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Solutions of the Relativistic and Non-Relativistic Wave Equations with $l \neq 0$ for Modified Hylleraas Plus Attractive Radial Molecular Potential using Nikiforov-Uvarov Method

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I. Introduction

It is now well known and widely understood that the exact solutions of the Schrodinger, Klein-Gordon and Dirac wave equations are only possible only for a handful of potentials of physical interest in a few cases such as the harmonic oscillator, Coulomb, pseudoharmonic potentials and others [1-4]. The total wave function of any quantum mechanical system basically provides implicitly the important information about the physical behavior of the system. Bound state solutions in most cases provides negative energies because usually, the energy of the particle is less than the maximum potential energy therefore, causing the particle to be trapped within the potential well [5]. The search of analytical bound state solutions of the Schrodinger equation for Hylleraas and as well as attractive radial potentials, have been of great interest as shown by many cases of specific potentials studied by means of different approaches to the centrifugal term and by using various methods such as the Nikiforov-Uvarov [4], asymptotic iteration [5], supersymmetric quantum mechanics [6], the path integral approach [7], numerical calculations [8] and many others.

We attempt to find the analytical approximate bound state solutions of Schrodinger and Klein-Gordon equations with modified Hylleraas plus attractive radial molecular potentials including the energy spectrum. Ever since Hylleraas proposed this potential [9] no much work has been reported on the bound state solution. The purpose of this paper is to use the Pekeris-like approximation [2, 3]to the centrifugal term to study the Schrodinger and Klein-Gordon equations for modified Hyllerras plus attractive potentials. The attractive radial potential which has been studied by many researchers [10,11] is one of the most important exponential-type potential in physics and chemical physics whereas Hylleraas potential can be used to study diatomic molecules [12,13].

II. Section 2: Theoretical Approach

a) Review of parametric Nikiforov-Uvarov Method

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The Nikiforov-Uvarov method has been successfully applied to relativistic, nonrelativistic quantum mechanical problems and other fields of study as well [3, 14].The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi''_n(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi'_n(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\Psi_n(s) = 0 \qquad (1)$$

Where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials at most second degree and $\tilde{\tau}(s)$ is first degree polynomials.The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation.

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$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1-c_3 s)}\Psi'(s) + \frac{1}{s^2(1-c_3 s)^2}[-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3]\Psi(s) = 0 \tag{2}$$

Thus eqn. (1) can be solved by comparing it with equation (2) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2 s),\ \sigma(s) = s(1-c_3 s),\ \overline{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \tag{3}$$

The parameters obtainable from equation (3) serve as important tools for finding the energy eigen value and eigen functions. They satisfy the following sets of equation respectively.

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \tag{4}$$

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \tag{5}$$

While the wave function is given as

$$\Psi_n(s) = N_{n,l} S^{c_{12}}(1-c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s) \tag{6}$$

Where

$$c_4 = \frac{1}{2}(1-c_1),\ c_5 = \frac{1}{2}(c_2 - 2c_3),\ c_6 = c_5{}^2 + \epsilon_1,\ c_7 = 2c_4 c_5 - \epsilon_2,\ c_8 = c_4{}^2 + \epsilon_3,$$

$$c_9 = c_3 c_7 + c_3{}^2 c_8 + c_6,\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},\ c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c3\sqrt{c_8}\right)$$

$$c_{12} = c_4 + \sqrt{c_8},\ c_{13} = c_5 - \left(\sqrt{c_9} + c3\sqrt{c_8}\right) \tag{7}$$

and P_n are the orthogonal polynomials.

