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Matrix Expression of a Complete Flag of Riemannian Extensions on a Manifold

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MATRIXEXPRESSIONOFACOMPLETEFLAGOFRIEMANNIANEXTENSIONSONAMANIFOLD

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I. Introduction

We note above all about that the idea of this paper came to us from the proof of structure theorem of complete Riemannian flags of H. Diallo being in [7].

Let $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$, ..., $(M_q, \mathcal{F}_q)$ be q codimension 1 transversaly orientable Riemannian foliations, let $M = M_1 \times M_2 \times ... \times M_q$, let $\widetilde{\mathcal{F}}_k = \mathcal{F}_1 \times \mathcal{F}_2 \times ... \times \mathcal{F}_k \times M_{k+1} \times M_2 \times ... \times M_q$, let $(U_i^k, f_i^k, T^k, \gamma_{ij}^k)_{i \in I^k}$ be a foliated cocycle defining $(M_k, \mathcal{F}_k)$, $\widetilde{T}^k = T^1 \times T^2 \times ... \times T^k$, let p_s be the projection of M on M_s, let $\overline{p}_s$ be the projection of $\widetilde{T}^q$ on T^s, let $\widetilde{U}_i^k = U_i^1 \times U_i^2 \times ... \times U_i^k \times M_{k+1} \times M_2 \times ... \times M_q$, let $\widetilde{f}_i^k = (f_i^1 \circ p_1, f_i^2 \circ p_2, ..., f_i^k \circ p_k)$ and let $\widetilde{\gamma}_{ij}^k = (\gamma_{ij}^1 \circ \overline{p}_1, \gamma_{ij}^2 \circ \overline{p}_2, ..., \gamma_{ij}^k \circ \overline{p}_k)$.

We easily verify that $\left(\widetilde{U}_i^k, \widetilde{f}_i^k, \widetilde{T}^k, \widetilde{\gamma}_{ij}^k\right)$ is a foliated cocycle defining the codimension k Riemannian foliation $\widetilde{\mathcal{F}}_k$.

We have $\widetilde{\mathcal{F}}_q \subset \widetilde{\mathcal{F}}_{q-1} \subset ... \subset \widetilde{\mathcal{F}}_1$. We say that the sequence $\mathcal{D}_{\widetilde{\mathcal{F}}_q} = \left(\widetilde{\mathcal{F}}_{q-1}, ..., \widetilde{\mathcal{F}}_1\right)$ is a completed flag of Riemannian extension of Riemannian foliation $\widetilde{\mathcal{F}}_q$ on M.

Specifically, being given a codimension q foliation $\mathcal{F}_q$ on a manifold M, a flag of extensions of a foliation $\mathcal{F}_q$ is a sequence $\mathcal{D}_{\mathcal{F}_q}^k = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_k)$ of foliations on M such as $\mathcal{F}_q \subset \mathcal{F}_{q-1} \subset \mathcal{F}_{q-2} \subset ... \subset \mathcal{F}_k$ and each foliation $\mathcal{F}_s$ is a codimension s foliation.

For $k = 1$, the flag of extensions $\mathcal{D}_{\mathcal{F}_q}^k$ will be called complete and will be noted $\mathcal{D}_{\mathcal{F}_q}$.

If each foliation $\mathcal{F}_s$ is Riemannian, the flag of extensions $\mathcal{D}_{\mathcal{F}_q}^k$ will be called flag of Riemannian extensions of $\mathcal{F}_q$.

That said, is denoted by X_k the unitary field of T^k orienting T^k and $(X_k)^{h_k}$ the lifted of X_k on the tangent bundle $T\widetilde{T}^q$ of $\widetilde{T}^q$.

One checks easily [10] that $\left[(X_k)^{h_k}, (X_s)^{h_s}\right] = 0$ for $k \neq s$. There is thus obtained a coordinates system $(x_1, ..., x_q)$ on $\widetilde{T}^q$ such as $\frac{\partial}{\partial x_k} = (X_k)^{h_k}$. As each γ_{ij}^k is an local isometry of T^k then relative to the coordinates system $(x_1, ..., x_q)$ the Jacobian matrix $J_{\widetilde{\gamma}_{ij}^k}$ of $\widetilde{\gamma}_{ij}^k$ checked:

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Ref

10. R. Nasri and M. Djaa, 2006. *"Sur la courbure des variétés riemanniennes produits"*, Laboratoire de géométrie.Centre universitaire de Saida, Algérie. M.S.C.2000:53C50-53C42. Sciences et Technologie A-N°24; Decembre, pp.15-20.

$$J_{\gamma_{ij}^k} = \begin{pmatrix} \varepsilon_{ij}^1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \varepsilon_{ij}^2 & 0 & \dots & \dots & 0 \\ 0 & 0 & \varepsilon_{ij}^3 & 0 & \dots & \vdots \\ \vdots & \vdots & 0 & . & . & \vdots \\ \vdots & \vdots & \vdots & . & . & 0 \\ 0 & 0 & 0 & \dots & 0 & \varepsilon_{ij}^k \end{pmatrix} \text{ where } \varepsilon_{ij}^r = \pm 1.$$

From the foregoing the foliation $\widetilde{\mathcal{F}}_k$ will be said Riemannian transversaly diagonal.

Specifically a foliation $\mathcal{F}$ on a manifold N is said transversely diagonal if and only if it is defined by a foliated cocycle $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ such as the opens U_i are $\mathcal{F}$ -distinguished and on each open $f_i(U_i)$ it exists a local $\mathcal{F}$-transverse coordinates system $\left(y_q^i, y_{2,\dots,}^i y_1^i\right)$ such as relatively to local $\mathcal{F}$-transverse coordinates systems $\left(\left(y_q^i, y_{2,\dots,}^i y_1^i\right)\right)_{i\in I}$ on the opens $f_i(U_i)$,

the Jacobian matrix $J_{\gamma_{ij}}$ of γ_{ij} is diagonal. In the case where there exists a metric h_T on the transverse manifold T such as the γ_{ij}^s are local isometries for this transverse metric, we say that $\mathcal{F}$ is Riemannian transversaly diagonal.

The primary purpose of this paper is to show that the existense of a transversaly diagonal foliation $\mathcal{F}$ on a manifold implies the existence of a complete flag of extensions of $\mathcal{F}$. The second purpose of this paper is to prove the existence of Riemannian transversely diagonal foliation *nontrivial*. Indeed, we show that if $\mathcal{F}_q$ is a codimension q Riemannian foliation having a complete flag of Riemannian extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, \dots, \mathcal{F}_1)$ on a connected manifold N and if there exists a metric h that is bundlelike for any foliation $\mathcal{F}_s$ of this flag then each foliation $\mathcal{F}_s$ is Riemannian transversely diagonal.

In all that follows, the manifolds considered are supposed connected and differentiability is C^∞.

II. Reminders

In this paragraph, we reformulate in the direction that is helpful to us some definitions and theorems that are in ([2],[3], [5], [7], [8], [10]).

Definition 2.1 Let M be a manifold.

