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Quasi-Hadamard Product of Certain Starlike and Convex P-Valent Functions

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QUASIHADAMARDPRODUCTOFCERTAINSTARLIKEANDCONVEXPVALENTFUNCTIONS

Strictly as per the compliance and regulations of:

Quasi-Hadamard Product of Certain Starlike and Convex P-Valent Functions

R. M. El-Ashwah [α], A. Y. Lashin [σ] & A. E. El-Shirbiny [ρ]

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I. Introduction

let $T(p)$ denote of the class of functions of the form:

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \qquad (p \in \mathbb{N} = \{1, 2, ...\}, a_1 \geq 0, a_{n+p} \geq 0), \tag{1}$$

$$f_r(z) = a_{p,r} z^p - \sum_{n=1}^{\infty} a_{n+p,r} z^{n+p} \qquad (r \in \mathbb{N}, a_{p,r} > 0, a_{n+p,r} \geq 0), \tag{2}$$

$$g(z) = b_p z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \qquad (p \in \mathbb{N} = \{1, 2, ...\}, b_p \geq 0, b_{n+p} \geq 0), \tag{3}$$

and

$$g_j(z) = b_{p,j} z^p - \sum_{n=1}^{\infty} b_{n+p,j} z^{n+p} \qquad (j \in \mathbb{N}, b_p \geq 0, b_{n+p,j} \geq 0), \tag{4}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C},\ |z| < 1\}$. We write $T(1) = T$, the class of analytic functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_1 \geq 0, a_n \geq 0), \tag{5}$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C},\ |z| < 1\}$.

Let S^* be the subclass of functions T consisting of starlike functions in U. It is well known that $f \in S^*$ if and only if

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$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (z \in U).$$

and C be the subclass of functions T consisting of convex functions in U. It is well known that $f \in C$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (z \in U).$$

Let S_S^* be the subclass of T consisting of functions of the form (5) satisfying

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \qquad (z \in U).$$

These functions are called starlike with respect to symmetric points and introduced by Sakaguchi [13] (see also Robertson [12], Stankiewics [14], Wu [16] and Owa et al. [8]).

Khairnar and Rajas (see [4], with $\delta = 0$) introduced the class $S_s^*(p, \alpha, \beta)$ consisting of functions of the form (1) and satisfying the following condition

$$\left|\frac{zf'(z)}{f(z) - f(-z)} - p\right| < \beta\left|\frac{\alpha zf'(z)}{f(z) - f(-z)} + p\right|, 0 \leq \alpha < 1, 0 < \beta \leq 1.$$

Let $S_c^*(p, \alpha, \beta)$ denote the class of functions of the form (1) for which $zf'(z) \in S_s^*(p, \alpha, \beta)$.

We note that $S_s^*(1, \alpha, \beta) = S_s^*(\alpha, \beta)$ (see [15] and [1]) and $S_c^*(1, \alpha, \beta) = S_c^*(\alpha, \beta)$ (see [3]).

By using the technique of Khairnar and Rajas (see [4], with $\delta = 0$), we get the following theorem.

Theorem 1. Let the function $f(z)$ defined by (1) then

(i) $f(z) \in S_s^*(p, \alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty}\left[\left(\frac{n+p}{p}\right)(1+\alpha\beta) + (1-\beta)\left((-1)^{n+p} - 1\right)\right]a_{n+p} \leq (\beta[\alpha + (1-(-1)^p)] + (-1)^p)\,a_p$$

(ii) $f(z) \in S_c^*(p, \alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)\left[\left(\frac{n+p}{p}\right)(1+\alpha\beta) + (1-\beta)\left((-1)^{n+p} - 1\right)\right]a_{n+p} \leq (\beta[\alpha + (1-(-1)^p)] + (-1)^p)\,a_p$$

(iii) $f(z) \in S_{s,h}^*(p, \alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^h\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)\left((-1)^{p+n} - 1\right)\right]a_{n+p}$$

$$\leq \quad (\beta[\alpha + (1-(-1)^p)] + (-1)^p)\,a_p;$$

where $0 \leq \alpha < 1$, $0 < \beta < 1$, $0 < \frac{2(1-\beta)}{1+\alpha\beta} < 1$, $z \in U$ and h is a nonnegative real number.

