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A Numerical Approach to the Solution of the System of Second-Order Boundary-Value Problems

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ANUMERICALAPPROACHTOTHESOLUTIONOFTHESYSTEMOFSECONDORDERBOUNDARYVALUEPROBLEMS

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A Numerical Approach to the Solution of the System of Second-Order Boundary-Value Problems

Muhaiminul Islam Adnan [α] & Ashek Ahmed [σ]

Abstract- In this paper, Galerkin method is presented to obtain the approximate solutions of the system of second order boundary value problems using piecewise continuous and differentiable Bernstein polynomials. Derivation of rigorous matrix formulations is exploited to solve the system of second order boundary value problems where, given boundary conditions are satisfied by Bernstein polynomials. The derived formulation is applied to solve the system of second order boundary value problems numerically. Results of numerical approximate solutions converge to the exact solutions monotonically with desired large significant accuracy.

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I. Introduction

Ordinary differential systems appear recurrently in various applications in engineering, physics, biology and other fields. Due to importance of their frequent appearance in various applied field focus on study of ordinary differential systems have been increased. Ordinary differential systems are essential gear in solving real-world problems. A wide variety of natural phenomena are modeled by second Order differential systems. However, various conventional numerical methods used to solve second order initial value problems which methods cannot be used to solve linear second order boundary value problems. There are few valid methods to obtain numerical solutions for a system of second order boundary value problems. The authors discussed the existence of solutions to second order systems, together with the approximation of solutions through finite difference equations in [1, 5]. T. Valanarasu and N. Ramanujan recommended a method to solve a system of singularly linear second order ordinary differential equations [6]. Geng et al. are represented a new method to obtain the analytical and approximate solutions of linear and non-linear system of second order boundary value problems [7].

There are numerous numerical methods such as least square method, finite difference method, Sinc-Galerkin method, and also others methods using polynomial and non-polynomial spline functions to solve second order boundary value problems (BVPs), recently Bhatti and Bracken [8] used Bernstein polynomials for solving two point BVPs by the Galerkin method, but few of them are used to solve system of second order

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boundary value problems. This paper is concentrated on Galerkin method which is used to solve system of second order BVPs with dirichlet boundary condition of the type

$$\left.\begin{aligned} a_1(x)u'' + a_2(x)v'' + a_3(x)u' + a_4(x)v' + a_5(x)u + a_6(x)v = f_1(x) \\ b_1(x)u'' + b_2(x)v'' + b_3(x)u' + b_4(x)v' + b_5(x)u + b_6(x)v = f_2(x) \end{aligned}\right\}$$

Subject to the boundary conditions

$$u(a) = u(b) = 0, \quad v(a) = v(b) = 0$$

where $a < x < b$, $a_i(x)$, $b_i(x)$, $f_1(x)$ and $f_2(x)$ are given functions, and $a_i(x)$, $b_i(x)$ are continuous.

Notes

II. Formulation

Let us consider the one-dimensional system of second order differential equations

$$\left.\begin{aligned} -u''(x) + q(x)\,u(x) + r(x)\,v(x) = f(x) \\ -v''(x) + s(x)\,v(x) + t(x)\,u(x) = g(x) \end{aligned}\right\} \tag{1}$$

for the pair of functions $u(x)$ and $v(x)$ in $0 < x < 1$. Since each equation is of second order, two boundary conditions are required to specify each of the solution components $u(x)$ and $v(x)$ uniquely. For convenience, we assume homogeneous Dirichlet data at the ends as boundary conditions

$$u(0) = u(1) = v(0) = v(1) = 0 \tag{2}$$

The data include the prescribed functions f, g, q, r, s and t, which are assumed to be bounded and sufficiently smooth to ensure subsequent variational integrals are well defined and the problem is "well posed". Let consider two trial approximate solutions for the pair of functions $u(x)$ and $v(x)$ of system (1) given by

$$\left.\begin{aligned} \tilde{u}(x) = \sum_{i=1}^{n} a_i\, p_i(x)\,, n \geq 1 \\ \tilde{v}(x) = \sum_{i=1}^{n} b_i\, p_i(x)\,, n \geq 1 \end{aligned}\right\} \tag{3}$$

where a_i and b_i are parameter, $p_i(x)$ are co-ordinate function which satisfies boundary condition (2). Now apply Galerkin Method in system (1) we get weighted residual system of equations

$$\left.\begin{aligned} \int_0^1 (-\tilde{u}''(x) + q\tilde{u}(x) + r\,\tilde{v}(x))\; p_i(x)\,dx = \int_0^1 f\; p_i(x)\,dx \\ \int_0^1 (-\tilde{v}''(x) + s\,\tilde{v}(x) + t\,\tilde{v}(x))\, p_i(x)\,dx = \int_0^1 g\; p_i(x)\,dx \end{aligned}\right\} \tag{4}$$