An extension of a codimension q foliation $(M, \mathcal{F})$ is a codimension q' foliation $(M, \mathcal{F}')$ such that $0 < q' < q$ and $(M, \mathcal{F}')$ leaves are $(M, \mathcal{F})$ leaves meetings (it is noted $\mathcal{F} \subset \mathcal{F}'$).

We show ([2], [6]) that if $(M, \mathcal{F}')$ is a simple extension of a simple foliation $(M, \mathcal{F})$ and if $(M, \mathcal{F})$ and $(M, \mathcal{F}')$ are defined respectively by submersions $\pi : M \to T$ and $\pi' : M \to T'$, then there exists a submersion $\theta : T \to T'$ such that $\pi' = \theta \circ \pi$.

We say that the submersion θ is a bond between the foliation $(M, \mathcal{F})$ and its extension foliation $(M, \mathcal{F}')$.

It is shown in [3] that if the foliation $(M, \mathcal{F})$ and its extension $(M, \mathcal{F}')$ are defined respectively by the cocycles $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ and $(U_i, f'_i, T', \gamma'_{ij})_{i\in I}$ then we have

$$f'_i = \theta_i \circ f_i \text{ and } \gamma'_{ij} \circ \theta_j = \theta_i \circ \gamma_{ij}$$

where θ_s is a bond between the foliation $(U_s, \mathcal{F})$ and its extension foliation $(U_s, \mathcal{F}')$.

Proposition 2.2 Given a foliation $(M,\mathcal{F})$ having T for model transverse foliation, let T' be a dimension $q'>0$ manifold. If the local diffeomorphisms of transition of $\mathcal{F}$ preserve the fibers of a submersion of T on T', then the foliation $\mathcal{F}$ admits a codimension q' extension having T' for model transverse manifold.

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages"*. Thèse unique, Université de Cocody, Abidjan.

The following theorem is demonstrated in the same way that the structure theorem of complete Riemannian flags being in [7] :

Theorem 2.3 *Let $(M; h)$ be a connected Riemannian manifold not necessarily compact and let $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$ be a complete flag of riemannian extensions of a codimension q Riemannian foliation $\mathcal{F}_q$ having T^q for transverse manifold and having h for bundlelike metric.*

If the metric h is bundlelike for any foliation of $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$ then:

1) Each foliation $\mathcal{F}_k$ is transversaly parallelizable. The vector fields of parallelism $\mathcal{F}_k-$transverse $\left(\overline{Y}_s\right)_{0\leq s\leq k-1}$ are orthogonal. For $s\neq 0$, each vector field $\overline{Y}_s$ is an unitary section of $(T\mathcal{F}_{s+1})^{\perp} \cap (T\mathcal{F}_s)$ and $\overline{Y}_0$ is an unitary section of $(T\mathcal{F}_1)^{\perp}$ where $(T\mathcal{F}_k)^{\perp}$ is the orthogonal bundle of $T\mathcal{F}_k$. Additionally each vector field $\overline{Y}_s$ directs the flow it generates.

2) The induced parallelism $(Y_s)_{0\leq s\leq q-1}$ of T^q by $\left(\overline{Y}_s\right)_{0\leq s\leq q-1}$ satisfies the equality $[Y_s, Y_r] = k_{sr}Y_s$ for $q-1 \geq s > r \geq 0$. Functions k_{sr} are called structure functions of $\mathcal{D}_{\mathcal{F}_q}$.

Note that the parallelism $(Y_s)_{0\leq s\leq q-1}$ of T^q will be said parallelism $\mathcal{F}_q-$transverse of Diallo associated to $\mathcal{D}_{\mathcal{F}_q}$.

We note also, relatively to the transverse induced metric h_T by h on T^q, that vectors fields Y_s are unitary and orthogonal two by two.

We end these reminders by the following proposition being in [10]. It will allow us to construct local coordinate systems in the proof of the theorem 3.2 which is the main theorem of this paper in the following paragraph.

Proposition 2.4 *Let $M \times N$ be the product of two manifolds M and N, let $X_i \in \mathcal{X}(M)$ and let $Y_j \in \mathcal{X}(N)$ then*

$$\left[X_1^{h_1}, X_2^{h_1}\right] = [X_1, X_2]^{h_1}, \left[Y_1^{h_2}, Y_2^{h_2}\right] = [Y_1, Y_2]^{h_2} \text{ and } \left[X_1^{h_1}, Y_2^{h_2}\right] = 0$$

where

$$\begin{array}{cccc} R^{h_1}: & T_xM & \rightarrow & T_{(x,y)}M\times N \\ & u & \mapsto & u^{h_1} = (u,0) \end{array} \text{ and } \begin{array}{cccc} R^{h_2}: & T_xN & \rightarrow & T_{(x,y)}M\times N \\ & w & \mapsto & w^{h_2} = (0,w) \end{array}.$$

III. Matrix Expression of a Complete Flag of Riemannian Extensions on a Manifold

There is a link between transversaly diagonal foliations and complete flags of extensions.

Specifically we have:

Proposition 3.1 *Let $\mathcal{F}_q$ be a codimension q transversaly diagonal foliation on a manifold M.*

Then $\mathcal{F}_q$ admits a complete flag of extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$.

Proof. Let $(U_i^q, f_i^q, T^q, \gamma_{ji}^q)_{i\in I}$ be a foliated cocyle defining the transversaly diagonal foliation $\mathcal{F}_q$ and let $\left((y_q^i, y_{q-1,...,}^i y_1^i)\right)_{i\in I}$ be $\mathcal{F}_q-$transverse coordinates systems on opens $f_i^q\left(U_i^q\right)$ such as the Jacobian matrix $J_{\gamma_{ij}^q}$ is diagonal. Let also

$$P^{qi}(x) = \overset{q-1}{\underset{k=1}{\oplus}} < \frac{\partial}{\partial y_k^i}(x) >$$

be the integrable differential system on $f_i^q(U_i^q)$ and let $\mathcal{U}_i^{q-1}$ be a leaf of this differential system.

It is clear that the foliation defined by the integrable differential system $x \rightarrow P^{qi}(x)$ is transverse for the flow $\mathcal{F}_{\partial y_q^i}$ of $\frac{\partial}{\partial y_q^i}$.

Ref

10. R. Nasri and M. Djaa, 2006. *"Sur la courbure des variétés riemanniennes produits"*, Laboratoire de géométrie.Centre universitaire de Saida, Algérie. M.S.C.2000:53C50-53C42. Sciences et Technologie A-N° 24; Decembre, pp.15-20.

That said, quits to reduce the "size" of opens U_i^q, it may be considered an open recovering $(U_i^q)_{i\in I}$ of M such as in each $f_i^q\left(U_i^q\right)$ the flow $\mathcal{F}_{\partial y_q^i}$ of $\frac{\partial}{\partial y_q^i}$ and the integrable differential system $x\rightarrow P^{qi}\left(x\right)$ define simple foliations so that $\mathcal{U}_i^{q-1}$ is diffeomorphic to quotient manifold of simple foliation $\mathcal{F}_{\partial y_q^i}$.

