(a + b + (n - 1 + j)c) \Gamma(1 + c)} \] (1.1)

where \( n \) is a positive integer, \( a, b \) and \( c \) are the complex number such that

\[ Re(a) > 0, Re(b) > 0, Re(c) > Max \left\{ \frac{1}{n}, \frac{Re(a)}{n - 1}, \frac{Re(b)}{n - 1} \right\} \]

We refer the reader to Forrester and Warnaar's exposition [2] for the history and importance of the Selberg integral.

In this document, we evaluate a generalized Selberg integral involving the product of the multivariable I-function defined by Prasad [4], modified multivariable H-function defined by Prasad and Singh [5] and class of multivariable polynomials defined by Srivastava [7].

The generalized multivariable polynomials defined by Srivastava [7], is given in the following manner:

\[ S_{N_1, \ldots, N_v}^{o_m, \ldots, o_m} [y_1, \ldots, y_v] = \sum_{K_1=0}^{[N_1/o_{m_1}]} \cdots \sum_{K_v=0}^{[N_v/o_{m_v}]} \frac{(-1)^{K_1} K_1!}{K_1!} \cdots \frac{(-1)^{K_v} K_v!}{K_v!} A[N_1, K_1; \ldots; N_v, K_v] y_1^{K_1} \cdots y_v^{K_v} \] (1.2)

where \( m_1, \ldots, m_v \) are arbitrary positive integers and the coefficients \( A[N_1, K_1; \ldots; N_v, K_v] \) are constants Real or complex. On suitably specializing the quantities, \( A[N_1, K_1; \ldots; N_v, K_v], S_{N_1, \ldots, N_v}^{m_1, \ldots, m_v} [y_1, \ldots, y_v] \) yields a Number of known polynomials, the
Laguerre polynomials, the Jacobi polynomials, and several other ([10], page. 158-161]. We shall note,

$$a_v = \frac{(-N_1)m_1K_1}{K_1!} \cdots \frac{(-N_v)m_vK_v}{K_v!} A[N_1,K_1,\cdots;N_v,K_v]$$  \hspace{1cm} (1.3)

The multivariable I-function of $r$-variables is defined in term of multiple Mellin-Barnes types integral:

$$I(z_1, z_2, \cdots, z_r) = \int_{\mathcal{L}_1} \cdots \int_{\mathcal{L}_r} \phi(s_1, \cdots, s_r) \prod_{i=1}^r \phi_i(s_i) z_i^s ds_1 \cdots ds_r$$  \hspace{1cm} (1.4)

where

$$\phi_i(s_i) = \frac{\prod_{j=1}^{n(i)} \Gamma(1-a_{j(i)}s_i) \prod_{j=m(i)+1}^{n(i)} \Gamma(1-a_{j(i)}s_i) + \alpha_{j(i)}s_i)}{\prod_{j=m(i)+1}^{n(i)+1} \Gamma(1-b_{j(i)}s_i) \prod_{j=n(i)+1}^{n(i)+1} \Gamma(1-a_{j(i)}s_i) - \alpha_{j(i)}s_i}, \quad i = 1, \cdots, r$$  \hspace{1cm} (1.5)

and

$$\phi(s_1, \cdots, s_r) = \prod_{j=n+1}^{n+1} \Gamma(1-a_{j(i)}s_i) \prod_{j=m+1}^{n} \Gamma(1-a_{j(i)}s_i) \prod_{j=m+1}^{n+1} \Gamma(1-a_{j(i)}s_i) \cdots

\frac{\prod_{j=n+1}^{n+1} \Gamma(1-a_{j(i)}s_i) + \alpha_{j(i)}s_i)}{\prod_{j=m+1}^{n+1} \Gamma(1-b_{j(i)}s_i) \prod_{j=n+1}^{n+1} \Gamma(1-a_{j(i)}s_i) - \alpha_{j(i)}s_i)} \cdots

\frac{1}{\prod_{j=m+1}^{n+1} \Gamma(1-b_{j(i)}s_i) \prod_{j=n+1}^{n+1} \Gamma(1-a_{j(i)}s_i) - \alpha_{j(i)}s_i)}$$  \hspace{1cm} (1.6)

