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An Unified Study of Some Multiple Integrals with Multivariable Gimel-Function

By Frédéric Ayant

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ANUNIFIEDSTUDYOFSOMEMULTIPLEINTEGRALSWITHMULTIVARIABLEGIMELFUNCTION

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An Unified Study of Some Multiple Integrals with Multivariable Gimel-Function

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Abstract- In this paper, we first evaluate a unified and general finite multiple integral whose integrand involves the product of the functions ${}_PF_Q, S_N^M$ and the multivariable Gimel-function occurring in the integrand involve the product of factors of the form $x^{\rho-1}(a-x)^{\sigma}\left[1+(bx)^{l}\right]^{-\lambda}$ while that of ${}_PF_Q$ occurring herein involves a finite series of such finite series of such factors. On account of the most general nature of the functions occurring in the integrand of our main integral, a large number of new and known integrals can easily be obtained from ot merely by specializing the functions and parameters involved therein. At the end of this study, we illustrate a new integral whose integrand involves a product of the Jacobi polynomial, the Appell's function F_3 and the Bessel function J_v

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I. Introduction And Preliminaries

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers, and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\qquad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

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$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

For more details, see Ayant [2].

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r),\quad \Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$ to the right of the contour L_k, and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)|<\frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)}=\sum_{j=1}^{m^{(k)}}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n^{(k)}}C_j^{(k)}\gamma_j^{(k)}-\tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}}D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)}+\sum_{j=n^{(k)}+1}^{p_i^{(k)}}C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}}A_{2ji_2}\alpha_{2ji_2}^{(k)}+\sum_{j=1}^{q_{i_2}}B_{2ji_2}\beta_{2ji_2}^{(k)}\right)-\cdots-\tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}}A_{rji_r}\alpha_{rji_r}^{(k)}+\sum_{j=1}^{q_{i_r}}B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\to 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\to\infty \text{ where } i=1,\cdots,r:$$

Ref

2. F. Ayant, An expansion formula for multivariable Gimel-function involving generalized Legendre Associated function, International Journal of Trends and Technology (IJMTT), 56(4) (2018), 223-228.

$$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [6].

Remark 3.
If $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [5].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [11,12].

Srivastava ([7],p. 1, Eq. 1) has defined the general class of polynomials

$$S_N^M(x) = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K} x^K \tag{1.5}$$

On suitably specializing the coefficients $A_{N,K}$, $S_N^M(x)$ yields some known polynomials, these include the Jacobi polynomials, Laguerre polynomials, and others polynomials ([12],p. 158-161).

II. Main Integral

In this section, we evaluate a unified multiple finite integrals involving the multivariable Gimel-function with general arguments.

Theorem.

$$\int_0^{a_1} \cdots \int_0^{a_s} \prod_{j=1}^{s}\left[x_j^{\rho_j-1}(a_j-x_j)^{\sigma_j}\left[1+(b_jx_j)^{l_j}\right]^{-\lambda_j}\right] \prod_{k=1}^{t} S_{N_k}^{M_k}\left[Y_k \prod_{j=1}^{s}\left[x_j^{e_j^{(k)}}(a_j-x_j)^{f_j^{(k)}}\left[1+(b_jx_j)^{l_j}\right]^{-g_j^{(k)}}\right]\right]$$

$$\gimel\left[\begin{array}{c} \mathrm{z}_1 \prod_{j=1}^{s}\left[x_j^{u_j^{(1)}}(a_j-x_j)^{v_j^{(1)}}\left[1+(b_jx_j)^{l_j}\right]^{-w_j^{(1)}}\right] \\ \cdot \\ \cdot \\ \mathrm{z}_r \prod_{j=1}^{s}\left[x_j^{u_j^{(r)}}(a_j-x_j)^{v_j^{(r)}}\left[1+(b_jx_j)^{l_j}\right]^{-w_j^{(r)}}\right] \end{array}\right]$$

$${}_PF_Q\left[(E_P);(F_Q);\sum_{j=1}^{s} B_j\left[x_j^{\mu_i}(a_i-x_j)^{\upsilon_j}\left[1+(b_jx_j)^{l_j}\right]^{-\omega_j}\right]\right] \mathrm{d}x_1 \cdots \mathrm{d}x_r = \frac{\prod_{j=1}^{Q}\Gamma(F_j)}{\prod_{j=1}^{P}\Gamma(E_j)} \prod_{j=1}^{s} a_j^{\rho_j+\sigma_j}$$

