Quasi-Hadamard Product of Certain Starlike and Convex P-Valent Functions

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Abstract- In this paper, we obtained some results using the quasi-Hadamard product for two classes of p-valent functions related to starlike and convex with respect to symmetric points.

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1. Introduction

Let $T(p)$ denote the class of functions of the form:

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}, a_1 \geq 0, a_{n+p} \geq 0), \quad (1)$$

$$f_r(z) = a_{p,r} z^p - \sum_{n=1}^{\infty} a_{n+p,r} z^{n+p} \quad (r \in \mathbb{N}, a_{p,r} > 0, a_{n+p,r} \geq 0), \quad (2)$$

$$g(z) = b_p z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, \ldots\}, b_p \geq 0, b_{n+p} \geq 0), \quad (3)$$

and

$$g_j(z) = b_{p,j} z^p - \sum_{n=1}^{\infty} b_{n+p,j} z^{n+p} \quad (j \in \mathbb{N}, b_p \geq 0, b_{n+p,j} \geq 0), \quad (4)$$

which are analytic and p-valent in the open unit disc $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$. We write $T(1) = T$, the class of analytic functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_1 \geq 0, a_n \geq 0), \quad (5)$$

which are analytic in the open unit disc $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$.

Let $S^*$ be the subclass of functions $T$ consisting of starlike functions in $U$. It is well known that $f \in S^*$ if and only if

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and \( C \) be the subclass of functions \( T \) consisting of convex functions in \( U \). It is well known that \( f \in C \) if and only if

\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad (z \in U).
\]

Let \( S_s^* \) be the subclass of \( T \) consisting of functions of the form (5) satisfying

\[
\Re \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad (z \in U).
\]

These functions are called starlike with respect to symmetric points and introduced by Sakaguchi [13] (see also Robertson [12], Stankiewics [14], Wu [16] and Owa et al. [8]). Khairnar and Rajas (see [4], with \( \delta = 0 \)) introduced the class \( S_s(p; \alpha, \beta) \) consisting of functions of the form (1) and satisfying the following condition

\[
\left| \frac{zf'(z)}{f(z) - f(-z)} - p \right| < \beta \left| \frac{zf'(z)}{f(z) - f(-z)} - p \right|, \quad 0 \leq \alpha < 1, 0 < \beta \leq 1.
\]

Let \( S_c(p; \alpha, \beta) \) denote the class of functions of the form (1) for which \( zf'(z) \in S_s^*(p; \alpha, \beta) \).

We note that \( S_s^*(1, \alpha, \beta) = S_s(\alpha, \beta) \) (see [15] and [1]) and \( S_c^*(1, \alpha, \beta) = S_c(\alpha, \beta) \) (see [3]).

By using the technique of Khairnar and Rajas (see [4], with \( \delta = 0 \)), we get the following theorem.

**Theorem 1.** Let the function \( f(z) \) defined by (1) then

(i) \( f(z) \in S_s^*(p; \alpha, \beta) \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{n+p}{p} \right) (1 + \alpha \beta + (1 - \beta) (-1)^{n+p} - 1) a_{n+p} \leq (\beta [\alpha + (1 - (-1)^p)] + (-1)^p) a_p
\]

(ii) \( f(z) \in S_c^*(p; \alpha, \beta) \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{n+p}{p} \right) (1 + \alpha \beta + (1 - \beta) (-1)^{n+p} - 1) a_{n+p} \leq (\beta [\alpha + (1 - (-1)^p)] + (-1)^p) a_p
\]

(iii) \( f(z) \in S_s^*(p; \alpha, \beta) \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{n+p}{p} \right)^h (1 + \alpha \beta + (1 - \beta) (-1)^{p+n} - 1) a_{n+p} \leq (\beta [\alpha + (1 - (-1)^p)] + (-1)^p) a_p,
\]

where \( 0 \leq \alpha < 1, 0 < \beta < 1, 0 < \frac{2(1-\beta)}{1+\alpha\beta} < 1, z \in U \) and \( h \) is a nonnegative real number.
We note that for a nonnegative real number \( h \) the class \( S_{s,h}^*(p, \alpha, \beta) \) is nonempty as the function of the form