About the above integrals and these existence and convergence conditions, see Prasad [4] for more details. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. We have:

$$|arg z_i| < \frac{1}{2} \Omega_i \pi,$$ where

$$\Omega_i = \sum_{k=1}^{n(i)} \alpha_k^{(i)} - \sum_{k=m(i)+1}^{n(i)} \alpha_k^{(i)} + \sum_{k=1}^{m(i)} \beta_k^{(i)} - \sum_{k=m(i)+1}^{n(i)} \beta_k^{(i)} + \left( \sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{n} \alpha_{2k}^{(i)} \right) + \cdots$$
where $i = 1, \ldots, r$

The complex numbers $z_i$ are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function. We may establish the asymptotic expansion in the following convenient form:

$$I(z_1, \ldots, z_r) = 0(|z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r}), \max(|z_1|, \ldots, |z_r|) \to 0$$

$$I(z_1, \ldots, z_r) = 0(|z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r}), \min(|z_1|, \ldots, |z_r|) \to \infty$$

where $k = 1, \ldots, r$, $\alpha_k = \min(\Re(b_j^{(k)}/\beta_j^{(k)})), j = 1, \ldots, m^{(k)}$ and $\beta_k = \max(\Re((a_j^{(k)})^{(k)})/\alpha_j^{(k)}), j = 1, \ldots, n^{(k)}$

If all the poles of (1.7) are simples, then the integral (1.6) can be evaluated with the help of the residue theorem to give

$$I(z_1, \ldots, z_r) = \sum_{G_i=1}^{m^{(i)}} \sum_{g_i=0}^{\infty} \phi_i \prod_{i=1}^{r} \frac{\eta_i}{\delta_{g_i}} \prod_{i=1}^{r} \frac{(-)^{\sum_{i=1} g_i}}{g_i!}$$

where

$$\phi = \phi(\eta_{G_1, g_1}, \ldots, \eta_{G_r, g_r}), \phi_i = \phi_i(\eta_{G_1, g_1}), i = 1, \ldots, r$$

$$\eta_{G_i, g_i} = \frac{d_i^{(i)} + G_i}{\delta_{g_i}}$$

which is valid under the following conditions: $\epsilon_j^{(i), (i)} p_j^{(i)} + p_j^{(i)} \neq \epsilon_j^{(i), (i)} p_j^{(i)} + g_j^{(i)}$, $\phi_i$ and $\phi$ are are given by (1.5) and (1.6) respectively.

The modified H-function studied by Prasad and Singh [5] generalizes the multivariable H-function defined by Srivastava and Panda [8,9]. It is defined in term of multiple Mellin-Barnes types integral:

$$H(z_1^{'}, \ldots, z_s^{'}) = H_{m,n,p,q}: \left( \begin{array}{c} z_1^{'} \\ \vdots \\ z_s^{'} \end{array} \right) = (a_j^{(1)}, \ldots, a_j^{(s)})_{1,p}: \left( \begin{array}{c} c_j^{(1)}, \gamma_j^{(1)} \\ \vdots \\ c_j^{(s)}, \gamma_j^{(s)} \end{array} \right)_{1,q}:$$

$$\left( \begin{array}{c} \alpha_j^{(1)} \\ \vdots \\ \alpha_j^{(s)} \end{array} \right)_{1,p}: \left( \begin{array}{c} \beta_j^{(1)} \\ \vdots \\ \beta_j^{(s)} \end{array} \right)_{1,q}:$$

$$= \frac{1}{(2\pi \omega)^s} \int_{L_1} \cdots \int_{L_s} \theta(t_1, \ldots, t_s) \prod_{i=1}^{s} \theta_i(t_i) z_i^{\omega} dt_1 \cdots dt_s$$

where \( \theta(t_1, \cdots, t_s), \theta_i(t_i), i = 1, \cdots, s \) are given by:

\[
\theta(t_1, \cdots, t_s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \sum_{i=1}^{s} \beta_j^{(i)} t_i) \prod_{j=1}^{n} \Gamma \left( 1 - a_j + \sum_{i=1}^{s} \alpha_j^{(i)} t_i \right)}{\prod_{j=n+1}^{p} \Gamma \left( a_j - \sum_{i=1}^{s} \alpha_j^{(i)} t_i \right) \prod_{j=m+1}^{q} \Gamma \left( 1 - b_j + \sum_{i=1}^{s} \beta_j^{(i)} t_j \right)}
\]

\[
\theta_i(t_i) = \frac{\prod_{j=1}^{m} \Gamma(c_j + \sum_{i=1}^{s} u_j^{(i)} g_j^{(i)} t_i)}{\prod_{j=1}^{p} \Gamma(t_j + \sum_{i=1}^{s} U_j^{(i)} f_j^{(i)} t_i)}
\]

The integrals (1.14) converges absolutely if

\[
|arg z_i| < \frac{1}{2} U_i \pi \quad (i = 1, \cdots, s)
\]

with

\[
U_i = \sum_{j=1}^{m} \beta_j^{(i)} - \sum_{j=m+1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{m} \delta_j^{(i)} - \sum_{j=m+1}^{q} \delta_j^{(i)} + \sum_{j=1}^{n} \gamma_j^{(i)} - \sum_{j=n+1}^{p} \gamma_j^{(i)} + \sum_{j=1}^{R} g_j^{(i)} - \sum_{j=1}^{R} f_j^{(i)} > 0 (i = 1, \cdots, s)
\]

For more details, see Prasad and Singh [5].

\section{Required Integral}

We have the following integrals, see Andrew and R. Askey for more details ([1], p. 402).

\begin{equation}
\int_{0}^{1} \cdots \prod_{i=1}^{k} x_i \prod_{i=1}^{n} x_i^{a-1} (1-x_i)^{b-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^2 \prod_{i=1}^{k} \frac{(a+(n-i)c)}{(a+b+(2n-i-1)c)} S_n(a, b, c)
\end{equation}

Where \( Re(a) > 0, Re(b) > 0, Re(c) > Max \left\{ -\frac{1}{n}, -\frac{Re(a)}{n-1}, -\frac{Re(b)}{n-1} \right\} \) and \( k \leq n. S_n(a, b, c) \) is defined by (1.1).

\section{Main Integral}

Let

\begin{equation}
X_{u,v,w}(x_1, \cdots, x_n) = \prod_{i=1}^{n} x_i^{u} (1-x_i)^{v} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2w}
\end{equation}

\begin{equation}
X = m_1, n_1; \cdots; m_s, n_s
\end{equation}

\begin{equation}
Y = p_1, q_1; \cdots; p_s, q_s
\end{equation}
In this section, we establish the general Selberg integral about the product of a class of multivariable polynomials, multivariable I-function and modified H-function of several variables.

**Theorem.**

\[
\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{s-1} (1-x_i)^{h-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} S_{N_1, \ldots, N_n}^{m_1, \ldots, m_n} \left( y_1 X_{\alpha_1, \beta_1, \gamma_1}(x_1, \ldots, x_n) \right) \cdots \left( y_n X_{\alpha_n, \beta_n, \gamma_n}(x_1, \ldots, x_n) \right) \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n} = \\
H_{m,n+2k+2n+2;R,X}^{p+2n+2k,q+2n+2k;R,Y} \left( z_1', \ldots, z_n' \right) \left[ A_1, A_2, A_3, A_4, A_5, A : C \right) \\
\left[ B_1, B_2, B_3, B_4 : D \right)
\]

where

\[
A_1 = \left[ 1 - a - \sum_{i=1}^v K_i \alpha_i - \sum_{i=1}^r \eta_i \gamma_i, \delta_i - j(c + \sum_{i=1}^v \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_i, \gamma_i); \epsilon_1 + j \zeta_1, \ldots, \epsilon_s + j \zeta_s \right]_{0,n-1}
\]