Let $\theta_i^q : f_i^q\left(U_i^q\right)\rightarrow\mathcal{U}_i^{q-1}$ be the projection on $\mathcal{U}_i^{q-1}$ following the flow $\mathcal{F}_{\partial y_q^i}$ of $\frac{\partial}{\partial y_q^i}$.

The manifold T^q can be regarded as a disjoint union of $f_i^q\left(U_i^q\right)$. Therefore we can say that submersions θ_i^q defines a submersion θ^q on T^q whose restriction to each $f_i^q\left(U_i^q\right)$ is θ_i^q.

Note that $J_{\gamma_{ji}^q}=\left(\lambda_{jirs}^q\right)_{rs}$ being a invertible and diagonal matrix has all its diagonal elements non-zero.

As

$$\left(\gamma_{ji}^q\right)_*\left(\frac{\partial}{\partial y_q^i}\right)=\left(\lambda_{jirs}^q\right)_{rs}\left(\frac{\partial}{\partial y_q^i}\right)=\lambda_{ji11}^q.\frac{\partial}{\partial y_q^j}\quad\text{and } \lambda_{ji11}^q\neq 0$$

then the γ_{ji}^q preserve the fibers of the submersion θ^q.

It follows from this ([3], [5])(*cf.prop.2.2*) that the codimension q foliation $\mathcal{F}_q$ have an codimension $q-1$ extension $\mathcal{F}_{q-1}$.

We set

$$f_i^{q-1}=\theta_i^q\circ f_i^q \text{ and } T^{q-1}=\underset{i\in I}{\cup}\mathcal{U}_i^{q-1}\text{ and } f_r^{q-1}\left(U_r^q\cap U_s^q\right)=\mathcal{V}_{rs}^{q-1}$$

for $U_r^q\cap U_s^q\neq\emptyset$.

As γ_{ji}^q preserves the fibers of the submersion θ^q then γ_{ji}^q induces a local diffeomorphism $\gamma_{ji}^{q-1}:\mathcal{V}_{ij}^{q-1}\rightarrow\mathcal{V}_{ji}^{q-1}$ and this diffeomorphism checks ([3], [5]) (*cf def.2.1*) the equality

$$\gamma_{ji}^{q-1}\circ\theta_i^q=\theta_j^q\circ\gamma_{ji}^q.$$

We easily verify that $(U_i^q,f_i^{q-1},T^{q-1},\gamma_{ji}^{q-1})_{i\in I}$ is a foliated cocycle defining the extension $\mathcal{F}_{q-1}$ of $\mathcal{F}_q$.

We now show that the foliation $\mathcal{F}_{q-1}$ is transversaly diagonal.

We have $\left[\frac{\partial}{\partial y_q^i},\frac{\partial}{\partial y_k^i}\right]=0$ for all k. Hence the $q-1$ vectors fields $\frac{\partial}{\partial y_{q-1}^i},\frac{\partial}{\partial y_{q-2}^i},...,\frac{\partial}{\partial y_1^i}$ are foliated for the flow of $\frac{\partial}{\partial y_q^i}$. Therefore $\left(\theta^q\right)_*\left(\frac{\partial}{\partial y_k^i}\right)$ is a vectors field on $\mathcal{U}_i^{q-1}$ for all $k\leq q-1$.

But the $q-1$ vector fields $\frac{\partial}{\partial y_{q-1}^i},\frac{\partial}{\partial y_{q-2}^i},...,\frac{\partial}{\partial y_1^i}$ are tangent to $\mathcal{U}_i^{q-1}$ at any point in $\mathcal{U}_i^{q-1}$ so for $k\neq q$ and $a\in f_i^q\left(U_i^q\right)$ we have $\left(\theta_i^q\right)_{*a}\left(\frac{\partial}{\partial y_k^i}\right)=\frac{\partial}{\partial y_k^i}\left(\theta_i^q\left(a\right)\right)$.

Therefore the $q-1$ vector fields $\frac{\partial}{\partial y_{q-1}^i},\frac{\partial}{\partial y_{q-2}^i},...,\frac{\partial}{\partial y_1^i}$ define a coordinates system $\mathcal{F}_{\partial y_q^i}$-transverse on $\mathcal{U}_i^{q-1}$ and this coordinates system is the restriction to $\mathcal{U}_i^{q-1}$ of $\left(y_q^i,y_{q-1,...,}^iy_1^i\right)$. So it will be noted yet $\left(y_{q-1}^i,...,y_1^i\right)$.

For clarity in the presentation we note for following $\left(\theta_i^q\right)_*\left(\frac{\partial}{\partial y_k^i}\right)=\frac{\partial}{\partial y_k^i}{}_{/\mathcal{U}_i^{q-1}}$.

Using equality $\gamma_{ji}^{q-1}\circ\theta_i^q=\theta_j^q\circ\gamma_{ji}^q$ we obtain for $k\neq q$,

$$\begin{aligned}\left(\gamma_{ji}^{q-1}\right)_*\left(\frac{\partial}{\partial y_k^i}{}_{/\mathcal{U}_i^{q-1}}\right) &= \left(\gamma_{ji}^{q-1}\right)_*\circ\left(\theta_i^q\right)_*\left(\frac{\partial}{\partial y_k^i}\right)\\ &= \left(\theta_j^q\right)_*\circ\left(\gamma_{ji}^q\right)_*\left(\frac{\partial}{\partial y_k^i}\right)\end{aligned}$$

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages"*. Thèse unique, Université de Cocody, Abidjan.

$$= \left(\theta_j^q\right)_* \left(\lambda^q_{ij(q-k+1)(q-k+1)} \cdot \frac{\partial}{\partial y_k^j}\right)$$

$$= \lambda^q_{ij(q-k+1)(q-k+1)} \left(\theta_j^q\right)_* \left(\frac{\partial}{\partial y_k^j}\right)$$

$$= \lambda^q_{ij(q-k+1)(q-k+1)} \cdot \frac{\partial}{\partial y^j_k}_{/\mathcal{U}_j^{q-1}} .$$

Notes

Thus the foliation $\mathcal{F}_{q-1}$ is transversaly diagonal.

Before closing we note that equality

$$\gamma_{ji}^{q-1} \circ \theta_i^q = \theta_j^q \circ \gamma_{ji}^q \text{ show that } J_{\gamma_{ji}^{q-1}} . J_{\theta_i^q} = J_{\theta_j^q} . J_{\gamma_{ji}^q}$$

where $J_{\theta_s^q}$ is the Jacobian matrix of θ_s^q. But

$$J_{\theta_s^q} = \begin{pmatrix} 0 & 0 & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & 0 \\ 0 & 0 & 1 & 0 & ... & : \\ : & : & 0 & . & . & : \\ : & : & : & . & . & 0 \\ 0 & 0 & 0 & .... & 0 & 1 \end{pmatrix}$$

so $J_{\gamma_{ij}^{q-1}}$ is obtained by removing the first line and the first collonne of $J_{\gamma_{ij}^q}$.

We are constructing $\mathcal{F}_{q-2}$, $\mathcal{F}_{q-3}$, ... , $\mathcal{F}_1$ using the same technical of construction of $\mathcal{F}_{q-1}$.