\[
f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{\beta [\alpha + (1 - (-1)^p)] + (-1)^p}{(n+p) - \left(1 + \frac{p+n}{p} (1 + \alpha \beta + (1 - \beta) ((-1)^{n+p} - 1) \right]} a_{n+p} z^{n+p},
\]

where \( a_p > 0, \lambda_{n+p} \geq 0 \) and \( \sum_{n=1}^{\infty} \lambda_{n+p} \leq 1 \) satisfy the inequality (6). It is evident that \( S_{s,1}^*(p, \alpha, \beta) = S_c^*(p, \alpha, \beta) \) and for \( h = 0 \), \( S_{s,0}^*(p, \alpha, \beta) \) is identical to \( S_c^*(p, \alpha, \beta) \). Further, \( S_{s,h}^*(p, \alpha, \beta) \subseteq S_{s,m}^*(p, \alpha, \beta) \) for \( h > m \), the containment being proper. Hence for any positive integer \( h \), the inclusion relation

\[
S_{s,h}^*(p, \alpha, \beta) \subseteq S_{s,h-1}^*(p, \alpha, \beta) \subseteq \ldots \subseteq \ldots S_{s,m}^*(p, \alpha, \beta) \subseteq \ldots \subseteq S_{s,2}^*(p, \alpha, \beta) \subseteq S_c^*(p, \alpha, \beta) \subseteq S_s^*(p, \alpha, \beta).
\]

The quasi-Hadamard product of two or more functions has recently defined by Darwich et al. [2], Kumar [5, 6, 7], Owa [9, 10, 11], and others. Accordingly, the quasi-Hadamard product of two functions \( f(z) \) and \( g(z) \)

\[
f(z) * g(z) = a_p b_p z^p - \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p}.
\]

In this paper, we obtained some results concerning the quasi-Hadamard product for two classes of \( p \)-valent functions related to starlike and convex with respect to symmetric points.

II. Main Results

Theorem 2 A functions \( f_r(z) \) defined by (2) in the \( S_c^*(p, \alpha, \beta) \) for each \( r = 1, 2, \ldots, u \) then we get quasi-Hadamard product

\[
f_1(z) * f_2(z) \ast \ldots \ast f_u(z) \in S_{s,2(u-1)+1}^*(p, \alpha, \beta).
\]

Proof. To prove the theorem, we need to show that

\[
\sum_{n=1}^{\infty} \frac{(n+p)^{2(u-1)+1}}{p} \left\{ \frac{p+n}{p} (1 + \alpha \beta + (1 - \beta) ((-1)^{n+p} - 1) \right\} \prod_{r=1}^{u} a_{n+p,r} \\
\leq (\beta [\alpha + (1 - (-1)^p)] + (-1)^p) \prod_{r=1}^{u} a_{p,r}.
\]

Since \( f_r(z) \in S_c^*(p, \alpha, \beta) \), we have

\[
\sum_{n=1}^{\infty} \frac{(n+p)^{2(u-1)+1}}{p} \left\{ \frac{p+n}{p} (1 + \alpha \beta + (1 - \beta) ((-1)^{n+p} - 1) \right\} a_{n+p,r} \leq [\beta [\alpha + (1 - (-1)^p)] + (-1)^p] a_{p,r}.
\]
for each $r = 1, \ldots, u$. Therefore
\[
\left(\frac{n+p}{p}\right)\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)((-1)^{n+p} - 1)\right] a_{n+p,r} \leq \left[\beta(\alpha+(1-(-1)^p)) + (-1)^p\right] a_{p,r}
\]
or
\[
a_{n+p,r} \leq \left\{\frac{\beta(\alpha+(1-(-1)^p)) + (-1)^p}{\left(\frac{n+p}{p}\right)\left(1+\alpha\beta + (1-\beta)((-1)^{n+p} - 1)\right)}\right\} a_{p,r}
\]
for each $r = 1, 2, \ldots, u$. The right hand expression of this last inequality is not greater than
\[
\left(\frac{n+p}{p}\right)^{-2} a_{p,r}
\]
hence
\[
a_{n+p,r} \leq \left(\frac{n+p}{p}\right)^{-2} a_{p,r} \quad r = 1, \ldots, u. \quad (8)
\]
By (8) for each $r = 1, 2, \ldots, u - 1$ and (7) for $r = u$, we get
\[
\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^{2(u-1)+1} \left[\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)((-1)^{n+p} - 1)\right] \prod_{r=1}^{u} a_{n+p,r} \leq \left(\frac{n+p}{p}\right)^{2(u-1)+1} \prod_{r=1}^{u-1} a_{n+p,r}
\]
\[
= \left[\prod_{r=1}^{u-1} a_{p,r}\right] \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right) \left[\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)((-1)^{n+p} - 1)\right] a_{n+p,u}
\]
\[
\leq \left(\beta(\alpha+(1-(-1)^p)) + (-1)^p\right) \prod_{r=1}^{u} a_{p,r}
\]
hence $f_1 * f_2 * \cdots * f_u \in S^*_{2(u-1)+1}(p, \alpha, \beta)$. This completes the proof of Theorem 2.