\[
A_2 = \left[ 1 - b - \sum_{i=1}^v K_i \beta_i - \sum_{i=1}^r \eta_i \gamma_i, \delta_i - j(c + \sum_{i=1}^v \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_i, \gamma_i); \eta_1 + j \zeta_1, \ldots, \eta_s + j \zeta_s \right]_{0,n-1}
\]

\[
A_3 = \left[ -j(c + \sum_{i=1}^v \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_i, \gamma_i); j \zeta_1, \ldots, j \zeta_s \right]_{1,n}
\]

\[
A_4 = \left[ -a - \sum_{i=1}^v K_i \alpha_i - \sum_{i=1}^r \eta_i \gamma_i, \delta_i - (n - j)(c + \sum_{i=1}^v \gamma_i K_i + \sum_{i=1}^r \phi_i \eta_i, \gamma_i); \right.
\]

\[
\epsilon_1 + (n - j) \zeta_1, \ldots, \epsilon_s + (n - j) \zeta_s \right]_{1,k}
\]

\[
A_5 = \left[ 1 - a - b - \sum_{i=1}^v K_i (\alpha_i + \beta_i) - \sum_{j=1}^r \eta_j \gamma_j, \delta_j + \psi_j - (2n - j - 1)(c + \sum_{i=1}^v \gamma_i K_i + \sum_{i=1}^r \phi_i \gamma_i, \gamma_i); \right.
\]

\[
\epsilon_1 + (n - j) \zeta_1, \ldots, \epsilon_s + (n - j) \zeta_s \right]_{1,k}
\]
\[ \epsilon_1 + \eta_1 + (2n - j - 1)\zeta_1, \cdots, \epsilon_s + (2n - j - 1)\zeta_s \]_1,k \tag{3.11}

\[ B_1 = \left( -c - \sum_{i=1}^{r} K_i \gamma_i - \sum_{i=1}^{r} \phi_i \eta_{G_i, G_i} ; \zeta_1, \cdots, \zeta_s \right) \cdots \cdots \left( -c - \sum_{i=1}^{r} \gamma_i R_i - \sum_{i=1}^{r} \phi_i \eta_{G_i, G_i} ; \zeta_1, \cdots, \zeta_s \right) \tag{3.12} \]

\[ B_2 = \left[ 1 - a - \sum_{i=1}^{r} K_i \alpha_i - \sum_{i=1}^{r} \eta_{G_i, G_i} \delta_i - (n-j)(c + \sum_{i=1}^{r} \gamma_i K_i + \sum_{i=1}^{r} \phi_i \eta_{G_i, G_i}) \right] \]

\[ \epsilon_1 + (n-j)\zeta_1, \cdots, \epsilon_s + (n-j)\zeta_s \]_1,k \tag{3.13}

\[ B_3 = \left[ -a - b - \sum_{i=1}^{r} K_i (\alpha_i + \beta_i) - \sum_{i=1}^{r} \eta_{G_i, G_i} (\delta_i + \psi_i) - (2n-j-1)(c + \sum_{i=1}^{r} \gamma_i K_i + \sum_{i=1}^{r} \phi_i \eta_{G_i, G_i}) \right] \]

\[ \epsilon_1 + \eta_1 + (2n-j-1)\zeta_1, \cdots, \epsilon_s + \eta_s + (2n-j-1)\zeta_s \]_1,k \tag{3.14}

\[ B_4 = \left[ 1 - a - b - \sum_{i=1}^{r} K_i (\alpha_i + \beta_i) - \sum_{i=1}^{r} \eta_{G_i, G_i} (\delta_i + \psi_j) - (n+j-1)(c + \sum_{i=1}^{r} \gamma_i K_i + \sum_{i=1}^{r} \phi_i \eta_{G_i, G_i}) \right] \]

\[ \epsilon_1 + \eta_1 + (n+j-1)\zeta_1, \cdots, \epsilon_s + \eta_s + (n+j-1)\zeta_s \]_0,\,n-1 \tag{3.15}