Ref

13. K. Sakaguchi, On certain univalent mapping, J. Math. Soc. Japen., 11 (1959), 72-75.

We note that for a nonnegative real number h the class $S^*_{s,h}(p,\alpha,\beta)$ is nonempty as the function of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{\beta\left[\alpha + (1-(-1)^p)\right] + (-1)^p}{\left(\frac{n+p}{p}\right)^h \left[\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)\left((-1)^{n+p}-1\right)\right]} a_p \lambda_{n+p} z^{n+p}, \qquad (6)$$

where $a_p > 0, \lambda_{n+p} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{n+p} \leq 1$ satisfy the inequality (6). It is evident that $S^*_{s,1}(p,\alpha,\beta) = S^*_c(p,\alpha,\beta)$ and for $h = 0$, $S^*_{s,0}(p,\alpha,\beta)$ is identical to $S^*_s(p,\alpha,\beta)$. Further more $S^*_{s,h}(p,\alpha,\beta) \subset S^*_{s,m}(p,\alpha,\beta)$ *for* $h > m$, the containment being proper. Hence for any positive integer h, the inclusion relation

$$S^*_{s,h}(p,\alpha,\beta) \subset S^*_{s,h-1}(p,\alpha,\beta) \subset \ldots \subset S^*_{s,m}(p,\alpha,\beta) \subset \ldots \subset S^*_{s,2}(p,\alpha,\beta) \subset S^*_c(p,\alpha,\beta) \subset S^*_s(p,\alpha,\beta).$$

The quasi-Hadamard product of two or more functions has recently defined by Darwich et al.[2], Kumar[5, 6, 7], Owa [9, 10, 11], and others. Accordingly, the quasi-Hadmard product of two functions $f(z)$ and $g(z)$

$$f(z) * g(z) = a_p b_p z^p - \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p}.$$

In this paper, we obtained some results concerning the quasi-Hadamard product for two classes of $p-$ valent functions related to starlike and convex with respect to symmetric points.

II. Main Results

Theorem 2 *A functions* $f_r(z)$ *defined by* (2) *in the* $S^*_c(p,\alpha,\beta)$ *for each* $r = 1, 2, \ldots, u$. *then we get quasi-Hadamard prouduct*

$$f_1(z) * f_2(z) * \ldots * f_u(z) \in S^*_{s,2(u-1)+1}(p,\alpha,\beta).$$

Proof. To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{2(u-1)+1} \left\{\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)\left((-1)^{n+p}-1\right)\right\} \prod_{r=1}^{u} a_{n+p,r}$$

$$\leq \left(\beta[\alpha + (1-(-1)^p)] + (-1)^p\right) \prod_{r=1}^{u} a_{p,r}.$$

Since $f_r(z) \in S^*_c(p,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right) \left\{\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)\left((-1)^{n+p}-1\right)\right\} a_{n+p,r} \leq [\beta[\alpha+(1-(-1)^p)]+(-1)^p] a_{p,r} \qquad (7)$$

Ref

2. H. E. Darwish, A. Y. Lashin and A. N. Alnayyef, Quasi Hadmeard product of certain starlike and convex functions, Global Journal of Science Frontier Research, 15 (10) (2015), no.1, 1-7.

for each $r = 1, ..., u$. Therefore

$$\left(\frac{n+p}{p}\right)\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p,r} \leq [\beta[\alpha+(1-(-1)^p)]+(-1)^p]a_{p,r}$$

or

$$a_{n+p,r} \leq \left\{\frac{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}{\left(\frac{n+p}{p}\right)\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]}\right\}a_{p,r}$$

for each $r = 1, 2, ..., u$. The right hand expression of this last inequality is not greater than $\left(\frac{n+p}{p}\right)^{-2} a_{p,r}$, hence

$$a_{n+p,r} \leq \left(\frac{n+p}{p}\right)^{-2} a_{p,r} \qquad r = 1, ..., u. \tag{8}$$

By (8) for each $r = 1, 2, ..., u-1$ and (7) for $r = u$, we get

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^{2(u-1)+1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\prod_{r=1}^{u}a_{n+p,r} \leq$$

$$\sum_{n=1}^{\infty}\left\{\left[\left(\frac{n+p}{p}\right)^{2(u-1)+1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\right]\left(\frac{n+p}{p}\right)^{-2(u-1)}\prod_{r=1}^{u-1}a_{p,r}\right\}a_{n+p,u}$$

$$= \left[\prod_{r=1}^{u-1}a_{p,r}\right]\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p,u}$$

$$\leq \left(\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p\right)\prod_{r=1}^{u}a_{p,r}$$

hence $f_1 * f_2 * ... * f_u \in S^*_{2(u-1)+1}(p, \alpha, \beta)$. This completes the proof of Theorem 2.

Theorem 3 A functions $f_r(z)$ defined by (2) in the class $S^*_s(p, \alpha, \beta)$ for each $r = 1, 2, ..., u$. Then, we get the quasi-Hadamard product

$$f_1(z) * f_2(z) * ... * f_u(z) \in S^*_{s,(u-1)}(p, \alpha, \beta).$$

Proof. Using $f_r(z) \in S^*_s(p, \alpha, \beta)$, we have

$$\sum_{n=1}^{\infty}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p,r} \leq \beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p a_{p,r} \tag{9}$$

for each $r = 1, 2, ..., u$, therefore,

$$a_{n+p,r} \leq \left(\frac{\beta[\alpha+(1-(-1)^p)]+(-1)^p}{\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)}\right)a_{p,r}$$