**Theorem 3** A function $f_r (z)$ defined by (2) in the class $S^*_s(p, \alpha, \beta)$ for each $r = 1, 2, \ldots, u$. Then, we get the quasi-Hadamard product
\[
f_1(z) * f_2(z) * \cdots * f_u (z) \in S^*_s(u-1)(p, \alpha, \beta).
\]

**Proof.** Using $f_r (z) \in S^*_s(p, \alpha, \beta)$, we have
\[
\sum_{n=1}^{\infty} \left[\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)((-1)^{n+p} - 1)\right] a_{n+p,r} \leq \beta[\alpha+(1-(-1)^p)] + (-1)^p a_{p,r}
\]
for each $r = 1, 2, \ldots, u$, therefore,
\[
a_{n+p,r} \leq \left(\frac{\beta[\alpha+(1-(-1)^p)] + (-1)^p}{\left(\frac{p+n}{p}\right)(1+\alpha\beta) + (1-\beta)((-1)^{n+p} - 1)}\right) a_{p,r}
\]


and hence
\[ a_{n+p,r} \leq \left( \frac{n+p}{p} \right)^{-1} a_{p,r}, \quad r = 1, 2, \ldots, u. \] (10)

By (10) for \( r = 1, 2, \ldots, (u - 1) \) and (9) for \( r = u \), we get
\[
\sum_{n=1}^{\infty} \left\{ \left( \frac{n+p}{p} \right)^{(u-1)} \left[ \left( \frac{p+n}{p} \right) (1+\alpha\beta) + (1-\beta) \left( (-1)^{n+p} - 1 \right) \right] \prod_{r=1}^{u} a_{n+p,r} \right\} \leq \\
\prod_{r=1}^{u-1} a_{p,r} \sum_{n=1}^{\infty} \left\{ \left( \frac{p+n}{p} \right)^{(u-1)} \left[ (1+\alpha\beta) + [(-1)(1-(-1)^{n+p})] a_{n+p,u} \right] \right\} \\
\leq (\beta[\alpha + (1 - (-1)^p)] + (-1)^p) \prod_{r=1}^{u} a_{p,r}
\]

Hence \( f_1(z) \ast f_2(z) \ast \ldots \ast f_u(z) \in S_{s,u-1}^*(p, \alpha, \beta) \). This completes the proof of Theorem 3.

**Theorem 4** A function \( f_r(z) \) defined by (2) in the class \( S_c^*(p, \alpha, \beta) \) for each \( r = 1, 2, \ldots, u \) and the functions \( g_j(z) \) defined by (1.4) in the class \( S_c^*(p, \alpha, \beta) \) for \( j = 1, 2, \ldots, q \). Then, we get the quasi-Hadamard product:
\[ f_1(z) \ast f_2(z) \ast \ldots \ast f_u(z) \ast g_1(z) \ast g_2(z) \ast \ldots \ast g_q(z) \in S_{s,2u+q-1}^*(p, \alpha, \beta). \]