Provided

\[ \min \{ \alpha_i, \beta_i, \delta_j, \psi_j, \phi_j, \epsilon_i, \eta_i, \zeta_i \} > 0, \, i = 1, \cdots, v, \, j = 1, \cdots, r, \, l = 1, \cdots, s ; a, b, c \in \mathbb{C} \]

\[ A = \text{Re} \left( a + \sum_{i=1}^{r} \delta_i \eta_{G_i, G_i} \right) + \sum_{i=1}^{s} \epsilon_i \min_{1 \leq j \leq m_i} \text{Re} \left( \frac{d_{ij}^{(i)}}{\delta_{ij}^{(i)}} \right) > 0 \]

\[ B = \text{Re} \left( b + \sum_{i=1}^{r} \psi_i \eta_{G_i, G_i} \right) + \sum_{i=1}^{s} \eta_i \min_{1 \leq j \leq m_i} \text{Re} \left( \frac{d_{ij}^{(i)}}{\delta_{ij}^{(i)}} \right) > 0 \]

\[ C = \text{Re} \left( c + \sum_{i=1}^{r} \phi_i \eta_{G_i, G_i} \right) + \sum_{i=1}^{s} \zeta_i \min_{1 \leq j \leq m_i} \text{Re} \left( \frac{d_{ij}^{(i)}}{\delta_{ij}^{(i)}} \right) > \text{Max} \left\{ -\frac{1}{n}, -\frac{A}{n-1}, -\frac{B}{n-1} \right\} \text{ and } k \leq n. \]

\[ |\text{arg}(z_i X_{3i, \epsilon_i, \phi_i, \epsilon_1, \cdots, x_n})| < \frac{1}{2} \Omega_i \pi, \text{ where } \Omega_i \text{ is defined by (1.7) for } i = 1, \cdots, r \]

\[ |\text{arg}(z_i X_{r, n, \zeta_i, \epsilon_1, \cdots, x_n})| < \frac{1}{2} U_i \pi, \text{ where } U_i \text{ is defined by (1.15) for } i = 1, \cdots, s \]

the multiple series on the left-hand side of (3.6) converges absolutely.

\textbf{Proof}

To evaluate the integrals (3.6), first, we replace the class of multivariable polynomials \( S^{d_{ij}}_{N_i, \cdots, N_s} [\cdot] \) and multivariable I-function occurring on the left-hand side of your integral in term of series with the help of (1.2) and (1.8) Respectively. Now we express the modified multivariable H-function in Mellin-Barnes contour integrals by using (1.11). Nest, we change the order of the \((t_1, \cdots, t_s)\)-integrals and \((x_1, \cdots, x_n)\)-integrals, (which is justified under the conditions stated), we obtain the following result (say L.H.S.):
Now we evaluate the inner \((x_1, \ldots, x_n)\)-integrals with the help of the lemma and reinterpret the Mellin-Barnes contour integrals thus obtained regarding modified \(H\)-function of \(s\)-variables, we arrive at the required result after algebraic manipulations.

### IV. Special Cases

The multivariable polynomial vanishes and the multivariable \(I\)-function reduces to \(H\)-function defined by Fox [3], we obtain:

**Corollary 1.**

\[
\int_0^1 \cdots \int_0^1 \prod_{i=1}^k x_i \prod_{i=1}^n x_i^{a_i-1}(1-x_i)^{b_i-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} H( \ z_1 X_{\eta_1, \phi_1}(x_1, \ldots, x_n) \ )
\]

where

\[
\begin{align*}
L.H.S. &= \sum_{K_1=0}^{N_1/2m_1} \cdots \sum_{K_v=0}^{N_v/2m_v} \sum_{r_i=0}^{m^{(1)}_r} \phi \prod_{i=1}^{m^{(1)}_r} \phi_i x_i^{\eta, \sigma, \phi} (-)^{\sum_{i=1}^{s_i} \sigma_i} \prod_{i=1}^{m^{(1)}_r} \delta_{G_i} \prod_{i=1} \phi_i x_i^{g_i} \\
&= \frac{1}{(2\pi i)^s} \int_{L_1} \cdots \int_{L_s} \theta(t_1, \ldots, t_s) \prod_{i=1}^{s_i} \theta_i(t_i) \frac{\prod_{i=1}^{s_i} \theta_i(t_i) z_i^{\eta, \sigma, \phi}}{\prod_{i=1}^{m^{(1)}_r} \delta_{G_i} \prod_{i=1} \phi_i x_i^{g_i}} \end{align*}
\]