Ref

1. F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

$$\prod_{k=1}^{t}\sum_{K_k=0}^{[N_k/M_k]}\frac{(-N_k)_{M_kK_k}A_{N_k,K_k}\left[Y_k\prod_{j=1}^{s}a_j^{e_j^{(k)}+f_j^{(k)}}\right]^{K_k}}{K_k!}$$

$$\gimel_{X;P+3s+n+p_{i_r},Q+2s+q+q_{i_r},\tau_{i_r}:R_r:Y}^{U;0,P+3s+n+n_r:V}\left(\begin{array}{c|c} \mathrm{z}_1\prod_{j=1}^{s}a_j^{u_j^{(1)}+v_j^{(1)}} & \mathbb{A};\mathrm{A}_s,\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\prod_{j=1}^{s}a_j^{u_j^{(r)}+v_j^{(r)}} & \cdot \\ \text{-}\mathrm{B}_1a_1^{\mu_1+\upsilon_1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \text{-}\mathrm{B}_ra_r^{\mu_r+\upsilon_r} & \cdot \\ (\mathrm{a}_1b_1)^{l_1} & \cdot \\ \cdot & \mathbb{B};\mathbf{B},\mathrm{B}_s:B;\underbrace{(0,1;1);\cdots;(0,1;1)}_{2s} \\ \cdot & \\ (\mathrm{a}_sb_s)^{l_s} & \end{array}\right) \quad (2.1)$$

where

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \quad (2.2)$$

$$A_s=(1-E_j;\underbrace{0,\cdots,0}_{r},\underbrace{1,\cdots,1}_{s},\underbrace{0,\cdots,0}_{s};1),\left(1-\rho_1-\sum_{k=1}^{t}e_1^{(k)}K_k;u_1^{(1)},\cdots,u_1^{(r)},u_1,\underbrace{0,\cdots,0}_{s-1},l_1,\underbrace{0,\cdots,0}_{s-1};1\right)$$

$$,\cdots,\left(1-\rho_s-\sum_{k=1}^{t}e_s^{(k)}K_k;u_s^{(1)},\cdots,u_s^{(r)},\underbrace{0,\cdots,0}_{s-1},\mu_s,\underbrace{0,\cdots,0}_{s-1},l_s;1\right),$$

$$\left(-\sigma_1-\sum_{k=1}^{t}f_1^{(k)}K_k;v_1^{(1)},\cdots,v_1^{(r)},\upsilon_1,\underbrace{0,\cdots,0}_{2s-1};1\right),\cdots,\left(-\sigma_s-\sum_{k=1}^{t}f_s^{(k)}K_k;v_s^{(1)},\cdots,v_s^{(r)},\underbrace{0,\cdots,0}_{s-1},\upsilon_s,\underbrace{0,\cdots,0}_{s-1};1\right)$$

$$\left(1-\lambda_1-\sum_{k=1}^{t}g_1^{(k)}K_k;w_1^{(1)},\cdots,w_1^{(r)},\omega_1,\underbrace{0,\cdots,0}_{s-1},1,\underbrace{0,\cdots,0}_{s-1};1\right),\cdots,$$

$$\left(1-\lambda_s-\sum_{k=1}^{t}g_s^{(k)}K_k;w_s^{(1)},\cdots,w_s^{(r)},\underbrace{0,\cdots,0}_{s-1},\omega_s,\underbrace{0,\cdots,0}_{s-1},1;1\right) \quad (2.3)$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)},\underbrace{0,\cdots,0}_{2s};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)},\underbrace{0,\cdots,0}_{2s};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \quad (2.4)$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

Notes

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{2.5}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{2.6}$$

Notes

$$\mathbf{B}=\ [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)},\underbrace{0,\cdots,0}_{2s};B_{rji_r})_{1,q_{i_r}}] \tag{2.7}$$

$$B_s=(1-F_j;\underbrace{0,\cdots,0}_{r},\underbrace{1,\cdots,1}_{s},\underbrace{0,\cdots,0}_{s};1)_{1,p},$$

$$\left(-\rho_1-\sigma_1-\sum_{k=1}^{t}(e_1^{(k)}+f_1^{(k)})K_k;u_1^{(1)}+v_1^{(1)},\cdots,u_1^{(r)}+v_1^{(r)},;\mu_1+\upsilon_1,\underbrace{0,\cdots,0}_{s-1},l_1,\underbrace{0,\cdots,0}_{s-1};1\right),\cdots,$$

$$\left(-\rho_s-\sigma_s-\sum_{k=1}^{t}(e_s^{(k)}+f_s^{(k)})K_k;u_s^{(1)}+v_s^{(1)},\cdots,u_s^{(r)}+v_s^{(r)},\underbrace{0,\cdots,0}_{s-1},\mu_s+\upsilon_s,\underbrace{0,\cdots,0}_{s-1},l_s;1\right)$$

$$\left(1-\lambda_1-\sum_{k=1}^{t}g_1^{(k)}K_k;w_1^{(1)},\cdots,w_1^{(r)},\omega_1,\underbrace{0,\cdots,0}_{2s-1};1\right),\cdots,$$

$$\left(1-\lambda_s-\sum_{k=1}^{t}g_s^{(k)}K_k;w_s^{(1)},\cdots,w_s^{(r)},\underbrace{0,\cdots,0}_{s-1},\omega_s,\underbrace{0,\cdots,0}_{s};1\right) \tag{2.8}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{2.9}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)};\underbrace{(1,0);\cdots;(1,0)}_{2s} \tag{2.10}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};$$

$$Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)};\underbrace{(0,1);\cdots;(0,1)}_{2s} \tag{2.11}$$