Integrating by parts and setting $p_i(x) = 0$ at the boundary $x = 0$ and $x = 1$, then we obtain system of weighted residual equations

$$\left.\begin{aligned} \int_0^1 \left(\tilde{u}'(x)p_i'(x) + q\,\tilde{u}(x)\,p_i(x) + r\,\tilde{v}(x)\,p_i(x)\right) dx = \int_0^1 f\; p_i(x)\,dx \\ \int_0^1 (\tilde{v}'(x)p_i'(x) + s\,\tilde{v}(x)\,p_i(x) + t\;\tilde{u}(x)\,p_i(x)\,)\;dx = \int_0^1 g\; p_i(x)\,dx \end{aligned}\right\} \tag{5}$$

Now putting the representation (3) into (5) we get

$$\int_0^1 \Big(\sum_{j=1}^{n} a_j\, p_j'(x)\ p_i'(x) + \mathrm{q} \sum_{j=1}^{n} a_j\, p_j(x)\ p_i(x) + \mathrm{r} \sum_{j=1}^{n} b_j\, p_j(x)\ p_i(x)\Big)\ dx = \int_0^1 f\ p_i(x)\, dx$$

$$\int_0^1 \Big(\sum_{j=1}^{n} b_j\, p_j'(x)\ p_i'(x) + \mathrm{s} \sum_{j=1}^{n} b_j\, p_j(x)\ p_i(x) + \mathrm{t} \sum_{j=1}^{n} a_j\, p_j(x)\ p_i(x)\ \Big)\, dx = \int_0^1 g\ p_i(x)\, dx$$

Notes

We can write above equations as

$$\sum_{j=1}^{n}\left\{\int_0^1 \left[\left(p_j'(x)\, p_i'(x)\right) a_j + \left(q\, p_j(x)\, p_i(x)\right) a_j + \left(r\, p_j(x)\, p_i(x)\right) b_j\ \right] dx\right\} = \int_0^1 f\ p_i(x) dx$$

$$\sum_{j=1}^{n}\left\{\int_0^1 \left[\left(p_j'(x)\, p_i'(x)\right) b_j + \left(s\, p_j(x)\, p_i(x)\right) b_j + \left(t\, p_j(x)\, p_i(x)\right) a_j\ \right] dx\right\} = \int_0^1 g\ p_i(x) dx$$

where, $i = 1,2,3, \dots, n$

Equivalently,

$$\left.\begin{aligned} \sum_{j=1}^{n} \{A_{j,i}\, a_j + B_{j,i}\, b_j\} &= F_i \\ \sum_{j=1}^{n} \{C_{j,i}\, b_j + D_{j,i}\, a_j\} &= G_i \end{aligned}\right\} \qquad (6)$$

where, $i = 1,2,3, \dots, n$

Where

$$A_{j,i} = \int_0^1 [\left(p_j'(x)\, p_i'(x)\right) + \left(q\, p_j(x)\, p_i(x)\right)] dx$$

$$B_{j,i} = \int_0^1 \left(r\, p_j(x)\, p_i(x)\right) dx\, , F_i = \int_0^1 f\ p_i(x) dx$$

$$C_{j,i} = \int_0^1 [\left(p_j'(x)\, p_i'(x)\right) + \left(s\, p_j(x)\, p_i(x)\right)] dx$$

$$D_{j,i} = \int_0^1 \left(t\, p_j(x)\, p_i(x)\right) dx\, , G_i = \int_0^1 g\ p_i(x) dx$$

where, $i = 1,2,3, \dots, n$

for $i = 1,2, \dots, n$ we get n system of equations, which involve parameters a_i and b_i and which can be obtained by solving system (6). System (6) can be assembled by element matrix contribution.