Given that
$$P_n{}^{(\alpha,\beta)} = \sum_{r=0}^{n} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+r+1)\Gamma(n+\beta-r+1)(n-r)!r!}\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r} \tag{8}$$

This can also be expressed in terms of the Rodriguez's formula

$$P_n{}^{(\alpha,\beta)}(x) = \frac{1}{2^n n!}(x-1)^{-\alpha}(x+1)^{-\beta}\left(\frac{d}{dx}\right)^n\left((x-1)^{n+\alpha}(x+1)^{n+\beta}\right)$$

III. Section 3: Bound State Solutions

a) *Schrodinger Equation with Modified Hylleraas Plus Attractive Radial Molecular Potential*

The l-State Schrodinger Equation with vector V(r), potential is given as [15]

$$\frac{d^2R(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[(E - V(r)) + \frac{l(l+1)\hbar^2}{2\mu r^2}\right]R(r) = 0 \tag{9}$$

Where E is the eigen energy value, l is the angular momentum quantum number

The Modified Hylleras Potential is given as [9]

$$V(s) = \frac{V_o}{b}\left(\frac{a-s}{1-s}\right) \tag{10}$$

Where $S = e^{-2\alpha r}$, α is the screening parameter and determines the range of the potential, and V_o, b, a are the coupling parameters describing the depth of the potential well.

The Attractive Radial Potential [14]

$$V(s) = \frac{V_1 s^2}{(1-s)^2} + \frac{V_2 s}{(1-s)^2} + \frac{V_3}{(1-s)^2} \tag{11}$$

Where, screening parameter αdetermines the range of the potential, and $V_1, V_2, V_3$are the coupling parameters describing the depth of the potential well Making the transformation $s = e^{-2\alpha r}$ the sum of the potentials (MHARMP) in equations (2) and (3) becomes

$$V(s) = \left(\frac{V_1 s^2}{(1-s)^2} + \frac{V_2 s}{(1-s)^2} + \frac{V_3}{(1-s)^2} + \frac{V_o}{b}\left(\frac{a-s}{1-s}\right)\right) \tag{12}$$

The Pekeris-like approximation [15-17] is applied to the inverse square term, $\frac{1}{r^2} = \frac{4\alpha^2}{(1-s)^2}$ in eq. (12) to enable us to completely solve eq. (9).

Again, applying the transformation $s = e^{-2\alpha r}$ to get the form that parametric Nikiforov-Uvarov (NU) method is applicable, equation (9) gives a generalized hypergeometric-type equation as.

$$\frac{d^2R(s)}{ds^2}+\frac{(1-s)}{(1-s)s}\frac{dR(s)}{ds}+\frac{1}{(1-s)^2s^2}[(2\beta^2-J-P)s^2+(J+R-H-4\beta^2)s+(2\beta^2-R-B+\lambda)]R(s)=0 \tag{13}$$

Where

$$-\beta^2=\left(\frac{\mu E}{4\alpha^2\hbar^2}\right),\ B=\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_3,\ \lambda=l(l+1),\ P=\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_1,\ H=\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_2,\ J=\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_o,R=\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_o a \tag{14}$$

$$c_1=c_2=c_3=1,c_4=0,c_5=-\frac{1}{2},c_6=\frac{1}{4}+2\beta^2-J-P,c_7=-4\beta^2-H+J+R,$$

$$c_8=2\beta^2-R-B+\lambda,$$

$$c_9=\frac{1}{4}+\lambda-H-B-P,$$

$$c_{10}=1+2\sqrt{2\beta^2-R-B+\lambda},$$

$$c_{11}=2+2\left(\sqrt{\frac{1}{4}+\lambda-H-B-P}+\sqrt{2\beta^2-R-B+\lambda}\right),$$

$$c_{12}=\sqrt{2\beta^2+P},$$

$$c_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\lambda-H-B-P}+\sqrt{2\beta^2-R-B+\lambda}\right),$$

$$\varepsilon_1=2\beta^2-J-P,$$

$$\varepsilon_2=4\beta^2+H-J-R,\varepsilon_3=2\beta^2-R-B+\lambda \tag{15}$$

Now using equations (6), (14) and (15) we obtain the energy eigen spectrum of the MHARMP as

$$\beta^2=\left[\frac{(P+H+2B)-(J+2\lambda)-\left(n^2+n-\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}+\lambda-H-B-P}}{\left(n+\frac{1}{2}\right)+2\sqrt{\frac{1}{4}+\lambda-H-B-P}}\right]^2-(R+B-\lambda) \tag{16}$$