Provided

$$\lambda_j,e_j^{(k)},f_j^{(k)},g_j^{(k)},u_j^{(i)},v_j^{(i)},w_j^{(i)},\mu_j,\upsilon_i,\omega_j>0;(j=1,\cdots,s);(k=1,\cdots,t);(i=1,\cdots,r)$$

$$Re(\rho_k)+\sum_{i=1}^{r}u_k^{(i)}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0\,;Re(\sigma_k+1)+\sum_{i=1}^{r}v_k^{(i)}\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$\left|arg\left(z_i\prod_{j=1}^{s}\left[x_j^{u_j^{(i)}}(a_j-x_j)^{v_j^{(i)}}\left[1+(b_jx_j)^{l_j}\right]^{-w_j^{(i)}}\right]\right)\right|<\frac{1}{2}\left(A_i^{(k)}-\sum_{j=1}^{s}(u_j^{(i)}+v_j^{(i)}+w_j^{(i)})\right)\pi$$

where $A_i^{(k)}$ is defined by (1.4).

Proof

First, we replace the polynomials $S_N^M(x)$ in series with the help of (1.5). We interchange the orders of series and the $(x_1, \cdots, x_s)$-integrals. Next, we express the generalized hypergeometric function of one variable regarding of generalized Kampé de Fériet function of s variables with the help of ([9], p.39, Eq. 30), and express this Kampé de Fériet function regarding of H-function of s variables with the help of ([10], p.272, Eq. 4.7). Next, we express the H-function of s-variables and the Gimel-function regarding of their respective Mellin-Barnes multiple integrals contour. Now we change the order of the ζ_i, η_j and x_j-integrals $(i = 1, \cdots, r; j = 1, \cdots, s)$ which is permissible under the stated conditions. Finally, on evaluating the x_j-integrals , after algebraic manipulations; we obtain (say I)

$$I = \frac{\prod_{j=1}^{Q} \Gamma(F_j)}{\prod_{j=1}^{P} \Gamma(E_j)} \prod_{k=1}^{t} \sum_{K_k=0}^{[N_k/M_k]} \frac{(-N_k)_{M_k K_k} A_{N_k,K_k} Y_k^{K_k}}{K_k!} \frac{1}{(2\pi\omega)^{r+s}} \int_{L_1} \cdots \int_{L_r} \int_{L_{r+1}} \cdots \int_{L_{r+s}}$$

$$\psi(s_1, \cdots, s_r) \prod_{i=1}^{r} \theta_i(s_i) z_i^{\zeta_i} \frac{\prod_{j=1}^{P} \Gamma\left((E_j + \sum_{j=1}^{s} \eta_j\right)}{\prod_{j=1}^{Q} \Gamma\left((F_j + \sum_{j=1}^{s} \eta_j\right)} \prod_{j=1}^{s} [\Gamma(\eta_j)(-B_j)^{\eta_j}$$

$$a_j^{\left(\rho_j + \sigma_j + \sum_{k=1}^{t} (e_j^{(k)} + f_j^{(k)} K_k + \sum_{i=1}^{r} [u_j^{(i)} + v_j^{(i)} \zeta_i + (\mu_i + \upsilon_i)\eta_i\right)} \frac{\Gamma\left(1 + \sigma_j + \sum_{k=1}^{t} f_j^{(k)} K_k + \sum_{i=1}^{r} v_j^{(i)} \zeta_i + \upsilon_i \eta_j\right)}{\Gamma\left(\lambda_j + + \sum_{k=1}^{t} g_j^{(k)} K_k + \sum_{i=1}^{r} v w_j^{(i)} \zeta_i + \omega_i \eta_j\right)}$$

$$H_2^1 \left[(\mathrm{b}_j a_j)^l \left| \begin{array}{c} (1\text{-}\lambda_j - \sum_{k=1}^{t} g_j^{(k)} k_k + \sum_{i=1}^{r} v_j^{(i)} \zeta_i - \omega_j \eta_j, 1; 1), (1 - \rho_j - \sum_{k=1}^{t} e_j^{(k)} K_k + \sum_{i=1}^{r} u_j^{(i)} \zeta_i - \mu_j \eta_j, 1; 1) \\ . \\ (0,1;1), (-\rho_j - \sigma_j - \sum_{k=1}^{t} (e_j^{(k)} + f_j^{(k)}) K_k - \sum_{i=1}^{r} (u_j^{i)} + v_j^{(i)}) \zeta_i - (\mu_j - \upsilon_j) \eta_j, l_j; 1) \end{array} \right. \right] \quad (2.12)$$