We note that each foliation $\mathcal{F}_k$ will be defined by $(U_i^q, f_i^k, T^k, \gamma_{ji}^k)_{i\in I}$ a foliated cocycle where θ_i^k is defined in the same way as θ_i^q, $f_i^{k-1} = \theta_i^k \circ f_i^k$, $T^k = \bigcup_{i\in I} \mathcal{U}_i^k$ with $\mathcal{U}_i^k$ defined in the same way as $\mathcal{U}_i^{q-1}$ and $\gamma_{ji}^{k-1} \circ \theta_i^k = \theta_j^k \circ \gamma_{ji}^k$.

We note that if the transversaly diagonal foliation $\mathcal{F}_q$ is Riemannian relatively to a transverse metric h_T and if local vector fields $\frac{\partial}{\partial y_k^i}$ are Killing vector fields for h_T for all i and all k then the foliations of complete flag of extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$ are Riemannian transversaly diagonal and there existe a common metric bundlelike for any foliation $\mathcal{F}_k$. Indeed in this case the submersions θ_i^k are Riemannian submersions for all i and all k. And, the equality $\gamma_{ji}^{k-1} \circ \theta_i^k = \theta_j^k \circ \gamma_{ji}^k$ implies that γ_{ji}^{k-1} is a local isometry once γ_{ji}^k is a local isometry.

Under certain conditions specified in Theorem 3.2, the previous proposal admits a reciprocal.

In the proof of the following result is given matrix expression of a complete flag of Riemannian extensions of a Riemannian foliation .

Theorem 3.2 *Let $\mathcal{F}_q$ be a codimension q Riemannian foliation on a connected manifold M and h a metric $\mathcal{F}_q-$bundlelike on M.*

If $\mathcal{F}_q$ admits a complete flag of Riemannian extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, ..., \mathcal{F}_1)$ such as the métric h is bundlelike for any Riemannian foliation $\mathcal{F}_k$ then each foliation $\mathcal{F}_k$ is Riemannian transversaly diagonal.

Proof. We assume that the foliation $\mathcal{F}_q$ admits a complete flag of Riemannian extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$ such as the metric h is bundlelike for any Riemannian foliation $\mathcal{F}_k$.

We denote by T^q a transverse manifold of $\mathcal{F}_q$.

According to the theorem 2.3 each foliation $\mathcal{F}_k$ is transversaly parallelizable. The vector fields of $\mathcal{F}_k-$transverse parallelism $\left(\overline{Y}_s\right)_{0\leq s\leq k-1}$ are orthogonal. For s$\neq$ 0, each vector fields $\overline{Y}_s$ is an unitary section of $(T\mathcal{F}_{s+1})^{\perp}\cap(T\mathcal{F}_s)$ and $\overline{Y}_0$ is an unitary section of $(T\mathcal{F}_1)^{\perp}$ where $(T\mathcal{F}_1)^{\perp}$ is the orthogonal bundle of $T\mathcal{F}_1$. Additionally each vector field $\overline{Y}_s$ directs the flow it generates and the induced parallelism $(Y_s)_{0\leq s\leq q-1}$ of T^q by $\left(\overline{Y}_s\right)_{0\leq s\leq q-1}$ satisfies the equality $[Y_s, Y_r] = k_{sr}Y_s$ for $q-1\geq s>r\geq 0$.

We have $T\left(T^q\right)=\underset{s=0}{\overset{q-1}{\oplus}} < Y_s >$ where $T\left(T^q\right)$ is the tangent bundle of T^q and $< Y_s >$ is the tangent bundle of flow of Y_s.

The differential system

$$S^k\left(x\right)=\underset{\substack{s=0\\ s\neq k}}{\overset{q-1}{\oplus}} < Y_s\left(x\right) >$$

is integrable because $[Y_s, Y_r] = k_{sr}Y_s$ for $s>r$. It then defines a foliation $\mathcal{S}_k$.

For $k\neq r$, one checks easily that $\mathcal{S}_r$ is an extension of flow $\mathcal{F}_{Y_k}$ of unitary vector field Y_k. On the other side $\mathcal{F}_{Y_k}$ is transverse for the foliation $\mathcal{S}_k$ because for all $x\in T^q$, $T_x\left(T^q\right)=T_x\mathcal{S}_k\oplus < Y_k\left(x\right) >$.

That said, for all $x_i\in T^q$ there exists an open V_i of T^q containing x_i, distinguished for each foliation $\mathcal{S}_k$ and for each flow $\mathcal{F}_{Y_k}$.

Let s_k^i be a submersion defining $\mathcal{S}_k$ on V_i .

As $\mathcal{F}_{Y_k}$ is transverse to the foliation $\mathcal{S}_k$ and $\dim\left(\mathcal{F}_{Y_k}\right)=co\dim\left(\mathcal{S}_k\right)$ then the open V_i can be chosen such that for all maximal plaques (for the natural relation of inclusion) $L^i_{Y_k}$ and $P^i_{Y_k}$ for $\mathcal{F}_{Y_k}$ contained in V_i we have $s_k^i\left(L^i_{Y_k}\right)=s_k^i\left(V_i\right)=s_k^i\left(P^i_{Y_k}\right)$.

Thus, we can assume that $s_k^i\left(V_i\right)=L^i_{Y_k}$ because $s_k^i\left(L^i_{Y_k}\right)$ is diffeomorphic to $L^i_{Y_k}$.

It follows from the foregoing that the application $\mu^i : V_i\rightarrow L^i_{Y_{q-1}}\times...\times L^i_{Y_0}$ such as for all $x\in V_i$, $\mu^i\left(x\right)=\left(s^i_{q-1}\left(x\right), s^i_{q-1}\left(x\right),..., s^i_0\left(x\right)\right)$ is a diffeomorphism.

It is easy to verify that the leaves of $\left(\left(\mu^i\right)^{-1}\right)^*\left(\mathcal{F}_{Y_k/V_i}\right)$ are of the form $\{p\}\times L^i_{Y_k}\times\{p'\}$ where $p\in L^i_{Y_{q-1}}\times L^i_{Y_{q-2}}\times...\times L^i_{Y_{k+1}}$ and $p'\in L^i_{Y_{k-1}}\times L^i_{Y_{k-2}}\times...\times L^i_{Y_0}$.

This being, is considered in what follows a recovery $(U_i)_{i\in I}$ of opens of the manifold M such as each open U_i is $\mathcal{F}_k-$distinguished for each k and if $(U_i, f^q_i, T^q, \gamma^q_{ji})_{i\in I}$ is a foliated cocycle defining $\mathcal{F}_q$ then there exists a diffeomorphism $\mu^i_q : f^q_i\left(U_i\right)\rightarrow L^i_{Y_{q-1}}\times L^i_{Y_{q-2}}\times...\times L^i_{Y_0}$ such as the leaves of $\left(\left(\mu^i_q\right)^{-1}\right)^*\left(\mathcal{F}_{Y_k/V_i}\right)$ are of the form $\{p\}\times L^i_{Y_k}\times\{p'\}$ where $p\in L^i_{Y_{q-1}}\times L^i_{Y_{q-2}}\times...\times L^i_{Y_{k+1}}$ and $p'\in L^i_{Y_{k-1}}\times L^i_{Y_{k-2}}\times...\times L^i_{Y_0}$.