Notes

and hence

$$a_{n+p,r} \leq \left(\frac{n+p}{p}\right)^{-1} a_{p,r}, \qquad r = 1, 2, ..., u. \tag{10}$$

By (10) *for* $r = 1, 2, ..., (u-1)$ *and* (9) *for* $r = u$, *we get*

Notes

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{(u-1)}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]\right]\prod_{r=1}^{u} a_{n+p,r}\right\} \leq$$

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{(u-1)}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\left[\left(\frac{n+p}{p}\right)^{-(u-1)}\prod_{r=1}^{u-1} a_{p,r}\right]\right\} a_{n+p,u}$$

$$= \left[\prod_{r=1}^{u-1} a_{p,r}\right]\sum_{n=1}^{\infty}\left\{\left(\frac{p+n}{p}\right)(1+\alpha\beta)+\left[(\beta-1)\left(1-(-1)^{n+p}\right)\right] a_{n+p,u}\right\}$$

$$\leq \left(\beta[\alpha+(1-(-1)^{p}]+(-1)^{p}\right)\prod_{r=1}^{u} a_{p,r}$$

Hence $f_1(z) * f_2(z) * ... * f_u(z) \in S^*_{s,u-1}(p, \alpha, \beta)$. *This completes the proof of Theorem 3.*

Theorem 4 *A functions* $f_r(z)$ *defined by* (2) *in the class* $S^*_c(p, \alpha, \beta)$ *for each* $r = 1, 2, ..., u$ *and the functions* $g_j(z)$ *defined by* (1.4) *in the class* $S^*_s(p, \alpha, \beta)$ *for* $j = 1, 2, ..., q$. *Then, we get the quasi-Hadamard product;*

$$f_1(z) * f_2(z) * ... * f_u(z) * g_1(z) * g_2(z) * ... * g_q(z) \in S^*_{s,2u+q-1}(p, \alpha, \beta).$$

Proof. *We denote the quasi-Hadamard produc* $f_1(z) * f_2(z) * ... * f_u(z) * g_1(z) * g_2(z) * ... * g_q(z)$ *by function* $h(z)$, *for the sake of the convenience. Clearly*

$$h(z) = \left[\prod_{r=1}^{u} a_{p,r}.\prod_{j=1}^{q} b_{p,j}\right] z^p - \sum_{n=1}^{\infty}\left[\prod_{r=1}^{u} a_{n+p,r}.\prod_{j=1}^{q} b_{n+p,j}\right] z^{n+p}.$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+q-1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\left[\prod_{r=1}^{u} a_{n+p,r}.\prod_{j=1}^{q} b_{n+p,j}\right]\right\}$$

$$\leq \left(\beta[\alpha+(1-(-1)^{p})]+(-1)^{p}\right)\left[\prod_{r=1}^{u} a_{p,r}.\prod_{j=1}^{q} b_{p,j}\right] \tag{11}$$

Since $f_r(z) \in S^*_c(p, \alpha, \beta)$, *the inequalities* (7) *and* (8) *hold for every* $r = 1, 2, ..., u$. *Further, since* $g_j(z) \in S^*_s(p, \alpha, \beta)$, *the inequality* (10) *holds for each* $j = 1, 2, ..., q$. *By* (8) *for* $r = 1, 2, ..., u$, (10) *for* $j = 1, 2, ..., q-1$ *and* (9) *for* $j = q$, *we get*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+q-1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]\left[\prod_{r=1}^{u} a_{n+p,r}.\prod_{j=1}^{q} b_{n+p,j}\right]\right]\right\}$$

$$\leq \sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+q-1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\right\}\times$$

$$\left(\frac{n+p}{p}\right)^{-2u}\prod_{r=1}^{u}a_{p,r}.\prod_{j=1}^{q}b_{n+p,j}\Bigg)$$

$$\leq \sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+(q-1)}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]\right]\right\}\times$$

$$\left[\prod_{r=1}^{u}a_{p,r}.\prod_{j=1}^{q-1}b_{p,j}\left(\frac{n+p}{p}\right)^{-2u}.\left(\frac{n+p}{p}\right)^{-(q-1)}\right]b_{n+p,q}$$

$$=\left[\prod_{r=1}^{u}a_{p,r}.\prod_{j=1}^{q-1}b_{p,j}\right]\sum_{n=1}^{\infty}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]b_{n+p,q}\right]$$

$$\leq \left(\beta[\alpha+(1-(-1)^{p})]+(-1)^{p}\right)\left[\prod_{r=1}^{u}a_{p,r}\prod_{j=1}^{q}b_{p,j}\right].$$

Hence, $h(z)\in S^{*}_{s,2u+q-1}(p,\alpha,\beta)$. *This completes the proof of Theorem 4.* ■

Remark 5 *Putting* $p=1$ *in the above results, we obtain the results obtained by Darwish et al. [2].*

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