Now we express the s-product of the H-functions of one variable occurring in the above expression regarding of their respectives Mellin-Barnes integral contour and we interpret the resulting multiple integrals contour with the help of (1.1) in term of the Gimel-function of $(r + 2s)$-variables, after algebraic manipulations, we obtain the theorem.

III. Special Cases

We consider the particular case studied by Gupta and Jain ([4], 9. 79-80, Eq. (3.1)) Taking $s = r = 2, l_1 = l_2 = t = 1$ in the result (2.1) and further reduce the general polynomial S_N^M into Jacobi polynomials $P_n^{(\alpha,\beta)}(1 - 2x)$, the Gimel-function of two variables into Appell's function F_3 and the generalized hypergeometric function ${}_PF_Q$ into the Bessel's function J_υ with the help of following results ([13], p. 159, Eq. (1.6)), ([8] p.18, Eq. (2.6.3), (2.6.5)), we obtain the following double integral after algebraic manipulations :

$$\int_0^{a_1} \int_0^{a_2} \prod_{j=1}^{2} \left[x_j^{\rho_j - 1} (a_j - x_j)^{\sigma_j} (1 + b_j x_j)^{-\lambda_j} \right] P_n^{(\alpha,\beta)} (1 - 2y x_1^{e_1} x_2^{e_2}) F_3(k_1, k_2, h_1, h_2; L; z_1 x_1^{u_1}, z_2 x_2^{u_2})$$

$$[B_1 x_1^{\mu_1} + B_2 x_2^{m^{u_2}}]^{-\frac{\upsilon}{2}} J_\upsilon \left[2\sqrt{B_1 x_1^{\mu_1} + B_2 x_2^{\mu_2}} \right] \mathrm{d}x_1 \mathrm{d}x_2 = \frac{\Gamma(L)\Gamma(1 + \sigma_1)\Gamma(1 + \sigma_2) a_1^{\rho_1 + \sigma_1} a_2^{\rho_2 + \sigma_2}}{\Gamma(k_1)\Gamma(k_2)\Gamma(h_1)\Gamma_(h_2)\Gamma(\lambda_1)\Gamma(\lambda_2)}$$

$$\sum_{R=0}^{n} \frac{(-n)_R \binom{\alpha+n}{n} (\alpha + \beta + n + 1)_R (y a_1^{e_1} a_2^{e_2})^R}{R! (\alpha + 1)_R} H^{0,2:1,2;1,2;1,0;1,0;1,1;1,1}_{2,4:2,1;2,1;0,1;0,1;1,1;1,1}$$

Notes

$$\left(\begin{array}{c|l} \text{-z}_1 a_1^{u_1} & \\ \text{-z}_2 a_2^{u_2} & (1\text{-}\rho_1 - e_1 R; u_1, 0, \mu_1, 0, 1, 0), (1-\rho_2 - e_2 R; 0, u_2, 0, \mu_2, 0, 1) : (1-k_1, 1), (1-k_2, 1); \\ \text{B}_1 a_1^{\mu_1} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \text{B}_2 a_2^{\mu_2} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\ \text{a}_1 b_1 & (\text{-}\upsilon; 0, 0, 1, 1, 0, 0), (-\rho_1 - \sigma_1 - e_1 R; u_1, 0, \mu_1, 0, 1, 0), (-\rho_2 - \sigma_2 - e_2 R; 0, u_2, 0, \mu_2, 0, 1), \\ \text{a}_2 b_2 & \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-h}_1,), (1-h_2, 1); -; -; (1-\lambda_1, 1); (1-\lambda_2, 1) \\ . \\ . \\ (1\text{-L};1,1,0,0,0,0):(0,1)\ ;\ (0,1);\ (0,1),(0,1);(0,1);(0,1) \end{array}\right) \tag{3.1}$$

The validity conditions mentioned above are verified.

Remarks :

We obtain easily the same relations with the functions defined in section 1.

Gupta and Jain. [4] have obtained the same relations about the multivariable H-function.

IV. Conclusion

The importance of our results lies in their manifold generality. Firstly, given of unified multiple integrals with general classes of polynomials, generalized hypergeometric function with general arguments utilized in this study, we can obtain a large variety of single, double or multiple simpler integrals specializing the coefficients and the parameters in these functions. Secondly by specializing the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving a remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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