III. Bernstein Polynomials

The general form of the Bernstein polynomials of n[th] degree over the interval [a,b] is defined by [8-10]

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i\, (b-x)^{n-i}}{(b-a)^n}\, , \qquad a \le x \le b$$

where, $i = 1,2,3, \dots, n$

Note that each of these $n+1$ polynomials having degree n satisfies the following properties:

i. $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$

ii. $\sum_{i=0}^{n} B_{i,n}(x) = 1$

iii. $B_{i,n}(a) = B_{i,n}(b) = 0$, $1 \le i \le n$

The first 11 Bernstein polynomials of degree ten over the interval [0,1], are given below:

i. $B_{0,10}(x) = (1-x)^{10}$

ii. $B_{1,10}(x) = 10(1-x)^9 x$

iii. $B_{2,10}(x) = 45(1-x)^8 x^2$

iv. $B_{3,10}(x) = 120(1-x)^7 x^3$

v. $B_{4,10}(x) = 210(1-x)^6 x^4$

vi. $B_{5,10}(x) = 252(1-x)^5 x^5$

vii. $B_{6,10}(x) = 210(1-x)^4 x^6$

viii. $B_{7,10}(x) = 120(1-x)^3 x^7$

ix. $B_{8,10}(x) = 45(1-x)^2 x^8$

x. $B_{9,10}(x) = 10(1-x)\, x^9$

xi. $B_{10,10}(x) = x^{10}$

All these polynomials satisfy dirichlet boundary conditions. These polynomials and combination of polynomials can be used as trial approximate solutions.

IV. Numerical Example

In this section, we apply the formulation discussed above to solve the system of linear second order BVPs [11]. Consider the following system of equations

$$\left.\begin{array}{l} u''(x) + x\,u(x) + x\,v(x) = f(x) \\ v''(x) + 2x\,v(x) + 2x\,u(x) = g(x) \end{array}\right\} \tag{7}$$

Subject to the boundary conditions

$$u(0) = u(1) = 0, \;\; v(0) = v(1) = 0 \tag{8}$$

where, $0 < x < 1$, $f(x) = 2$ and $g(x) = -2$. The solutions of system (7) are $u(x) = x^2 - x$ and $v(x) = x - x^2$, respectively.

We use combinations of nine Bernstein polynomials as trial approximate solution to solve the system (7). Consider trial approximation solutions of the system (7) are

$$\left.\begin{array}{l} \tilde{u}(x) = \sum_{i=1}^{n} a_i B_{i,10}(x) \\ \tilde{v}(x) = \sum_{i=1}^{n} b_i\, B_{i,10}(x) \end{array}\right\} \tag{9}$$

where, $i = 1,2,3, \dots, n$

Which satisfy boundary condition (8). Where a_i and b_i are unknown parameter. Solving the system (7) by using derived formula in section (2), we get the values of parameters a_i and b_i By putting these parameters in (9) we obtain our desire approximate solutions.

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11. Jungfeng Lu - Variational Iteration Method for Solving a nonlinear System of Second-order Boundary value problems, Computer and Mathematics with Applications, 54(2007), 1133-1138.

Notes

Table 1: Comparison of approximation solution $\tilde{u}\,(x)$ and exact solution $u\,(x)$ of system (7)

Values of x	Exact solution	Approximate solution $\tilde{u}(x)$	Absolute error $\lvert\tilde{u}(x) - u(x)\rvert$
0	0.00000	0.00000	0.00000
0.1	-0.09000	-0.09000	1.44875×10^{-12}
0.2	-0.16000	-0.16000	5.68325×10^{-12}
0.3	-0.21000	-0.21000	9.00676×10^{-12}
0.4	-0.24000	-0.24000	1.64499×10^{-11}
0.5	-0.25000	-0.25000	3.71680×10^{-11}
0.6	-0.24000	-0.24000	6.82071×10^{-11}
0.7	-0.21000	-0.21000	8.63545×10^{-11}
0.8	-0.16000	-0.16000	6.54453×10^{-11}
0.9	-0.09000	-0.09000	1.86649×10^{-11}
1.0	0.00000	0.00000	0.00000