\[
\prod_{i=1}^{n} x_i^{a_i-1} + \sum_{i=1}^{s_i} K_i \alpha_i + \sum_{i=1}^{s_i} \eta_i \alpha_i + \sum_{i=1}^{s_i} \phi_i \alpha_i (1-x_i)^{b_i-1} + \sum_{i=1}^{s_i} \delta_i + \sum_{i=1}^{s_i} \psi_i + \sum_{i=1}^{s_i} \gamma_i t_i
\]

\[
\prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} \left( x_1 X_{\eta_1, \phi_1}(x_1, \ldots, x_n) \right)
\]

\[
H^{m, n}_{p, q} : R^X \rightarrow R^Y
\]

\[
H^{m, n}_{p+3n+2k} : R^X \rightarrow R^Y
\]

\[
\begin{pmatrix}
 z'_1 \\
 A'_1, A'_2, A'_3, A'_4, A'_5, A'_6, A'_7, B'_1, B'_2, B'_3, B'_4
\end{pmatrix} : D
\]

\[
\begin{pmatrix}
 z'_1 \\
 z'_2, B'_2, B'_3, B'_4
\end{pmatrix} : D
\]

Provided

\[ \min(\delta_1, \psi_1, \phi_1, \epsilon_i, \eta_i) > 0, \quad l = 1, \cdots, s; \quad a, b, c \in \mathbb{C} \]

\[ A' = \text{Re} (a + \delta_1 \eta_{G, \sigma}) + \sum_{i=1}^{s} \epsilon_i \min_{1 \leq j \leq m_i} \text{Re} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0 \]

\[ B' = \text{Re} (b + \psi_1 \eta_{G, \sigma}) + \sum_{i=1}^{s} \eta_i \min_{1 \leq j \leq m_i} \text{Re} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > 0 \]

\[ C' = \text{Re} (c + \phi_1 \eta_{G, \sigma}) + \sum_{i=1}^{s} \zeta_i \min_{1 \leq j \leq m_i} \text{Re} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) > \text{Max} \left\{ -\frac{1}{n}, \frac{A'}{n-1}, \frac{B'}{n-1} \right\} \quad \text{and} \quad k \leq n. \]

\[ |\arg(z_1 X_{\delta_1, \psi_1, \phi_1}(x_1, \cdots, x_n))| < \frac{1}{2} \Omega_1 \pi, \quad \text{where} \quad \Omega_1 = \sum_{k=1}^{n_{(1)}} \alpha_k^{(1)} - \sum_{k=n_{(1)}+1}^{p_{(1)}} \alpha_k^{(1)} + \sum_{k=1}^{m_{(1)}} \beta_k^{(1)} - \sum_{k=m_{(1)}+1}^{q_{(1)}} \beta_k^{(1)} \]

\[ |\arg(z'_i X_{\epsilon_i, \eta_i, \zeta_i}(x_1, \cdots, x_n))| < \frac{1}{2} U_i \pi, \quad \text{where} \quad U_i \text{ is defined by (1.15) for } i = 1, \cdots, s \]

the series on the left-hand side of (4.1) converges absolutely.

Consider the above corollary, now the modified multivariable H-function reduces to H-function of one variable defined by Fox [3], we have.