The above equation can be solved explicitly and the energy eigen spectrum of MHARMP becomes

$$E=\frac{4\alpha^2\hbar^2}{\mu}\left\{\left[\frac{\left(\left(\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_1+\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_2\right)+\left(\frac{\mu}{\alpha^2\hbar^2}\right)V_2\right)-\left(\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_o+2l(l+1)\right)-\left(n^2+n-\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}+l(l+1)-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_2}-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_3-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_1}{(2n+1)+2\sqrt{\frac{1}{4}+l(l+1)-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_2}-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_3-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_1}\right]\right\}-$$
$$\left(\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_o a+\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_3-l(l+1)\right) \tag{17}$$

b) Klein-Gordon Edquation with Modified Hylleraas Plus Attractive Radial Molecular Potential

The Klein-Gordon Equation with vector V(r), potential in atomic units ($\hbar = c = 1$) is given as [3]

$$\frac{d^2R(r)}{dr^2}+\left[(E^2-M^2)-2(E+M)V(r)+\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{18}$$

Where $E, M, V(r), l$ are the relativistic energy, reduced mass, potential and angular momentum respectively. From eq. (12) we have the expression of the superposed or mixed potentials, MHARMP given as

$$V(s) = \left(\frac{V_1 s^2}{(1-s)^2} + \frac{V_2 s}{(1-s)^2} + \frac{V_3}{(1-s)^2} + \frac{V_o}{b}\left(\frac{a-s}{1-s}\right)\right) \tag{12}$$

Similarly, the Pekeris-like approximation [15-17] is applied to the inverse square term, $\frac{1}{r^2} = \frac{4\alpha^2}{(1-s)^2}$ in eq. (12) to enable us to completely solve eq. (18).

Again, applying the transformation $s = e^{-2\alpha r}$ to get the form that parametric Nikiforov-Uvarov (NU) method is applicable, equation (18) gives a generalized hypergeometric-type equation as

$$\frac{d^2R(s)}{ds^2} + \frac{(1-s)}{(1-s)s}\frac{dR(s)}{ds} + \frac{1}{(1-s)^2 s^2}\left[(\beta^2 - P - J)s^2 + (-2\beta^2 + J + R - B)s + (\beta^2 - H - R + \lambda)\right]R(s) = 0 \tag{19}$$

Where

$$\beta^2 = \left(\frac{E^2+M_o^2}{4\alpha^2}\right),\ B = \left(\frac{E+M_o}{2\alpha^2}\right)V_2,\ \lambda = l(l+1), H = \left(\frac{E+M_o}{2\alpha^2}\right)V_3, R = \left(\frac{E+M_o}{2b\alpha^2}\right)V_o a, J = \left(\frac{E+M_o}{2b\alpha^2}\right)V_o,\ P = \left(\frac{E+M_o}{2\alpha^2}\right)V_1 \tag{20}$$

$$c_1 = c_2 = c_3 = 1, c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \beta^2 - J - P, c_7 = -2\beta^2 - B + J + R,$$

$$c_8 = \beta^2 - H - R + \lambda, c_9 = \frac{1}{4} - B - H - P + \lambda, c_{10} = 1 + 2\sqrt{\beta^2 - H - R + \lambda},$$

$$c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} - B - H - P + \lambda} + \sqrt{\beta^2 - H - R + \lambda}\right),$$

$$c_{12} = \sqrt{\beta^2 - H - R + \lambda}, c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} - B - H - P + \lambda} + \sqrt{\beta^2 - H - R + \lambda}\right),$$

$$\varepsilon_1 = \beta^2 - P - J, \varepsilon_2 = 2\beta^2 + B - J - R, \varepsilon_3 = \beta^2 - H - R + \lambda \tag{21}$$

Now using equations (6), (20) and (21) we obtain the energy eigen spectrum of the MHARMP as

$$\beta^2 = \left[\frac{(R - 2\lambda - J) + (2H + B) - \left(n^2 + n - \frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - B - H - P + \lambda}}{(2n+1) + 2\sqrt{\frac{1}{4} - B - H - P + \lambda}}\right]^2 - (H + R - \lambda) \tag{22}$$