Let h_T be the metric $\mathcal{F}_q$-transverse associated to the metric $\mathcal{F}_q-$bundlelike h and let Y^i_{Lk} be the unitary vector field tangent to the leaf $L^i_{Y_k}$ induced by the unitary vector field Y_k of $\mathcal{F}_q-$transverse parallelism $(Y_s)_{0\leq s\leq q-1}$ of Diallo of T^q.

We note in passing that the vector fields Y^i_{Lk} are orthogonal two by two relatively to the metric h_T.

We now consider $a=\left(a^i_{q-1},..., a^i_0\right)\in L^i_{Y_{q-1}}\times...\times L^i_{Y_0}$.

We set

$$\begin{array}{rccc} R^{h_k}_a : & T_{a^i_k}L^i_{Y_k} & \rightarrow & T_{a^i_{q-1}}L^i_{Y_{q-1}}\times...\times T_{a^i_0}L^i_{Y_0}\\ & X^i_{a^i_k} & \mapsto & \left(X^i_{a^i_k}\right)^{h_k}_{a^{qi}}=\left(0_{a^i_{q-1}},...,0_{a^i_{k+1}}, X^i_{a^i_k}, 0_{a^i_{k-1}},...,0_{a^i_0}\right)\end{array}$$

Notes

where $0_{a_t^i}$ is the null vector of $T_{a_t^i}L^i_{Y_t}$.

Let $\mathcal{X}\left(L^i_{Y_k}\right)$ be the Lie algebra of vector fields tangent to $L^i_{Y_k}$ and let $X_k^i \in \mathcal{X}\left(L^i_{Y_k}\right)$.

By varying $a_k^i \in L^i_{Y_k}$, the lifted $R_a^{h_k}\left(\left(X_k^i\right)_{a_k^i}\right)$ of $\left(X_k^i\right)_{a_k^i}$ in $TL^i_{Y_{q-1}} \times ... \times TL^i_{Y_0}$ define a vector field on $L^i_{Y_{q-1}} \times ... \times L^i_{Y_0}$. We will note it by $\left(X_k^i\right)^{h_k}$. So, we have $\left(X_k^i\right)_a^{h_k} = \left(X_k^i\right)^{h_k}(a) = R_a^{h_k}\left(\left(X_k^i\right)_{a_k^i}\right)$.

Is shown in [10] $(cf.prop.2.4)$ that for $X_k^i \in \mathcal{X}\left(L^i_{Y_k}\right)$ and $X_s^i \in \mathcal{X}\left(L^i_{Y_s}\right)$ we have $\left[\left(Y_k^i\right)^{h_k}, \left(Y_s^i\right)^{h_s}\right] = 0$ for $k \neq s$. It follows from this, Y^i_{Lr} being the unitary vector field on $L^i_{Y_r}$ induced by Y_r of $\mathcal{F}_q-$transverse parallelism $\left(Y_s\right)_{0\leq s\leq q-1}$ of Diallo, that $\left[\left(Y^i_{Lk}\right)^{h_k}, \left(Y^i_{Ls}\right)^{h_s}\right] = 0$ for $k \neq s$.

Thus, $\mu_q^i : f_i^q\left(U_i\right) \to L^i_{Y_{q-1}} \times L^i_{Y_{q-2}} \times ... \times L^i_{Y_0}$ being a diffeomorphism,the vector fields $\left(\left(\mu_q^i\right)^{-1}\right)^*\left(\left(Y^i_{Ls}\right)^{h_s}\right)$ define on $f_i^q\left(U_i\right)$ a coordinated system $\mathcal{F}_q-$transverse.

In the following $\left(\left(\mu_q^i\right)^{-1}\right)^*\left(\left(Y^i_{Ls}\right)^{h_s}\right)$ is noted $\frac{\partial}{\partial y_s^i}$.

We note in passing that if we denote by Y_s^i the restriction of Y_s at $f_i^q\left(U_i\right)$ and $p_k^i : L^i_{Y_{q-1}} \times L^i_{Y_{q-2}} \times ... \times L^i_{Y_0} \to L^i_{Y_k}$ the projection on $L^i_{Y_k}$ then:

1) We have $\left(p_k^i\right)_*\left(\left(Y_k^i\right)^{h_k}\right) = Y^i_{Lk}$ and $\left(p_k^i\right)_*\left(\left(Y_s^i\right)^{h_s}\right) = 0$ for $s \neq k$.

2) The projections $\left(p_s^i\right)_*$ not being necessarily Riemannian submersions. Therefore the vector fields $\left(Y_s^i\right)^{h_s}$ are not necessarily unitary (relatively to the metric $\left(\left(\mu_q^i\right)^{-1}\right)^* h_T$ on the product $L^i_{Y_{q-1}} \times ... \times L^i_{Y_0}$) despite the fact that Y^i_{Ls} is unitary on $L^i_{Y_s}$.

3) For all $p \in L^i_{Y_{q-1}} \times ... \times L^i_{Y_{s+1}}$ and $p' \in L^i_{Y_{s-1}} \times ... \times L^i_{Y_0}$ we have $\left(\mu_q^i\right)_*\left(Y_s^i\right)$ which is tangent to $\{p\} \times L^i_{Y_s} \times \{p'\}$. Hence the vector fields $\left(Y^i_{Ls}\right)^{h_s}$ and $\left(\mu_q^i\right)_*\left(Y_s^i\right)$ are collinear. And that implies that $\frac{\partial}{\partial y^i_s}$ and Y_s^i are also collinear.

4) Y_s^i coincides with $\frac{\partial}{\partial y_s^i}$ on $L^i_{Y_s}$ however, the values of these two fields on $V_i^q \backslash L^i_{Y_s}$ are not necessarily identical where $f_i^q\left(U_i\right) = V_i^q$.

One checks easily that the vector fields $\frac{\partial}{\partial y_s^i}$ on $f_i^q\left(U_i\right) = V_i^q$ are orthogonal two by two.

Let $\mathcal{F}_k^i = \mathcal{F}_{k/U_i}$ and $\mathcal{D}_{\mathcal{F}_q^i} = \left(\mathcal{F}^i_{q-1}, \mathcal{F}^i_{q-2}, ..., \mathcal{F}^i_1\right)$ the restriction of $\mathcal{D}_{\mathcal{F}_q} = \left(\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1\right)$ at the open U_i.

The flag $\mathcal{D}_{\mathcal{F}_q^i} = \left(\mathcal{F}^i_{q-1}, \mathcal{F}^i_{q-2}, ..., \mathcal{F}^i_1\right)$ is projected on $f_i^q\left(U_i\right) = V_i^q$ following the leaves of $\mathcal{F}_q^i$ in a complete Riemannian flag $\overline{\mathcal{D}}_i = \left(\overline{\mathcal{F}}^i_1, \overline{\mathcal{F}}^i_2, ..., \overline{\mathcal{F}}^i_{q-1}\right)$ where $\overline{\mathcal{F}}^i_k = f_i^q\left(\mathcal{F}^i_{q-k}\right)$.