**Corollary 2.**

\[
\int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} x_i \prod_{i=1}^{n} x_i^{-1}(1-x_i)^{b-1} \prod_{1 \leq j < k \leq n} |x_j - x_k|^{2c} H\left( z_1 X_{\delta_1, \psi_1, \phi_1}(x_1, \cdots, x_n) \right)
\]
\[ H_{p_1,q_1}^{m_1,n_1}\left( z_1^m X_{e_1,m_1}(x_1, \cdots, x_n) \right) dx_1 \cdots dx_n = \sum_{G=1}^{\infty} \sum_{g=0}^{G} \phi_1 \frac{\eta_1^{m G} (-)^g}{\delta_{GG!} G!} \]

\[ H_{p_1+3n+2k,q_1+2n+2k}^{m_1,n_1+2n+2k} \left( \begin{array}{c} A'_{1}^{n}, A'_{2}^{n}, A'_{3}^{n}, A'_{4}^{n}, A'_{5}^{n}, A'_{6}^{n}, (\epsilon_j^{(1)}, \gamma_j^{(1)})_{1,p_1}, \epsilon_1, \gamma_1 \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right) \]

where

\[ A'_1 = [1 - a - \eta_{G,G} \delta_1 - j(c + \phi_1 \eta_{G,G}); \epsilon_1 + j \zeta_1]_{0,n-1} \]

\[ A'_2 = [1 - b - \eta_{G,G} \psi_1 - j(c + \phi_1 \eta_{G,G}); \eta_1 + j \zeta_1]_{0,n-1} \]

\[ A'_3 = [-j(c + \phi_1 \eta_{G,G}); j \zeta_1]_{1,n} \]

\[ A'_4 = [\eta_{G,G} \delta_1 - (n - j)(c + \phi_1 \eta_{G,G}); \epsilon_1 + (n - j) \zeta_1]_{1,k} \]

\[ A'_5 = [1 - b - \eta_{G,G} \psi_1 -(2n - j - 1)(c + \phi_1 \eta_{G,G}); \epsilon_1 + \eta_1 + (2n - j - 1) \zeta_1]_{1,k} \]

\[ B'_1 = (-c - \phi_1 \eta_{G,G}; \zeta_1),\cdots, (-c - \phi_1 \eta_{G,G}; \zeta_1) \]

\[ B'_2 = [1 - a - \eta_{G,G} \delta_1 - (n - j)(c + \phi_1 \eta_{G,G}); \epsilon_1 + (n - j) \zeta_1]_{1,k} \]

\[ B'_3 = [-a - b - \eta_{G,G} \psi_1 -(2n - j - 1)(c + \phi_1 \eta_{G,G}); \epsilon_1 + \eta_1 + (2n - j - 1) \zeta_1]_{1,k} \]

\[ B'_4 = [1 - a - b - \eta_{G,G} \psi_1 -(n + j - 1)(c + \phi_1 \eta_{G,G}); \epsilon_1 + \eta_1 + (n + j - 1) \zeta_1]_{0,n-1} \]

Provided

\[ \min \{ \delta_1, \psi_1, \phi_1, \epsilon_1, \eta_1, \zeta_1 \} > 0; a, b, c \in \mathbb{C} \]

\[ A'' = \text{Re} \left( a + \psi_1 \eta_{G,G} \right) + \epsilon_1 \min_{1 \leq j \leq m_1} \text{Re} \left( \frac{d_j^{(1)}}{\delta_j^{(1)}} \right) > 0 \]

\[ B'' = \text{Re} \left( b + \psi_1 \eta_{G,G} \right) + \eta_1 \min_{1 \leq j \leq m_1} \text{Re} \left( \frac{d_j^{(1)}}{\delta_j^{(1)}} \right) > 0 \]

\[ C'' = \text{Re} \left( c + \phi_1 \eta_{G,G} \right) + \zeta_1 \min_{1 \leq j \leq m_1} \text{Re} \left( \frac{d_j^{(1)}}{\delta_j^{(1)}} \right) > \max \left\{ -\frac{1}{n}, -\frac{A''}{n-1}, -\frac{B''}{n-1} \right\} \quad \text{and} \quad k \leq n. \]
the series on the left-hand side of (4.11) converges absolutely.

V. Conclusion

In this paper, we have evaluated a general Selberg integral involving the product of an expansion of multivariable I function defined by Prasad [5], modified multivariable H-function defined by Prasad and Singh [6] and class of multivariable polynomials defined by Srivastava [9] with general arguments. The formulae evaluated in this paper are very general nature. Thus, the results established in this research work would serve as a formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

References Références Referencias