The above equation can be solved explicitly and the energy eigen spectrum of MHARMP becomes

$$E^2 - M^2 = -4\alpha^2\left\{\left[\frac{\left(\left(\frac{E+M_o}{2b\alpha^2}\right)V_o a - 2l(l+1) - \left(\frac{E+M_o}{2b\alpha^2}\right)V_o\right) + \left(\left(\frac{E+M_o}{\alpha^2}\right)V_3 + \left(\frac{E+M_o}{2\alpha^2}\right)V_2\right) - \left(n^2+n-\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - \left(\frac{E+M_o}{2\alpha^2}\right)V_3 - \left(\frac{E+M_o}{2\alpha^2}\right)V_2 - \left(\frac{E+M_o}{2\alpha^2}\right)V_1}}{(2n+1) + \sqrt{\frac{1}{4} + (l(l+1)) - \left(\frac{E+M_o}{2\alpha^2}\right)V_1} - \left(\frac{E+M_o}{2\alpha^2}\right)V_3 - \left(\frac{E+M_o}{2\alpha^2}\right)V_2}\right]\right\} - \left(\left(\frac{E+M_o}{2b\alpha^2}\right)V_o a + \left(\frac{E+M_o}{2\alpha^2}\right)V_3 - l(l+1)\right) \tag{23}$$

IV. Section 4: Discussion

In this section, special case of potential considerations to the obtained bound state eigenenergy for Schrodinger and Klein-Gordon equations are considered.

a) Schrodinger Equation

Case I: From eq. (12) *when* $V_o = 0$, eq. (17) is reduced to Schrodinger equation with Attractive radial potential given as:

$$E = \frac{4\alpha^2\hbar^2}{\mu}\left\{\left[\frac{\left(\left(\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_1+\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_2\right)+\left(\frac{\mu}{\alpha^2\hbar^2}\right)V_2\right)-(2l(l+1))-\left(n^2+n-\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}+l(l+1)-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_2}-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_3-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_1}{(2n+1)+2\sqrt{\frac{1}{4}+l(l+1)-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_2}-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_3-\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_1}\right]\right\}-$$

$$\left(+\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_3 - l(l+1)\right) \quad (24)$$

Eq. (24) is similar to the bound state solution obtained in ref. [14]

Case II: From eq. (12) $when\ V_1 = V_2 = V_3 = 0$, eq. (17) is reduced to Schrodinger equation with Modified Hylleraas potential given as:

$$E = \frac{4\alpha^2\hbar^2}{\mu}\left\{\left[\frac{-\left(\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_o+2l(l+1)\right)-\left(n^2+n-\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}+l(l+1)}}{(2n+1)+2\sqrt{\frac{1}{4}+l(l+1)}}\right]\right\}-\left(\left(\frac{\mu}{2\alpha^2\hbar^2}\right)V_o a - l(l+1)\right) \quad (25)$$

b) Klein-Gordon Equation

Case I: From eq. (12) $when\ V_o = 0$, eq. (23) is reduced to Schrodinger equation with Attractive radial potential given as:

$$E^2 - M^2 = -4\alpha^2\left\{\left[\frac{(2l(l+1))+\left(\left(\frac{E+M_o}{\alpha^2}\right)V_3+\left(\frac{E+M_o}{2\alpha^2}\right)V_2\right)-\left(n^2+n-\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}-\left(\frac{E+M_o}{2\alpha^2}\right)V_3-\left(\frac{E+M_o}{2\alpha^2}\right)V_2-\left(\frac{E+M_o}{2\alpha^2}\right)V_1}}{(2n+1)+\sqrt{\frac{1}{4}+(l(l+1))-\left(\frac{E+M_o}{2\alpha^2}\right)V_1}-\left(\frac{E+M_o}{2\alpha^2}\right)V_3-\left(\frac{E+M_o}{2\alpha^2}\right)V_2}\right]\right\}-\left(\left(\frac{E+M_o}{2\alpha^2}\right)V_3 - l(l+1)\right) \quad (26)$$