We have $T\overline{\mathcal{F}}^i_k = \overset{q-1}{\underset{s=k}{\oplus}} < Y_s^i >= \overset{q-1}{\underset{s=k}{\oplus}} < \frac{\partial}{\partial y_s^i} >$. Thus on each open U_i the submersion $p^{iq}_{k-1} \circ \mu_q^i \circ f_i^q$ defines the foliation $\mathcal{F}_k^i$ where p^{iq}_{k-1} is the projection of $L^i_{Y_{q-1}} \times ... \times L^i_{Y_0}$ on $L^i_{Y_{k-1}} \times L^i_{Y_{k-2}} \times ... \times L^i_{Y_0}$.

Let θ_i^{qk} be the Riemannian bond between the Riemannian foliations $\mathcal{F}_q^i$ and $\mathcal{F}_k^i$, let θ_i^{k-1} be the Riemannian bond between the Riemannian foliations $\mathcal{F}_k^i$ and $\mathcal{F}^i_{k-1}$ and let $p^{i(k-1)}_{k-2}$ be the projection of $L^i_{Y_{k-1}} \times L^i_{Y_{k-2}} \times ... \times L^i_{Y_0}$ on $L^i_{Y_{k-2}} \times ... \times L^i_{Y_0}$.

Ref

10. R. Nasri and M. Djaa, 2006. *"Sur la courbure des variétés riemanniennes produits"*, Laboratoire de géométrie.Centre universitaire de Saida, Algérie. M.S.C.2000:53C50-53C42. Sciences et Technologie A-N°24; Decembre, pp.15-20.

We note [3] that fibers of θ_i^{qk} are leaves of foliation $\overline{\mathcal{F}}_k$ and fibers of θ_i^{k-1} are leaves of flow $\mathcal{F}_{\partial y_{k-1}^i}$ of $\frac{\partial}{\partial y_{k-1}^i}$.

For all s, it follows from the above that there exists a diffeomorphism $\mu_s^i : f_i^s(U_i) \to L^i_{Y_{s-1}} \times L^i_{Y_{s-2}} \times ... \times L^i_{Y_0}$ making the following diagram commutative

$$\begin{array}{ccccc} U_i & \overset{f_i^q}{\rightarrow} & f_i^q(U_i) & \overset{\mu_q^i}{\rightarrow} & L^i_{Y_{q-1}} \times ... \times L^i_{Y_0} \\ Id \downarrow & & \downarrow \theta_i^{qk} & & \downarrow p_{k-1}^{iq} \\ U_i & \overset{f_i^k}{\rightarrow} & f_i^k(U_i) & \overset{\mu_k^i}{\rightarrow} & L^i_{Y_{k-1}} \times ... \times L^i_{Y_0} \\ Id \downarrow & & \downarrow \theta_i^{k-1} & & \downarrow p_{k-2}^{i(k-1)} \\ U_i & \overset{f_i^{k-1}}{\rightarrow} & f_i^{k-1}(U_i) & \overset{\mu_{k-1}^i}{\rightarrow} & L^i_{Y_{k-2}} \times ... \times L^i_{Y_0} \end{array} \quad (*).$$

Let $a_k^i \in L^i_{Y_{q-1}} \times ... \times L^i_{Y_k}$ and $\tau_{a_k^i}$ the immersion of $L^i_{Y_{k-1}} \times ... \times L^i_{Y_0}$ in $L^i_{Y_{q-1}} \times ... \times L^i_{Y_0}$ such as $\tau_{a_k^i}(b) = a_k^i \times b$ for all $b \in L^i_{Y_{k-1}} \times ... \times L^i_{Y_0}$.

We have $\left(\mu_q^i\right)^{-1} \circ \tau_{a_k^i} \circ \mu_k^i$ which is an immersion of $f_i^k(U_i)$ in $f_i^q(U_i)$.

Thus, quits to replace f_i^k by $\left(\mu_q^i\right)^{-1} \circ \tau_{a_k^i} \circ \mu_k^i \circ f_i^k$, we can suppose that $f_i^k(U_i)$ is an immersed submanifold of $f_i^q(U_i)$ and this submanifold is a leaf of differential integrable system $P^k(x) = \overset{k-1}{\underset{s=0}{\oplus}} < Y_s^i(x) >= \overset{k-1}{\underset{s=0}{\oplus}} < \frac{\partial}{\partial y_s^i}(x) >$ on $f_i^q(U_i)$.

Using the same arguments one can assume that $f_i^1(U_i) \subset f_i^2(U_i) \subset .... \subset f_i^q(U_i)$.

That said, it is assumed in what follows that $f_i^1(U_i) \subset f_i^2(U_i) \subset .... \subset f_i^q(U_i)$ and $f_i^k(U_i)$ is a leaf of differential integrable system $x \to P^k(x)$ on $f_i^q(U_i)$.

We have $\left[\frac{\partial}{\partial y_s^i}, \frac{\partial}{\partial y_k^i}\right] = 0$ for all (k, s) and $T\overline{\mathcal{F}}_k^i = \overset{q-1}{\underset{s=k}{\oplus}} < \frac{\partial}{\partial y_s^i} >$. Hence the vector fields $\frac{\partial}{\partial y_s^i}$ are foliate for the foliation $\overline{\mathcal{F}}_k^i$ for $s < k$. Therefore $\left(\theta_i^{qk}\right)_* \left(\frac{\partial}{\partial y_s^i}\right)$ is a vector field on $f_i^k(U_i)$ for all $s < k$ because θ_i^{qk} is the projection of $f_i^q(U_i)$ on $f_i^k(U_i)$ following the leaves of $\overline{\mathcal{F}}_k^i$.

But for $s < k$ the vector fields $\frac{\partial}{\partial y_s^i}$ are tangent to $f_i^k(U_i)$ at any point of $f_i^k(U_i)$ so for $s < k$ and $a \in f_i^q(U^q)$ we have $\left(\theta_i^{qk}\right)_{*a} \left(\frac{\partial}{\partial y_s^i}\right) = \frac{\partial}{\partial y_s^i}\left(\theta_i^{qk}(a)\right)$.

Therefore for $s < k$ the vector fields $\frac{\partial}{\partial y_s^i}$ define a coordinate system $\overline{\mathcal{F}}_k^i$-transverse on $f_i^k(U_i)$ and this coordinates system is the restriction of $\left(y_{q-1}^i, ..., y_0^i\right)$ to $f_i^k(U_i)$. It will be noted therefore $\left(y_{k-1}^i, ..., y_0^i\right)$.

We can write that $\left(\theta_i^{qk}\right)\left(y_{q-1}^i, ..., y_0^i\right) = \left(y_{k-1}^i, ..., y_0^i\right)$.