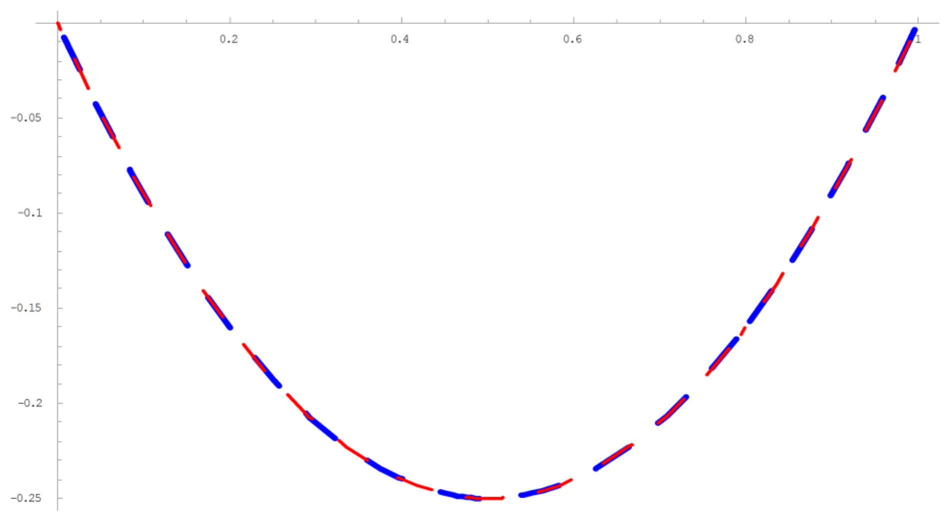

Figure 1: Graphical comparison of Exact solution $\boldsymbol{u}(\boldsymbol{x})$ (- - -) and approximate solution $\tilde{\boldsymbol{u}}\,(\boldsymbol{x})$ (- - -)

Table 2: Comparison of approximation solution $\tilde{\boldsymbol{v}}\,(\boldsymbol{x})$and exact solution $\boldsymbol{v}(\boldsymbol{x})$ of system (7)

Values of x	Exact solution	Approximate solution $\tilde{v}(x)$	Absolute error $\lvert\tilde{v}(x) - v(x)\rvert$
0	0.00000	0.00000	0.00000
0.1	0.09000	0.09000	1.44875×10^{-12}
0.2	0.16000	0.16000	5.68325×10^{-12}
0.3	0.21000	0.21000	9.00676×10^{-12}
0.4	0.24000	0.24000	1.64499×10^{-11}
0.5	0.25000	0.25000	3.71680×10^{-11}
0.6	0.24000	0.24000	6.82071×10^{-11}
0.7	0.21000	0.21000	8.63545×10^{-11}
0.8	0.16000	0.16000	6.54453×10^{-11}
0.9	0.09000	0.09000	1.86649×10^{-11}
1.0	0.00000	0.00000	0.00000

Table 1 and 2 shows the numerical solution and comparison between exact solutions [11] of system of second order BVPs (7). Figure 1 and 2 display comparison with exact and approximate numerical solution. Our numerical approximate solution gives better accuracy compare with other method [11].

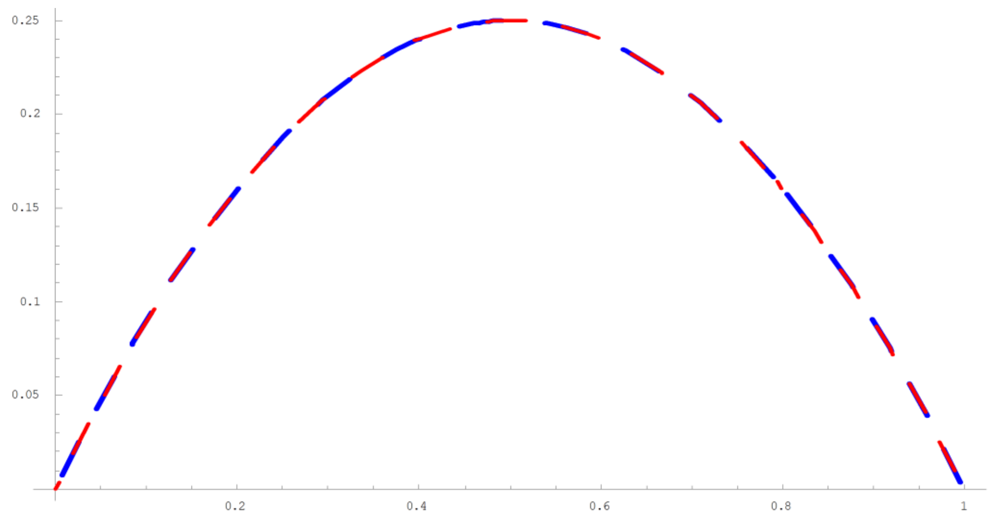

Figure 2: Graphical comparison of Exact solution $\boldsymbol{v}(\boldsymbol{x})$ (- - -) and approximate solution $\widetilde{\boldsymbol{v}}(\boldsymbol{x})$ (- - -).

V. Conclusion

In this paper, a numerical method is developed to solve the system of the system of second order boundary value problems by using the Galerkin method. Developed matrix formulation is a general method which used to solve problems. We introduce a problem in matrix formulation which gives us a better result. The results obtained are very encouraging and this method performs better than other methods.

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