Case II: From eq. (12) $when\ V_1 = V_2 = V_3 = 0$, eq. (23) is reduced to Schrodinger equation with Modified Hylleraas potential given as:

$$E^2 - M^2 = -4\alpha^2\left\{\left[\frac{\left(\left(\frac{E+M_o}{2b\alpha^2}\right)V_o a-2l(l+1)-\left(\frac{E+M_o}{2b\alpha^2}\right)V_o\right)-\left(n^2+n-\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}+(l(l+1)}}{(2n+1)+\sqrt{\frac{1}{4}+(l(l+1))}}\right]\right\}-\left(\left(\frac{E+M_o}{2b\alpha^2}\right)V_o a - l(l+1)\right) \quad (27)$$

V. Conclusion

In this work, using the parametric generalization of the NU method, we have obtained approximately energy eigenvalues of the Schrodinger and Klein-Gordon equations for Modified Hylleraas and attractive radial molecular potential. Interestingly, the Dirac equation with the arbitrary angular momentum values for this potential can be solved by this method. The resulting eigen energy equations can be used to study the spectroscopy of some selected diatomic atoms and molecules.

References Références Referencias

1. Louis H., Ita B.I., Nelson N.A., I. Joseph., Amos P.I and Magu T.O; …*International Research Journal Of Pure and Applied Physics*. Vol. 5, No. 3, pp 27-32, 2017
2. Louis Hitler, Benedict Iserom Ita, Pigweh Amos Isa, Nzeata-Ibe Nelson, Innocent Joseph, Opara Ivan, Thomas Odey Magu. *World Journal of Applied Physics*.Vol. 2, No. 4, 2017, pp. 109-112. doi: 10.11648/j.wjap.20170204.13.
3. Louis Hitler, Benedict Iserom Ita, Pigweh Amos Isa, Innocent Joseph, Nzeata-Ibe Nelson, Thomas Odey Magu, Opara Ivan. *World Journal of Applied Physics*. Vol. 2, No. 4, 2017, pp. 101-108. doi: 10.11648/j.wjap.20170204.12
4. Nikiforov A F and Uvarov V B 1988 Special Functions of Mathematical Physics (Birkhauser, Bassel) [5] Cifci H, Hall R L and Saad N 2003 J. Phys A: Math. Gen. 36 1807
5. Morales D A. 2004 Chem. Phys. Lett. 394 68
6. Diaf A, Chouchaoui A and Lombard R J 2005 Annals of Physics 317 354
7. Lucha W and Schöberl F F 1999 Int. J. of Mod. Phys. C 10 607
8. E. A. Hylleraas, J.Chem.Phys., 3, 595 (1935). Y.P.Varshni, Rev.Mod.Phys., 29, 664 (1957).
9. A. P. Zhang, W. C. Qiang, Y. W. Ling, Chin.Phys. Lett. 26(10) (2006) 100302
10. G. F. Wei, C. Y. Long, X. Y. Duan and S. H. Dong, Phys.Scr. 77(2008) 035001
11. Y. P. Varshni, Rev.Mod.Phys.29 (4) (1957) 664
12. E. A. Hylleraas, J.Chem.Phys. 3(1935) 595

13. B.I. Ita, A.I. Ikeuba, H. Louisand P. Tchoua (2015): *Journal of Theoretical Physics and Cryptography*. IJTPC, Vol. 10, December, 2015. www. IJTPC.org
14. B.I. Ita, H. Louis, P.I. Amos, I. Joseph, N.A. Nzeata-Ibe, T.O. Magu and H. Disho: *World Scientific News 89 (2017) 64 – 70.*
15. B.I. Ita**,** H. Louis., P.I. Amos., T.O. Magu and N.A. Nzeata-Ibe (2017): *Physical Science International Journal*. 15(3): 1-6, 2017.
16. Nzeata-Ibe Nelson, Benedict Iserom Ita, Innocent Joseph, Pigwe Amos Isa, Thomas Odey Magu and Louis Hitler* *World Journal of Applied Physics*, Vol. 2, N0. 5, 2017, pp. 77-84. doi: 10.11648/j.wjap.20170205.13