Using the diagram $(*)$ it is easy to verify that $\theta_i^{q(k-1)} = \theta_i^{k-1} \circ \theta_i^{qk}$. Which causes that

$$\begin{aligned} \theta_i^{k-1}\left(y_{k-1}^i, y_{k-2}^i, ..., y_0^i\right) &= \left(\theta_i^{k-1} \circ \theta_i^{qk}\right)\left(y_{q-1}^i, ..., y_0^i\right) \\ &= \theta_i^{q(k-1)}\left(y_{q-1}^i, ..., y_0^i\right) \\ &= \left(y_{k-2}^i, ..., y_0^i\right) . \end{aligned}$$

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages"*. Thèse unique, Université de Cocody, Abidjan.

For $U_i \cap U_j \neq \emptyset$, we set:

$$\gamma^k_{ij}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^{ki}_{k-1}, \gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right).$$

We have [3] the equality,

$$f_i^{k-1} = \theta_i^{k-1} \circ f_i^k \text{ and } \gamma^{k-1}_{ij} \circ \theta_j^{k-1} = \theta_i^{k-1} \circ \gamma^k_{ij}$$

that is to say that the following diagram is commutative

$$\begin{array}{ccccc} U_i \cap U_j & \overset{f_j^k}{\rightarrow} & f_j^k(U_i \cap U_j) & \overset{\theta_j^{k-1}}{\rightarrow} & f_j^{k-1}(U_i \cap U_j) \\ {}^{Id_{U_i \cap U_j}}\downarrow & & \downarrow \gamma^k_{ij} & & \downarrow \gamma^{k-1}_{ij} \\ U_i \cap U_j & \overset{f_i^k}{\rightarrow} & f_i^k(U_i \cap U_j) & \overset{\theta_i^{k-1}}{\rightarrow} & f_i^{k-1}(U_i \cap U_j) \end{array}.$$

From where the equality $\theta_i^{k-1}\left(y^i_{k-1}, y^i_{k-2}, ..., y^i_0\right) = \left(y^i_{k-2}, ..., y^i_0\right)$ causes that:

$$\begin{aligned} \gamma^{k-1}_{ij}\left(y^j_{k-2}, ..., y^j_0\right) &= \gamma^{k-1}_{ij} \circ \theta_j^{k-1}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) \\ &= \theta_i^{k-1} \circ \gamma^k_{ij}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) \\ &= \theta_i^{k-1}\left(\gamma^{ki}_{k-1}, \gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right) \\ &= \left(\gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right) \end{aligned}$$

but

$$\gamma^{k-1}_{ij}\left(y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^{(k-1)i}_{k-2}, ..., \gamma^{(k-1)i}_0\right) \text{ so } \left(\gamma^{(k-1)i}_{k-2}, ..., \gamma^{(k-1)i}_0\right) = \left(\gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right).$$

Which implies $\gamma^{si}_r = \gamma^{ti}_r$ for all r, s and t.

We can set

$$\gamma^k_{ij}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^i_{k-1}, \gamma^i_{k-2}, ..., \gamma^i_0\right).$$

The equality

$$\gamma^{k-1}_{ij}\left(y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right) = \left(\gamma^i_{k-1}, \gamma^i_{k-2}, ..., \gamma^i_0\right)$$

show that

$$\gamma^k_{ij}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^i_{k-1}, \gamma^i_{k-2}, ..., \gamma^i_0\right)$$

with

$$\left(\gamma^i_{k-s-1}, \gamma^i_{k-s-2}, ..., \gamma^i_0\right) = \gamma^{k-s}_{ij}\left(y^j_{k-s-1}, ..., y^j_0\right) \text{ for } s \in \{1, ..., k-1\}.$$

It follows from the foregoing that the Jacobian matrix $J_{\gamma^k_{ij}}$ of γ^k_{ij} satisfies the equality:

$$J_{\gamma^k_{ij}} = \begin{pmatrix} \frac{\partial \gamma^i_{k-1}}{\partial y^j_{k-1}} & \frac{\partial \gamma^i_{k-1}}{\partial y^j_{k-2}} & ... & ... & ... & \frac{\partial \gamma^i_{k-1}}{\partial y^j_0} \\ 0 & \frac{\partial \gamma^i_{k-2}}{\partial y^j_{k-2}} & \frac{\partial \gamma^i_{k-2}}{\partial y^j_{k-3}} & ... & ... & \frac{\partial \gamma^i_{k-2}}{\partial y^j_0} \\ 0 & 0 & \frac{\partial \gamma^i_{k-3}}{\partial y^j_{k-3}} & ... & ... & \frac{\partial \gamma^i_{k-3}}{\partial y^j_0} \\ \vdots & \vdots & 0 & . & & \vdots \\ \vdots & \vdots & \vdots & . & . & \vdots \\ 0 & 0 & 0 & .... & 0 & \frac{\partial \gamma^i_0}{\partial y^j_0} \end{pmatrix}.$$

Therefore

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages"*. Thèse unique, Université de Cocody, Abidjan.

$$\left(\gamma_{ij}^k\right)_*\left(\frac{\partial}{\partial y_{k-r}^j}\right)=\sum_{t=1}^r\frac{\partial\gamma_{k-t}^i}{\partial y_{k-r}^j}.\frac{\partial}{\partial y_{k-t}^i}.$$

The matrix $J_{\gamma_{ij}^k}$ is invertible and triangular. From where all diagonal element of $J_{\gamma_{ij}^k}$ is nonzero that is to say that $\frac{\partial\gamma_{k-r}^i}{\partial y_{k-r}^j}\neq 0$ for all r.

As γ_{ij}^k is an isometric for the transverse metric h_T and as the vector fields $\frac{\partial}{\partial y_{k-1}^j}$ are orthogonal two by two then

$$\begin{aligned}0 &= h_T\left(\frac{\partial}{\partial y_{k-1}^j},\frac{\partial}{\partial y_{k-2}^j}\right)\\ &= h_T\left(\left(\gamma_{ij}^k\right)_*\frac{\partial}{\partial y_{k-1}^j}\right),\left(\gamma_{ij}^k\right)_*\frac{\partial}{\partial y_{k-2}^j}\right)\right)\\ &= \frac{\partial\gamma_{k-1}^i}{\partial y_{k-2}^j}.\frac{\partial\gamma_{k-1}^i}{\partial y_{k-1}^j}h_T\left(\frac{\partial}{\partial y_{k-1}^i},\frac{\partial}{\partial y_{k-1}^i}\right).\end{aligned}$$

But

$$\frac{\partial\gamma_{k-1}^i}{\partial y_{k-1}^j}\neq 0 \text{ and } h_T\left(\frac{\partial}{\partial y_{k-1}^i},\frac{\partial}{\partial y_{k-1}^i}\right)\neq 0 \text{ so } \frac{\partial\gamma_{k-1}^i}{\partial y_{k-2}^j}=0.$$

Let $r_0\in\{1,2,...,k\}$. Suppose by recurrence that for all $r\leq r_0$ and all $s<r$, $\frac{\partial\gamma_{k-s}^i}{\partial y_{k-r}^j}=0$.