**Proof.** We denote the quasi-Hadamard product \( f_1(z) \ast f_2(z) \ast \ldots \ast f_u(z) \ast g_1(z) \ast g_2(z) \ast \ldots \ast g_q(z) \) by function \( h(z) \), for the sake of the convenience. Clearly
\[ h(z) = \left[ \prod_{r=1}^{u} a_{p,r} \cdot \prod_{j=1}^{q} b_{p,j} \right] z^p - \sum_{n=1}^{\infty} \left[ \prod_{r=1}^{u} a_{n+p,r} \cdot \prod_{j=1}^{q} b_{n+p,j} \right] z^{n+p}. \]

To prove the theorem, we need to show that
\[
\sum_{n=1}^{\infty} \left\{ \left( \frac{n+p}{p} \right)^{2u+q-1} \left[ \left( \frac{p+n}{p} \right) (1+\alpha\beta) + (1-\beta) \left( (-1)^{n+p} - 1 \right) \right] \prod_{r=1}^{u} a_{n+p,r} \cdot \prod_{j=1}^{q} b_{n+p,j} \right\} \\
\leq (\beta[\alpha + (1 - (-1)^p)] + (-1)^p) \prod_{r=1}^{u} a_{p,r} \cdot \prod_{j=1}^{q} b_{p,j}
\]

(11)

Since \( f_r(z) \in S_c^*(p, \alpha, \beta) \), the inequalities (7) and (8) hold for every \( r = 1, 2, \ldots, u \). Further, since \( g_j(z) \in S_c^*(p, \alpha, \beta) \), the inequality (10) holds for each \( j = 1, 2, \ldots, q \). By (8) for \( r = 1, 2, \ldots, u \), (10) for \( j = 1, 2, \ldots, q - 1 \) and (9) for \( j = q \), we get
\[
\sum_{n=1}^{\infty} \left\{ \left( \frac{n+p}{p} \right)^{2u+q-1} \left[ \left( \frac{p+n}{p} \right) (1+\alpha\beta) + (1-\beta) \left( (-1)^{n+p} - 1 \right) \right] \prod_{r=1}^{u} a_{n+p,r} \cdot \prod_{j=1}^{q} b_{n+p,j} \right\} \\
\leq (\beta[\alpha + (1 - (-1)^p)] + (-1)^p) \prod_{r=1}^{u} a_{p,r} \cdot \prod_{j=1}^{q} b_{p,j}
\]

(11)
\[ \leq \sum_{n=1}^{\infty} \left\{ \left( \frac{n+p}{p} \right)^{2u+q-1} \left[ \left( \frac{p+n}{p} \right) \left( 1 + \alpha \beta \right) + (1 - \beta) \left( (-1)^{n+p} - 1 \right) \right] \right\} \times \\
\left( \frac{n+p}{p} \right)^{-2u} \prod_{r=1}^{u} a_{p,r} \prod_{j=1}^{q} b_{n+p,j} \]

\[ \leq \sum_{n=1}^{\infty} \left\{ \left( \frac{n+p}{p} \right)^{2u+(q-1)} \left[ \left( \frac{p+n}{p} \right) \left( 1 + \alpha \beta \right) + (1 - \beta) \left[ (-1)^{n+p} - 1 \right] \right] \right\} \times \\
\left[ \prod_{r=1}^{u} a_{p,r} \prod_{j=1}^{q} b_{p,j} \right] \sum_{n=1}^{\infty} \left[ \left( \frac{p+n}{p} \right) \left( 1 + \alpha \beta \right) + (1 - \beta) \left[ (-1)^{n+p} - 1 \right] b_{n+p,q} \right] \\
= \left[ \prod_{r=1}^{u} a_{p,r} \prod_{j=1}^{q} b_{p,j} \right] \sum_{n=1}^{\infty} \left[ \left( \frac{p+n}{p} \right) \left( 1 + \alpha \beta \right) + (1 - \beta) \left[ (-1)^{n+p} - 1 \right] b_{n+p,q} \right] \\
\leq (\beta[\alpha + (1 - (-1)^p)] + (-1)^p) \left[ \prod_{r=1}^{u} a_{p,r} \prod_{j=1}^{q} b_{p,j} \right]. \]

Hence, \( h(z) \in S_{s,2u+q-1}^*(p, \alpha, \beta). \) This completes the proof of Theorem 4.

**Remark 5** Putting \( p = 1 \) in the above results, we obtain the results obtained by Darwish et al. [2].

**References Références Referencias**