We have for all $s<r_0+1$

$$\begin{aligned}0 &= h_T\left(\frac{\partial}{\partial y_{k-s}^j},\frac{\partial}{\partial y_{k-(r_0+1)}^j}\right)\\ &= h_T\left(\left(\gamma_{ij}^k\right)_*\frac{\partial}{\partial y_{k-s}^j}\right),\left(\gamma_{ij}^k\right)_*\left(\frac{\partial}{\partial y_{k-(r_0+1)}^j}\right)\right)\\ &= h_T\left(\frac{\partial\gamma_{k-s}^i}{\partial y_{k-s}^j}.\frac{\partial}{\partial y_{k-s}^i},\sum_{t=1}^{r_0+1}\frac{\partial\gamma_{k-t}^i}{\partial y_{k-(r_0+1)}^j}.\frac{\partial}{\partial y_{k-t}^i}\right)\\ &= \frac{\partial\gamma_{k-s}^i}{\partial y_{k-(r_0+1)}^j}.\frac{\partial\gamma_{k-s}^i}{\partial y_{k-s}^j}.h\left(\frac{\partial}{\partial y_{k-s}^i},\frac{\partial}{\partial y_{k-s}^i}\right).\end{aligned}$$

But

$$\frac{\partial\gamma_{k-s}^i}{\partial y_{k-s}^j}\neq 0 \text{ and } h_T\left(\frac{\partial}{\partial y_{k-s}^i},\frac{\partial}{\partial y_{k-s}^i}\right)\neq 0$$

so

$$\frac{\partial\gamma_{k-s}^i}{\partial y_{k-(r_0+1)}^j}=0 \text{ for all } s<r_0+1.$$

Notes

In conclusion

$$\frac{\partial \gamma^i_{k-s}}{\partial y^j_{k-r}} = 0 \text{ for all } s < r \text{ and } \frac{\partial \gamma^i_{k-r}}{\partial y^j_{k-r}} \neq 0 \text{ for all } r.$$

It follows from the foregoing that

$$J_{\gamma^k_{ij}} = \begin{pmatrix} \frac{\partial \gamma^i_{k-1}}{\partial y^j_{k-1}} & 0 & \dots & \dots & \dots & 0 \\ 0 & \frac{\partial \gamma^i_{k-2}}{\partial y^j_{k-2}} & 0 & \dots & \dots & 0 \\ 0 & 0 & \frac{\partial \gamma^i_{k-3}}{\partial y^j_{k-3}} & 0 & \dots & : \\ : & : & 0 & . & . & : \\ : & : & : & . & . & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{\partial \gamma^i_0}{\partial y^j_0} \end{pmatrix} \text{ with } \frac{\partial \gamma^i_{k-r}}{\partial y^j_{k-r}} \neq 0 \text{ for all } r.$$

Thus $\mathcal{F}_k$ is Riemannian transversaly diagonal.

We will say that the covering of opens $(U_i)_{i\in I}$ of the manifold M and the local $\mathcal{F}_k-$transverse coordinates system $\left((y^i_{k-1}, ..., y^i_0)\right)_{i\in I}$ are compatible with the flag $\mathcal{D}_{\mathcal{F}_q}$.

We show in [4] that if $\mathcal{F}_q$ is a foliation with dense leaves then the local compatible $\mathcal{F}_k-$transverse coordinates system with $\mathcal{D}_{\mathcal{F}_q}$ is a global $\mathcal{F}_k-$transverse coordinates system on any $\mathcal{F}_k-$transverse manifold. And, in this case it is not necessary to specify in the previous theorem the fact that the $\mathcal{F}_q-$bundlelike metric h is bundlelike for each foliation $\mathcal{F}_k$ of $\mathcal{D}_{\mathcal{F}_q}$ for $k < q$.

We note that for any $\mathcal{F}_k-$transverse manifold T^k there exists k foliations $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$ and $\dim(\mathcal{H}_r) = 1$ for each r (just write $\mathcal{H}_{r+1} = \mathcal{F}_{Y_r}$ where $\mathcal{F}_{Y_r}$ is the flow of unitary field Y_r) such as:

i) $T\left(T^k\right) = \overset{k}{\underset{r=0}{\oplus}} T\mathcal{H}_r$ and each foliation $\mathcal{H}_r$ is invariant by the changers γ^k_{ij} of $\mathcal{F}_k-$transverse coordinates,

ii) the differential system $S^{(k,t)}(x) = \overset{k}{\underset{\substack{r=1\\ r\neq t}}{\oplus}} T_x\mathcal{H}_r$ is integrable.

We say that a n dimension manifold N is *almost produces p-type multi-foliate* if and only if there exists p foliations $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_p$ where $p \leq n$ such as $TN = \overset{k}{\underset{r=0}{\oplus}} T\mathcal{H}_r$ and the differential system $S^{(k,t)}(x) = \overset{k}{\underset{\substack{r=1\\ r\neq t}}{\oplus}} T_x\mathcal{H}_r$ is integrable.

A such manifold is locally diffeomorphic to $L_1 \times L_2 \times ... \times L_p$ where L_r is a plaque of $\mathcal{H}_r$ (*Cf. proof of theorem* 3.2).

That said, we now consider a codimension q foliation $\mathcal{F}$ having N^q for $\mathcal{F}-$transverse manifold.

The proofs of proposition 3.1 and of theorem 3.2 allow us to see that $\mathcal{F}$ is transversaly diagonal if and only if N^q is *almost produces n-type multi-foliate* and the foliations $\mathcal{H}_r$ allowing the decomposition of TN^q are invariant by the changers γ_{ij} of $\mathcal{F}-$transverse coordinates.

In the case where the $\mathcal{F}-$transverse manifold N^q is *almost produces p-type multi-foliate* with $p < n$, if the foliations $\mathcal{H}_r$ allowing the decomposition of TN^q are invariant by the changers γ_{ij} of $\mathcal{F}-$transverse coordinates then using Proposition 2.4 we can construct as in the proof of Theorem 3.2 a family of local $\mathcal{F}-$transverse coordinates system $\left((y^i_n, ..., y^i_1)\right)_{i\in I}$ on N^q and following this family we have

Ref

4. C. Dadi and A Codjia, 2O16. *"Riemannian foliation with dense leaves on a compact manifold"* International Journal of Mathematics and Computer Science, 11, no 2, paper accepted.

$$J_{\gamma_{ij}} = \begin{pmatrix} J_{ij}^1 & 0 & \dots & \dots & \dots & 0 \\ 0 & J_{ij}^2 & 0 & \dots & \dots & 0 \\ 0 & 0 & . & 0 & \dots & \vdots \\ \vdots & \vdots & 0 & . & . & \vdots \\ \vdots & \vdots & \vdots & . & . & 0 \\ 0 & 0 & 0 & \dots & 0 & J_{ij}^p \end{pmatrix}$$

where $J_{\gamma_{ij}}$ is the Jacobian matrix of γ_{ij} and J_{ij}^r is a square matrix of order n_r where $n_r = \dim(\mathcal{H}_r)$.

We say in this case that the foliation $\mathcal{F}$ is *transversaly diagonal by block.*

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Notes